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Categorical Lagrangian Grassmannians and Brauer–Picard groups of pointed fusion categories

Dmitri Nikshych *, Brianna Riepel

Department of Mathematics and Statistics, University of New Hampshire, Durham, NH 03824, USA

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ABSTRACT

We analyze the action of the Brauer–Picard group of a pointed fusion category on the set of Lagrangian subcategories of its center. Using this action we compute the Brauer–Picard groups of pointed fusion categories associated to several classical finite groups. As an application, we construct examples of weakly group-theoretical fusion categories.

Contents

* Corresponding author.
E-mail addresses: nikshych@math.unh.edu (D. Nikshych), briepel@wildcats.unh.edu (B. Riepel).

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1. Introduction

Let $\mathcal{A}$ be a fusion category. The *Brauer–Picard* group $\mathsf{BrPic}(\mathcal{A})$ of $\mathcal{A}$ consists of equivalence classes of semisimple invertible $\mathcal{A}$-bimodule categories (see [12]). Brauer–Picard groups play an important role in the theory of fusion categories. In particular, they are used in the classification of graded extensions of fusion categories [12]. In the case when $\mathcal{A}$ is the category of representations of a Hopf algebra the group $\mathsf{BrPic}(\mathcal{A})$ is known as the *strong Brauer group*, see [1].

Computing Brauer–Picard groups for concrete examples of fusion categories is an important task. A number of special results of this type were obtained in the literature, see, e.g., [2,3,16]. In this paper we develop techniques that allow to compute explicitly Brauer–Picard groups of pointed (and, hence, group-theoretical) fusion categories. We use the following characterization of Brauer–Picard groups established in [12]. For any fusion category $\mathcal{A}$ there is a canonical isomorphism:

$$\Phi : \mathsf{BrPic}(\mathcal{A}) \to \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathcal{A})\big), \tag{1}$$

where $\mathcal{Z}(\mathcal{A})$ is the *Drinfeld center* of $\mathcal{A}$ and $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$ is the group of braided autoequivalences of $\mathcal{Z}(\mathcal{A})$. The latter group has a distinct geometric flavor (e.g., when $\mathcal{A}$ is the representation category of a finite Abelian group A, the group $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$ is the split orthogonal group $O(A \oplus \widehat{A})$). This suggests the use of "categorical-geometric" methods for computation of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$ (which is identified with $\mathsf{BrPic}(\mathcal{A})$ via isomorphism (1)).

In this paper we analyze the action of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$, where $\mathcal{A} = \mathsf{Vec}_G$ is the category of vector spaces graded by a finite group G, on the categorical Lagrangian Grassmannian $\mathbb{L}(G)$ associated to it. By definition, the latter is the set of Lagrangian subcategories of $\mathcal{Z}(\mathcal{A})$. The set $\mathbb{L}(G)$ was described in group-theoretical terms in [20]. We determine the point stabilizers for this action and explicitly compute the corresponding permutation groups in a number of concrete examples. Note that Mombelli in [16] studied the group $\mathsf{BrPic}(\mathsf{Vec}_G)$ using methods different from ours.

Module categories over a braided fusion category $\mathcal{C}$ can be regarded as $\mathcal{C}$-bimodule categories. In this case the group $\mathsf{BrPic}(\mathcal{C})$ contains a subgroup $\mathsf{Pic}(\mathcal{C})$, called the *Picard group* of $\mathcal{C}$, consisting of invertible $\mathcal{C}$-module categories [12]. This group is isomorphic to the group of Morita equivalence classes of Azumaya algebras in $\mathcal{C}$ (the latter group was introduced in [22]).

One defines a homomorphism

$$\partial : \mathsf{Pic}(\mathcal{C}) \to \mathsf{Aut}^{br}(\mathcal{C}), \tag{2}$$

in a way parallel to (1). It was shown in [12] that (2) is an isomorphism for every non-degenerate braided fusion category $\mathcal{C}$. One has $\mathsf{Pic}(\mathcal{Z}(\mathcal{A})) \cong \mathsf{BrPic}(\mathcal{A})$ for any fusion category $\mathcal{A}$.

The paper is organized as follows.

In Section 3 we collect results about finite group cohomology that will be used for computations. Section 4 contains definitions and basic facts about fusion categories and their Brauer–Picard groups.

In Section 5 we present a useful parameterization of the group $\mathsf{BrPic}(\mathsf{Vec}_G)$ previously obtained by Davydov in [5]. This parameterization allows one easily recognize involutions in $\mathsf{BrPic}(\mathsf{Vec}_G)$ (see Corollary 5.6).

In Section 6 we describe, following [12], the construction of isomorphism (1) between the Brauer–Picard group of a fusion category and the group of braided autoequivalences of its center. This allows us to concentrate on the computation of the latter group. For a braided fusion category $\mathcal{C}$ we find the subgroup of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ stabilizing the subcategory $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$ (see Proposition 6.8 and Corollary 6.9).

The action of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$ on the Lagrangian Grassmannian $\mathbb{L}(G)$ (which is, by definition, the set of Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_G)$) is studied in Section 7. In general, this action is not transitive. We show in Proposition 7.6 that the orbit of this action containing the canonical subcategory $\mathsf{Rep}(G) \subset \mathcal{Z}(\mathsf{Vec}_G)$ is precisely the set $\mathbb{L}_0(G)$ of subcategories of $\mathcal{Z}(\mathsf{Vec}_G)$ braided equivalent to $\mathsf{Rep}(G)$.

Sections 8 through 11 illustrate our techniques. They contain explicit computations of groups $\mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$ for several classes of finite groups G. The common feature of these examples is that in each case it is possible to describe the set $\mathbb{L}_0(G)$ and the corresponding action of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$. Combining information about this action with previously developed machinery we determine groups $\mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$. As a byproduct, we obtain interesting examples of non-integral weakly group-theoretical fusion categories, see Examples 8.1, 9.3, and 10.3.

