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Asymptotics of H -identities for associative algebras with an H -invariant radical[☆]

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ABSTRACT

We prove the existence of the Hopf PI-exponent for finite dimensional associative algebras A with a generalized Hopf action of an associative algebra H with 1 over an algebraically closed field of characteristic 0 assuming only the invariance of the Jacobson radical $J(A)$ under the H -action and the existence of the decomposition of $A/J(A)$ into the sum of H -simple algebras. As a consequence, we show that the analog of Amitsur's conjecture holds for G -codimensions of finite dimensional associative algebras over a field of characteristic 0 with an action of an arbitrary group G by automorphisms and anti-automorphisms and for differential codimensions of finite dimensional associative algebras with an action of an arbitrary Lie algebra by derivations.

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1. Introduction

Amitsur's conjecture on asymptotic behaviour of codimensions of ordinary polynomial identities was proved by A. Giambruno and M.V. Zaicev [10, Theorem 6.5.2] in 1999.

When an algebra is endowed with a grading, an action of a group G by automorphisms and anti-automorphisms, an action of a Lie algebra by derivations or a structure of an H -module algebra for some Hopf algebra H , it is natural to consider, respectively, graded, G -, differential or

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H -identities [4–6,14]. The analog of Amitsur's conjecture for finite dimensional associative algebras with a \mathbb{Z}_2 -action was proved by A. Giambruno and M.V. Zaicev [10, Theorem 10.8.4] in 1999. In 2010–2011, E. Aljadeff, A. Giambruno, and D. La Mattina [1,2,9] obtained the validity of the analog of Amitsur's conjecture for associative PI-algebras with an action of a finite Abelian group by automorphisms as a particular case of their result for graded algebras.

In 2012, the analog of the conjecture was proved [12,13] for finite dimensional associative algebras with a rational action of a reductive affine algebraic group by automorphisms and anti-automorphisms, with an action of a finite dimensional semisimple Lie algebra by derivations or an action of a semisimple Hopf algebra. These results were obtained as a consequence of [12, Theorem 5] and [13, Theorem 6], where the authors considered finite dimensional associative algebras with a generalized Hopf action of an associative algebra H with 1. In the proof, they required the existence of an H -invariant Wedderburn–Mal'cev and Wedderburn–Artin decompositions. Here we remove the first restriction. This enables us to prove the analog of Amitsur's conjecture for G -codimensions of finite dimensional associative algebras with an action of an arbitrary group G by automorphisms and anti-automorphisms and for differential codimensions of finite dimensional associative algebras with an action of an arbitrary Lie algebra by derivations.

2. Polynomial H -identities and their codimensions

Let H be a Hopf algebra over a field F . An algebra A over F is an H -module algebra or an algebra with an H -action, if A is endowed with a homomorphism $H \rightarrow \text{End}_F(A)$ such that $h(ab) = (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$, $a, b \in A$. Here we use Sweedler's notation $\Delta h = h_{(1)} \otimes h_{(2)}$ where Δ is the comultiplication in H .

In order to embrace an action of a group by anti-automorphisms, we consider a generalized Hopf action [6, Section 3].

Let H be an associative algebra with 1 over F . We say that an associative algebra A is an algebra with a generalized H -action if A is endowed with a homomorphism $H \rightarrow \text{End}_F(A)$ and for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that

$$h(ab) = \sum_i ((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a)) \quad \text{for all } a, b \in A. \quad (1)$$

Choose a basis $(\gamma_\beta)_{\beta \in \Lambda}$ in H and denote by $F\langle X|H \rangle$ the free associative algebra over F with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$. Let $x_i^h := \sum_{\beta \in \Lambda} \alpha_\beta x_i^{\gamma_\beta}$ for $h = \sum_{\beta \in \Lambda} \alpha_\beta \gamma_\beta$, $\alpha_\beta \in F$, where only finite number of α_β are nonzero. Here $X := \{x_1, x_2, x_3, \dots\}$, $x_j := x_j^1$, $1 \in H$. We refer to the elements of $F\langle X|H \rangle$ as H -polynomials. Note that here we do not consider any H -action on $F\langle X|H \rangle$.

