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Some new characterizations of *PST*-groups

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ABSTRACT

Let H and B be subgroups of a finite group G such that $G = N_G(H)B$. Then we say that H is *quasipermutable* (respectively, *S-quasipermutable*) in G provided H permutes with B and with every subgroup (respectively, with every Sylow subgroup) A of B such that $(|H|, |A|) = 1$. In this paper we analyze the influence of S-quasipermutable and quasipermutable subgroups on the structure of G. As an application, we give new characterizations of soluble *PST*-groups.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a subset of the set $\mathbb{P}$ of all primes; $\pi(G)$ denotes the set of all primes dividing $|G|$.

A subgroup H of G is said to be *quasinormal* or *permutable* in G if H permutes with every subgroup A of G, that is, $HA = AH$; H is said to be *S-permutable* in G if H permutes with every Sylow subgroup of G.

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A group G is called a *PT-group* if permutability is a transitive relation on G, that is, every permutable subgroup of a permutable subgroup of G is permutable in G. A group G is called a *PST-group* if S-permutability is a transitive relation on G.

As well as T-groups, *PT*-groups and *PST*-groups possess many interesting properties (see Chapter 2 in [1]). The general description of *PT*-groups and *PST*-groups was first obtained by Zacher [2] and Agrawal [3], for the soluble case, and by Robinson in [4], for the general case. Nevertheless, in the further publications, authors (see for example the recent papers [5–16]) have found out and described many other interesting characterizations of soluble *PT*- and *PST*-groups.

In this paper we give new "Hall"-characterizations of soluble *PST*-groups on the basis of the following

Definition 1.1. We say that a subgroup H is *quasipermutable* (respectively S*-quasipermutable*) in G provided there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with B and with every subgroup (respectively with every Sylow subgroup) A of B such that $(|H|, |A|) = 1$.

Examples and some applications of quasipermutable subgroups were discussed in the papers [17] and [18] (see also remarks in Section 7 below). In this paper, we prove the following result, which we consider as one more motivation for introducing the concept of quasipermutability.

Theorem A. *Let $D = G^{\mathfrak{N}}$ and $\pi = \pi(D)$. Then the following statements are equivalent*:

(i) *D is a Hall subgroup of G and every Hall subgroup of G is quasipermutable in G.*
(ii) *G is a soluble PST-group.*
(iii) *Every subgroup of G is quasipermutable in G.*
(iv) *Every π-subgroup of G and some minimal supplement of D in G are quasipermutable in G.*

We prove Theorem A in Section 6 on the basis of quasipermutability properties which we study in Sections 2–5. In particular, we use in the proof of this theorem the following three results.

A subgroup S of G is called a *Gaschütz* subgroup of G (L.A. Shemetkov [19, IV, 15.3]) if S is supersoluble and for any subgroups $K \leqslant H$ of G, where $S \leqslant K$, the number $|H : K|$ is not prime.

Theorem B. *The following statements are equivalent*:

(I) *G is soluble, and if S is a Gaschütz subgroup of G, then every Hall subgroup H of G satisfying $\pi(H) \subseteq \pi(S)$ is quasipermutable in G.*
(II) *G is supersoluble and the following hold*:
  (a) *$G = DC$, where $D = G^{\mathcal{N}}$ is an abelian complemented subgroup of G and C is a Carter subgroup of G*;
  (b) *$D \cap C$ is normal in G and $(p, |D/D \cap C|) = 1$ for all prime divisors p of $|G|$ satisfying $(p-1, |G|) = 1$*;
  (c) *For any non-empty set π of primes, every π-element of any Carter subgroup of G induces a power automorphism on the Hall π'-subgroup of D.*
(III) *Every Hall subgroup of G is quasipermutable in G.*

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. A subgroup H of G is said to be an $\mathfrak{F}$*-covering subgroup* of G provided $H \in \mathfrak{F}$ and $E = E^{\mathfrak{F}}H$ for any subgroup E of G containing H. By the Gaschütz theorem [20, VI, 9.5.4 and 9.5.6], for any saturated formation $\mathfrak{F}$, every soluble group G has an $\mathfrak{F}$-covering subgroup and any two $\mathfrak{F}$-covering subgroups of G are conjugate.

Theorem C. *Let $\mathfrak{F}$ be a saturated formation containing all nilpotent groups. Suppose that G is soluble and let $\pi = \pi(C) \cap \pi(G^{\mathfrak{F}})$, where C is an $\mathfrak{F}$-covering subgroup of G. If every maximal subgroup of every Sylow p-subgroup of G is S-quasipermutable in G for all $p \in \pi$, then $G^{\mathfrak{F}}$ is a Hall subgroup of G.*

Theorem D. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and $\pi = \pi(F^*(G^{\mathfrak{F}}))$. If $G^{\mathfrak{F}} \neq 1$, then for some $p \in \pi$ some maximal subgroup of a Sylow p-subgroup of G is not S-quasipermutable in G.*

In this theorem $F^*(G^{\mathfrak{F}})$ denotes the generalized Fitting subgroup of $G^{\mathfrak{F}}$, that is, the product of all normal quasinilpotent subgroups of $G^{\mathfrak{F}}$.

We prove Theorem B in Section 4. But before, in Section 3, we describe groups with a quasipermutable supersoluble Hall subgroup.

Theorems C and D are proved in Section 5. And the main tool for that is the following

Proposition. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $|P| > p$.*

(i) *If every member V of some fixed $\mathcal{M}_\phi(P)$ is S-quasipermutable in G, then E is p-supersoluble.*
(ii) *If every maximal subgroup of P is S-quasipermutable in G, then every chief factor of G between E and $O_{p'}(E)$ is cyclic.*
(iii) *If every maximal subgroup of every Sylow subgroup of E is S-quasipermutable in G, then every chief factor of G below E is cyclic.*

In this proposition we write $\mathcal{M}_\phi(G)$, by analogy with [21], to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathcal{M}_\phi(G)$.

Note that Proposition may be independently interesting because this result unifies and generalizes many known results, and in particular, Theorems 1.1–1.5 in [21] (see Section 7). In Section 7 we also discuss some further applications of the results.

All unexplained notation and terminology are standard. The reader is referred to [19,22], or [23] if necessary.

2. Basic lemmas

Let H be a subgroup of G. Then we say, following [17], that H is *propermutable* (respectively, *S-propermutable*) in G provided there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with all subgroups (respectively, with all Sylow subgroups) of B.

Lemma 2.1. *Let $H \leqslant G$ and N a normal subgroup of G. Suppose that H is quasipermutable (S-quasipermutable) in G.*

(1) *If either H is a Hall subgroup of G or for every prime p dividing $|H|$ and for every Sylow p-subgroup H_p of H we have $H_p \nleqslant N$, then HN/N is quasipermutable (S-quasipermutable, respectively) in G/N.*
(2) *If $\pi = \pi(H)$ and G is π-soluble, then H permutes with some Hall π'-subgroup of G.*
(3) *H permutes with some Sylow p-subgroup of G for every prime p dividing $|G|$ such that $(p, |H|) = 1$.*
(4) *$|G : N_G(H \cap N)|$ is a π-number, where $\pi = \pi(N) \cup \pi(H)$.*
(5) *If H is propermutable (S-propermutable) in G, then HN/N is propermutable (S-propermutable, respectively) in G/N.*
(6) *If H is S-propermutable in G, then H permutes with some Sylow p-subgroup of G for any prime p dividing $|G|$.*
(7) *Suppose that G is π-soluble. If H is a Hall π-subgroup of G, then H is propermutable (S-propermutable, respectively) in G.*

Proof. By hypothesis, there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with B and with all subgroups (with all Sylow subgroups, respectively) A of B such that $(|H|, |A|) = 1$.