2. Conventions and notation

Throughout this paper we work over an algebraically closed field k of characteristic 0. All categories considered in this paper are finite, Abelian, semisimple, and k-linear. All functors are additive and k-linear. We freely use the language and basic results of the theory of fusion categories and module categories over them [11,12,10]. We will denote Vec the fusion category of finite-dimensional k-vector spaces.

For a finite group G we denote $\mathsf{Aut}(G)$ the group of automorphisms of G and by $\mathsf{Out}(G)$ the group of (congruence classes of) outer automorphisms of G. For a G-module A we denote by $Z^n(G, A)$ the group of n-cocycles on G with values in A and by $H^n(G, A)$ the corresponding nth cohomology group. We will often identify cohomology classes with cocycles representing them.

For any (not necessarily Abelian) group G we denote $\widehat{G} = \mathsf{Hom}(G, k^\times)$ the group of linear characters of G.

We can view the multiplicative group $k^\times$ as a G-module with the trivial action. There is an obvious action of $\mathsf{Aut}(G)$ on $H^n(G,k^\times)$. This action factors through the subgroup of inner automorphisms and, hence, gives rise to an action of $\mathsf{Out}(G)$. For a subgroup $L \subset G$, a cocycle $f \in Z^n(L,k^\times)$, and an automorphism $\theta \in \mathsf{Aut}(G)$ denote

$$f^\theta = f \circ \left(\theta^{-1} \times \cdots \times \theta^{-1}\right) \in Z^n\left(\theta(L),k^\times\right). \tag{3}$$

It is clear that the cohomology class of f^θ in $H^n(L,k^\times)$ is well defined. When θ is the inner automorphism $x \mapsto gxg^{-1}$, $x \in G$, we denote f^θ by f^g.

For any positive integer n we denote $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ the dihedral group of order $2n$, S_n the symmetric group of degree n, and A_n the alternating group of degree n. More generally, for any set Ω we denote $\mathsf{Sym}(\Omega)$ the symmetric group of Ω.

Finally, for a finite group G we denote by Vec_G the fusion category of finite-dimensional G-graded vector spaces and by $\mathsf{Rep}(G)$ the symmetric fusion category of finite-dimensional representations of G.

3. Some facts about cohomology of finite groups

Let G be a finite group.

Remark 3.1. Let A be a G-module. It is well known that $H^1(G,A)$ classifies homomorphisms $G \to A \rtimes G$ which are right inverse to the standard projection $A \rtimes G \to G$, up to a conjugation by elements of A.

Proposition 3.2. *Let G be a finite group and let A be a finite G-module such that the orders $|G|$ and $|A|$ are relatively prime. Then $H^n(G,A)=0$ for all n.*

The following result is taken from [23] and [15, Theorem 2.2.5]. It can also be proved by means of the Hochschild–Serre spectral sequence.

Theorem 3.3. *Let $G = N \rtimes T$ and let $\tilde{M}(G) \subset H^2(G,k^\times)$ be the kernel of the restriction homomorphism $H^2(G,k^\times) \to H^2(T,k^\times)$. Then*

$$H^2\left(G,k^\times\right) \cong H^2\left(T,k^\times\right) \times \tilde{M}(G) \tag{4}$$

and there is an exact sequence

$$0 \to H^1(T,\widehat{N}) \to \tilde{M}(G) \xrightarrow{res} H^2\left(N,k^\times\right)^T \to H^2(T,\widehat{N}), \tag{5}$$

where the homomorphism $res : \tilde{M}(G) \to H^2(N,k^\times)^T$ is induced by the restriction $H^2(G,k^\times) \to H^2(N,k^\times)$.

The following result [15, Theorem 2.1.2] will be useful for our computations.

Theorem 3.4. *Let G be a finite group and let P be a Sylow p-subgroup of G. The restriction map $H^2(G, k^\times) \to H^2(P, k^\times)$ is injective on the p-primary component of $H^2(G, k^\times)$.*

4. Fusion categories and their Brauer–Picard groups

For a fusion category $\mathcal{A}$ let $\mathsf{Aut}(\mathcal{A})$ denote the group of isomorphism classes of tensor autoequivalences of $\mathcal{A}$. It is known that this group is finite [11].

A fusion category is called *pointed* if all its simple objects are invertible with respect to the tensor product. A most general example of a pointed fusion category is the category Vec_G^ω of vector spaces graded by a finite group G with the associativity constraint given by a 3-cocycle $\omega \in Z^3(G, k^\times)$. In this paper we only consider the case when ω is cohomologically trivial, i.e., we work with pointed categories of the form Vec_G. Let δ_g, $g \in G$, denote simple objects of Vec_G. We have $\delta_g \otimes \delta_h \cong \delta_{gh}$. In particular, the unit object of Vec_G is δ_1.

The following result is well known.

Proposition 4.1. *Let G be a finite group. Then $\mathsf{Aut}(\mathsf{Vec}_G) \cong H^2(G, k^\times) \rtimes \mathsf{Aut}(G)$.*

For $\zeta \in H^2(G, k^\times)$ and $a \in \mathsf{Aut}(G)$ the corresponding autoequivalence $F_{(a,\zeta)}$ of Vec_G is defined as follows. As a functor, $F_{(a,\zeta)}(\delta_g) = \delta_{a(g)}$, while the tensor structure of $F_{(a,\zeta)}$ is given by

$$\zeta(g,h)\,\mathrm{id}_{\delta_{a(gh)}} : F_{(a,\zeta)}(\delta_g) \otimes F_{(a,\zeta)}(\delta_h) \xrightarrow{\sim} F_{(a,\zeta)}(\delta_{gh}), \quad g, h \in G.$$

Let $\mathcal{A}$ be a fusion category. The notion of a tensor product $\boxtimes_\mathcal{A}$ of $\mathcal{A}$-bimodule categories was introduced in [12]. With respect to this product equivalence classes of $\mathcal{A}$-bimodule categories form a monoid. The unit of this monoid is the regular $\mathcal{A}$-bimodule category $\mathcal{A}$. A semisimple $\mathcal{A}$-bimodule category $\mathcal{M}$ is called *invertible* if there is an $\mathcal{A}$-bimodule category $\mathcal{N}$ such that $\mathcal{M} \boxtimes_\mathcal{A} \mathcal{N} \cong \mathcal{A}$ and $\mathcal{N} \boxtimes_\mathcal{A} \mathcal{M} \cong \mathcal{A}$. By definition, the *Brauer–Picard* group of $\mathcal{A}$ is the group $\mathsf{BrPic}(\mathcal{A})$ of equivalence classes of invertible $\mathcal{A}$-bimodule categories.