Let A be an associative algebra with a generalized H -action. Any map $\psi : X \rightarrow A$ has a unique homomorphic extension $\bar{\psi} : F\langle X|H \rangle \rightarrow A$ such that $\bar{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H -polynomial $f \in F\langle X|H \rangle$ is an H -identity of A if $\bar{\psi}(f) = 0$ for all maps $\psi : X \rightarrow A$. In other words, $f(x_1, x_2, \dots, x_n)$ is an H -identity of A if and only if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_i \in A$. In this case we write $f \equiv 0$. The set $\text{Id}^H(A)$ of all H -identities of A is an ideal of $F\langle X|H \rangle$. Note that our definition of $F\langle X|H \rangle$ depends on the choice of the basis $(\gamma_\beta)_{\beta \in \Lambda}$ in H . However such algebras can be identified in the natural way, and $\text{Id}^H(A)$ is the same.

Denote by P_n^H the space of all multilinear H -polynomials in x_1, \dots, x_n , $n \in \mathbb{N}$, i.e.

$$P_n^H = \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \dots x_{\sigma(n)}^{h_n} \mid h_i \in H, \sigma \in S_n \rangle_F \subset F\langle X|H \rangle.$$

Then the number $c_n^H(A) := \dim(\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)})$ is called the n th codimension of polynomial H -identities or the n th H -codimension of A .

The analog of Amitsur's conjecture for H -codimensions of A can be formulated as follows.

Conjecture. There exists $\text{Plexp}^H(A) := \lim_{n \rightarrow \infty} \sqrt[n]{c_n^H(A)} \in \mathbb{Z}_+$.

We call $\text{Plexp}^H(A)$ the *Hopf PI-exponent* of A .

Example 1. Every algebra A is an H -module algebra for $H = F$. In this case the H -action is trivial and we get ordinary polynomial identities and their codimensions.

Example 2. Let A be an associative algebra with an action of a group G by automorphisms and anti-automorphisms. Then A is an algebra with a generalized H -action where $H = FG$. We introduce the *free G -algebra* $F\langle X|G \rangle := F\langle X|H \rangle$, the ideal of polynomial G -identities $\text{Id}^G(A) := \text{Id}^H(A)$, and G -codimensions $c_n^G(A) := c_n^H(A)$.

Example 3. If $H = U(\mathfrak{g})$ where $U(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra \mathfrak{g} , then an H -module algebra is an algebra with a \mathfrak{g} -action by derivations. The corresponding H -identities are called *differential identities* or *polynomial identities with derivations*.

Theorem 1. Let A be a finite dimensional non-nilpotent associative algebra with a generalized H -action where H is an associative algebra with 1 over an algebraically closed field F of characteristic 0. Suppose that the Jacobson radical $J := J(A)$ is an H -submodule. Let

$$A/J = B_1 \oplus \cdots \oplus B_q \quad (\text{direct sum of } H\text{-invariant ideals})$$

where B_i are H -simple algebras and let $\varkappa : A/J \rightarrow A$ be any homomorphism of algebras (not necessarily H -linear) such that $\pi \varkappa = \text{id}_{A/J}$ where $\pi : A \rightarrow A/J$ is the natural projection. Then there exist constants $C_1, C_2 > 0, r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^H(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}$$

where

$$d = \max(B_{i_1} \oplus B_{i_2} \oplus \cdots \oplus B_{i_r} \mid r \geq 1, \\ (H\varkappa(B_{i_1}))A^+ (H\varkappa(B_{i_2}))A^+ \cdots (H\varkappa(B_{i_{r-1}}))A^+ (H\varkappa(B_{i_r})) \neq 0) \quad (2)$$

and $A^+ := A + F \cdot 1$.

Remark. If A is nilpotent, i.e. $x_1 x_2 \cdots x_p \equiv 0$ for some $p \in \mathbb{N}$, then $P_n^H \subseteq \text{Id}^H(A)$ and $c_n^H(A) = 0$ for all $n \geq p$.

Corollary. The analog of Amitsur's conjecture holds for such codimensions.

Remark. The existence of the map \varkappa follows from the ordinary Wedderburn–Mal'cev theorem.

Theorem 1 will be proved in Sections 4 and 5.

3. Applications

Here we list some important corollaries from Theorem 1.