(1) It is clear that

$$G/N = \bigl(N_G(H)N/N\bigr)(BN/N) = N_{G/N}(HN/N)(BN/N).$$

Let K/N be any subgroup (any Sylow subgroup, respectively) of BN/N such that $(|HN/N|, |K/N|) = 1$. Then $K = (K \cap B)N$. Let B_0 be a minimal supplement of $K \cap B \cap N$ to $K \cap B$. Then $K/N = (K \cap$

$B)N/N = B_0(K \cap B \cap N)N/N = B_0N/N$ and $K \cap B \cap N \cap B_0 = N \cap B_0 \leqslant \Phi(B_0)$. Therefore $\pi(K/N) = \pi(K \cap B/K \cap B \cap N) = \pi(B_0)$, so $(|HN/N|, |B_0|) = 1$. Suppose that some prime $p \in \pi(B_0)$ divides $|H|$, and let H_p be a Sylow p-subgroup of H. We shall show that $H_p \nleqslant N$. In fact, we may suppose that H is a Hall subgroup of G. But in this case, H_p is a Sylow p-subgroup of G. Therefore, since $p \in \pi(B_0) \subseteq \pi(G/N)$, $H_p \nleqslant N$. Hence p divides $|HN/N|$, a contradiction. Thus $(|H|, |B_0|) = 1$, so in the case, when H is quasipermutable in G, we have $HB_0 = B_0H$ and hence HN/N permutes with $K/N = B_0N/N$. Thus HN/N is quasipermutable in G/N.

Finally, suppose that H is S-quasipermutable in N. In this case, B_0 is a p-subgroup of B, so for some Sylow p-subgroup B_p of B we have $B_0 \leqslant B_p$ and $(|H|, p) = 1$. Hence $K/N = B_0N/N \leqslant B_pN/N$, which implies that $K/N = B_pN/N$. But H permutes with B_p by hypothesis, so HN/N permutes with K/N. Therefore HN/N is S-quasipermutable in G/N.

(2) By [20, VI, 4.6], there are Hall π'-subgroups E_1, E_2 and E of $N_G(H)$, B and G, respectively, such that $E = E_1E_2$. Then H permutes with all Sylow subgroups of E_2 by hypothesis, so

$$\begin{aligned} HE &= H(E_1E_2) = (HE_1)E_2 = (E_1H)E_2 \\ &= E_1(HE_2) = E_1(E_2H) = (E_1E_2)H = EH \end{aligned}$$

by [22, A, 1.6].

(3) See the proof of (2).

(4) Let p be a prime such that $p \notin \pi$. Then by (3), there is a Sylow p-subgroup P of G such that $HP = PH$ is a subgroup of G. Hence $HP \cap N = H \cap N$ is a normal subgroup of HP. Thus p does not divide $|G : N_G(H \cap N)|$.

(5) See the proof of (1).

(6) See the proof of (2).

(7) Since G is π-soluble, B is π-soluble. Hence by [20, VI, 1.7], $B = B_\pi B_{\pi'}$ where B_π is a Hall π-subgroup of B and $B_{\pi'}$ is a Hall π'-subgroup of B. By [20, VI, 4.6], there are Hall π-subgroups N_π, B_π and G_π of $N_G(H)$, B and G, respectively, such that $G_\pi = N_\pi B_\pi$. But since $H \leqslant N_\pi$, N_π is a Hall π-subgroup of G. Therefore $G_\pi = N_\pi B_\pi = N_\pi$, so $B_\pi \leqslant N_\pi$. Hence $G = N_G(H)B = N_G(H)B_\pi B_{\pi'} = N_G(H)B_{\pi'}$, so H is propermutable (S-propermutable, respectively) in G. □

Lemma 2.2. *Let H and B be subgroups of G. If $G = N_G(H)B$ and $HV^b = V^bH$ for some subgroup V of B and for all $b \in B$, then $HV^x = V^xH$ for all $x \in G$.*

Proof. Since $G = N_G(H)B$ we have $x = nb$ for some $n \in N_G(H)$ and $b \in B$. Hence $HV^x = HV^{nb} = Hn(V^b)n^{-1} = n(V^b)n^{-1}H = V^xH$. □

Lemma 2.3. *Suppose that for the subgroups A and B of G we have $AB = BA$ and $G = N_G(A)B$. Then*

(1) $A^G = A(A^G \cap B)$.
(2) *If A permutes with all Sylow p-subgroups of B, then A permutes with all Sylow p-subgroups of $A^G \cap B$.*
(3) *If p is a prime dividing $|A^G|$ such that p does not divide $|A|$ and A permutes with all Sylow p-subgroups of B, then A permutes with all Sylow p-subgroups of A^G.*

Proof. (1) Since $AB = BA$, AB is a subgroup of G and so $A^G = A^{N_G(A)B} = A^B \leqslant \langle A, B\rangle = AB$. Hence $A^G = A^G \cap AB = A(A^G \cap B)$.

(2) By (1) we have $A^G = A(A^G \cap B)$. Let P be any Sylow p-subgroup of $A^G \cap B$ and $P \leqslant B_p$ where B_p is a Sylow p-subgroup of B. Then $AB_p = B_pA$ and $P = A^G \cap B \cap B_p = A^G \cap B_p$. Hence $AB_p \cap A^G = A(B_p \cap A^G) = AP = PA$.

(3) Let P be any Sylow p-subgroup of A^G. Then, since p does not divide $|A|$, for some $x \in A$, P^x is a Sylow p-subgroup of $A^G \cap B$ by (1). Let B_p be a Sylow p-subgroups of B such that $P^x \leqslant B_p$. Then $AB_p = B_pA$ is a subgroup of G, so $AB_p \cap A^G = A(B_p \cap A^G) = AP^x = P^xA$. Now, since $x \in A$, we have $(AP^x)^{x^{-1}} = A^{x^{-1}}P = AP = PA$. □

Lemma 2.4. *(See Kegel [25].) Let A and B be subgroups of G such that $G \neq AB$ and $AB^x = B^x A$, for all $x \in G$. Then G has a proper normal subgroup N such that either $A \leqslant N$ or $B \leqslant N$.*

Lemma 2.5. *(See Knyagina and Monakhov [24].) Let H, K and N be subgroups of G. If N is normal in G, H permutes with K and H is a Hall subgroup of G, then*

$$N \cap HK = (N \cap H)(N \cap K).$$

A group G is said to be a C_π*-group* provided G has a Hall π-subgroup and any two Hall π-subgroups of G are conjugate.

Lemma 2.6. *Let H be a Hall S-quasipermutable subgroup of G. If $\pi = \pi(|G : H|)$, then G is a C_π-group.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Then $|\pi| > 1$. By hypothesis there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with B and with every Sylow subgroup A of B such that $(|H|, |A|) = 1$.

(1) $H^G = G = HB$.

By Lemma 2.3, $H^G = H(H^G \cap B)$ and H permutes with all Sylow p-subgroups of $H^G \cap B$ for all primes $p \in \pi$. Hence H is S-quasipermutable in H^G. Suppose that $H^G \neq G$. Then H^G is a C_π-group by the choice of G. On the other hand, G/H^G is a π-group since H is a Hall π'-subgroup of G. Therefore G is a C_π-group by [19, IV, 18.12], contrary to the choice of G. Hence $G = H^G = H(H^G \cap B) = HB$.

(2) *If Q is a Sylow q-subgroup of G, where q is a prime dividing $|G|$ such that $q \in \pi$, then $Q^G \neq G$.*

Since $|\pi| > 1$, $HQ \neq G$. On the other hand, by (1) and Lemma 2.2, $HQ^x = Q^x H$ for all $x \in G$. Hence in view of (1) and Lemma 2.4 we have $Q^G \neq G$.

(3) *Q^G is a C_π-group.*

By Lemma 2.5, $Q^G = (Q^G \cap H)(Q^G \cap B)$, where $Q^G \cap H$ is a Hall π'-subgroup of Q^G. Let R be a Sylow r-subgroup of $Q^G \cap B$, where $r \in \pi$. Then for some Sylow r-subgroup B_r of B we have $HB_r = B_r H$ and

$$R = B_r \cap \left(Q^G \cap B\right) = B_r \cap Q^G.$$

Therefore by Lemma 2.5,

$$Q^G \cap HB_r = \left(Q^G \cap H\right)\left(Q^G \cap B_r\right) = \left(Q^G \cap H\right)R = R\left(Q^G \cap H\right).$$

Thus the hypothesis holds for $(Q^G, Q^G \cap H)$. Therefore we have (3) by (2) and the choice of G.