The Brauer–Picard group is an important invariant of a fusion category. It is used, in particular, in the classification of extensions of fusion categories [12]. Let G be a finite group. By a *G-extension* of a fusion category $\mathcal{A}$ we mean a faithfully G-graded fusion category

$$\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g, \quad \text{with } \mathcal{B}_e \cong \mathcal{A}. \tag{6}$$

Such extensions are parameterized by group homomorphisms $c : G \to \mathsf{BrPic}(\mathcal{A})$ and certain cohomological data associated to G (provided that certain obstructions vanish, see [12] for details). One has $c(g) = \mathcal{B}_g$ for all $g \in G$. We say that an extension (6) is *non-trivial* if $\mathcal{B}_g \ncong \mathcal{A}$ (as a left $\mathcal{A}$-module category) for some $g \in G$.

Remark 4.2. In a particularly simple situation when $|G|$ and the Frobenius–Perron dimension of $\mathcal{A}$ are relatively prime, for any fixed homomorphism $c : G \to \mathsf{BrPic}(\mathcal{A})$ extensions (6) exist and are parameterized by a torsor over $H^3(G, k^\times)$, see [12, Theorem 9.5].

5. Parameterization of $\mathsf{BrPic}(\mathsf{Vec}_G)$

Let G be a finite group. In this section we recall a group-theoretical parameterization of the Brauer–Picard group of Vec_G. This description was obtained by Davydov in [5] (in terms of equivalences of centers, cf. isomorphism (1)). We provide an alternative argument for the reader's convenience.

Recall [21] that indecomposable Vec_G-module categories are parameterized by pairs (L, μ), where $L \subset G$ is a subgroup and $\mu \in Z^2(L, k^\times)$. Namely, the category $\mathcal{M}(L, \mu)$ corresponding to such a pair consists of vector spaces graded by the set of cosets G/L with the action of Vec_G induced by the translation action of G on G/L and the module category structure induced by μ.

Two Vec_G-module categories $\mathcal{M}(L, \mu)$ and $\mathcal{M}(L', \mu')$ are equivalent if and only if there is $g \in G$ such that $L' = gLg^{-1}$ and 2-cocycles μ' and μ^g are cohomologous in $H^2(L', k^\times)$.

Fix a subgroup L of G. Let E denote the group of isomorphism classes of right Vec_G-module autoequivalences of $\mathcal{M}(L, \mu)$ isomorphic to the identity as an additive functor. It follows from [19] that there is a group isomorphism

$$\iota : E \to \widehat{L} : F \mapsto \iota_F \tag{7}$$

such that the Vec_G-module functor structure $\delta_x \otimes F(L) \xrightarrow{\sim} F(\delta_x \otimes L) = F(xL)$ is given by $\iota_F(x)\,\mathrm{id}_{F(xL)}$ for all $x \in L$.

Let G_1, G_2 be a pair of normal subgroups of G centralizing each other. Let us define

$$L_1 := G_1 \cap L, \qquad L_2 := G_2 \cap L.$$

Any 2-cocycle $\mu \in Z^2(L, k^\times)$ determines a group homomorphism

$$a : L_1 \to \widehat{L_2} : g \mapsto a_g, \quad \text{where } a_g(h) := \frac{\mu(g,h)}{\mu(h,g)},\ h \in L_2. \tag{8}$$

Similarly, μ determines a group homomorphism $L_2 \to \widehat{L_1}$.

Lemma 5.1. *Let G_1, G_2 be a pair of commuting normal subgroups of G such that $G_1L = G_2L = G$. For $g \in G_1$ let F_g denote the functor of left tensor multiplication by δ_g on $\mathcal{M}(L, \mu)$. Then F_g is equivalent to the identity as a left Vec_{G_2}-module autoequivalence of $\mathcal{M}(L, \mu)$ if and only if $g \in L_1$ and $a_g = 1$ on L_2.*

Proof. It is clear that F_g is isomorphic to $\mathrm{id}_{\mathcal{M}(L,\mu)}$ as an additive functor if and only if $g \in L_1$. Let $C = \mathsf{Hom}_L(G, k^\times)$ and let $\tilde{\mu} \in Z^2(G, C)$ be a 2-cocycle such that the cohomology class of $\tilde{\mu}$ in $H^2(G, C)$ is identified with the class of μ in $H^2(L, k^\times)$ via Shapiro's lemma, i.e., $\mu(h_1, h_2) = \tilde{\mu}(h_1, h_2)(L)$ for all $x_1, x_2 \in L$.

Then the Vec_{G_2}-module structure on F_g is given by

$$\tfrac{\tilde{\mu}(g,g_2)(xL)}{\tilde{\mu}(g_2,g)(xL)}\, \mathrm{id}_{g_2xL} : F_g(\delta_{g_2} \otimes xL) \xrightarrow{\sim} \delta_{g_2} \otimes F_g(xL), \quad x, g_2 \in L_2.$$

Using isomorphism (7) we conclude that for $g \in L_1$ one has $F_g \cong \mathrm{id}$ as a left Vec_{G_2}-module functor if and only if $a_g = 1$, as required. □

Let μ be a 2-cocycle on G. Let $L_1, L_2 \subset G$ be a pair of subgroups centralizing each other. It is straightforward to check that the function

$$Alt(\mu) : L_1 \times L_2 \to k^\times : (x_1, x_2) \mapsto \frac{\mu(x_1, x_2)}{\mu(x_2, x_1)}, \tag{9}$$

is a bicharacter, i.e., is multiplicative in both arguments.

