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A finitely generated branch group of exponential growth without free subgroups

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ABSTRACT

We will give an example of a branch group G that has exponential growth but does not contain any non-abelian free subgroups. This answers question 16 from Bartholdi et al. (2003) [1] positively. The proof demonstrates how to construct a non-trivial word $w_{a,b}(x,y)$ for any $a, b \in G$ such that $w_{a,b}(a,b) = 1$. The group G is not just infinite. We prove that every normal subgroup of G is finitely generated as an abstract group and every proper quotient soluble. Further, G has infinite virtual first Betti number but is not large.

1. Introduction

Groups acting on infinite rooted trees have provided remarkable examples in the last decades. Starting with Grigorchuk's group in [4] of intermediate growth branch groups received more and more attention. Let T be a rooted tree and $\mathrm{Aut}(T)$ the group of automorphisms acting on T. A branch group H is a subgroup of $\mathrm{Aut}(T)$ which fulfills a certain stabilizing condition. A standard introduction to this topic is the survey [1] by Bartholdi, Grigorchuk and Sunik.

In their section on open questions the authors of [1] ask whether there exist branch groups which have exponential word growth but do not contain any non-abelian free subgroups. We answer this question affirmatively by constructing explicit words $w_{a,b}(x,y)$ for any $a, b \in G$ such that $w_{a,b}(a,b) = 1$. It is a result by Grigorchuk and Zuk [6] that the weakly branch Basilica group has exponential growth but no free subgroups. Sidki and Wilson constructed in [10] branch groups that contain free subgroups and hence have exponential growth. Nekrashevych proved in [8] that branch groups containing free subgroups fall into one of two cases. These cases are defined via the action of the group on the boundary of the rooted tree.

The group G in this paper will depend on an infinite sequence of primes. In order to establish that G has exponential growth and no free subgroups we have to make restrictions on this sequence.

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If we weaken those assumptions we can prove by other means that G is not large. We do not know whether these restrictions are necessary. We also do not know whether our group G is amenable. Motivated by a result of Brieussel [3], we suspect that this could hold at least if the sequence of primes grows slowly. Consideration of the abelianization of certain normal subgroups shows that G has infinite virtual first Betti number.

Most of the examples studied in the literature are groups acting on regular, rooted, spherically transitive trees. In this paper we look at finitely generated automorphism groups of an irregular rooted tree. A similar class of examples was first mentioned by Segal in [9]. A related construction was investigated by Woryna [11] and Bondarenko [2] where the authors describe generating sets of infinite iterated wreath products.

2. Rooted trees and automorphisms

In this section we will recall some of the notation and definitions from [1] and [9].

2.1. Trees

A *tree* is a connected graph which has no non-trivial cycles. If T has a distinguished *root* vertex r it is called a *rooted tree*. The distance of a vertex v from the root is given by the length of the path from r to v and called the *norm* of v. The number

$$d_v = \left|\left\{e \in E(T)\colon\ e = (v_1, v_2),\ v = v_1 \text{ or } v = v_2\right\}\right|$$

is called the *degree* of $v \in V(T)$. The tree is called *spherically homogeneous* if vertices of the same norm have the same degree. Let $\Omega(n)$ denote the set of vertices of distance n from the root. This set is called the *n-th level* of T. A spherically homogeneous tree T is determined by a finite or infinite sequence $\bar{l} = \{l_n\}_{n=1}$ where $l_n + 1$ is the degree of the vertices on level n for $n \geqslant 1$. The root has degree l_0. Hence each level $\Omega(n)$ has $\prod_{i=0}^{n-1} l_i$ vertices. Let us denote this number by $m_n = |\Omega(n)|$. We denote such a tree by $T_{\bar{l}}$. A tree is called *regular* if $l_i = l_{i+1}$ for all $i \in \mathbb{N}$. Given a spherically homogeneous tree T we denote by $T[n]$ the finite tree where all vertices have norm less or equal to n and write T_v for the subtree of T with root v. For all vertices $v, u \in \Omega(n)$ we have that $T_u \simeq T_v$. Denote a tree isomorphic to T_v for $v \in \Omega(n)$ by T_n. This will be the tree with defining sequence $(l_n, l_{n+1}, \ldots)$. To each sequence $\bar{l}$ we associate a sequence $\{X_n\}_{n\in\mathbb{N}}$ of alphabets where $X_n = \{v_1^{(n)}, \ldots, v_{l_n}^{(n)}\}$ is an l_n-tuple so that $|X_n| = l_n$. A path beginning at the root of length n in $T_{\bar{l}}$ is identified with the sequence $x_1, \ldots, x_i, \ldots, x_n$ where $x_i \in X_i$ and infinite paths are identified in a natural way with infinite sequences. Vertices will be identified with finite strings in the alphabets X_i. Vertices on level n can be written as elements of $Y_n = X_0 \times \cdots \times X_{n-1}$. Alphabets induce the lexicographic order on the paths of a tree and therefore the vertices.

2.2. Automorphisms

An *automorphism* of a rooted tree T is a bijection from $V(T)$ to $V(T)$ that preserves edge incidence and the distinguished root vertex r. The set of all such bijections is denoted by $\mathrm{Aut}(T)$. This group acts as an imprimitive permutation group on the set $\Omega(n)$ of vertices on level n for each $n \geqslant 2$. Consider an element $g \in \mathrm{Aut}(T)$. Let y be a letter from Y_n, hence a vertex of $T[n]$ and z a vertex of T_n. Then $g(y)$ induces a vertex permutation g_y of Y_n. If we denote the image of z under g_y by $g_y(z)$ then

$$g(yz) = g(y)g_y(z).$$

With any group $G \leqslant \mathrm{Aut}(T)$ we associate the subgroups

$$\mathrm{St}_G(u) = \left\{g \in G\colon\ g(u) = u\right\},$$

the *stabilizer* of a vertex u. Then the subgroup

$$\mathrm{St}_G(n) = \bigcap_{u \in \Omega(n)} \mathrm{St}_G(u)$$

is called the *n-th level stabilizer* and it fixes all vertices on the n-th level. Another important class of subgroups associated with $G \leqslant \mathrm{Aut}(T)$ consists of the *rigid vertex stabilizers*

$$\mathrm{rst}_G(u) = \{g \in G\colon \forall v \in V(T) \setminus V(T_u)\colon\ g(v) = v\}.$$

The subgroup

$$\mathrm{rst}_G(n) = \prod_{u \in \Omega(n)} \mathrm{rst}_G(u)$$

is called the *n-th level rigid stabilizer*. Obviously $\mathrm{rst}_G(n) \leqslant \mathrm{St}_G(n)$.

Definition 2.1. Let G be a subgroup of $\mathrm{Aut}(T)$ where T is a spherically homogeneous rooted tree. We say that G acts on T as a *branch group* if it acts transitively on the vertices of each level of T and $\mathrm{rst}_G(n)$ has finite index for all $n \in \mathbb{N}$.

The definition implies that branch groups are infinite and residually finite groups. We can specify an automorphism g of T that fixes all vertices of level n by writing $g = (g_1, g_2, \dots, g_{m_n})_n$ with $g_i \in \mathrm{Aut}(T_n)$ where the subscript n of the brackets indicates that we are on level n. Each automorphism can be written as $g = (g_1, g_2, \dots, g_{m_n})_n \cdot \alpha$ with $g_i \in \mathrm{Aut}(T_n)$ and α an element of $\mathrm{Sym}(l_{n-1}) \wr \cdots \wr \mathrm{Sym}(l_0)$. Automorphisms acting only on level 1 by permutation are called *rooted automorphisms*. We can identify those with elements of $\mathrm{Sym}(l_0)$.

3. The construction

In this subsection we describe the main construction of the group. The trees in this paper will have a defining sequence $\{l_i\}_{i\in\mathbb{N}}$ where all l_i are pairwise distinct primes greater or equal than 7. We further, without loss of generality, assume the sequence $\{l_i\}$ to be ascending. This sequence of valencies will prove to be the key to the exponential growth and the non-existence of non-abelian free subgroups. The group G constructed here is finitely generated, but recursively presented. We shall prove that for every normal subgroup $N \neq 1$, N is finitely generated as an abstract group and that G/N is soluble.