Final contradiction. By Lemma 2.1 the hypothesis holds for G/Q^G. Therefore G/Q^G is a C_π-group by the choice of G. Hence G is a C_π-group by (3) and [19, IV, 18.12]. This final contradiction completes the proof. □

Lemma 2.7. *Let E be a normal subgroup of G and H a Hall π-subgroup of E. If H is nilpotent and S-quasipermutable in G, then E is π-soluble.*

Proof. See proof of Lemma 2.6 and use the Kegel–Wielandt theorem on solubility of the product of nilpotent groups [20, VI, 4.3]. □

Lemma 2.8. *Let A and B be subgroups of G. If $A^x B = BA^x$ for all $x \in G$, then $AB^x = B^x A$ for all $x \in G$.*

Proof. From $A^{x^{-1}} B = BA^{x^{-1}}$ we get $AB^x = (A^{x^{-1}} B)^x = (BA^{x^{-1}})^x = B^x A$. □

3. Groups with a Hall quasipermutable subgroup

A group G is said to be *π-separable* if every chief factor of G is either a π-group or a π'-group. Every π-separable group G has a series

$$1 = P_0(G) \leqslant M_0(G) < P_1(G) < M_1(G) < \cdots < P_t(G) \leqslant M_t(G) = G$$

such that

$$M_i(G)/P_i(G) = O_{\pi'}\big(G/P_i(G)\big)$$

$(i = 0, 1, \ldots, t)$ and

$$P_{i+1}(G)/M_i(G) = O_{\pi}\big(G/M_i(G)\big)$$

$(i = 1, \ldots, t)$.

The number t is called the *π-length* of G and denoted by $l_\pi(G)$ (see [34, p. 249]).

In this section we prove the following result.

Theorem 3.1. *Let H be a Hall subgroup of G and $\pi = \pi(H)$. Suppose that H is quasipermutable in G.*

(I) *If $p > q$ for all primes p and q such that $p \in \pi$ and q divides $|G : N_G(H)|$, then H is normal in G.*
(II) *If H is supersoluble, then G is π-soluble.*
(III) *If G is π-separable, then the following holds:*
  (i) *$H' \leqslant O_\pi(G)$. If, in addition, $N_G(H)$ is nilpotent, then $G' \cap H \leqslant O_\pi(G)$.*
  (ii) *$l_\pi(G) \leqslant 2$ and $l_{\pi'}(G) \leqslant 2$.*
  (iii) *If for some prime $p \in \pi'$ a Hall π'-subgroup E of G is p-supersoluble, then G is p-supersoluble.*

Let $\mathfrak{M}$ and $\mathfrak{H}$ be non-empty formations. Then the *Gaschütz product* $\mathfrak{M} \circ \mathfrak{H}$ of these formations is the class of all groups G such that $G^{\mathfrak{H}} \in \mathfrak{M}$. It is well-known that such an operation on the set of all non-empty formations is associative (W. Gaschütz). The symbol $\mathfrak{M}^t$ denotes the product of t copies of $\mathfrak{M}$.

We shall need the following well-known lemma of Gaschütz and Shemetkov [26, Corollary 7.13].

Lemma 3.2. *The product of any two non-empty saturated formations is also a saturated formation.*

Lemma 3.3. *The class $\mathcal{F}$ of all π-separable groups G with $l_\pi(G) \leqslant t$ is a saturated formation.*

Proof. It is not difficult to show that for any non-empty set $\omega \subseteq \mathbb{P}$ the class $\mathfrak{G}_\omega$ of all ω-groups is a saturated formation and that $\mathfrak{F} = (\mathfrak{G}_{\pi'} \circ \mathfrak{G}_\pi)^t \circ \mathfrak{G}_{\pi'}$. Hence $\mathfrak{F}$ is a saturated formation by Lemma 3.2. □

Lemma 3.4. *Suppose that G is separable. If Hall π-subgroups of G are abelian, then $l_\pi(G) \leqslant 1$.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Let N be a minimal normal subgroup of G. Since G is π-separable, N is a π-group or a π'-group. It is clear that the hypothesis holds for G/N, so $l_\pi(G/N) \leqslant 1$ by the choice of G. By Lemma 3.3, the class of all π-soluble groups with $l_\pi(G) \leqslant 1$ is a saturated formation. Therefore N is a unique minimal normal subgroup of G, $N \nleqslant \Phi(G)$ and N is not a π'-group. Hence N is a π-group and $N = C_G(N)$ by [22, A, 15.2]. Therefore $N \leqslant H$, where H is a Hall π-subgroup of G. But since H is abelian, $N = H$ is a Hall π-subgroup of G. Hence $l_\pi(G) \leqslant 1$. □

A group G is called *π-closed* provided G has a normal Hall π-subgroup.

Lemma 3.5. *Let H be a Hall π-subgroup of G. If G is π-separable and $H \leqslant Z(N_G(H))$, then G is π'-closed.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Then $G \neq H$. The class $\mathfrak{F}$ of all π'-closed groups coincides with the product $\mathfrak{G}_{\pi'} \circ \mathfrak{G}_{\pi}$. Hence $\mathfrak{F}$ is a saturated formation by Lemma 3.2. Let N be a minimal normal subgroup of G. Since G is π-separable, N is a π-group or a π'-group. Moreover, G is a C_π-group by [34, 9.1.6], so the hypothesis holds for G/N. Hence G/N is π'-closed by the choice of G. Therefore N is the only minimal normal subgroup of G, $N \nleqslant \Phi(G)$ and N is a π-group. Therefore $N \leqslant H$ and $N = C_G(N)$ by [22, A, 15.2]. Since $H \leqslant Z(N_G(H))$ and H is a Hall π-subgroup of G, $N = H$. Therefore $N \leqslant Z(G)$, which implies that $N = H = G$. This contradiction completes the proof of the lemma. □

The following lemma is well-known (see for example Lemma 2.1.6 in [1]).

Lemma 3.6. *If G is p-supersoluble and $O_{p'}(G) = 1$, then p is the largest prime dividing $|G|$, G is supersoluble and $F(G) = O_p(G)$ is a Sylow p-subgroup of G.*

Finally, we also need the following elegant result of V.S. Monakhov.

Lemma 3.7. *(See Monakhov [28].) If $G = AB$, where A is a supersoluble subgroup of G and B is a Sylow p-subgroup of G for some odd prime p, then G is soluble.*

Proof of Theorem 3.1. Suppose that this theorem is false and let G be a counterexample of minimal order. By hypothesis, there is a subgroup B of G such that $G = N_G(H)B$ and H permutes with B and with every subgroup A of B such that $(|H|, |A|) = 1$. By Lemma 2.3, $H^G = H(H^G \cap B)$ and H permutes with every subgroup A of $H^G \cap B$ such that $(|H|, |A|) = 1$. Therefore H is quasipermutable in H^G.

(I) Suppose that this assertion is false.

(1) *$V = H^G \cap B$ is a $C_{\pi'}$-group. Hence $H^G = HB_0$, where B_0 is a Hall π'-subgroup of V.*

Let $H_0 = H \cap V$. Since H is a Hall π-subgroup of H^G, H_0 is a Hall π-subgroup of V. Now let A be a subgroup of V such that $(|H_0|, |A|) = 1$. Then $(|H|, |A|) = 1$, which implies that $HA = AH$. Hence $HA \cap V = A(H \cap V) = AH_0 = H_0A$. Therefore H_0 is quasipermutable in V. Thus V is a $C_{\pi'}$-group by Lemma 2.6.

(2) *The hypothesis holds for (H^G, H).*

By (1), $H^G = HB_0$, where B_0 is a Hall π'-subgroup of $H^G \cap B$, so H permutes with all subgroups of B_0. Now let p be a prime dividing $|H^G : N_{H^G}(H)|$. Let $P \leqslant G_p$, where P and G_p are Sylow p-subgroups of H^G and G, respectively. Suppose that there is a prime $q \in \pi$ such that $p > q$. Then, by hypothesis, for some $x \in G$ we have $(G_p)^x \leqslant N_G(H)$. Hence $(G_p)^x \cap H^G = (G_p \cap H^G)^x = P^x \leqslant N_{H^G}(H)$. Hence p does not divide $|H^G : N_{H^G}(H)|$. This contradiction shows that for any primes p and q, where $p \in \pi$ and q divides $|H^G : N_{H^G}(H)|$ we have $p > q$. Therefore the hypothesis holds for (H^G, H).

(3) *$H^G = HB_0 = G$, where B_0 is a Hall π'-subgroup of B.*

Suppose that $H^G \neq G$. Then H is normal, and so also characteristic in H^G, by (2) and the choice of G. Hence H is normal in G, a contradiction. Thus we have (3) by (1).

In view of (3), we assume without loss of generality that $B_0 = B$ is a Hall π'-subgroup of G.