Note that a Vec_G-bimodule category is the same thing as a $\mathsf{Vec}_{G\times G^{\mathrm{op}}}$-module category, where G^{op} is G with the opposite multiplication.

Proposition 5.2. *Let G be a finite group, let L be a subgroup of $G \times G^{\mathrm{op}}$, and let $\mu \in Z^2(L, k^\times)$ be a 2-cocycle. Then Vec_G-bimodule category $\mathcal{M}(L, \mu)$ is invertible if and only if the following three conditions are satisfied:*

(i) $L(G \times \{1\}) = L(\{1\} \times G^{\mathrm{op}}) = G \times G^{\mathrm{op}}$,
(ii) $L_1 := L \cap (G \times \{1\})$ *and* $L_2 := L \cap (\{1\} \times G^{\mathrm{op}})$ *are Abelian groups,*
(iii) *bicharacter* $Alt(\mu) : L_1 \times L_2 \to k^\times$ *defined in* (9) *is non-degenerate.*

Proof. Let us denote $\mathcal{M} := \mathcal{M}(L, \mu)$. Condition (i) is equivalent to $(G \times G^{\mathrm{op}})/L$ being transitive as both left and right G-set, i.e., to $\mathcal{M}$ being indecomposable as left and right Vec_G-module category. It implies that L_1 is a normal subgroup of G and L_2 is a normal subgroup of G^{op}.

For $g \in G$ let $L(g)$ (respectively, $R(g)$) denote the additive endofunctor of $\mathcal{M}$ given by the action of $\delta_g \boxtimes 1$ (respectively, $1 \boxtimes \delta_g$). By [12] $\mathcal{M}$ is invertible if and only if the functors $\mathsf{Vec}_G \to \mathrm{Fun}_{\mathsf{Vec}_G}(\mathcal{M}, \mathcal{M}) : g \mapsto R(g)$ (respectively, $\mathsf{Vec}_G \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})_{\mathsf{Vec}_G} : g \mapsto L(g)$) are equivalences. Since those functors are tensor, the above conditions are equivalent to $L(g) \ncong \mathrm{id}_{\mathcal{M}}$ as a right Vec_G-module functor (respectively, to $R(g) \ncong \mathrm{id}_{\mathcal{M}}$ as a left Vec_G-module functor) for all $g \neq 1$.

We apply Lemma 5.1 with G replaced by $G \times G^{\mathrm{op}}$, $G_1 = G \times \{1\}$ and $G_2 = \{1\} \times G^{\mathrm{op}}$. It follows that the above conditions are satisfied if and only if group homomorphisms defined as in (8), i.e.,

$$L_1 \to \widehat{L_2} : x \mapsto a_x, \quad \text{where } a_x(h) := \frac{\mu(x,h)}{\mu(h,x)},\ h \in L_2, \tag{10}$$

and

$$L_2 \to \widehat{L_1} : y \mapsto a'_y, \quad \text{where } a'_y(g) := \frac{\mu(y,g)}{\mu(g,y)},\ g \in L_1, \tag{11}$$

are injective. This is equivalent to L_1, L_2 being Abelian and $Alt(\mu)$ being non-degenerate on $L_1 \times L_2$. $\square$

Remark 5.3. For an invertible Vec_G-bimodule category $\mathcal{M}(L,\mu)$ the subgroups $L_1 \subset G$ and $L_2 \subset G^{\mathrm{op}}$ are normal and restrictions $\mu|_{L_1\times L_1}$ and $\mu|_{L_2\times L_2}$ are G-invariant.

Remark 5.4. It is easy to describe a one-sided restriction of the Vec_G-module category $\mathcal{M}(L,\mu)$ from Proposition 5.2. Namely, as a left Vec_G-module category it is equivalent to $\mathcal{M}(L_1, \mu|_{L_1\times L_1})$.

There is a convenient way to determine which of the categories $\mathcal{M}(L,\mu)$ described in Proposition 5.2 are involutions in the Brauer–Picard group.

Remark 5.5. Let G be a finite group, let L be a subgroup of $G\times G^{\mathrm{op}}$, and let $\mu \in Z^2(L, k^\times)$ be a 2-cocycle satisfying conditions of Proposition 5.2. Then the inverse of $\mathcal{M}(L,\mu)$ in $\mathsf{BrPic}(\mathsf{Vec}_G)$ is $\mathcal{M}(L^\vee, (\mu^\vee)^{-1})$, where

$$L^\vee = \{(x_2,x_1) \mid (x_1,x_2) \in L\},$$
$$\mu^\vee\big((x_1,x_2),(y_1,y_2)\big) = \mu\big(\big(x_2^{-1},x_1^{-1}\big),\big(y_2^{-1},y_1^{-1}\big)\big).$$

Indeed, it was shown in [12] that the inverse of a bimodule category is given by taking its opposite.

Corollary 5.6. *The category $\mathcal{M}(L,\mu)$ has order $\leqslant 2$ in $\mathsf{BrPic}(\mathsf{Vec}_G)$ if and only if there is $g \in G\times G^{\mathrm{op}}$ such that $L^\vee = gLg^{-1}$ and μ^g and $(\mu^\vee)^{-1}$ are cohomologous in $H^2(L^\vee, k^\times)$.*

6. Braided autoequivalences of centers

Let $\mathcal{C}$ be a braided fusion category with braiding $c_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$. Let $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$. The *centralizer* of $\mathcal{D}$ in $\mathcal{C}$ [17] is the fusion subcategory $\mathcal{D}' \subset \mathcal{C}$ consisting of objects X such that $c_{YX} \circ c_{XY} = \mathrm{id}_{X\otimes Y}$ for all objects Y in $\mathcal{C}$. A braided fusion category $\mathcal{C}$ is *symmetric* if $\mathcal{C} = \mathcal{C}'$ and *non-degenerate* if $\mathcal{C}' = \mathsf{Vec}$. A symmetric fusion category is called *Tannakian* if it is equivalent to $\mathsf{Rep}(G)$, the category of representations of a finite group G.