3.1. The generators

Let $\{l_i\}_{i\in\mathbb{N}}$ be a sequence as described above and $\{A_i\}_{i\in\mathbb{N}}$ a sequence of finite cyclic groups of pairwise coprime orders $l_i = |A_i|$ and we assume $l_i \geqslant 3$ for the rest of this paper unless stated otherwise. Fix a generator a_i for each A_i. Let us consider the rooted tree with defining sequence $\{l_i\}_{i\in\mathbb{N}}$ as described in Subsection 2.1.

We study the group

$$G = \langle a_0, b\rangle$$

where a_0 is the chosen generator of A_0 acting as rooted automorphism and b is recursively defined on each level $n \geqslant 0$ by

$$b_n = (b_{n+1}, a_{n+1}, 1, \dots, 1)_{n+1}$$

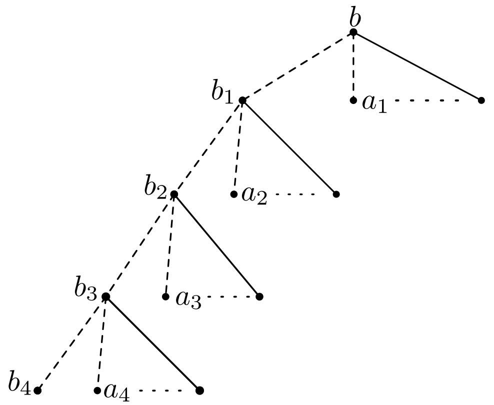

Fig. 1. The automorphism b.

where a_{n+1} is the generator of the group A_{n+1}. This means the action on the first vertex of level n is given by b_{n+1} and the action on the second vertex by the rooted automorphism a_{n+1}. Fig. 1 shows the action of the automorphism b on the tree. The action of b on all unlabeled vertices v in the figure will be given by the identity on T_v.

Proposition 3.1. *G acts as the iterated wreath product $A_{n-1} \wr \cdots \wr A_1 \wr A_0$ on the set $\Omega(n)$ of m_n vertices of each level n.*

Proof. We argue by induction. The action on level 1 is given by A_0. Now assume that the action of G on $\Omega(n-1)$ is given by $A_{n-2} \wr \cdots \wr A_0$. The automorphism $b^{m_{n-1}}$ acts as $a_{n-1}^{m_{n-1}}$ on $v \in \Omega(n-1)$ and trivially above level $n-1$. There exists an integer q such that $a_{n-1}^{q \cdot m_{n-1}} = a_{n-1}$ because l_{n-1} and m_{n-1} are coprime. Hence for all $a_{n-1}^k \in A_{n-1}$ there exists a $g = b^{q \cdot m_{n-1}} \in G$ such that $g|_{T_v} = a_{n-1}^k$. This holds for any vertex of level $n-1$ by the transitivity of $A_{n-2} \wr \cdots \wr A_0$. Therefore G induces the action of $A_{n-1} \wr \cdots \wr A_0$ on $\Omega(n)$. $\square$

Corollary 3.2. $G/\operatorname{St}_G(n) = A_{n-1} \wr \cdots \wr A_0$.

We denote conjugation by $x^y = y^{-1}xy$ and commutators by $[x, y] = x^{-1}y^{-1}xy$. Define with $b = b_0$ the following automorphisms and groups:

$$b_n(i) = b_n^{a_n^{k-1}} \quad \text{for } k \equiv i \bmod l_n,\ i \in \mathbb{Z}$$

and

$$B_n = \langle b_n(1), \ldots, b_n(l_{n-1}) \rangle$$

for $n \geqslant 0$ and similarly to G the groups

$$G_n = \langle a_n, b_n \rangle$$

for $n \geqslant 0$. Write $G_0 = G$, $A = A_0$, $B = B_0$ and $b(i)$ short for $b_0(i)$ and we see from the definition of $b_n(i)$ that $b_n(i) = b_n(j)$ whenever $i \equiv j \bmod l_n$.

Write Γ' for the derived subgroup $[\Gamma, \Gamma]$ of a group Γ and by $\Gamma^{(n)}$ for $n \geqslant 1$ the n-th derived subgroup $\Gamma^{(n)} = [\Gamma^{(n-1)}, \Gamma^{(n-1)}]$ where $\Gamma^{(0)} = \Gamma$. We denote by $G_n \times \cdots \times G_n$ the subgroup of $\operatorname{Aut}(T)$ which acts as G_n on each vertex of level n and similarly for B_n and later the subgroups N_n.

Proposition 3.3. *With the above definitions we get the following statements*:

(a) $G = B \rtimes A$ *and so* $G' = B' \cdot \langle [B, A] \rangle$.
(b) $\mathrm{St}_G(1) \leqslant G_1 \times \cdots \times G_1$.
(c) $B = \mathrm{St}_G(1)$.

Proof.

(a) Clearly $B \cap A = 1$ and $B \lhd G$.
(b) $\mathrm{St}_G(1)$ is generated by a-conjugates of $b_0 = (b_1, a_1, 1, \dots, 1)_1$. But b_1 and a_1 are in G_1, hence $\mathrm{St}_G(1) \leqslant G_1 \times \cdots \times G_1$.
(c) We see that $B \leqslant \mathrm{St}_G(1)$. For the other inclusion we use $G = B \cdot A$ and the modular law with $B \leqslant \mathrm{St}_G(1)$. We get $\mathrm{St}_G(1) = B(A \cap \mathrm{St}_G(1)) = B$ because $A \cap \mathrm{St}_G(1) = 1$. □

Lemma 3.4. *The subgroup B defined above satisfies $B'B^{l_1} \leqslant \mathrm{rst}_G(1)$.*

Proof. We first prove $B' \leqslant \mathrm{rst}_G(1)$ and claim that

$$\bigl[b(i), b(j)\bigr] = \begin{cases} (1, \dots, 1, [a_1, b_1], 1, \dots, 1)_1 & \text{if } j \equiv i + 1 \bmod l_0, \\ (1, \dots, 1, [b_1, a_1], 1, \dots, 1)_1 & \text{if } i \equiv j + 1 \bmod l_0, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

We look at the action of $[b(i), b(j)] = b(i)^{-1} b(j)^{-1} b(i) b(j)$ on the first layer for the first and third case. The second one follows similarly. Denote by underbracing the positions of the respective elements.

- $j \equiv i + 1 \bmod l_0$:

$$\begin{aligned} b(i)^{-1} b(j)^{-1} b(i) b(j) &= \bigl(1, \dots, 1, \underbrace{b_1^{-1} b_1}_{i}, \underbrace{a_1^{-1} b_1^{-1} a_1 b_1}_{j=i+1}, \underbrace{a_1^{-1} a_1}_{j+1}, 1, \dots, 1\bigr)_1 \\ &= \bigl(1, \dots, 1, [a_1, b_1], 1, \dots, 1\bigr)_1. \end{aligned}$$

- $|i - j| \bmod l_0 > 1$:

$$b(i)^{-1} b(j)^{-1} b(i) b(j) = \bigl(1, \dots, 1, \underbrace{b_1^{-1} b_1}_{i}, \underbrace{a_1^{-1} a_1}_{i+1}, 1, \dots, 1, \underbrace{b_1^{-1} b_1}_{j}, \underbrace{a_1^{-1} a_1}_{j+1}, 1, \dots, 1\bigr)_1 = 1.$$

It remains to show $B^{l_1} \leqslant \mathrm{rst}_G(1)$.

$$\begin{aligned} b(k)^{l_1} &= \bigl(1, \dots, 1, \underbrace{b_1^{l_1}}_{k}, a_1^{l_1}, 1, \dots, 1\bigr)_1 \\ &= \bigl(1, \dots, 1, \underbrace{b_1^{l_1}}_{k}, 1, \dots, 1\bigr)_1 \in \mathrm{rst}_G(1) \quad \text{for } i = 1, \dots, l_0. \qquad \square \end{aligned}$$

3.2. Introducing N

In this subsection we define a normal subgroup N that will be proved to be equal to the derived group of G. However, this explicit construction and the explicit finite set of generators that we will obtain will be very useful.