(4) *$G = HP$ for some Sylow p-subgroup P of G such that $P \leqslant B$ and $p < q$ for all primes $q \in \pi$.*

Let P be a Sylow p-subgroup of G such that p divides $|G : N_G(H)|$. Then $p < q$ for all primes $q \in \pi$ by hypothesis, so for some $x \in G$ we have $P^x \leqslant B$. Then $HP = PH$ and H permutes with all subgroups of P by Lemma 2.2. Hence the hypothesis holds for HP. Suppose that $HP \neq G$. Then, H is normal in HP by the choice of G. Therefore p does not divide $|G : N_G(H)|$. This contradiction shows that $G = HP$.

(5) *The hypothesis holds for every subgroup E of G containing H.*

By (4), $E = H(E \cap P)$, H is quasipermutable in E and $p < q$ for all primes $q \in \pi$.

Final contradiction for (I).

Let V be a maximal subgroup of P. Then by (4) and (5) the hypothesis holds for HV, so H is normal in HV by the choice of G. On the other hand, since $|G : HV| = p$ and $p < q$ for all primes $q \in \pi$, HV is normal in G. Therefore H is normal in G, contrary to the choice of G. Hence we have (I).

(II) Suppose that this assertion is false. Then $H^G = G = HB$. Otherwise, since H is quasipermutable in H^G, H^G and G/H^G are π-soluble by the choice of G, which implies the π-solubility of G. Now let Q be a Sylow q-subgroup of G, where q is a prime dividing $|G|$ such that $q \notin \pi$. Then a Sylow q-subgroup Q of B is a Sylow subgroup of G and $HQ^x = Q^xH$ for all $x \in G$ by Lemma 2.2. Suppose that $HQ = G$. Since H is supersoluble, $q = 2$ by Lemma 3.7 since G is not soluble. Hence H is normal in G by Assertion (I). But then G is π-soluble, a contradiction. Therefore $HQ \neq G$, so $Q^G \neq G$ by Lemma 2.4. Now arguing similarly as in the proof of Lemma 2.6 one can show that the hypothesis holds for Q^G and so Q^G is π-soluble by the choice of G, which implies the π-solubility of G. This contradiction completes the proof of Assertion (II).

(III) Since G is π-separable, G is a $C_{\pi'}$-group by [34, 9.1.6]. Let E be a Hall π'-subgroup of G.

(i) Suppose that this assertion is false. Then:

(1) $O_\pi(N) = 1$ *for any normal subgroup* N *of* G.

Suppose that $O_\pi(G) \neq 1$. By Lemma 2.1(1), the hypothesis holds for $G/O_\pi(G)$. Hence Assertion (i) is true for $G/O_\pi(G)$ by the choice of G. Thus

$$H'O_\pi(G)/O_\pi(G) \leqslant \big(H/O_\pi(G)\big)' \leqslant O_\pi\big(G/O_\pi(G)\big) = 1,$$

and if $N_G(H)$ is nilpotent, then

$$\begin{aligned}\big(G/O_\pi(G)\big)' \cap \big(H/O_\pi(G)\big) &= \big(G'O_\pi(G)/O_\pi(G)\big) \cap \big(H/O_\pi(G)\big) \\ &= O_\pi(G)\big(G' \cap H\big)/O_\pi(G) \leqslant O_\pi\big(G/O_\pi(G)\big) = 1.\end{aligned}$$

Hence we have $H' \leqslant O_\pi(G)$ in the former case, and $G' \cap H \leqslant O_\pi(G)$ in the case, when $N_G(H)$ is nilpotent. Thus Assertion (i) is true for G, a contradiction. Therefore $O_\pi(G) = 1$. Finally, since $O_\pi(N)$ is characteristic in N, $O_\pi(N) \leqslant O_\pi(G) = 1$. Hence we have (1).

(2) *H is not abelian.*

Suppose that H is abelian and $N_G(H)$ is nilpotent. Then $H \leqslant Z(N_G(H))$, so G is π'-closed by Lemma 3.5. Hence E is normal in G. Since H is abelian, $G' \leqslant E$. Therefore $G' \cap H = 1 \leqslant O_\pi(G)$, contrary to our assumption on G. Hence we have (2).

(3) $C_G(O_{\pi'}(G)) \leqslant O_{\pi'}(G) \neq 1$.

By (1), $O_\pi(G) = 1$. Therefore, since G is π-separable, $O_{\pi'}(G) \neq 1$ and $C_G(O_{\pi'}(G)) \leqslant O_{\pi'}(G)$ by [27, 6, 3.2].

(4) $H^G = G$ *and* $G = HB$.

Suppose that $H^G \neq G$. Since $H^G = H(H^G \cap B)$ and H is quasipermutable in H^G, it follows that $H' \leqslant O_\pi(H^G)$ by the choice of G. But by (1), $O_\pi(H^G) = 1$. Therefore $H' = 1$, so H is abelian, which contradicts (2). Thus $H^G = G$ and $G = HB$.

(5) *G is not supersoluble.*

Suppose that G is supersoluble. Then $G' \leqslant F(G)$, so $G' \cap H \leqslant O_\pi(G) = 1$. Hence $H \simeq G'H/G'$ is abelian, contrary to (2). Hence we have (5).

(6) $HO_{\pi'}(G) = G$.

Suppose that $E = HO_{\pi'}(G) \neq G$. By (4) we have $O_{\pi'}(G) \leqslant B$. Hence H is quasipermutable in E. Thus $H' \leqslant O_\pi(E)$ by the choice of G. Therefore $H' \leqslant C_G(O_{\pi'}(G)) \leqslant O_{\pi'}(G)$ by (3). Hence H is abelian, which contradicts (2). Thus $HO_{\pi'}(G) = G$.

(7) Final contradiction for (i). Let V be any subgroup of $O_{\pi'}(G)$. Then by (4) for any $x \in G$ we have $HV^x = V^xH$. Hence $VH^x = H^xV$ for all $x \in G$ by Lemma 2.8. Now note that $V = H^xV \cap O_{\pi'}(G)$ is normal in H^xV, so $H^x \leqslant N_G(V)$. But then, by (4), $G = H^G \leqslant N_G(V)$. Therefore every subgroup of $O_{\pi'}(G)$ is normal in G. Hence every chief factor of G below $O_{\pi'}(G)$ is cyclic. But by (3) we have $C_G(O_{\pi'}(G)) \leqslant O_{\pi'}(G)$, so $G/O_{\pi'}(G)$ is supersoluble by the Schmid–Shemetkov theorem on $\mathfrak{U}$-stable groups of automorphisms. [19, II, 9.2]. But then G is supersoluble, contrary to (5). Therefore Assertion (i) is true for G.

(ii) Suppose that this assertion is false. Since by (i) we have $H' \leqslant O_\pi(G)$, the Hall π-subgroup $HO_\pi(G)/O_\pi(G)$ of $G/O_\pi(G)$ is abelian. Hence, by $l_\pi(G/O_\pi(G)) \leqslant 1$ by Lemma 3.4. But then $l_\pi(G) \leqslant 2$.

It is clear that $l_{\pi'}(G/O_\pi(G)) = l_{\pi'}(G)$. Hence in the case when $O_\pi(G) \neq 1$ the choice of G implies that $l_{\pi'}(G) \leqslant 2$, contrary to our assumption on G. Therefore $O_\pi(G) = 1$. But by (i) we have $H' \leqslant O_\pi(G)$, so H is abelian. Hence by Lemma 3.4, $l_\pi(G) \leqslant 1$. Thus $l_{\pi'}(G) \leqslant 2$.