Let $\mathsf{Aut}^{br}(\mathcal{C})$ denote the group of isomorphism classes of braided autoequivalences of $\mathcal{C}$. The following result is well known.

Proposition 6.1. *Let G be a finite group. We have $\mathsf{Aut}^{br}(\mathsf{Rep}(G)) \cong \mathsf{Out}(G)$.*

Proof. By the result of Deligne [7], every braided tensor functor $F : \mathsf{Rep}(G) \to \mathsf{Vec}$ is isomorphic to the obvious forgetful functor. Furthermore, the group G_F of tensor automorphisms of F is isomorphic to G. Hence, a braided tensor autoequivalence $\alpha \in \mathsf{Aut}^{br}(\mathsf{Rep}(G))$ induces a group automorphism $\iota(\alpha) \in \mathsf{Aut}(G_F)$. The assignment $V \mapsto F(V)$ is a braided tensor equivalence between $\mathsf{Rep}(G)$ and $\mathsf{Rep}(G_F)$. Under this equivalence α corresponds to the autoequivalence of $\mathsf{Rep}(G_F)$ induced by the automorphism $\iota(\alpha)$.

Hence, every braided autoequivalence of $\mathsf{Rep}(G)$ is induced by an automorphism of G. It is straightforward to verify that the above autoequivalence $\iota(\alpha)$ is isomorphic to the identity tensor functor if and only if the corresponding group automorphism is inner. This implies the result. □

For any fusion category $\mathcal{A}$ let $\mathcal{Z}(\mathcal{A})$ denote its *center*. The objects of $\mathcal{Z}(\mathcal{A})$ are pairs (Z, γ) where Z is an object of $\mathcal{A}$ and $\gamma = \{\gamma_X\}_{X \in \mathcal{A}}$, where

$$\gamma_X : X \otimes Z \xrightarrow{\sim} Z \otimes X,$$

is a natural isomorphism satisfying certain compatibility conditions. We will usually simply write Z for (Z, γ). It is known that $\mathcal{Z}(\mathcal{A})$ is a non-degenerate braided fusion category [18,10].

Let $\mathcal{A}$ be a fusion category and let $\mathcal{M}$ be an invertible $\mathcal{A}$-bimodule category. One assigns to $\mathcal{M}$ a braided autoequivalence $\Phi_{\mathcal{M}}$ of $\mathcal{Z}(\mathcal{A})$ as follows. Note that $\mathcal{Z}(\mathcal{A})$ can be identified with the category of $\mathcal{A}$-bimodule endofunctors of $\mathcal{M}$ in two ways: via the functors $Z \mapsto Z \otimes -$ and $Z \mapsto - \otimes Z$. Define $\Phi_{\mathcal{M}}$ in such a way that there is an isomorphism of $\mathcal{A}$-bimodule functors

$$Z \otimes - \cong - \otimes \Phi_{\mathcal{M}}(Z) \tag{12}$$

for all $Z \in \mathcal{Z}(\mathcal{A})$.

The following result was established in [12].

Theorem 6.2. *Let $\mathcal{A}$ be a fusion category. The assignment*

$$\mathcal{M} \mapsto \Phi_{\mathcal{M}} \tag{13}$$

gives rise to an isomorphism

$$\mathsf{BrPic}(\mathcal{A}) \simeq \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathcal{A})\big). \tag{14}$$

Remark 6.3. The inverse to the above isomorphism (13) is constructed as follows (see [12, Section 5]). Let $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ be the right adjoint of the forgetful functor

$F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$. For a braided autoequivalence $\alpha \in \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{A}))$ consider the commutative algebra $A := \alpha^{-1}(I(\mathbf{1}))$ in $\mathcal{Z}(\mathcal{A})$. Let $\mathcal{M}_\alpha$ be any indecomposable component of the category of $F(A)$-modules in $\mathcal{A}$. It has a structure of invertible $\mathcal{A}$-bimodule category and the assignment $\alpha \mapsto \mathcal{M}_\alpha$ is the inverse of (13).

Let A be a finite Abelian group. Then Theorem 6.2 implies that

$$\mathsf{BrPic}(\mathsf{Vec}_A) \cong O(A \oplus \widehat{A}, q), \tag{15}$$

where $O(A \oplus \widehat{A}, q)$ is the group of automorphisms of $A \oplus \widehat{A}$ preserving the canonical quadratic form

$$q(a, \chi) = \chi(a), \quad a \in A,\ \chi \in \widehat{A}.$$

For any fusion category $\mathcal{A}$ there is an induction homomorphism

$$\Delta : \mathsf{Aut}(\mathcal{A}) \to \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathcal{A})\big) : \alpha \mapsto \Delta_\alpha, \tag{16}$$

where $\Delta_\alpha(Z, \gamma) = (\alpha(Z), \gamma^\alpha)$ and γ^α is defined by the following commutative diagram

$$\begin{tikzcd}
X \otimes \alpha(Z) \arrow[r, "\gamma^\alpha_X"] \arrow[d] & \alpha(Z) \otimes X \arrow[d] \\
\alpha(\alpha^{-1}(X)) \otimes \alpha(Z) \arrow[d, "J_{\alpha^{-1}(X),Z}"'] & \alpha(Z) \otimes \alpha(\alpha^{-1}(X)) \arrow[d, "J_{Z,\alpha^{-1}(X)}"] \\
\alpha(\alpha^{-1}(X) \otimes Z) \arrow[r, "\alpha(\gamma_{\alpha^{-1}(X)})"] & \alpha(Z \otimes \alpha^{-1}(X)).
\end{tikzcd} \tag{17}$$

Here α^{-1} is a quasi-inverse of α and $J_{X,Y} : \alpha(X) \otimes \alpha(Z) \xrightarrow{\sim} \alpha(X \otimes Z)$ is the tensor functor structure of α.