Theorem 2. Let A be a finite dimensional non-nilpotent associative H -module algebra for a Hopf algebra H over a field F of characteristic 0. Suppose that the antipode of H is bijective and the Jacobson radical $J(A)$ is an H -submodule. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^H(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let $K \supset F$ be an extension of the field F . Then

$$(A \otimes_F K)/(J \otimes_F K) \cong (A/J) \otimes_F K$$

is again a semisimple algebra and $J \otimes_F K$ is nilpotent and $H \otimes_F K$ -invariant.

Now we notice that H -codimensions do not change upon an extension of the base field. The proof is analogous to the case of ordinary codimensions [10, Theorem 4.1.9]. Hence we may assume F to be algebraically closed. By [13, Lemma 1], $A/J = B_1 \oplus \cdots \oplus B_q$ (direct sum of H -invariant ideals) for some H -simple algebras B_i . Now we apply Theorem 1. \square

Theorem 3. Let A be a finite dimensional non-nilpotent associative algebra over a field F of characteristic 0 with an action of a Lie algebra \mathfrak{g} by derivations. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^{U(\mathfrak{g})}(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.$$

Proof. By [7, Lemma 3.2.2], the Jacobson radical (which coincides with the prime radical) of a finite dimensional associative algebra is invariant under all derivations. Hence we may apply Theorem 2. \square

Theorem 4. Let A be a finite dimensional non-nilpotent associative algebra over a field F of characteristic 0 with an action of a group G by automorphisms and anti-automorphisms. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} d^n \leq c_n^G(A) \leq C_2 n^{r_2} d^n \quad \text{for all } n \in \mathbb{N}.$$

Proof. Again, G -codimensions do not change upon an extension of the base field. Hence we may assume F to be algebraically closed. The radical is invariant under all automorphisms and anti-automorphisms. Now we apply [13, Lemma 2] and Theorem 1. \square

The algebra in the example below has no G -invariant Wedderburn–Mal'cev decomposition, however it satisfies the analog of Amitsur's conjecture.

Example 4 (Yuri Bahturin). Let F be a field of characteristic 0 and let

$$A = \left\{ \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \mid C, D \in M_m(F) \right\} \subseteq M_{2m}(F),$$

$m \geq 2$. Consider $\varphi \in \text{Aut}(A)$ where

$$\varphi \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C & C+D \\ 0 & 0 \end{pmatrix}.$$

Then A is an algebra with an action of the group $G = \langle \varphi \rangle \cong \mathbb{Z}$ by automorphisms. There is no G -invariant Wedderburn–Mal'cev decomposition for A , however there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} m^{2n} \leq c_n^G(A) \leq C_2 n^{r_2} m^{2n} \quad \text{for all } n \in \mathbb{N}.$$

Proof. Note that

$$J(A) = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid D \in M_m(F) \right\}. \quad (3)$$

Suppose $A = B \oplus J(A)$ (direct sum of G -invariant subspaces) for some maximal semisimple subalgebra B . Since $\varphi(a) - a \in J(A)$ for all $a \in A$, we have $\varphi(a) = a$ for all $a \in B$. Thus $B \subseteq J(A)$ and we get a contradiction. Therefore, there is no G -invariant Wedderburn–Mal'cev decomposition for A .

Again, G -codimensions do not change upon an extension of the base field. Moreover, upon an extension of F , A remains an algebra of the same type. Thus without loss of generality we may assume F to be algebraically closed.

Note that $A/J \cong M_m(F)$ is a simple algebra. Hence $\text{Plexp}^G(A) = \dim M_m(F) = m^2$ by Theorems 1 and 4. \square

Example 5. Let A be the associative algebra from Example 4. Denote by \mathfrak{g} the corresponding Lie algebra with the commutator $[x, y] = xy - yx$ and consider the adjoint action of \mathfrak{g} on A by derivations. Then there is no \mathfrak{g} -invariant Wedderburn–Mal'cev decomposition for A , however there exist constants $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that

$$C_1 n^{r_1} m^{2n} \leq c_n^{U(\mathfrak{g})}(A) \leq C_2 n^{r_2} m^{2n} \quad \text{for all } n \in \mathbb{N}.$$

Proof. Suppose $A = B \oplus J(A)$ (direct sum of \mathfrak{g} -submodules) for some maximal semisimple associative subalgebra B . Then B is a Lie ideal of \mathfrak{g} . By (3), $J(A)$ is Abelian as a Lie algebra. Thus the center of \mathfrak{g} contains $J(A)$, which is not true. We get a contradiction. Hence there is no \mathfrak{g} -invariant Wedderburn–Mal'cev decomposition for A .