(iii) Suppose that this assertion is false. Let N be a minimal normal subgroup of G. Then the hypothesis holds for G/N, so G/N is p-supersoluble by the choice of G. Therefore $O_{p'}(G) = 1$. In particular, $N \nleqslant H$ and p divides $|N|$, which implies that $N \leqslant E$ and N is a p-group since E is p-supersoluble. Moreover, since the class of all p-supersoluble groups is a saturated formation, N is the only minimal normal subgroup of G and $N \nleqslant \Phi(G)$. Hence $N \leqslant H^G$, and $N = C_G(N)$ by [22, A, 15.2]. Since $H^G = H(H^G \cap B)$, it follows that $N \leqslant B$. Thus H permutes with all subgroups of N. Since E is p-supersoluble, N has a maximal subgroup V such that V is normal in E. On the other hand, $HV \cap N = V$ is normal in HV. Hence $G = HE \leqslant N_G(V)$, which in view of the minimality of N implies that $V = 1$. Hence $|N| = p$, so $G/N = G/C_G(N)$ is a cyclic group of exponent dividing $p - 1$. But then G is supersoluble. This contradiction completes the proof of Assertion (iii). The theorem is proved. □

4. Proof of Theorem B

(I) $\Rightarrow$ (II). Suppose that this is false and let G be a counterexample of minimal order. Then G is not nilpotent, so $D = G^{\mathfrak{N}} \neq 1$. Since G is soluble, the class of all Gaschütz subgroups of G coincides with the set of all $\mathfrak{U}$-covering subgroups of G by [19, 15.1]. Therefore by [34, VI, 9.5.4 and 9.5.6], G has a Gaschütz subgroup and any two Gaschütz subgroups of G are conjugate, since the class of all supersoluble groups is a saturated formation. Let S be a Gaschütz subgroup of G and $\pi = \pi(S)$. Let C be a Carter subgroup of G, a any π-element of C and E the Hall π'-subgroup of D.

(1) *The hypothesis holds for any quotient G/N of G.*

It is clear that G/N is soluble. Now, let S_1/N be a Gaschütz subgroup of G/N and $\pi_1 = \pi(S_1/N)$. Let W be a Gaschütz subgroup of S_1. Then $S_1/N = WN/N \simeq W/W \cap N$. Moreover, in view of [19, IV, 15.1], W is a Gaschütz subgroup of G, so $\pi_1 \subseteq \pi(W) = \pi(S) = \pi$. Let H_1/N be any Hall subgroup of G/N with $\pi_2 = \pi(H_1/N) \subseteq \pi_1$. Let E be a Hall π_2-subgroup of H_1. Then E is a Hall π_2-subgroup of G and $H_1/N = EN/N$. Hence E is quasipermutable in G by hypothesis and so H_1/N is quasipermutable in G/N by Lemma 2.1. Hence the hypothesis holds for G/N.

(2) *G is supersoluble and $G = DC$, where C is a Carter subgroup of G.*

First we shall show that G is supersoluble. Suppose that this is false. Let N be a minimal normal subgroup of G. Since G is soluble, N is a p-group for some prime p. Moreover, the hypothesis holds for G/N by (1). Hence G/N is supersoluble by the choice of G. Therefore $G = NS = N \rtimes S$ since G is not supersoluble. Moreover, N is the only minimal normal subgroup of G and $|N| > p$. Let E be a Hall p'-subgroup of S. Then E is quasipermutable in G. Let B be a subgroup of G such that $G = N_G(E)B$ and E permutes with all subgroups A of B satisfying $(|E|, |A|) = 1$. By Lemma 2.3, $E^G = E(E^G \cap B)$. It is clear that $N \leqslant E^G$, which implies that $N \leqslant B$ and so E permutes with all subgroups of N. Let V be a maximal subgroup of N such that V is normal in a Sylow p-subgroup G_p of G. Then $V \neq 1$ and $EV = VE$ is a subgroup of G. Therefore $N \cap VE = V$ is normal in VE. Hence $G = G_pE \leqslant N_G(V)$, which contradicts the minimality of N. Hence G is supersoluble.

Finally, since G is supersoluble, $D \leqslant F(G)$ by [20, VI, 9.1]. Therefore for any Carter subgroup C of G we have $G = DC$ by [20, VI, 12.3].

(3) *D is a p-group for some prime p.*

Suppose that $|\pi(D)| > 1$. We shall show that under this hypothesis Assertions (a)–(c) are true for G. Let P and Q be the Sylow p-subgroup and the Sylow q-subgroup of D, respectively, where $p \neq q$ are primes dividing $|D|$. In view of (1), the hypothesis holds for G/P and for G/Q. Hence Assertions (a)–(c) are true for G/P and for G/Q by the choice of G. By Assertion (a), $D/P = (G/P)^{\mathcal{N}}$ and $D/Q = (G/Q)^{\mathcal{N}}$ are abelian. Hence D is abelian since $D \simeq D/P \cap Q$ is isomorphic to a subgroup of the direct product $(D/P) \times (D/Q)$, and so D is complemented in G by [22, IV, 5.18]. Thus, in view of (2), Assertion (a) is true for G.

By [20, IV, 11.3], PC/P is a Carter subgroup of G/P and so every π-element of PC/P induces a power automorphism on the Hall π'-subgroup PE/P of D/P. Thus for any subgroup H of E we have $a \in N_G(HP)$. Similarly, $a \in N_G(HQ)$. Thus $a \in N_G(HP \cap HQ) = N_G(H)$, so Assertion (b) is true for G.

Since D is abelian and $G = DC$, the subgroup $C \cap D$ is normal in G. Finally, let r be any prime satisfying $(r-1, |G|) = 1$. Then $(r-1, |G/P|) = 1$ and $(r-1, |G/Q|) = 1$. Hence r does not divide $|D/P : (D/P) \cap (PC/P)| = |D : P(D \cap C)|$ since Assertion (c) is true for G/P. Similarly, we deduce that r does not divide $|D : Q(D \cap C)|$. Hence $(r, |D/D \cap C|) = 1$ since $|D : D \cap C| = |P(D \cap C) : D \cap C||D : P(D \cap C)| = |Q(D \cap C) : D \cap C||D : Q(D \cap C)|$ and $p \neq q$. Therefore Assertion (c) is true for G. But then Assertions (a)–(c) are true for G, contrary to the choice of G. Hence we have (3).

(4) *A Sylow p-subgroup P of G is normal in G.*

Indeed, since P/D is a Sylow p-subgroup of the nilpotent group $G/D = G/G^{\mathcal{N}}$, P is normal in G.

(5) $(p-1, |G|) \neq 1$. *Hence $p > 2$.*

Suppose that $(p-1, |G|) = 1$. Since $G \neq C$ and $G = DC$, $D \cap C \neq D$. Let H/K be any chief factor of G such that $D \cap C \leqslant K < H \leqslant D$. Since G is supersoluble, $|H/K| = p$. Hence $G/C_G(H/K)$ is a cyclic group of exponent dividing $p-1$. But then $C_G(H/K) = G$ since $(p-1, |G|) = 1$. Therefore C covers H/K, that is, $(H \cap C)K = H$ by [20, VI, 13.4 and 11.10]. But since $D \cap C \leqslant K$, we have $H \cap C \leqslant K$ and hence $(H \cap C)K = K$. This contradiction shows that $(p-1, |G|) \neq 1$.

Now let W be a Hall p'-subgroup of G contained in C. Let B be a subgroup of G such that $G = N_G(W)B$ and W permutes with all subgroups A of B satisfying $(|W|, |A|) = 1$. Then $W^G = W(W^G \cap B)$ by Lemma 2.3.

(6) *Let $B_0 = W^G \cap P$. Then $W^G = WB_0$ and $D \leqslant B_0 \leqslant B$.*

Indeed, $W^G = W^G \cap PW = W(W^G \cap P) = WB_0$. By [20, VI, 12.2], C is abnormal in G. Hence $CW^G = CWB_0 = CB_0 = G$. Hence $G/B_0 = CB_0/B_0 \simeq C/C \cap B_0$ is nilpotent. Therefore $D \leqslant B_0$. Finally, since $B_0 \leqslant W^G \leqslant WB$ and B_0 is normal in G, we get $B_0 \leqslant B$.

(7) *$W^G \leqslant N_G(H)$ for any subgroup H of D. Hence every element of W induces a power automorphism on D.*

Since $D \leqslant B_0 \leqslant B$ and D is normal in G, $WH^x = H^xW$ is a subgroup of G for all $x \in G$. Hence $HW^x = W^xH$ for all $x \in G$ by Lemma 2.8. Therefore $H = D \cap W^xH$ is normal in W^xH. Thus $W^G \leqslant N_G(H)$.

(8) *The group D is abelian.*

First note that in view of (6) and (7), D is a Dedekind group. On the other hand, by (5), $p > 2$ and hence D is abelian.

Final contradiction for the implication (I) $\Rightarrow$ (II). In view of (3) we may assume that $E = D$. Then for some $x \in G$ we have $a \in W^x$, so a induces a power automorphism on D by (7). Therefore all Assertions (a)–(c) are true for G, which contradicts the choice of G.