Example 6.4. Let $\mathcal{A}$ be a fusion category and let $\alpha \in \mathsf{Aut}(\mathcal{A})$. Consider an invertible $\mathcal{A}$-bimodule category $\mathcal{A}_\alpha$, where $\mathcal{A}_\alpha = \mathcal{A}$ and the actions of $\mathcal{A}$ on $\mathcal{A}_\alpha$ are given by

$$(X, V) \mapsto \alpha(X) \otimes V, \qquad (V, Y) \mapsto V \otimes Y \tag{18}$$

for all $X, Y \in \mathcal{A}$ and $V \in \mathcal{A}_\alpha$. Under isomorphism (14) this category $\mathcal{A}_\alpha$ corresponds to the induced autoequivalence Δ_α, i.e.,

$$\Phi_{\mathcal{A}_\alpha} = \Delta_\alpha.$$

For a finite group G let $\mathsf{Inn}(G) \subset \mathsf{Aut}(G)$ denote the normal subgroup of inner automorphisms of G and let $\mathsf{Out}(G) = \mathsf{Aut}(G)/\mathsf{Inn}(G)$.

Proposition 6.5. *The kernel of induction homomorphism*

$$\Delta : \mathsf{Aut}(\mathsf{Vec}_G) \to \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathsf{Vec}_G)\big)$$

is $\mathsf{Inn}(G)$*.*

Proof. By Theorem 6.2 and Example 6.4, the kernel of Δ consists of all autoequivalences $\alpha \in \mathsf{Aut}(\mathsf{Vec}_G)$ such that the Vec_G-bimodule category $(\mathsf{Vec}_G)_\alpha$ is equivalent to the regular Vec_G-bimodule category Vec_G.

Let $\alpha = F_{(a,\zeta)}$, $a \in \mathsf{Aut}(G)$, $\zeta \in H^2(G, k^\times)$ (we use notation from Section 4). It follows from definition of $(\mathsf{Vec}_G)_\alpha$ (see (18)) that any right Vec_G-module equivalence between $(\mathsf{Vec}_G)_\alpha$ and Vec_G is of the form $\delta_g \mapsto \delta_x \otimes \delta_g$, $g \in G$, for some invertible $x \in G$. This autoequivalence is compatible with the left Vec_G-module structure of $(\mathsf{Vec}_G)_\alpha$ if and only if a is equal to the conjugation by x and ζ is the trivial cohomology class. □

Let $\mathcal{C}$ be a braided fusion category. Then $\mathcal{C}$ is embedded into $\mathcal{Z}(\mathcal{C})$ via $X \mapsto (X, c_{-,X})$, where c denotes the braiding of $\mathcal{C}$. In what follows we will identify $\mathcal{C}$ with a fusion subcategory of $\mathcal{Z}(\mathcal{C})$ (the image of this embedding). Left $\mathcal{C}$-module categories can be viewed as $\mathcal{C}$-bimodule categories (analogously to how modules over a commutative ring can be viewed as bimodules). Invertible left $\mathcal{C}$-module categories form a subgroup $\mathsf{Pic}(\mathcal{C}) \subset \mathsf{BrPic}(\mathcal{C})$ called the *Picard* group of $\mathcal{C}$. Note that the action of the group $\mathsf{Aut}^{br}(\mathcal{C})$ on $\mathsf{Pic}(\mathcal{C})$ factors through $\mathsf{Out}(\mathcal{C})$.

Remark 6.6. The restriction of the induction homomorphism (16) to $\mathsf{Aut}^{br}(\mathcal{C})$ is injective.

Let $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C}) \subset \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ be the subgroup consisting of braided autoequivalences of $\mathcal{Z}(\mathcal{C})$ that restrict to the trivial autoequivalence of $\mathcal{C}$.

The following result was established in [6].

Theorem 6.7. *The image of* $\mathsf{Pic}(\mathcal{C})$ *under isomorphism* (14) *is* $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$*.*

The group $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ acts on the lattice of fusion subcategories of $\mathcal{Z}(\mathcal{C})$. Let $\mathsf{Stab}(\mathcal{C})$ denote the stabilizer of the subcategory $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$ under this action.

Proposition 6.8. *For a braided fusion category* $\mathcal{C}$ *we have*

$$\mathsf{Stab}(\mathcal{C}) \cong \mathsf{Pic}(\mathcal{C}) \rtimes \mathsf{Aut}^{br}(\mathcal{C}). \tag{19}$$

Proof. Observe that the subgroup $N := \mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}); \mathcal{C})$ is normal in $\mathsf{Stab}(\mathcal{C})$ and $N \cong \mathsf{Pic}(\mathcal{C})$ by Theorem 6.7. Since the image of a braided autoequivalence of $\mathcal{C}$ under the induction homomorphism (16) belongs to $\mathsf{Stab}(\mathcal{C})$ we see from Remark 6.6 that $\mathsf{Stab}(\mathcal{C})$ contains a subgroup $H \cong \mathsf{Aut}^{br}(\mathcal{C})$.

Any $\alpha \in \mathsf{Stab}(\mathcal{C})$ restricts to a braided autoequivalence $\tilde{\alpha}$ of $\mathcal{C}$. Let $\beta = \Delta_{\tilde{\alpha}} \in H$ be the element of $\mathsf{Aut}^{br}(\mathcal{Z}(\mathcal{C}))$ induced from $\tilde{\alpha}$. The restriction of β on $\mathcal{C}$ is $\tilde{\alpha}$, hence, $\alpha \circ \beta^{-1} \in N$ restricts to a trivial autoequivalence of $\mathcal{C}$. This proves $\mathsf{Stab}(\mathcal{C}) = NH$, i.e., $\mathsf{Stab}(\mathcal{C})$ is the semi-direct product of N and H. □

We can apply Proposition 6.8 to centers of Tannakian categories. Let G be a finite group.