Again, without loss of generality we may assume F to be algebraically closed. Since $A/J \cong M_m(F)$ is a simple algebra, $\text{Plexp}^{U(\mathfrak{g})}(A) = \dim M_m(F) = m^2$ by Theorems 1 and 3. \square

Remark. The radical of Sweedler's algebra with an action of its dual is not H -invariant, however the analog of Amitsur's conjecture holds for its H -identities [12, Section 7.4].

4. S_n -cocharacters and upper bound

One of the main tools in the investigation of polynomial identities is provided by the representation theory of symmetric groups.

Let A be an associative algebra with a generalized H -action where H is an associative algebra with 1 over a field F of characteristic 0. The symmetric group S_n acts on the spaces $\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}$ by permuting the variables. Irreducible FS_n -modules are described by partitions $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ and their Young diagrams D_λ . The character $\chi_n^H(A)$ of the FS_n -module $\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}$ is called the n th cocharacter of polynomial H -identities of A . We can rewrite $\chi_n^H(A)$ as a sum

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m(A, H, \lambda) \chi(\lambda)$$

of irreducible characters $\chi(\lambda)$. Let $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} (\text{sign } \sigma) \sigma$, be Young symmetrizers corresponding to a Young tableau T_λ . Then $M(\lambda) = FSe_{T_\lambda} \cong FSe_{T_\lambda}^*$ is an irreducible FS_n -module corresponding to a partition $\lambda \vdash n$. We refer the reader to [3,8,10] for an account of S_n -representations and their applications to polynomial identities.

Theorem 5 below is a generalization of [6, Theorem 13 (b)] and the remark after [6, Theorem 14].

Theorem 5. *Let A be a finite dimensional associative algebra with a generalized H -action where H is an associative algebra with 1 over a field F of characteristic 0. Then there exist constants $C_3 > 0, r_3 \in \mathbb{N}$ such that*

$$\sum_{\lambda \vdash n} m(A, H, \lambda) \leq C_3 n^{r_3} \quad \text{for all } n \in \mathbb{N}.$$

Remark. Cocharacters do not change upon an extension of the base field F (the proof is completely analogous to [10, Theorem 4.1.9]), so we may assume F to be algebraically closed.

Proof of Theorem 5. Consider ordinary polynomial identities and cocharacters of A . We may define them as H -identities and H -cocharacters for $H = F$: $P_n := P_n^F$, $\chi_n(A) := \chi_n^F(A)$, $m(A, \lambda) := m(A, F, \lambda)$, $\text{Id}(A) := \text{Id}^F(A)$. By the Berele–Regev theorem (see e.g. [10, Theorem 4.9.3]),

$$\sum_{\lambda \vdash n} m(A, \lambda) \leq C_4 n^{r_4} \quad (4)$$

for some $C_4 > 0$ and $r_4 \in \mathbb{N}$.

Let $G_1 \subseteq G_2$ be finite groups and let W be an FG_2 -module. Denote by $W \downarrow_{G_1}$ the module W with the G_2 -action restricted to G_1 .

Let $\zeta : H \rightarrow \text{End}_F(A)$ be the homomorphism corresponding to the H -action, and let $(\zeta(\gamma_j))_{j=1}^m$, $\gamma_j \in H$, be a basis in $\zeta(H)$.

Consider the diagonal embedding $\varphi : S_n \rightarrow S_{mn}$,

$$\varphi(\sigma) := \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{array} \middle| \begin{array}{cccc} n+1 & n+2 & \dots & 2n \\ n+\sigma(1) & n+\sigma(2) & \dots & n+\sigma(n) \end{array} \middle| \dots \right)$$

and the S_n -homomorphism $\pi : (P_{mn} \downarrow \varphi(S_n)) \rightarrow P_n^H$ defined by $\pi(x_{n(i-1)+t}) = x_t^{\gamma_i}$, $1 \leq i \leq m$, $1 \leq t \leq n$. Note that $\pi(P_{mn} \cap \text{Id}(A)) \subseteq P_n^H \cap \text{Id}^H(A)$ and $x^h - \sum_{j=1}^m \alpha_j x^{\gamma_j} \in \text{Id}^H(A)$ for all $h \in H$ and $\alpha_j \in F$ such that $\zeta(h) = \sum_{j=1}^m \alpha_j \zeta(\gamma_j)$. Hence the FS_n -module $\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}$ is a homomorphic image of the FS_n -module $(\frac{P_{mn}}{P_{mn} \cap \text{Id}(A)}) \downarrow \varphi(S_n)$. Denote by $\text{length}(M)$ the number of irreducible components of a module M . Then