Let $F_{l_0} = \langle x_1, \dots, x_{l_0} \rangle$ be the free group on l_0 generators. The map

$$f : \begin{cases} F_{l_0} \to \mathbb{Z}, \\ x_i \mapsto 1 \quad \text{for all } i \in \{1, \dots, l_0\} \end{cases} \tag{2}$$

is surjective. Its kernel $K(x_1, \dots, x_{l_0}) = \ker(f)$ consists of all words in the generators where the sum over all exponents is 0.

Lemma 3.5. *The kernel of f is given by $K(x_1, \dots, x_{l_0}) = \langle x_i^{-1} x_j \mid i, j = 1, \dots, l_0 \rangle^F$.*

Proof. Define $X = \langle x_i^{-1} x_j \mid i, j = 1, \dots, l_0 \rangle^F$. We first show $F' \leqslant X$. We can write

$$x_i^{-1} x_j^{-1} x_i x_j = \left(x_j^{-1} x_i\right)^{x_i} \cdot x_i^{-1} x_j$$

which proves the claim. Clearly $X \leqslant K$. We observe that $K/F' = X/F'$ which yields that $K = X$. $\square$

Define

$$N_n = K\left(b_n(1), \dots, b_n(l_n)\right)$$

for $n \geqslant 0$ and write $N = N_0$ for the rest of this paper. The following lemma follows straight from the definition.

Lemma 3.6. *$N_n \leqslant B_n$ for $n \geqslant 0$.*

Lemma 3.7. *If $l_0 \geqslant 5$, then N is finitely generated. A set of generators is given by the elements $\{b(2)^{-1}b(1), b(3)^{-1}b(2), \dots, b(1)^{-1}b(l_0)\}$.*

The essential property used in this proof is that each generator of B commutes with most of the others. More precisely we have the identities $[b(i), b(k)] = 1$ if $|i-k| \not\equiv 1 \bmod l_0$.

Proof. Set $D = \langle b(2)^{-1}b(1), \dots, b(1)^{-1}b(l_0)\rangle$. We show that $(b(2)^{-1}b(1))^{b(k)} \in D$. Concatenation yields that $b(j)^{-1}b(i) \in D$ for all $i, j \in \{1, \dots, l_0\}$. Using that $l_0 \geqslant 5$ we can see that

$$b(i)b(i-1)^{-1} = b(i+2)^{-1}b(i) \cdot b(i-1)^{-1}b(i+2)$$

because $[b(i), b(j)] = 1$ if i, j are such that $|i-j| \bmod l_0 > 1$ by Lemma 3.4. Hence all elements $b(j)b(k)^{-1}$ for all j, k are in $\langle D \rangle$. We can write

$$\begin{aligned} \left(b(i)^{-1}b(i-1)\right)^{b(k)} &= b(k)^{-1}b(i)^{-1}b(i-1)b(k) \\ &= b(k)^{-1}b(i+2) \cdot b(i+2)b(i)^{-1} \cdot b(i+2)^{-1}b(i-1) \cdot b(i+2)^{-1}b(k) \end{aligned}$$

because $b(i+2)$ commutes with $b(i)$ and $b(i-1)$. The latter is a product of four elements of D. This yields $D^b \leqslant D$ for all $b \in B$ and so $D^B = D$ which gives $N = D$ and so N is finitely generated. $\square$

Proposition 3.8. *$G_n' = N_n$ for $n \geqslant 0$.*

Proof. N is the kernel of a map whose image is abelian hence $G' \leqslant N$. Looking at the generators of N we see that $N/G' = 1$ and hence the groups are equal. $\square$

Lemma 3.9. *$B' = N_1 \times \cdots \times N_1$ and so $B' \leqslant B_1 \times \cdots \times B_1$.*

Proof. We have $B = \mathrm{St}_G(1) \leqslant G_1 \times \cdots \times G_1$ and hence $B' \leqslant G_1' \times \cdots \times G_1' = N_1 \times \cdots \times N_1$ by Corollary 3.8. We now prove $N_1 \times \cdots \times N_1 \leqslant B'$. The group N_1 is generated by elements of the form $b_1(j+1)^{-1} b_1(j) = [b(1), b(2)]^{b(1)^{j-1}}$ and hence in B'. $\square$

Corollary 3.10. *$B'_{n-1} = N_n \times \cdots \times N_n \leqslant B_n \times \cdots \times B_n$ for $n \geqslant 1$.*

Lemma 3.11. *We have the following identities for the subgroups defined above for $n \geqslant 0$:*

(a) $N' = B'$.
(b) $N_n' = N_{n+1} \times \cdots \times N_{n+1}$ *with l_n factors in the direct product.*
(c) $G^{(n+1)} = G_n' \times \cdots \times G_n'$ *with m_n factors in the direct product.*
(d) $G^{(n+1)} \subseteq \mathrm{rst}_G(n)$.

Proof.

(a) Elementary commutator manipulation shows that

$$\bigl[b(2), b(1)\bigr] = \bigl[b(4)^{-1} b(2), b(2)^{-1} b(1)\bigr].$$

This implies $B' \leqslant N'$. The other inclusion follows straight from $N \leqslant B$.
(b) By Corollary 3.10 we have $N_n' = B_n' = N_{n+1} \times \cdots \times N_{n+1}$.
(c) We start with

$$G^{(n+1)} = \bigl(G'\bigr)^{(n)} = N^{(n)} = \bigl(N'\bigr)^{(n-1)} = (\underbrace{N_1 \times \cdots \times N_1}_{l_o \text{ times}})^{(n-1)} = N_1^{(n-1)} \times \cdots \times N_1^{(n-1)}$$

and apply (b) iteratively together with Proposition 3.8 and get

$$\underbrace{N_n \times \cdots \times N_n}_{m_n \text{ times}} = G_n' \times \cdots \times G_n'.$$

(d) The proof of (c) implies $G^{(n+1)} = N_n \times \cdots \times N_n \leqslant (G \cap G_n) \times \cdots \times (G \cap G_n) = \mathrm{rst}_G(n)$. $\square$

Corollary 3.12. *$B_n'' = B_{n+1}' \times \cdots \times B_{n+1}'$ and $B^{(n)} = B_{n-1}' \times \cdots \times B_{n-1}'$ for $n \geqslant 0$.*

Lemma 3.13. *$\mathrm{St}_G(n) = G \cap (G_n \times \cdots \times G_n)$ for $n \geqslant 0$.*

Proof. It is obvious that $G \cap (G_n \times \cdots \times G_n)$ is contained in $\mathrm{St}_G(n)$. The other inclusion is given by Proposition 3.3 for $n = 1$ and follows iteratively from $\mathrm{St}_G(n+1) \leqslant \mathrm{St}_{\mathrm{St}_G(n)}(1)$ for all $n \geqslant 1$. $\square$

Lemma 3.14. *$b^{m_{n+1}} = (b_n^{m_{n+1}}, 1, \ldots, 1)_n = (b_{n-1}^{m_{n+1}}, 1, \ldots, 1)_{n-1} \in G$ for $n \geqslant 0$.*

Proof. Every a_n has order l_n. Hence $(b_n, a_n, 1, \ldots, 1,)_n^{l_0 \ldots l_n} = (b_n^{l_0, \ldots, l_n}, 1, \ldots, 1)_n$. $\square$

Lemma 3.15. *The following statements hold for $n \geqslant 0$:*

(a) $B_n' \cdot B_n^{m_{n+1}} \leqslant G$ *where $B_n^{m_{n+1}} = \langle b_n(i)^{m_{n+1}} \rangle$ for $n \in \mathbb{N}$.*
(b) $B'_{n-1} B^{m_n}_{n-1} \times \cdots \times B'_{n-1} B^{m_n}_{n-1} \leqslant \mathrm{rst}_G(n)$.