Corollary 6.9. *We have*

$$\mathsf{Stab}\big(\mathsf{Rep}(G)\big) \cong H^2\big(G, k^\times\big) \rtimes \mathsf{Out}(G). \tag{20}$$

Proof. It was shown in [14] that $\mathsf{Pic}(\mathsf{Rep}(G)) \cong H^2(G, k^\times)$. Combining this with Proposition 6.1 we get the result. □

Let us describe the above stabilizer $\mathsf{Stab}(\mathsf{Rep}(G))$ in terms convenient for computations. Note that $\mathcal{Z}(\mathsf{Rep}(G)) \cong \mathcal{Z}(\mathsf{Vec}_G)$ and so we can consider the induction homomorphism

$$\Delta : \mathsf{Aut}(\mathsf{Vec}_G) \to \mathsf{Aut}^{br}\big(\mathcal{Z}\big(\mathsf{Rep}(G)\big)\big).$$

Lemma 6.10. $\mathsf{Stab}(\mathsf{Rep}(G)) = \Delta(\mathsf{Aut}(\mathsf{Vec}_G))$.

Proof. This follows from Propositions 4.1 and 6.5. □

For $a \in \mathsf{Out}(G)$ and $\zeta \in H^2(G, k^\times)$ let

$$\Delta_{(a,\zeta)} \in \mathsf{Aut}^{br}\big(\mathcal{Z}\big(\mathsf{Rep}(G)\big)\big) \cong \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathsf{Vec}_G)\big) \tag{21}$$

denote the braided autoequivalence induced from the tensor autoequivalence $F_{(a,\zeta)}$ of Vec_G introduced in Section 4. By Lemma 6.10, $\Delta_{(a,\zeta)} \in \mathsf{Stab}(\mathsf{Rep}(G))$.

Example 6.11. We can compute the effect of autoequivalence $\Delta_{a,\zeta}$ on objects of $\mathcal{Z}(\mathsf{Vec}_G)$. Recall that objects of $\mathcal{Z}(\mathsf{Vec}_G)$ can be viewed as G-equivariant vector bundles of G, i.e., G-graded vector spaces

$$V = \bigoplus_{g \in G} V_g$$

along with linear isomorphisms $\gamma(x, g) : V_g \to V_{xgx^{-1}}$, $x, g \in G$, satisfying

$$\gamma(xy, g) = \gamma\big(x, ygy^{-1}\big) \circ \gamma(y, g), \quad g, x, y \in G. \tag{22}$$

In particular, simple objects of $\mathcal{Z}(\mathsf{Vec}_G)$ are parameterized by pairs (K, χ), where K is a conjugacy class of G and χ is an irreducible character of the centralizer $C_G(g)$ of an element $g \in K$. Let $Z_{(K,\chi)}$ denote the corresponding simple object.

Let us denote $\bigoplus_{g\epsilon G}(V_g, \{\gamma(x,g)\}_{x,g\in G})$ a typical object in $\mathcal{Z}(\mathsf{Vec}_G)$.

We have

$$\Delta_{(a,\zeta)}\left(\bigoplus_{g\in G} V_g, \{\gamma(x,g)\}_{x,g\in G}\right) = \left(\bigoplus_{g\in G} V_{a^{-1}(g)}, \{\widetilde{\gamma(x,g)}\}_{x,g\in G}\right), \tag{23}$$

where

$$\widetilde{\gamma(x,g)} = \gamma\big(a^{-1}(x), a^{-1}(g)\big)\frac{\zeta(a^{-1}(x), a^{-1}(g))}{\zeta(a^{-1}(g), a^{-1}(x))}.$$

In particular, $\Delta_{(a,\zeta)}(Z_{(K,\chi)}) = Z_{(a(K),(\chi\circ a^{-1})\rho_a^g)}$, where ρ_a^g, $g \in a(K)$, is the linear character of $C_G(g)$ given by

$$\rho_a^g(x) = \frac{\zeta(a^{-1}(x), a^{-1}(g))}{\zeta(a^{-1}(g), a^{-1}(x))}.$$

7. Action on the categorical Lagrangian Grassmannian

Let $\mathcal{C}$ be a non-degenerate braided fusion category.

Definition 7.1. A fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is called *Lagrangian* if $\mathcal{D}$ is Tannakian and $\mathcal{D} = \mathcal{D}'$.

It was shown in [10] that $\mathcal{C}$ contains a Lagrangian subcategory if and only if $\mathcal{C}$ is braided equivalent to the center of a pointed fusion category.

Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_G)$ were classified in [20]. They are parameterized by pairs (N, μ) where N is a normal Abelian subgroup of G and μ is a G-invariant cohomology class in $H^2(N, k^\times)$. The Lagrangian subcategory $\mathcal{L}_{(N,\mu)}$ corresponding to the pair (N, μ) is identified with the subcategory of G-equivariant bundles $V = \bigoplus_{a\in N} V_a$ supported on N whose G-equivariant structure (22) satisfies

$$\gamma(x,a) = \frac{\mu(a,x)}{\mu(x,a)}\,\mathrm{id}_{V_a}$$

for all $a, x \in N$.

Example 7.2. The canonical subcategory $\mathsf{Rep}(G) \subset \mathcal{Z}(\mathsf{Vec}_G)$ consisting of vector bundles supported on the identity element of G is $\mathcal{L}_{(1,1)}$.