$$\sum_{\lambda \vdash n} m(A, H, \lambda) = \text{length}\left(\frac{P_n^H}{P_n^H \cap \text{Id}^H(A)}\right) \leq \text{length}\left(\left(\frac{P_{mn}}{P_{mn} \cap \text{Id}(A)}\right) \downarrow \varphi(S_n)\right).$$

Therefore, it is sufficient to prove that $\text{length}((\frac{P_{mn}}{P_{mn} \cap \text{Id}(A)}) \downarrow \varphi(S_n))$ is polynomially bounded. Replacing $|G|$ with m in [11, Lemmas 10 and 12] (or, alternatively, using the proof of [6, Theorem 13 (b)]), we derive this from (4) and [10, Theorem 4.6.2]. \square

In the next two lemmas we consider a finite dimensional associative algebra with a generalized H -action having an H -invariant nilpotent ideal J where H is an associative algebra with 1 over a field F of characteristic 0 and $J^p = 0$ for some $p \in \mathbb{N}$. Fix a decomposition $A/J = B_1 \oplus \dots \oplus B_q$ where B_i are some subspaces. Let $\kappa : A/J \rightarrow A$ be an F -linear map such that $\pi \kappa = \text{id}_{A/J}$ where $\pi : A \rightarrow A/J$ is the natural projection. Define the number d by (2).

Lemma 1. Let $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$. Then if $\sum_{k=d+1}^s \lambda_k \geq p$, we have $m(A, H, \lambda) = 0$.

Proof. It is sufficient to prove that $e_{T_\lambda}^* f \in \text{Id}^H(A)$ for all $f \in P_n$ and for all Young tableaux T_λ corresponding to λ .

Fix a basis in A that is a union of bases of $\kappa(B_1), \dots, \kappa(B_q)$ and J . Since $e_{T_\lambda}^* f$ is multilinear, it is sufficient to prove that $e_{T_\lambda}^* f$ vanishes under all evaluations on basis elements. Fix some substitution of basis elements and choose $1 \leq i_1, \dots, i_r \leq q$ such that all the elements substituted belong to $\kappa(B_{i_1}) \oplus \dots \oplus \kappa(B_{i_r}) \oplus J$, and for each k we have an element being substituted from $\kappa(B_{i_k})$. Then we may assume that $\dim(B_{i_1} \oplus \dots \oplus B_{i_r}) \leq d$, since otherwise $e_{T_\lambda}^* f$ is zero by the definition of d . Note that $e_{T_\lambda}^* = b_{T_\lambda} a_{T_\lambda}$ and b_{T_λ} alternates the variables of each column of T_λ . Hence if $e_{T_\lambda}^* f$ does not vanish, this implies that different basis elements are substituted for the variables of each column. Therefore, at least $\sum_{k=d+1}^s \lambda_k \geq p$ elements must be taken from J . Since $J^p = 0$, we have $e_{T_\lambda}^* f \in \text{Id}^H(A)$. \square

Lemma 2. If $d > 0$, then there exist constants $C_2 > 0$, $r_2 \in \mathbb{R}$ such that $c_n^H(A) \leq C_2 n^{r_2} d^n$ for all $n \in \mathbb{N}$. In the case $d = 0$, the algebra A is nilpotent.

Proof. Lemma 1 and [10, Lemmas 6.2.4, 6.2.5] imply

$$\sum_{m(A, H, \lambda) \neq 0} \dim M(\lambda) \leq C_3 n^{r_3} d^n$$

for some constants $C_3, r_3 > 0$. Together with Theorem 5 this inequality yields the upper bound. \square

5. Lower bound

As usual, in order to prove the lower bound, it is sufficient to provide a polynomial alternating on sufficiently many sufficiently large sets of variables. Lemma 3 below is a generalization of [12, Lemma 10].