Proof. Lemma 3.11 implies $B_n' \times \cdots \times B_n' = N_n' \times \cdots \times N_n' \leqslant N_n \times \cdots \times N_n = G^{(n+1)}$ and together with Lemma 3.14 this gives $B_n' \cdot B_n^{m_{n+1}} \leqslant G$ which proves both parts. $\square$

Lemma 3.16. $(G_{n+1} \times \cdots \times G_{n+1} \cap G)' = G_{n+1}' \times \cdots \times G_{n+1}'$ *for* $n \geqslant 0$.

Proof. We have $G_n' \times \cdots \times G_n' \leqslant (G_{n+1} \times \cdots \times G_{n+1}) \cap G$ and we get

$$\begin{aligned} G_{n+1}' \times \cdots \times G_{n+1}' = G_n'' \times \cdots \times G_n'' &\leqslant \big((G_{n+1} \times \cdots \times G_{n+1}) \cap G\big)' \\ &\leqslant \big(G_{n+1}' \times \cdots \times G_{n+1}'\big) \cap G' = G^{(n+2)} \cap G' = G^{(n+2)} = G_{n+1}' \times \cdots \times G_{n+1}'. \qquad \square \end{aligned}$$

Lemma 3.17. *The following statements hold*:

(a) $\mathrm{rst}_G(1) = B' \cdot B^{l_1}$.
(b) $\mathrm{rst}_G(n) \leqslant \prod_{i=1}^{m_{n-1}} B_{n-1}' \cdot B^{m_{n+1}}$ *for* $n \geqslant 1$.
(c) $\mathrm{rst}_G(n) \leqslant \mathrm{St}_G(n+1)$ *for* $n \geqslant 1$.

Proof. We first see that $\mathrm{rst}_G(1) = B' \cdot B^{l_1}$ because of Lemma 3.15 and $\mathrm{rst}_G(1) \leqslant B = \mathrm{St}_G(1)$. Hence $\mathrm{rst}_G(n) \leqslant \prod_{i=1}^{m_{n-1}} \mathrm{rst}_{G_{n-1}}(1) = \prod B_{n-1}' \cdot B^{m_{n+1}}$ which fixes layer $n+1$. $\square$

Proposition 3.18. $\mathrm{rst}_G(n)' = G^{(n+2)}$ *for* $n \geqslant 1$, *in particular* $\mathrm{rst}_G(n)'$ *is finitely generated.*

Proof. Lemma 3.17 states $\mathrm{rst}_G(1) = B' \cdot B^{l_1}$ and therefore $\mathrm{rst}_G(1)' = B'' \cdot [B', B^{l_1}](B^{l_1})'$. For the first group we have $B'' = B_1' \times \cdots \times B_1'$ and for the last one we see that $B^{l_1} \leqslant B_1 \times \cdots \times B_1$. It therefore remains to observe that $[B', B^{l_1}] \leqslant \prod B_1'$ which follows from $B' \leqslant B_1$ and $B^{l_1} \leqslant B_1$. This implies $\mathrm{rst}_G(1)' = B_1' \times \cdots \times B_1' = N_2 \times \cdots \times N_2$ by Corollary 3.10 which is finitely generated. It is now left to show that this implies $\mathrm{rst}_G(n)'$ is finitely generated for all $n \in \mathbb{N}$. By Lemma 3.17(c) we have the following inclusions:

$$\begin{aligned} \mathrm{rst}_G(n)' &\leqslant (G_{n+1} \times \cdots \times G_{n+1} \cap G)' \\ &= G_{n+1}' \times \cdots \times G_{n+1}' = \big(G_n' \times \cdots \times G_n'\big)' \leqslant \mathrm{rst}_G(n)' \end{aligned}$$

because $G_{n+1}' \times \cdots \times G_{n+1}' = G^{(n+2)} \leqslant G'$. So by this we have

$$N_{n+1} \times \cdots \times N_{n+1} = G_{n+1}' \times \cdots \times G_{n+1}' = \big(G_n' \times \cdots \times G_n'\big)' = \mathrm{rst}_G(n)'$$

which is therefore finitely generated by Lemma 3.7. $\square$

Theorem 3.19. *The group G is a branch group. Further the quotient* $\frac{\mathrm{St}_G(n)}{\mathrm{rst}_G(n)}$ *for* $n \geqslant 1$ *is abelian and has exponent dividing* $l_1 l_2 \ldots l_{n-1} l_n$.

Proof. In the case $n = 1$ we have $\mathrm{St}_G(1) = B$. We have $\mathrm{St}_G(n) \leqslant B_{n-1} \times \cdots \times B_{n-1}$ for $n > 1$ and so

$$\mathrm{St}_G(n)^{l_1 \ldots l_n} \leqslant (B_{n-1} \times \cdots \times B_{n-1})^{l_1 \ldots l_n} = (B_n \times \cdots \times B_n)^{l_1 \ldots l_n} \leqslant \mathrm{rst}_G(n)$$

by Lemma 3.14. Now Lemma 3.11 implies

$$\mathrm{St}_G(n)' = G' \cap \big(G_n' \times \cdots \times G_n'\big) = G' \cap G^{(n+1)} \leqslant \mathrm{rst}_G(n).$$

The quotient $\frac{\mathrm{St}_G(n)}{\mathrm{rst}_G(n)}$ is therefore abelian and has exponent dividing $l_1 l_2 \dots l_{n-1} l_n$. The n-th level stabilizers $\mathrm{St}_G(n)$ always have finite index, hence $\mathrm{rst}_G(n)$ is of finite index in G. $\square$

Lemma 3.20. $\frac{B_n}{N_n} \simeq \mathbb{Z}$ *for* $n \geqslant 0$.

Proof. Let $F_{l_0} = \langle x_1, \dots, x_{l_0} \rangle$ be the free group on l_0 generators and π the natural projection

$$\pi : \begin{cases} F_{l_0} \to B, \\ x_i \mapsto b(i) \in B \quad \text{for all } i = 1, \dots, l_0. \end{cases}$$

The map from Eq. (2) together with the natural injection

$$\iota : \begin{cases} N(x_1, \dots, x_{l_0}) \hookrightarrow F_{l_0}, \\ x_i \mapsto x_i \quad \text{for all } i = 1, \dots, l_0 \end{cases}$$

gives the following sequence:

$$\begin{array}{ccccccccc} 1 & \hookrightarrow & N(x_1, \dots, x_{l_0}) & \hookrightarrow & F_{l_0} & \twoheadrightarrow & \mathbb{Z} & \to & 0 \\ & & \downarrow \pi & & & & & & \\ & & N\pi \leqslant B. & & & & & & \end{array}$$

We see that $F_{l_0}/N \simeq \mathbb{Z}$ and hence its image B/N under π must be an infinite cyclic group. $\square$

3.3. Finite generation of normal subgroups

We quote a theorem by Grigorchuk [5].

Theorem 3.21. *Let* $\Gamma \leqslant \mathrm{Aut}(T)$ *be a spherically transitive subgroup of the full automorphism group on* T*. If* $1 \neq N \lhd \Gamma$*, then there exists an* n *such that* $\mathrm{rst}_\Gamma(n)' \leqslant N$*.*

Proposition 3.22. *Every proper quotient of* G *is soluble.*

Proof. This follows straight from Theorem 3.21 and Lemma 3.11. $\square$

Theorem 3.23. *In the group* G *defined above every normal subgroup is finitely generated.*

Proof. By Theorem 3.21 every normal subgroup $K \lhd G$ contains some $\mathrm{rst}'_G(n)$. Proposition 3.18 states that $\mathrm{rst}_G(n)'$ is finitely generated. So it suffices to show that $K/\mathrm{rst}_G(n)'$ is finitely generated. The group $K/\mathrm{rst}_G(n)'$ is a finite extension of the finitely generated abelian group $(K \cap \mathrm{rst}_G(n))/\mathrm{rst}'_G(n)$. $\square$

3.4. Congruence subgroup property

We recall that a branch group Γ has the *congruence subgroup property* if for every subgroup $H \leqslant \Gamma$ of finite index in Γ there exists an n such that $\mathrm{St}_\Gamma(n) \leqslant H$.