(II) $\Rightarrow$ (III). Let E be any Hall subgroup of G. We shall show that E is quasipermutable in G. Since G is supersoluble, $G = N_G(E)D$. Hence we have only to show that E permutes with any subgroup H of D. In fact, in view of [22, A, 1.6], it is enough to consider the case where H is a p-group and E is a Sylow q-group of G for some primes $q \neq p$. But in this case, by hypothesis, every element of E induces a power automorphism on D_p, where D_p is the Sylow p-subgroup of D and so $E \leqslant N_G(H)$. Hence $EH = HE$. Thus E is quasipermutable G.

(III) $\Rightarrow$ (I). This implication follows from Theorem 3.1.

The theorem is proved.

5. Proofs of Proposition and Theorems C and D

A chief factor H/K of G is called $\mathfrak{F}$-*central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_{\mathfrak{F}}(G)$ denotes the product of all normal subgroups E of G such that every chief factor of G below E is $\mathfrak{F}$-central [22, p. 389].

Lemma 5.1. *(See Theorem B in [29].) Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \leqslant Z_{\mathfrak{F}}(G)$, then $E \leqslant Z_{\mathfrak{F}}(G)$.*

The formation $\mathfrak{F}$ is said to be *hereditary* if $H \in \mathfrak{F}$ whenever $H \leqslant G \in \mathfrak{F}$.

Lemma 5.2. *(See Corollary 1.6 in [32].) Let $\mathfrak{F}$ be a hereditary saturated formation containing all nilpotent groups and E a normal subgroup of G. If $E/E \cap \Phi(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.*

The following lemma is well-known (see for example Lemma 2.2 in [29]).

Lemma 5.3. *Let E be a normal p-subgroup of G. If $E \leqslant Z_{\mathfrak{U}}(G)$, then $G/C_G(E)$ is an extension of some p-group by an abelian group of exponent dividing $p-1$.*

Lemma 5.4. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $(p-1, |G|) = 1$. If either P is cyclic or G is p-supersoluble, then E is p-nilpotent and $E/O_{p'}(E) \leqslant Z_\infty(G/O_{p'}(E))$.*

Proof. First note that in view of the proof of [34, 10.1.9], E is p-supersoluble. Let H/K be any chief factor of G such that $O_{p'}(E) \leqslant K < H \leqslant E$. Then $|H/K| = p$, so $G/C_G(H/K)$ divides $p-1$. But by hypothesis, $(p-1, |G|) = 1$. Hence $C_G(H/K) = G$. Thus $E/O_{p'}(E) \leqslant Z_\infty(G/O_{p'}(E))$. □

Proof of Proposition. Suppose that this proposition is false and let G be a counterexample with $|G| + |E|$ minimal.

(i) Suppose that this assertion is false. Let $V \in \mathcal{M}_\phi(P)$. By hypothesis there is a subgroup B of G that $G = N_G(V)B$ and V permutes with B and with every Sylow q-subgroup of B for all primes $q \neq p$ dividing $|B|$.

(1) *$V^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup of $V^G \cap B$ for all primes $q \neq p$ dividing* $|V^G \cap B|$ (this directly follows from Lemma 2.3).

(2) *$O_{p'}(N) = 1$ for every normal subgroup N of G contained in E.*

Suppose that for some normal subgroup N of G contained in E we have $O_{p'}(N) \neq 1$. Since $O_{p'}(N)$ is a characteristic subgroup of N, it is normal in G. On the other hand, by Lemma 2.1, the hypothesis holds for $(G/O_{p'}(N), E/O_{p'}(N))$. Hence $E/O_{p'}(N)$ is p-supersoluble by the choice of (G, E). Thus E is p-supersoluble, a contradiction.

(3) *If L is a minimal normal subgroup of G, then $L \nleqslant \Phi(P)$.*

Indeed, in the case, where $L \leqslant \Phi(P)$, we have $L \leqslant \Phi(E)$ and the hypothesis holds for $(G/L, E/L)$ by Lemma 2.1. Hence E/L is p-supersoluble by the choice of (G, E). Therefore E is p-supersoluble by Lemma 5.2, which contradicts to our assumption on E.

(4) *If D is a normal p-soluble subgroup of G contained in E, then D is supersoluble and p-closed.*

By (2), $O_{p'}(D) = 1$. Therefore $O_p = O_p(D) \neq 1$. Let N be a minimal normal subgroup of G contained in O_p. In view of (3) we have $N \nleqslant \Phi(P)$. Hence for some subgroup $W \in \mathcal{M}_\phi(P)$ we have $P = NW$. Let $S = N \cap W$. Then S is normal in P. On the other hand, by Lemma 2.1, for any prime $q \neq p$ dividing $|E|$, there are Sylow q-subgroups Q and E_q of G and E, respectively, such that $WQ = QW$ and $E_q = Q \cap E$. Hence $S = QW \cap N$ is a normal subgroup of QW and so $E_q \leqslant N_E(S)$. Thus S is normal in E. By Proposition 4.13(c) in [22, Chapter A], $N = N_1 \times \cdots \times N_t$, where $N_1, \ldots, N_t$ are minimal normal subgroups of E, and from the proof of this proposition we also know that $|N_i| = |N_j|$ for all i, j. Therefore there is a minimal normal subgroup L of E such that $N = SL$ and $S \cap L = 1$. Then $|L| = p$, so $N_1, \ldots, N_t$ are groups of order p by [22, A, 3.2]. Hence $P = L \rtimes W$, which implies by the Gaschütz theorem [20, I, 17.4] that L has a complement M in E. Thus $N \nleqslant \Phi(E)$. It is clear that $\Phi(E) \cap O_p$ is normal in G. Therefore $\Phi(E) \cap O_p = 1$. Hence every minimal normal subgroup of E contained in O_p is not contained in $\Phi(P)$. Therefore $O_p = L_1 \times \cdots \times L_r$, where $L_1, \ldots, L_r$ are minimal normal subgroups of E by [22, A, 13.8(b)]. Moreover, as above, it can be shown that $|L_i| = p$ for all $i = 1, \ldots, r$. Therefore every chief factor of E below O_p is cyclic by [22, A, 3.2]. It is clear that $C_E(O_p) = C_E(L_1) \cap \cdots \cap C_E(L_r)$. Hence $E/C_E(O_p)$ is an abelian group of exponent dividing $p-1$. Hence $DC_E(O_p)/C_E(O_p) \simeq D/C_E(O_p) \cap D$ is abelian. But since D is p-soluble and $O_{p'}(D) = 1$, $C_E(O_p) \cap D = C_D(O_p) \leqslant O_p$ by [27, 6, 3.2]. Hence D is supersoluble and O_p is a Sylow p-subgroup of D, by Lemma 3.6, since $O_{p'}(D) = 1$.

(5) *E is p-soluble.*

Assume that E is not p-soluble.

(a) *If $O_p(E) \neq 1$, then P is not cyclic.*

Suppose that P is cyclic. Let L be a minimal normal subgroup of G contained in $O_p(E) \leqslant P$. Suppose that $C_E(L) = E$, so $L \leqslant Z(E)$. Let $N = N_E(P)$. If $P \leqslant Z(N)$, then E is p-nilpotent by Burnside's theorem [20, IV, 2.6], which contradicts to our assumption on E. Hence $N \neq C_E(P)$. Let $x \in N \backslash C_E(P)$

with $(|x|, |P|) = 1$ and $K = P \rtimes \langle x \rangle$. By [20, III, 13.4], $P = [K, P] \times (P \cap Z(K))$. Since $L \leqslant P \cap Z(K)$ and P is cyclic, it follows that $P = P \cap Z(K)$ and so $x \in C_K(P)$. This contradiction shows that $C_E(L) \neq E$.

Since P is cyclic, $|L| = p$. Hence $G/C_G(L)$ is a cyclic group of order dividing $p - 1$. If $|P/L| > p$, then the hypothesis holds for $(G/L, E/L)$ by Lemma 2.1. Hence E/L is p-supersoluble by the choice of (G, E) and so E is p-soluble, a contradiction. Thus $|P/L| = p$ and hence $V = L$ is normal in G. Therefore the hypothesis holds for $(G, C_E(L))$, so $C_E(L)$ is p-supersoluble since $C_E(L) \neq E$. But then E is p-soluble since $E/C_E(L) = E/E \cap C_G(L) \simeq EC_G(L)/C_G(L)$ is cyclic. This contradiction shows that we have (a).