Lemma 3. Let A, J, κ, B_i , and d be the same as in Theorem 1. If $d > 0$, then there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist disjoint subsets $X_1, \dots, X_{2k} \subseteq \{x_1, \dots, x_n\}$, $k := \lfloor \frac{n-n_0}{2d} \rfloor$, $|X_1| = \dots = |X_{2k}| = d$ and a polynomial $f \in P_n^H \setminus \text{Id}^H(A)$ alternating in the variables of each set X_j .

Proof. Without loss of generality, we may assume that $d = \dim(B_1 \oplus B_2 \oplus \dots \oplus B_r)$ where $(H\kappa(B_1))A^+(H\kappa(B_2))A^+ \dots (H\kappa(B_{r-1}))A^+(H\kappa(B_r)) \neq 0$.

Since J is nilpotent, we can find maximal $\sum_{i=1}^r q_i$, $q_i \in \mathbb{Z}_+$, such that

$$\left(a_1 \prod_{i=1}^{q_1} j_{1i} \right) \gamma_1 \kappa(b_1) \left(a_2 \prod_{i=1}^{q_2} j_{2i} \right) \gamma_2 \kappa(b_2) \dots \left(a_r \prod_{i=1}^{q_r} j_{ri} \right) \gamma_r \kappa(b_r) \left(a_{r+1} \prod_{i=1}^{q_{r+1}} j_{r+1,i} \right) \neq 0$$

for some $j_{ik} \in J$, $a_i \in A^+$, $b_i \in B_i$, $\gamma_i \in H$. Let $j_i := a_i \prod_{k=1}^{q_i} j_{ik}$.
Then

$$j_1 \gamma_1 \kappa(b_1) j_2 \gamma_2 \kappa(b_2) \dots j_r \gamma_r \kappa(b_r) j_{r+1} \neq 0 \quad (5)$$

for some $b_i \in B_i$, $\gamma_i \in H$, however

$$j_1 \tilde{b}_1 j_2 \tilde{b}_2 \dots j_r \tilde{b}_r j_{r+1} = 0 \quad (6)$$

for all $\tilde{b}_i \in A^+(H\kappa(B_i))A^+$ such that $\tilde{b}_k \in J(H\kappa(B_i))A^+ + A^+(H\kappa(B_i))J$ for at least one k .

Recall that κ is a homomorphism of algebras. Moreover $\pi(h\kappa(a) - \kappa(ha)) = 0$ implies $h\kappa(a) - \kappa(ha) \in J$ for all $a \in A$ and $h \in H$. Hence, by (6), if we replace $\kappa(b_i)$ in the left-hand side of (5) with a product of $\kappa(b_i)$ and an expression involving κ , the map κ will behave like a homomorphism of H -modules. We will exploit this property further.

In virtue of [12, Theorem 7], there exist constants $m_t \in \mathbb{Z}_+$ such that for any k there exist multilinear polynomials

$$f_t = f_t(x_1^{(t,1)}, \dots, x_{d_1}^{(t,1)}; \dots; x_1^{(t,2k)}, \dots, x_{d_1}^{(t,2k)}; z_1^{(t)}, \dots, z_{m_t}^{(t)}; z_t) \in P_{2kd_t+m_t+1}^H$$

alternating in the variables from disjoint sets $X_\ell^{(t)} = \{x_1^{(t,\ell)}, x_2^{(t,\ell)}, \dots, x_{d_t}^{(t,\ell)}\}$, $1 \leq \ell \leq 2k$. There exist $\bar{z}_\alpha^{(t)} \in B_t$, $1 \leq \alpha \leq m_t$, such that

$$f_t(a_1^{(t)}, \dots, a_{d_t}^{(t)}; \dots; a_1^{(t)}, \dots, a_{d_t}^{(t)}; \bar{z}_1^{(t)}, \dots, \bar{z}_{m_t}^{(t)}; \bar{z}_t) = \bar{z}_t$$

for any $\bar{z}_t \in B_t$.