Theorem 3.24. *G does not have the congruence subgroup property.*

Proof. The quotient

$$\frac{\mathrm{rst}_G(n)}{\mathrm{rst}_G(n)' \, \mathrm{rst}_G(n)^p}$$

is an elementary abelian p-section for every prime p. By taking n large enough we can find p-sections of arbitrarily large rank in G. Because G is a branch group $\mathrm{rst}_G(n)$ has finite index. On the other hand any congruence quotient G/H is a quotient of $A_{k-1} \wr \cdots \wr A$ for some $k \in \mathbb{N}$. Hence its p-rank is finite and determined by the sequence of primes we chose. □

This implies that the profinite completion maps onto the congruence completion with non-trivial congruence kernel.

Theorem 3.25. *The rank of $\frac{\mathrm{St}_G(n+1)}{\mathrm{rst}_G(n)}$ is less than or equal to $m_{n+1} = \prod_{i=0}^{n} l_i$ for $n \geqslant 0$.*

Proof. The inclusions $\mathrm{St}_G(n+1) \leqslant \prod_{i=1}^{m_n} B_n$ and $N_n \times \cdots \times N_n = G^{(n+1)} \leqslant \mathrm{rst}_G(n)$ give that the quotient $\frac{\mathrm{St}_G(n+1)}{\mathrm{rst}_G(n)}$ is a section of $\frac{B_n \times \cdots \times B_n}{N_n \times \cdots \times N_n}$. Hence the first quotient has rank less than or equal to m_{n+1} by Lemma 3.20. □

4. Abelianization

This section is devoted to computing the abelianization $G^{ab} = G/G'$ of G where G' is the derived group. This will allow us to determine the abelianizations of the n-th level rigid stabilizers $\mathrm{rst}_G(n)$. Considering those we show that the virtual first Betti number of G is infinite.

4.1. Abelianization of G

The abelianization of G as a 2-generator group must be an image of the free abelian group $F_2^{ab} = \langle x_1, x_2 \rangle$ on two generators, in particular an image of $F_2^{ab} = C_\infty \times C_\infty$.

Theorem 4.1. $G^{ab} = C_{l_0} \times C_\infty$.

Proof. The abelianization can be presented as $G^{ab} = \langle a, b \mid a^{e_i} b^{d_i} = 1 \rangle$ for possibly infinitely many pairs of exponents $e_i, d_i \in \mathbb{Z}$. By construction the order of a is $o(a) = l_0$. We now show that the image of b has infinite order in the abelianization. Corollary 3.2 describes the quotients

$$\frac{G}{\mathrm{St}_G(n)} = A_{l_{n-1}} \wr \cdots \wr A_{l_0} =: W(n).$$

Consider the natural projections

$$\varphi : G \twoheadrightarrow \frac{G}{G'} \twoheadrightarrow \frac{W(n)}{W'(n)} = A_{l_{n-1}} \times \cdots \times A_{l_0}.$$

The image of b under the composite of these has order $o(\varphi(b)) = \prod_{i=0}^{n-1} l_i$. This order tends to infinity with n and must therefore be infinite in G^{ab}. □

Corollary 4.2. $G_n^{ab} = C_{l_n} \times C_\infty$ *for* $n \geqslant 1$.

4.2. Abelianization of subgroups

In this subsection we determine the abelianization of the subgroups B and $\mathrm{rst}_G(n)$. This will yield that G is not just infinite.

Proposition 4.3. $B^{ab} \simeq \prod_{i=1}^{l_0} \mathbb{Z}$.

Proof. The elements $b(i)^{l_1}$ are all in B. The image of each $b(i)$ in G^{ab} has infinite order by the proof of Theorem 4.1. The subgroup $H = \langle b(1)^{l_1}, \dots, b(l_0)^{l_1} \rangle \leqslant B$ is therefore free abelian of rank l_0, hence $H \cap B' = 1$. We get that

$$H \simeq \frac{H}{H \cap B'} \simeq \frac{HB'}{B'} \leqslant \frac{B}{B'}$$

and hence B^{ab} has rank at least l_0. But B is generated by l_0 elements and so $B^{ab} \simeq \prod_{i=1}^{l_0} \mathbb{Z}$. $\square$

Corollary 4.4. *$B_n^{ab} \simeq \prod_{i=1}^{l_n} \mathbb{Z}$ for $n \geqslant 1$.*

Theorem 4.5. *$\mathrm{rst}_G(n)^{ab} = \prod_{i=1}^{m_{n+1}} \mathbb{Z}$ for $n \geqslant 1$.*

Proof. Proposition 3.18 gives the equality $\mathrm{rst}_G(n)' = G'_{n+1} \times \cdots \times G'_{n+1} = N_{n+1} \times \cdots \times N_{i+1}$ and hence

$$\frac{\mathrm{rst}_G(n)}{\mathrm{rst}_G(n)'} = \frac{\mathrm{rst}_G(n)}{N_{n+1} \times \cdots \times N_{n+1}} \leqslant \frac{\prod B_n}{\prod B'_n} \simeq \prod_{i=1}^{m_n} \prod_{j=1}^{l_n} \mathbb{Z}.$$

It remains to prove that we have full rank $m_{n+1} = m_n \cdot l_n$. We observe that the elements $b_n(i)^{m_n}$ for $i = 1, \dots, l_n$ all lie in $\mathrm{rst}_G(n)$ and have disjoint support. The subgroup

$$\prod_{i=1}^{m_n} \bigl\langle b(1)^{m_{n+1}}, \dots, b(l_n)^{m_{n+1}} \bigr\rangle$$

of $\mathrm{rst}_G(n)$ therefore maps onto a rank m_{n+1} subgroup of $\mathrm{rst}_G(n)^{ab}$ which proves the claim. $\square$

Corollary 4.6. *G is not just infinite.*

Proof. Theorem 4.1 states that the quotient $G^{ab} = G/G'$ is infinite. $\square$

We recall that the first Betti number $b_1(\Gamma)$ of a group Γ is the dimension of $H_1(\Gamma; \mathbb{Z}) \otimes \mathbb{Q}$. This is the rank of Γ^{ab}. The virtual first Betti number of a group Γ is defined [7] to be

$$vb_1(G) = \sup\bigl\{b_1(H)\colon |G/H| < \infty\bigr\}.$$

Corollary 4.7. *$vb_1(G)$ is infinite.*

Proof. Theorem 4.5 states that the rank of $\mathrm{rst}_G(n)^{ab}$ is m_{n+1} for all n. $\square$

5. Growth

Denote for any finitely generated group $\Gamma = \langle X \rangle$ for any element $\alpha = \prod_{i=1}^{m_\alpha} x_{j_i}^{\pm 1}$, $x_{j_i} \in X$ in Γ by

$$\lambda(\alpha) = \min\left\{ m_\alpha \colon \alpha = \prod_{i=1}^{m_\alpha} x_{j_i}^{\pm 1},\ x_{j_i} \in X \right\}$$

the word length of α in the generators X of Γ. Write $\gamma_\Gamma(n) = |\{\alpha \in \Gamma \colon \lambda(\alpha) \leqslant n\}|$ for the growth function of Γ.

Proposition 5.1. *The group G does not have polynomial growth.*

Proof. The free abelian group F_n^{ab} of rank n embeds into G for all $n \in \mathbb{N}$ as the proof of Theorem 4.5 shows. $\square$

If we say a group G acts as G_n on a vertex v of level n, then we mean that the projection of G onto a vertex v of level n acts as G_n on v.