(b) *If $P \nleqslant V^G$, then V is normal in G.*

Indeed, since $P \nleqslant V^G \leqslant E$, V is a Sylow p-subgroup of V^G. On the other hand, by (1) we have that $V^G = V(V^G \cap B)$ and V is S-quasipermutable in V^G. Therefore V^G is p-soluble by Lemma 2.7. Thus V is normal in V^G by (4). Since V is a Sylow p-subgroup of V^G, V is characteristic in V^G. Hence $V = V^G$ is normal in G.

(c) *P is not cyclic.*

Suppose that P is cyclic. Then $\mathcal{M}_\phi(P) = \{V\}$, and by (a) and (b) we have $P \leqslant V^G = V(V^G \cap B)$ and V permutes with every Sylow q-subgroup of $V^G \cap B$ for all primes $q \neq p$ dividing $|V^G \cap B|$. Hence the hypothesis holds for (V^G, V^G). Assume that $V^G \neq G$. Then V^G is p-supersoluble by the choice of (G, E). Hence by (4), P is normal in G, which contradicts (a). Therefore $V^G = G$, which implies that $G = VB$ by (1). Hence $P = P \cap VB = V(P \cap B)$, so $P \leqslant B$ since P is cyclic. Therefore $B = G$, so V is S-permutable in G. Hence $V \leqslant P_E \leqslant O_p(E)$, which contradicts (a). Hence P is not cyclic.

(d) *P permutes with every Sylow q-subgroup Q of P^G for all primes $q \neq p$ dividing $|P^G|$.*

Let $D = P^G$. In view (c), there is a subgroup $W \in \mathcal{M}_\phi(P)$ such that $V \neq W$. Then $P = VW$. Hence we have only to show that V and W permute with Q. In view of (b) we may assume that $P \leqslant V^G$ and $P \leqslant W^G$. Then $D = P^G \leqslant V^G$ and so by (1), $D = V(D \cap B)$ and V permutes with every Sylow q-subgroup Q_1 of $D \cap B$. It is also clear that Q_1 is a Sylow q-subgroup of D. Therefore for some $x \in D$ we have $Q_1 = Q^x$. Hence V permutes with Q by Lemma 2.2. Similarly, it may be proved that W permutes with Q.

Final contradiction for (5). By (d), P is S-quasipermutable in P^G. Therefore by Lemma 2.7, P^G is p-soluble. Hence by (4), P is normal in G. Therefore E is p-soluble. This contradiction completes the proof of (5).

By (5), E is p-soluble. Hence E is supersoluble by (4). This contradiction completes the proof of (i).

(ii) Let $Z = Z_{\mathfrak{U}}(G)$. First we shall show that $O_{p'}(E) = 1$. Indeed, suppose that $O_{p'}(E) \neq 1$. It is clear that $O_{p'}(E)$ is normal in G. Moreover, the hypothesis holds for $(G/O_{p'}(E), E/O_{p'}(E))$ by Lemma 2.1. Therefore every chief factor of $G/O_{p'}(E)$ between $E/O_{p'}(E)$ and 1 is cyclic by the choice of (G, E). Hence every chief factor of G between E and $O_{p'}(E)$ is cyclic, a contradiction. Thus $O_{p'}(E) = 1$.

By (i), E is p-supersoluble. Hence in view of Lemma 3.6, E is supersoluble and $P = F(E)$. Therefore the hypothesis is true for (G, P). If $P \neq E$, then every chief factor of G below P is cyclic by the choice of (G, E). Hence every chief factor of G below E is cyclic by Lemma 5.1, contrary to the choice of (G, E). Hence $P = E$.

Let N be any minimal normal subgroup of G contained in P. Then every chief factor of G/N below P/N is cyclic. Indeed, if $|P/N| > p$, then the hypothesis holds for $(G/N, P/N)$, so this assertion is true by the choice of $(G, E) = (G, P)$. Thus $|N| > p$. Assume that $N \leqslant \Phi(P)$. Then, in view of Lemma 5.3 and [27, 5, 1.4], $G/C_G(P)$ is an extension of some p-group by an abelian group of exponent dividing $p - 1$. Therefore $G/C_G(N)$ is an abelian group of exponent dividing $p - 1$ since $O_p(G/C_G(N)) = 1$ by [22, A, 13.6]. Hence $|N| = p$ by [30, 1, 1.4]. This contradiction shows $\Phi(P) = 1$ and so P is elementary abelian. Let W be a maximal subgroup of N such that W is normal in a Sylow p-subgroup G_p of G. Let $V = WS$, where S is a complement of N in P. Then $W = V \cap N$ and V is S-quasipermutable in G by hypothesis. Hence by Lemma 2.1(4), $G = G_pN_G(W)$. Hence W is normal in G, so $W = 1$. This contradiction shows that we have (ii).

(iii) In view of (ii) we have $P \neq E$. Let p be the smallest prime dividing $|E|$ and P a Sylow p-subgroup of E. Then E is p-nilpotent. Indeed, if $|P| = p$, it follows directly from Lemma 5.4. If $|P| > p$, then E is p-supersoluble by (i), so E is p-nilpotent again by Lemma 5.4. Let $V = O_{p'}(E)$. Since V is characteristic in E, it is normal in G and the hypothesis holds for (G, V) and $(G/V, E/V)$

by Lemma 2.1. Since $P \neq E$, $V \neq 1$, so $E/V \leqslant Z_{\mathfrak{U}}(G/V)$ by the choice of (G, E). It is also clear that $V \leqslant Z_{\mathfrak{U}}(G)$. Hence $E \leqslant Z_{\mathfrak{U}}(G)$, a contradiction. The proposition is proved. □

Proof of Theorem C. Suppose that this theorem is false and let G be a counterexample of minimal order.

Let $D = G^{\mathfrak{F}}$. Then $G \neq D \neq 1$ and for any $\mathfrak{F}$-covering subgroup C of G we have $G = DC$. Let P be a Sylow p-subgroup of D such that $1 < P < G_p$ for a Sylow p-subgroup G_p of G. Then $|G_p| > p$ and $p \in \pi$.

(1) *The hypothesis holds on G/N for any minimal normal subgroup N of G.*

Indeed, let W/N be an $\mathfrak{F}$-covering subgroup of G/N and $\pi_0 = \pi(W/N) \cap \pi(DN/N) = \pi(W/N) \cap \pi((G/N)^{\mathcal{F}})$. Then, by [20, VI, 7.9 and 7.10], there is an $\mathfrak{F}$-covering subgroup C of G such that $W/N = CN/N \simeq C/C \cap N$. Hence $\pi_0 \subseteq \pi$. Let V/N be a maximal subgroup of a Sylow q-subgroup Q/N of G/N, where $q \in \pi_0$. Then for some Sylow q-subgroup G_q of G and for some maximal subgroup Q_1 of G_q we have $V/N = Q_1N/N$. Therefore by hypothesis, Q_1 is S-quasipermutable in G. Hence V/N is S-quasipermutable in G by Lemma 2.1. Hence we have (1).

(2) *If N is a minimal normal subgroup of G contained in D, then $N = O_p(D) = F(D)$ is a Sylow p-subgroup of D and every Sylow q-subgroup Q of D, where $q \neq p$, is a Sylow q-subgroup of G.*

Since G is soluble, N is an r-group for some prime r. By (1), the hypothesis holds on G/N, so $D/N = (G/N)^{\mathfrak{F}}$ is a Hall subgroup of G by the choice of G. Therefore, if S is a Sylow subgroup of D such that $(|S|, r) = 1$, then NS/N is a Sylow subgroup of G/N, which implies that S is a Sylow subgroup of G. Hence $r = p$, $N = O_p(D) = F(D) = P$ and Q is a Sylow q-subgroup of G.

(3) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$ and let R be a minimal normal subgroup of G contained in $O_{p'}(G)$. Then, in view (2), $R \cap D = 1$. Moreover, the hypothesis holds for G/R by (1). Therefore $(G/R)^{\mathcal{N}} = DR/R \simeq D$ is a Hall subgroup of G/R. But then $P = G_p$, a contradiction. Hence we have (3).

(4) *G is supersoluble, $D = N$ and G_p is normal in G.*

Since $|G_p| > p$ and $p \in \pi$, G is p-supersoluble by Proposition. Hence in view of (3) and Lemma 3.6, G is supersoluble and G_p is normal in G. Thus $D = N \leqslant F(G) = G_p$ by (3) since $\mathcal{F}$ contains all nilpotent groups by hypothesis.