Let $n_0 = 2r + 1 + \sum_{i=1}^r m_i$, $k = \lfloor \frac{n-n_0}{2d} \rfloor$, $\tilde{k} = \lfloor \frac{(n-2kd-n_0)-m_1}{2d_1} \rfloor + 1$. We choose f_t for B_t and k , $1 \leq t \leq r$. In addition, again by [12, Theorem 7], we take \tilde{f}_1 for B_1 and \tilde{k} . Let

$$\begin{aligned} \hat{f}_0 := & v_1 \gamma_1 \kappa(f_1(x_1^{(1,1)}, \dots, x_{d_1}^{(1,1)}; \dots; x_1^{(1,2k)}, \dots, x_{d_1}^{(1,2k)}; z_1^{(1)}, \dots, z_{m_1}^{(1)}; \\ & \tilde{f}_1(y_1^{(1)}, \dots, y_{d_1}^{(1)}; \dots; y_1^{(2\tilde{k})}, \dots, y_{d_1}^{(2\tilde{k})}; u_1, \dots, u_{m_1}; z_1))) v_2 \\ & \cdot \prod_{t=2}^r (\gamma_t \kappa(f_t(x_1^{(t,1)}, \dots, x_{d_t}^{(t,1)}; \dots; x_1^{(t,2k)}, \dots, x_{d_t}^{(t,2k)}; z_1^{(t)}, \dots, z_{m_t}^{(t)}; z_t))) v_{t+1}). \end{aligned}$$

The value of the multilinear function \hat{f}_0 under the substitution $x_\beta^{(t,\alpha)} = a_\beta^{(t)}$, $z_\beta^{(t)} = \bar{z}_\beta^{(t)}$, $v_t = j_t$, $z_t = b_t$, $y_\beta^{(\alpha)} = a_\beta^{(1)}$, $u_\beta = \bar{z}_\beta^{(1)}$ equals the left-hand side of (5), which is nonzero.

Note that f_i are multilinear in z_i and \tilde{f}_1 is multilinear in z_1 . Therefore we may assume that for a fixed i in all entries of z_i^h in f_i the element $h \in H$ is the same. Analogously, we may assume that in all entries of z_1^h in \tilde{f}_1 the element $h \in H$ is the same. Furthermore, we can hide these h inside the elements substituted for z_i in the case of \tilde{f}_1 and f_i , $i \geq 2$, and inside \tilde{f}_1 in the case of f_1 , and the value b of \hat{f}_0 is still nonzero under the substitution $x_\beta^{(t,\alpha)} = a_\beta^{(t)}$, $z_\beta^{(t)} = \bar{z}_\beta^{(t)}$, $v_t = j_t$, $z_t = h_t b_t$, $y_\beta^{(\alpha)} = a_\beta^{(1)}$, $u_\beta = \bar{z}_\beta^{(1)}$ for some $h_t \in H$.

As we have mentioned, κ is a homomorphism of algebras and, by (6), behaves like a homomorphism of H -modules. Note that we do not need the last property for z_i^h since we may assume that in all such entries we have $h = 1$. Hence the value of

$$\begin{aligned} f_0 := & v_1 (f_1(x_1^{(1,1)}, \dots, x_{d_1}^{(1,1)}; \dots; x_1^{(1,2k)}, \dots, x_{d_1}^{(1,2k)}; z_1^{(1)}, \dots, z_{m_1}^{(1)}; \\ & \tilde{f}_1(y_1^{(1)}, \dots, y_{d_1}^{(1)}; \dots; y_1^{(2\tilde{k})}, \dots, y_{d_1}^{(2\tilde{k})}; u_1, \dots, u_{m_1}; z_1)))^{\gamma_1} v_2 \\ & \cdot \prod_{t=2}^r (((f_t(x_1^{(t,1)}, \dots, x_{d_t}^{(t,1)}; \dots; x_1^{(t,2k)}, \dots, x_{d_t}^{(t,2k)}; z_1^{(t)}, \dots, z_{m_t}^{(t)}; z_t)))^{\gamma_t} v_{t+1}) \end{aligned}$$

under the substitution $x_\beta^{(t,\alpha)} = \kappa(a_\beta^{(t)})$, $z_\beta^{(t)} = \kappa(\bar{z}_\beta^{(t)})$, $v_t = j_t$, $z_t = \kappa(h_t b_t)$, $y_\beta^{(\alpha)} = \kappa(a_\beta^{(1)})$, $u_\beta = \kappa(\bar{z}_\beta^{(1)})$ is again $b \neq 0$. We denote this substitution by \mathcal{E} .