Lemma 5.2. *Let $v_i \in \Omega(1)$ be the i-th vertex on level 1. Then the projection of G acts as G_1 on v_i for $i \in \{1, \dots, l_0\}$.*

Proof. The action of the projection of G onto v_2 is given by $b(1) = (b_1, a_1, 1, \dots, 1)_1$ and $b(2) = a^{-1}ba = (1, b_1, a_1, \dots, 1)_1$. Hence $b(1)$ and $b(2)$ generate G_1 on v_2. The same follows for every vertex v_i on level 1 with $b(i-1)$ and $b(i)$. $\square$

The last Lemma 5.2 allows us to make assumptions about G akin to assumptions that can be made about self-similar groups. In the next Lemma 5.3 we only decorate a third of the vertices of level i with G_i and then distribute them among the m_i vertices.

Lemma 5.3. $\gamma_G(m_i(n+i)) \geqslant \gamma_{G_i}(n)^{\lfloor \frac{m_i}{3^i} \rfloor} \cdot (\lfloor \frac{2l_{i-1}}{3} \rfloor)^{\lfloor \frac{m_i}{3^i} \rfloor}$.

Proof. Assume w is a word in the generators a, b of G such that $w(a,b)$ is a generator of G_1 on the second vertex v_2 of level 1. Then w can be chosen to have length at most 3, because a_1 can be obtained from $b(1)$ and b_1 from $b(2) = a^{-1}ba$. Every word in the generators a_1, b_1 of G_1 on $v_i \in \Omega(1)$ can be obtained by one in the generators a_1, b_1 of G_1 on v_2 and then conjugated by at most $a^{\pm \lfloor \frac{l_0}{2} \rfloor}$, which adds l_0 to its word length in $\{a, b\}$. In order to get $\frac{l_0}{3}$ words $w_j \in G_1$ on positions $0 < j < l_0$ it is enough to construct $\prod_{j=1}^{\lfloor l_0/3 \rfloor} w_j \cdot a^{q_j}$ with $1 \leqslant q_j \leqslant l_0$, depending on where we place the $\lfloor \frac{l_0}{3} \rfloor$ words w_j and $\sum_{j=1}^{\lfloor l_0/3 \rfloor} q_j = l_0$. Hence we get a recursion

$$\gamma_G\left(\left\lfloor \frac{l_0}{3} \right\rfloor 3n + l_0 \right) \geqslant \gamma_{G_1}(n)^{\lfloor \frac{l_0}{3} \rfloor} \cdot \left(\left\lfloor \frac{2l_0}{3} \right\rfloor \right)^{\lfloor \frac{l_0}{3} \rfloor}.$$

We can estimate this expression by $\frac{l_0}{3} \cdot 3n + l_0 \leqslant l_0 n + l_0$. Iterating this we get $(l_0 n + l_0)l_1 + l_1 = l_0 l_1 n + l_0 l_1 + l_1$ and so for the i-th step we can see that $m_i(n+i)$ gives an upper bound for this expression. Hence we get

$$\gamma_G\big(m_i(n+i)\big) \geqslant \gamma_{G_i}(n)^{\lfloor \frac{m_i}{3^i} \rfloor} \cdot \left(\left\lfloor \frac{2l_{i-1}}{3} \right\rfloor \right)^{\lfloor \frac{l_{i-1}}{3} \rfloor \cdot \lfloor \frac{m_{i-1}}{3^{i-1}} \rfloor}. \qquad \square$$

We are now able to show that G has exponential growth under certain assumptions.

Theorem 5.4. *The 2-generator group G has exponential growth rate if $\{l_i\}_{i\in\mathbb{N}_0}$ is a sequence of distinct primes and is such that $l_i \geqslant C^{3^{i+1}\cdot(2+i)+1}$ for some $C > 1$.*

Proof. We want an estimate for $\gamma_G(r)$. We can assume that $r = m_i(1+i)$, as the latter term grows unboundedly with i. Lemma 5.3 then gives

$$\gamma_G(r) = \gamma_G\big(m_i(1+i)\big) \geqslant \gamma_{G_i}(1)^{\frac{m_i}{3^i}} \cdot \left(\frac{2l_{i-1}}{3}\right)^{\frac{l_{i-1}}{3}\cdot\frac{m_{i-1}}{3^{i-1}}}.$$

We want to find an $\alpha > 0$ such that $\gamma_G(r) \geqslant e^{\alpha r}$. With $\gamma_{G_i}(1) = 2$, we get that we will need

$$e^{\alpha m_i(1+i)} \leqslant 2^{\frac{m_i}{3^i}} \cdot \left(\frac{2l_{i-1}}{3}\right)^{\frac{m_i}{3^i}}$$

which can be transformed into

$$\alpha \cdot m_i(1+i) \leqslant \frac{m_i}{3^i}\log(2) + \frac{m_i}{3^i}\log\left(\frac{2l_{i-1}}{3}\right).$$

It is enough to find a bound for $\alpha > 0$, so we focus on the second term of the sum on the right hand side which gives

$$\alpha \leqslant \frac{\log(2) + \log(l_{i-1}) - \log(3)}{3^i \cdot (1+i)}$$

and it is further enough to have

$$\alpha \leqslant \frac{\log(l_{i-1}) - 1}{3^i \cdot (1+i)}.$$

We see that in order to be able to find such an $\alpha > 0$ independent of i we have to require that $l_{i-1} \geqslant C^{3^i\cdot(1+i)+1}$ for some $C > 1$ which gives that any $\alpha \leqslant \log(C)$ can be chosen. $\square$

6. Non-trivial words

Our object in this section is to show that if the defining sequence satisfies $l_i \geqslant 36^i$ for all $i \in \mathbb{N}$ then the group constructed above has no free subgroups of rank 2. Indeed, given any two elements g_1, g_2 of the group we construct explicitly a non-trivial word w_{g_1,g_2} in the free group of rank 2 such that $w_{g_1,g_2}(g_1, g_2) = 1$.

We can write every $g \in G$ as $g = a^r \prod_{i=1}^{t} b(k_i)^{q_i}$ with $r, q_i \in \mathbb{Z}$, $k_i \in \{1, \dots, l_0\}$ and $t \in \mathbb{N}$.

Definition 6.1. A *spine* $s = g^{-1}b^q g$ is a power of a g-conjugate of b with $g \in G$ and some $q \in \mathbb{Z} \setminus \{0\}$. Denote by

$$\xi(g) = \min\left\{t \;\middle|\; g = a^r \prod_{i=1}^{t} b(k_i)^{q_i}\right\}$$

the *number of spines* of g and by

$$\lambda(g) = r + \sum_{i=1}^{t} (q_i + 2k_i - 2)$$

the *length* of g as before. This is the usual notation for the word length of an element which can be seen by considering the definition of $b(i) = a^{-i+1}ba^{i-1}$.

Lemma 6.2. *$\lambda(gh) \leqslant \lambda(g) + \lambda(h)$ and hence $\lambda(g^h) \leqslant \lambda(g) + 2\lambda(h)$ for any $g, h \in G$.*

Proof. This follows immediately from the definition. $\square$

Let $g_1, g_2 \in G$ be fixed for the rest of this section. Recursively define commutators

$$c_1 = [g_1, g_2] \quad \text{and} \quad c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], \quad \text{for } i \geqslant 2 \text{ with } c_0 = g_1. \tag{3}$$

With those definitions we get the following lemma:

Lemma 6.3. *If $g_1, g_2 \in G$, then the length $\lambda(c_i)$ of the commutator c_i defined as above is bounded by $\lambda(c_i) \leqslant 5^i(\lambda(g_1) + \lambda(g_2))$ for all $i \geqslant 0$.*

Proof. We use induction. It follows from Lemma 6.2 that $\lambda(c_1) \leqslant 2\lambda(g_1) + 2\lambda(g_2) \leqslant 5(\lambda(g_1) + \lambda(g_2))$. Then by induction hypothesis

$$\begin{aligned} \lambda(c_i) &\leqslant 4\lambda(c_{i-1}) + 4\lambda(c_{i-2}) \\ &\leqslant 4 \cdot 5^{i-1}\big(\lambda(g_1) + \lambda(g_2)\big) + 4 \cdot 5^{i-2}\big(\lambda(g_1) + \lambda(g_2)\big) \leqslant 5^i\big(\lambda(g_1) + \lambda(g_2)\big). \qquad \square \end{aligned}$$

The strategy is to observe that the number of spines of the commutators c_i grows more slowly than the number of vertices on each level. We note the position of the spines of c_i and aim to move them by conjugation such that none of the conjugated spines is at an old position. This new element will then commute with c_i.