(5) *$\Phi(G_p) = 1$, that is, G_p is elementary abelian.*

Suppose that $\Phi(G_p) \neq 1$ and let R be a minimal normal subgroup of G contained in $\Phi(G_p)$. Then $R \leqslant \Phi(G)$. If $R = D = G^{\mathcal{F}}$, then $G \in \mathcal{F}$ since the formation $\mathcal{F}$ is saturated by hypothesis. But then $D = 1$, a contradiction. Hence $R \neq D$. It is also clear that $RD \neq G_p$. But the hypothesis holds for G/R by (1), so $DR/R = G_p/N$ by the choice of G. Thus $RD = G_p$. This contradiction shows that we have (5).

(6) *Every subgroup of G_p is normal in G.*

In view of (4) we only need show that every maximal subgroup V of G_p is normal in G. By hypothesis V is S-quasipermutable in G. Hence by Lemma 2.1 for any prime $q \neq p$ there is a Sylow q-subgroup Q of G such that $VQ = QV$, which implies that $V = VQ \cap G_p$ is normal in VQ. Hence $|G : N_G(V)|$ is a p-number, so V is normal in G.

Final contradiction. By the Maschke's theorem, $G_p = \langle a \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$, where $\langle a_i \rangle$ is a minimal normal subgroup of G, $\langle a \rangle = D$. Write $a_1 = aa_2 \ldots a_t$. Then since $\langle a_1 \rangle \cap \langle a_2 \rangle \ldots \langle a_t \rangle = 1$, we have $G_p = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$. Note that $\langle a_1 \rangle$ is normal in G. Therefore from the G-isomorphism $D\langle a_1 \rangle/D \simeq \langle a_1 \rangle$ we have $\langle a_i \rangle \leqslant Z_{\mathfrak{F}}(G)$. It is also clear that $\langle a_2 \rangle \times \cdots \times \langle a_t \rangle \leqslant Z_{\mathfrak{F}}(G)$. Hence $G_p \leqslant Z_{\mathfrak{F}}(G)$, which implies that $G \in \mathfrak{F}$, a contradiction. The theorem is proved. □

Proof of Theorem D. Suppose that this theorem is false and let G be a counterexample of minimal order. Then for every $p \in \pi$ every maximal subgroup of a Sylow p-subgroup of G is S-quasipermutable in G. We shall show that under this condition every chief factor of G below of $F^*(G^{\mathfrak{F}})$ is cyclic. First we show that $F^*(G^{\mathfrak{F}})$ is soluble. Let p be the smallest prime in π and P a Sylow p-subgroup of G. Then $F^*(G^{\mathfrak{F}})$ is p-nilpotent. Indeed, if $|P| = p$, then $F^*(G^{\mathfrak{F}})$ is p-nilpotent directly by Lemma 5.4. Otherwise, G is p-supersoluble by Proposition and so $F^*(G^{\mathfrak{F}})$ is p-nilpotent by Lemma 5.4 again. Hence $F^*(G^{\mathfrak{F}})$ is soluble by the Feit–Thompson theorem, which implies that $F^*(G^{\mathfrak{F}}) = F(G^{\mathfrak{F}})$ by

[31, X, 13.7]. Now let H/K be any chief factor of G below $F^*(G^{\mathfrak{F}})$. Suppose that $|H/K| = q^n$, where $n > 1$. Then G is q-supersoluble by Proposition. Hence H/K is cyclic, that is, $n = 1$, a contradiction. Thus every chief factor of G below of $F^*(G^{\mathfrak{F}})$ is cyclic, so every chief factor of G below of $G^{\mathfrak{F}}$ is also cyclic by Lemma 5.1. But then $G \in \mathfrak{F}$ since $\mathfrak{F}$ contains all supersoluble groups by hypothesis. Hence $D = 1$. This contradiction completes the proof of the theorem.

6. Proof of Theorem A

Recall that G is a soluble *PST*-group if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is an abelian Hall subgroup of G and every element $x \in M$ induces a power automorphism on D [3]. Therefore the implication (i) $\Rightarrow$ (ii) is a direct corollary of Theorem B.

Now suppose that $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$, is a soluble *PST*-group. Let H be any subgroup of G and S a Hall π'-subgroup of H. Since G is soluble, we may assume without loss of generality that $S \leqslant M$. Hence $H = (D \cap H)(M \cap H) = (D \cap H)S$ and $D \cap H$ is normal in G. Let $\pi_1 = \pi(S)$. Let A be a Hall π_1-subgroup of M and E a complement to A in M. Then $E \leqslant C_G(S)$. Therefore $G = DM = DAE = N_G(H)(DA)$ and every subgroup L of DA satisfying $(|H|, |L|) = 1$ is contained in D. Thus H is quasipermutable in G. Thus (ii) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (ii) By Theorems C and D, G is supersoluble and D is a Hall subgroup of G. Therefore $G = D \rtimes W$, where W is a Hall π'-subgroup of G. By hypothesis, W is quasipermutable in G. Now arguing similarly as in the proof of Theorem B one can show that D is abelian and every subgroup of D is normal in G. Therefore G is a *PST*-group.

7. Final remarks

1. The subgroup S_3 is quasipermutable, S-propermutable and not propermutable in S_4. If H is the subgroup of order 3 in S_3, then H is S-quasipermutable and not quasipermutable in S_4.

2. Arguing similarly to the proof of Theorem A one can prove the following fact.

Theorem 7.1. *Suppose that G is soluble and let $\pi = \pi(G^{\mathfrak{N}})$. Then G is a PST-group if and only if every subnormal π-subgroup and a Hall π'-subgroup of G are propermutable in G.*

3. If G is metanilpotent, that is $G/F(G)$ is nilpotent, then for every Hall subgroup E of G we have $G = N_G(E)F(G)$. Therefore, in this case, every characteristic subgroup of every Hall subgroup of G is S-propermutable in G. In particular, every Hall subgroup of every supersoluble group is S-propermutable. This observation makes natural the following question: *What is the structure of G under the hypothesis that every Hall subgroup of G is propermutable in G?* Theorem B gives an answer to this question.

4. Every maximal subgroup of a supersoluble group is quasipermutable. Therefore, in fact, Theorem A shows that the class of all soluble groups in which quasipermutability is a transitive relation coincides with the class of all soluble *PST*-groups.

5. We say that G is an *SQT-group* if S-quasipermutability is a transitive relation in G. Arguing similarly to the proof of Theorem A one can prove the following fact.

Theorem 7.2. *A soluble group G is an SQT-group if and only if $G = D \rtimes M$ is supersoluble, where D and M are Hall nilpotent subgroups of G and the index $|G : DN_G(H \cap D)|$ is a $\pi(H)$-number for every subgroup H of G.*

6. A subgroup H of G is called *SS-quasinormal* [21] (*semi-normal* [33]) in G provided G has a subgroup B such that $HB = G$ and H permutes with all Sylow subgroups (H permutes with all subgroups, respectively) of B.

It is clear that every SS-quasinormal subgroup is S-propermutable and every semi-normal subgroup is propermutable. Moreover, there are simple examples (consider, for example, the group $C_7 \rtimes \mathrm{Aut}(C_7)$, where C_7 is a group of order 7) which show that, in general, the class of all S-propermutable subgroups of G is wider than the class of all its SS-quasinormal subgroups and the class of all propermutable subgroups of G is wider than the class of all its semi-normal subgroups. Therefore Proposition covers the main results (Theorems 1.1–1.5) in [21].

7. We have already used Theorem 3.1 in the proof of Theorem B. From this result we also get

Corollary 7.3. *(See [35, Theorem 5.4].) Let H be a Hall semi-normal subgroup of G. If $p > q$ for all primes p and q such that p divides $|H|$ and q divides $|G : H|$, then H is normal in G.*

Corollary 7.4. *(See [36, Theorem].) Let P be a Sylow p-subgroup of G. If P is semi-normal in G, then the following statements hold*:

(i) *G is p-soluble and $P' \leqslant O_p(G)$.*
(ii) *$l_p(G) \leqslant 2$.*
(iii) *If for some prime $q \in p'$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.*

Corollary 7.5. *(See [37, Theorem 3].) If a Sylow p-subgroup P of G, where p is the largest prime dividing $|G|$, is semi-normal in G, then P is normal in G.*

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