Note that without additional manipulations a composition of H -polynomials is only a multilinear function but not an H -polynomial. However, using (1), we can always represent such function by an H -polynomial. Here we make such manipulations at the very end of the proof.

Let $X_\ell = \bigcup_{t=1}^r X_\ell^{(t)}$ and let Alt_ℓ be the operator of alternation on the set X_ℓ . Denote $\hat{f} := \text{Alt}_1 \text{Alt}_2 \dots \text{Alt}_{2k} f_0$. Note that the alternations do not change z_t , and f_t is alternating on each $X_\ell^{(t)}$. Hence the value of \hat{f} under the substitution \mathcal{E} equals $((d_1)!(d_2)!\dots(d_r)!)^{2k} b \neq 0$ since $\varkappa(B_1) \oplus \dots \oplus \varkappa(B_r)$ is a direct sum of (not necessarily H -invariant) ideals and if the alternation puts a variable from $X_\ell^{(t)}$ on the place of a variable from $X_\ell^{(t')}$ for $t \neq t'$, the corresponding $h\varkappa(a_\beta^{(t)})$, $h \in H$, annihilates $\varkappa(h_{t'}b_{t'})$.

Note that \tilde{f}_1 is a linear combination of multilinear monomials W , and one of the terms

$$\begin{aligned} & \text{Alt}_1 \text{Alt}_2 \dots \text{Alt}_{2k} v_1 (f_1(x_1^{(1,1)}, \dots, x_{d_1}^{(1,1)}; \dots; x_1^{(1,2k)}, \dots, x_{d_1}^{(1,2k)}; z_1^{(1)}, \dots, z_{m_1}^{(1)}; W))^{\gamma_1} v_2 \\ & \cdot \prod_{q=2}^r ((f_t(x_1^{(t,1)}, \dots, x_{d_t}^{(t,1)}; \dots; x_1^{(t,2k)}, \dots, x_{d_t}^{(t,2k)}; z_1^{(t)}, \dots, z_{m_t}^{(t)}; z_t))^{\gamma_q} v_{q+1}) \end{aligned}$$

in \hat{f} does not vanish under the substitution \mathcal{E} . Moreover,

$$\deg \hat{f} = 2kd + (2\tilde{k}d_1 + m_1) + n_0 > n$$

and $\deg W = \deg \tilde{f}_1 = 2\tilde{k}d_1 + m_1 + 1$. Let $W = w_1 w_2 \dots w_{2\tilde{k}d_1 + m_1 + 1}$ where w_i are variables from the set $\{y_1^{(1)}, \dots, y_{d_1}^{(1)}; \dots; y_1^{(2\tilde{k})}, \dots, y_{d_1}^{(2\tilde{k})}; u_1, \dots, u_{m_1}; z_1\}$ replaced under the substitution \mathcal{E} with $\bar{w}_i \in \varkappa(B_1)$. Let

$$\begin{aligned} f &:= \text{Alt}_1 \text{Alt}_2 \dots \text{Alt}_{2k} v_1 (f_1(x_1^{(1,1)}, \dots, x_{d_1}^{(1,1)}; \dots; x_1^{(1,2k)}, \dots, x_{d_1}^{(1,2k)}; z_1^{(1)}, \dots, z_{m_1}^{(1)}; \\ & w_1 w_2 \dots w_{n-2kd-n_0} z))^{\gamma_1} v_2 \\ & \cdot \prod_{q=2}^r ((f_t(x_1^{(t,1)}, \dots, x_{d_t}^{(t,1)}; \dots; x_1^{(t,2k)}, \dots, x_{d_t}^{(t,2k)}; z_1^{(t)}, \dots, z_{m_t}^{(t)}; z_t))^{\gamma_q} v_{q+1}) \end{aligned}$$

where z is an additional variable. Then using (1), we may assume $f \in P_n^H$. Note that f is alternating in X_ℓ , $1 \leq \ell \leq 2k$, and does not vanish under the substitution \mathcal{E} with $z = \bar{w}_{n-2kd-n_0+1} \dots \bar{w}_{2\tilde{k}d_1+m_1+1}$. Thus f satisfies all the conditions of the lemma. \square

Proof of Theorem 1. Now we repeat verbatim the proofs of [12, Lemma 11 and Theorem 5] using Lemmas 1 and 3 instead of [12, Lemma 10 and Theorem 6]. \square

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