Lemma 6.4. *For every $i \geqslant 1$ we have $c_i \in \mathrm{rst}_G(i-1) \leqslant \mathrm{St}_G(i)$.*

Proof. We have $c_2 \in G'' \leqslant \mathrm{rst}_G(1) \lhd G$. Hence $c_2^{c_1} \in \mathrm{rst}_G(1)$ and so

$$c_3 = \left[c_2, c_2^{c_1}\right] \in \mathrm{rst}_G(1)' \leqslant \mathrm{rst}_G(2).$$

Now assume $c_{n-1} \in \mathrm{rst}_G(n-2) \lhd G$. Then $c_{n-1}^{c_{n-2}} \in \mathrm{rst}_G(n-2)$ and hence again

$$c_n = \left[c_{n-1}, c_{n-1}^{c_{n-2}}\right] \in \mathrm{rst}_G(n-2)' \leqslant \mathrm{rst}_G(n-1).$$

The last statement $\mathrm{rst}_G(n-1) \leqslant \mathrm{St}_G(n)$ is given by Lemma 3.17. $\square$

We will now describe the commutators c_i. We will see below that after some finite level, their non-trivial decorations can only be one of two rather simple types of elements.

Proposition 6.5. *The commutators c_i have the recursive form $c_i = (d_{i,k,1}, \ldots, d_{i,k,m_k})_k$ on level k where each $d_{i,k,j}$ falls into one of the four cases:*

1. $d_{i,k,j} = 1$,
2. $d_{i,k,j} = b_k^t$ *for $t \in \mathbb{Z}$, b_k the generator of G_k, $b_k \in B_k$,*

3. $d_{i,k,j} = a_k^q \cdot z$ *with* $q \not\equiv 0 \bmod l_i$, $z \in B_k$, $a_k \in A_k$ *or*
4. $d_{i,k,j} = (d_{i,k+1,1+(j-1)l_i}, \dots, d_{i,k+1,j \cdot l_i})_{k+1}$.

Further, there exists some level n such that all $d_{i,n,j}$ *will have fallen into one of the first three cases.*

The indices in $d_{i,k,j}$ are as follows:

- i: means that $d_{i,k,j}$ is coming from the commutator c_i,
- k: describes the level k on which we look at the commutator c_i,
- j: indicates the position j on level k on which $d_{i,k,j}$ acts.

Write $\lfloor i/j \rfloor$ for the biggest integer q such that $q \leqslant \frac{i}{j}$.

Proof. From Lemma 6.4 we have $c_i = (d_{i,i,1}, \dots, d_{i,i,m_i})_i \in G_i \times \dots \times G_i$ with

$$d_{i,i,j} = a_i^{q_j} \prod_{k=0}^{u_j} b_i(r_{i,j,k})^{f_{i,j,k}}$$

and $q_j, f_{i,j,k} \in \mathbb{Z}$, $u_j \in \mathbb{N}$, $r_{i,j,k} \in \{1, \dots, l_i\}$. As an element of G_i, $d_{i,i,j}$ can only either be of the form $a_i^q z$ with $a_i \in A_i$, $z \in B_i$ which is the third case, or $d_{i,i,j} \in B_i$. If $d_{i,i,j}$ is an element of $B_i \times \dots \times B_i \leqslant G_{i+1} \times \dots \times G_{i+1}$ then it can be written as

$$d_{i,i,j} = \prod_{k=1}^{r} b_i(r_k).$$

Now assume that $r > 1$, hence $d_{i,i,j}$ is not of the form b_i^t for some $t \in \mathbb{Z}$, hence does not fall into the second case. Then $d_{i,i,j}$ can be expressed as

$$d_{i,i,j} = (d_{i,i+1,1+(j-1)l_i}, \dots, d_{i,i+1,j \cdot l_i})_{i+1} \in \mathrm{St}_G(i+1),$$

where each of the $d_{i,i+1,h}$ is an element of B_{i+1} for $h \in \{1, \dots, m_{i+1}\}$. Assume that at least one $d_{i,i+1,h}$ is again of this form, which is the fourth case. Then

$$d_{i,i+1,h} = a_{i+1}^{q_h} \prod_{s=1}^{y_h} b_{i+1}(f_{h,s})^{z_{h,s}}$$

with $q_h, z_{h,s} \in \mathbb{Z}$, $y_h \in \mathbb{N}$ and $f_{h,s} \in \{1, \dots, l_i\}$. We assume that not all $f_{h,s}$ are equal to 1 and that $q_h \equiv 0 \bmod l_i$ to eliminate cases 2 and 3. However, if there exists an $f_{h,s_0} \neq 1$ then $d_{i,i,j}$ was such that $b_{i+1}(f_{h,s_0}) = b_i(c)^{b_i(c-1)^q}$ for some $c \in \{1, \dots, l_{i-1}\}$ and some $q \in \mathbb{Z} \setminus \{0\}$. This yields that the word lengths satisfy $\lambda(d_{i,i+1,h}) < \lambda(d_{i,i,j}) - 1$ if $j = \lfloor h/l_i \rfloor$ and hence there exists a level n such that all $d_{i,n,m}$ fall into one of the first three cases. $\square$

The following proposition deduces from the decoration of a particular vertex v what the decoration of the vertex directly above v must look like.

Proposition 6.6. *Let x, y be elements of* $\mathrm{rst}_G(n-1)$. *If* $[x, y] \notin \mathrm{rst}_G(n+1)$ *then either* $x \notin \mathrm{rst}_G(n)$ *or* $y \notin \mathrm{rst}_G(n)$.

Proof. The elements x and y can also be seen as elements of G_{n-1}. The word

$$[x, y] = (h_1, \dots, h_{m_n})_n$$

is a commutator and hence by Proposition 6.5 we get that each h_j falls into one of the four cases above. Now assume that both x and y are of the form 1, 2 or 4. Then $x, y \in B_n \times \cdots \times B_n$ and so

$$[h, k] \in B_n' \times \cdots \times B_n' = N_n' \times \cdots \times N_n' = \mathrm{rst}_G(n)' \leqslant \mathrm{rst}_G(n+1) \leqslant \mathrm{St}_G(n+2)$$

and hence h_j cannot be of the third type because this case does not stabilize level n. □

Corollary 6.7. *For every $d_{i,i,j}$ in c_i that is of the second or third type we have that either*

$$d_{i,i-1,\lfloor j/l_i \rfloor} = a_{i-1}^q z \quad \text{or} \quad d_{i,i-1,\lfloor j/l_i \rfloor}^{d_{i-2,\lfloor j/(l_{i-1}l_i)\rfloor}} = a_{i-1}^q z$$

with $q \not\equiv 0 \bmod l_i$, $z \in B_{i-1}$, hence at least one of the two was of type 3.

Proof. This is an application of Proposition 6.6 with $x = d_{i,i-1,\lfloor j/l_i \rfloor}$ and $y = d_{i,i-1,\lfloor j/l_i \rfloor}^{d_{i,i-2,\lfloor j/(l_{i-1}l_i)\rfloor}}$. □

Corollary 6.7 allows us to deduce that if we have a vertex with non-trivial decoration, then there must be a decoration above it with a non-trivial rooted part, a power of some $a_i \in A_i$.

Theorem 6.8. *Assume that the defining sequence $\{l_i\}$ satisfies $l_i \geqslant 36^i$ where the l_i are pairwise coprime odd integers and we have $l_0 \geqslant 3$. Then G has no free subgroup of rank 2.*

Proof. Let $g_1, g_2 \in G$. From these we construct a non-trivial word $w_{g_1,g_2}(x, y)$ such that we have $w_{g_1,g_2}(g_1, g_2) = 1$. Find a level k such that

$$\lambda(c_k) \leqslant 6^k. \tag{4}$$

Such a k exists because we had that $\lambda(c_k) \leqslant 5^k(\lambda(g_1) + \lambda(g_2))$ by Lemma 6.3.

Write $c_k = (d_{k,k,1}, \dots, d_{k,k,m_k})_k$. Some of the $d_{k,k,t}$ might be of the fourth, recursive, case. Proposition 6.5 states that this case only occurs down to some finite level n. Every $d_{k,k,t}$ that is of the recursive case will then satisfy that there exists a level i_t such that d_{k,i_t,j_t} is of case 1, 2 or 3, decorating a vertex v_{j_t} of level i_t with $k \leqslant i_t \leqslant n$ and all j_t with $\lfloor j_t/(\prod_{r=k}^{i_t-1} l_r)\rfloor = t$. In this situation we have that d_{k,i_t,j_t} decorates a subtree $T_{v_{j_t}}$, with root $v_{j_t} \in \Omega(i_t)$. We need to look at those vertices which have a non-recursive decoration coming from c_k. Those will as just argued lie on different levels i_t between n and k.

We now aim to form commutator words. Let $v_{j_t} \in \Omega(i_t)$ be the vertex of level i_t on which d_{k,i_t,j_t} acts. By Corollary 6.7 we now have either

$$c_{k-1}|_{v_{j_t}} = a_{i_t}^q z \quad \text{or} \quad c_{k-1}^{c_{k-2}}|_{v_{j_t}} = a_{i_t}^q z \tag{5}$$

with $q \not\equiv 0 \bmod l_{i_t}$ and $z \in B_{i_t}$. We now shift the spines of $c_k|_{v_{j_t}}$ by conjugation such that their new position does not overlap with their previous one assuming l_{i_t} is large enough which we will justify in the second half of this proof. We are using the non-trivial power q of a_{i_t} from (5) to conjugate the spines of d_{k,i_t,j_t} to empty positions. This conjugated element of d_{k,i_t,j_t} will then commute with d_{k,i_t,j_t}. The non-trivially decorated subvertices of v_{j_t} can only have positions p with $-\lambda(c_k) < p < \lambda(c_k)$

because in order to decorate a subvertex of v_{j_t} at position p on level $i_t + 1$ we need to have a power $a_{i_t}^p$, which will add p to the word length of c_k. We have found 6^k to be an upper bound for the word length of c_k in (4). With these considerations we conclude that either

$$d = \left[d_{k,i_t,j_t}, d_{k,i_t,j_t}^{d_{k,i_t-1,\lfloor j/l_{i_t}\rfloor}^{6^k}}\right] \quad \text{or} \quad e = \left[d_{k,i_t,j_t}, d_{k,i_t,j_t}^{(d_{k,i_t-1,\lfloor j/l_{i_t}\rfloor}^{d_{k,i_t-2,\lfloor j/(l_k l_{i_t-1})\rfloor}})^{6^k}}\right] \tag{6}$$

is such that $d|_{v_{j_t}} = 1$ or $e|_{v_{j_t}} = 1$ for all vertices $v_{j_t} \in \Omega(i_t)$. We have the case that $d|_{v_{j_t}} = 1$ if we had $c_{k-1}|_{v_{j_t}} = a_{i_t}^q z$ in Eq. (5) and we have $e|_{v_{j_t}} = 1$ if we had $c_{k-1}^{c_{k-2}}|_{v_{j_t}} = a_{i_t}^q z$.

We now describe how this will yield a word $w_{g_1,g_2}(x, y) \in F(x, y)$. Similarly to the definition of c_k recursively define commutators in $F(x, y)$ as

$$\gamma_0 = x, \qquad \gamma_1 = [x, y], \qquad \gamma_i = \left[\gamma_{i-1}, \gamma_{i-1}^{\gamma_{i-2}}\right]$$

with $\gamma_i \in F(x, y)^{(i)}$, the i-th derived group of $F(x, y)$. We begin our word w by $w_1 = [\gamma_k, (\gamma_k^{\gamma_{k-1}})^{6^k}]$. This will give identity on the vertices v_j with $d|_{v_j} = 1$ with d as in (6). We will then proceed with $w_2 = [w_1, w_1^{(c_{k-1}^{c_{k-2}})}]$ to get identity for the vertices in which only the case $e|_{v_j} = 1$ applied.

The two cases in (6) are the only ones possible for any vertex v_j. Hence we either have $w_1(g_1, g_2)|_{v_j} = 1$ after the first step or $w_2(g_1, g_2)|_{v_{j_t}} = 1$ after the second step. Assume $w_1(g_1, g_2)|_{v_j} \neq 1$ for some vertex v_j. Then this means that $d_{k,i_t-1,\lfloor j/l_{i_t}\rfloor} \in B_{i_t-1}$, hence there is no rooted action and conjugating by $d_{k,i_t-1,\lfloor j/l_{i_t}\rfloor}$ will not move or add any spines. We then get that $w_2(g_1, g_2)|_{v_j} = 1$ in the second step.

We now have to justify that l_i is big enough in each step. We are conjugating by the power 6^k in (6). This applies to an element $a_{i_t}^q$, where $|q| < \lambda(c_k) \leqslant 6^k$. We get that this moves spines by at most $6^{2k} - 6^k$. Hence we need at most 6^{2k} places to fit these at most 6^k spines in and so each l_i for $i \geqslant k$ must be such that $6^{2k} \leqslant l_i$ for $i \geqslant k$. It is hence enough to require that the sequence $\{l_i\}_{i\in\mathbb{N}_0}$ is such that $l_i \geqslant 36^i$.

This yields that the procedure described above will result in a non-trivial word $w_{g_1,g_2}(x, y)$ in the free group $F(x, y)$ on the two generators x and y. This word has the form of a nested commutator and is an element of $F(x, y)^{(k)}$, the k-th derived group of $F(x, y)$, where k depends on the number of spines in the two chosen elements g_1 and g_2. This now implies that G cannot contain a non-abelian free subgroup under the given assumptions on the defining sequence $\{l_i\}$. □

This immediately implies that G cannot be large in this case. However, we can prove for any coprime sequence $\{l_i\}$ with $l_i \geqslant 3$ that G is not large:

Theorem 6.9. *The group G is not large.*

Proof. Assume for a contradiction that G is large. Then there exists a finite index subgroup H that maps onto the non-abelian free group of rank 2, hence also onto the alternating group A_5. Denote the kernel of the canonical map $H \to A_5$ by $N \leqslant H$. Then $N_0 = \bigcap_{g\in G} N^g$ is a proper normal subgroup of G. The quotient G/N_0 is soluble by Proposition 3.22 and hence cannot have a section isomorphic to A_5. □

Theorem 6.8 explicitly describes a relation that holds between two given elements of the group. In [8] Nekrashevych proves a theorem which describes which branch groups contain free subgroups.

Theorem 6.10. *(See [8].) Let Γ be a group acting faithfully on a locally finite rooted tree T. Then one of the following holds.*

1. *Γ has no free non-abelian subgroups,*
2. *there is a free non-abelian subgroup $F < \Gamma$ and a point $\omega \in \partial T$ such that the stabilizer F_ω is trivial,*
3. *there is a point $\omega \in \partial T$ such that the group of Γ-germs $\Gamma_{(\omega)}$ has a free non-abelian subgroup.*

Here the group of Γ-germs $\Gamma_{(\omega)}$ is the quotient of the stabilizer Γ_ω by the subgroup of automorphisms g of the tree T acting trivially on a neighborhood $U_g \subset \partial T$ of ω.

It might conceivably be possible to derive Theorem 6.8 using this criterion, but it does not seem to be a straightforward consequence. In particular, it can be shown that there exist subgroups with trivial stabilizer of every point in the boundary. This suggests, that excluding the second condition of Theorem 6.10 needs some similar considerations as in the proof of Theorem 6.8.

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