We have $\mathcal{L}_{(N,\mu)} \cong \mathsf{Rep}(G_{(N,\mu)})$ for some group $G_{(N,\mu)}$ such that $|G_{(N,\mu)}| = |G|$. The group $G_{(N,\mu)}$ is not isomorphic to G in general. It can be described as follows (see [19] for details). There exists a canonical homomorphism

$$H^2\big(N, k^\times\big)^G \to H^2(G/N, \widehat{N}). \tag{24}$$

Let $\nu \in H^2(G/N, \widehat{N})$ be the image of μ under this homomorphism. Then $G_{(N,\mu)}$ is an extension

$$1 \to \widehat{N} \to G_{(N,\mu)} \to G/N \to 1$$

corresponding to ν.

Remark 7.3.

(1) If $\mu \in H^2(N, k^\times)^G$ is trivial then $G_{(N,\mu)}$ is isomorphic to the semidirect product $\widehat{N} \rtimes G/N$.
(2) For non-degenerate μ the group $G_{(N,\mu)}$ first appeared in [4].

Definition 7.4. Let $\mathcal{C}$ be a non-degenerate braided fusion category. The set of Lagrangian subcategories of $\mathcal{C}$ will be called the *categorical Lagrangian Grassmannian* of $\mathcal{C}$.

Let $\mathbb{L}(G)$ denote the categorical Lagrangian Grassmannian of $\mathcal{Z}(\mathsf{Vec}_G)$. Let

$$\mathbb{L}_0(G) := \big\{\mathcal{L} \in \mathbb{L}(G) \;\big|\; \mathcal{L} \cong \mathsf{Rep}(G) \text{ as a braided fusion category}\big\}. \tag{25}$$

This set is non-empty since it contains the canonical subcategory $\mathsf{Rep}(G) \subset \mathcal{Z}(\mathsf{Vec}_G)$, see Example 7.2.

To simplify notation in what follows we will denote

$$\mathfrak{A}(G) := \mathsf{Aut}^{br}\big(\mathcal{Z}(\mathsf{Vec}_G)\big). \tag{26}$$

By Theorem 6.2 we have

$$\mathfrak{A}(G) \cong \mathsf{BrPic}(\mathsf{Vec}_G) = \mathsf{BrPic}\big(\mathsf{Rep}(G)\big). \tag{27}$$

Clearly, the group $\mathfrak{A}(G)$ acts on the set $\mathbb{L}(G)$ and leaves the subset $\mathbb{L}_0(G)$ invariant.

Let us denote $\mathcal{C} = \mathcal{Z}(\mathsf{Vec}_G)$.

Remark 7.5. For every category $\mathcal{L} \in \mathbb{L}_0(G)$ the algebra

$$A_{\mathcal{L}} := \mathrm{Fun}\big(G, k^\times\big) \in \mathcal{L} \cong \mathsf{Rep}(G) \tag{28}$$

is commutative and separable (i.e., is an *étale* algebra in terminology of [8]) and the fusion category $\mathcal{C}_{A_\mathcal{L}}$ of $A_\mathcal{L}$-modules in $\mathcal{C}$ is equivalent to Vec_G. Indeed, by [10], this category is pointed and has a faithful G-grading. Hence, $\mathcal{C}_{A_\mathcal{L}}$ is equivalent to Vec_G^ω for some $\omega \in Z^3(G, k^\times)$. By [8] $\mathcal{Z}(\mathcal{C}_{A_\mathcal{L}}) \cong \mathcal{Z}(\mathsf{Vec}_G)$ and, hence, $\mathcal{C}_{A_\mathcal{L}}$ and Vec_G are categorically Morita equivalent [12]. This implies that ω is cohomologically trivial.

Proposition 7.6. *The action of $\mathfrak{A}(G)$ on $\mathbb{L}_0(G)$ is transitive.*

Proof. Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{L}_0(G)$ be Lagrangian subcategories of $\mathcal{C}$ and let A_1 and A_2 be the corresponding étale algebras in $\mathcal{C}$ defined in (28). By Remark 7.5 $\mathcal{C}$-module categories $\mathcal{C}_{A_1}$ and $\mathcal{C}_{A_2}$ are equivalent to Vec_G. Pick a tensor equivalence

$$\phi : \mathcal{C}_{A_1} \xrightarrow{\sim} \mathcal{C}_{A_2}.$$

It follows from the results of [8] that there are braided equivalences

$$\Phi_i : \mathcal{C} \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}_{A_i}), \quad i = 1, 2.$$

Let $\alpha := \Phi_2^{-1} \circ \Delta_\phi \circ \Phi_1 \in \mathsf{Aut}^{br}(\mathcal{C})$, where $\Delta_\phi : \mathcal{Z}(\mathcal{C}_{A_1}) \to \mathcal{Z}(\mathcal{C}_{A_2})$ is the braided equivalence induced from ϕ. Then $\alpha(A_1) \cong A_2$. Note that A_i is isomorphic, as an object of $\mathcal{C}$, to the regular object in $\mathcal{L}_i$, $i = 1, 2$. Hence, $\alpha(\mathcal{L}_1) = \mathcal{L}_2$, which proves the statement. □

Thus, the image of $\mathfrak{A}(G)$ is a transitive subgroup of $\mathsf{Sym}(\mathbb{L}_0(G))$. Let

$$\mathfrak{A}_0(G) := \mathsf{Stab}\big(\mathsf{Rep}(G)\big) \tag{29}$$

denote the stabilizer of the canonical Lagrangian subcategory $\mathsf{Rep}(G) \subset \mathcal{C}$ in $\mathfrak{A}(G)$. By Corollary 6.9,

$$\mathfrak{A}_0(G) \cong H^2\big(G, k^\times\big) \rtimes \mathsf{Out}(G). \tag{30}$$

Since the cardinality of a transitive set is equal to the index of the stabilizer of a point, we have

$$\big[\mathfrak{A}(G) : \mathfrak{A}_0(G)\big] = \big|\mathbb{L}_0(G)\big|. \tag{31}$$

The next corollary allows to find the order of the Brauer–Picard group.

Corollary 7.7. *Let G be a finite group. Then*

$$\big|\mathfrak{A}(G)\big| = \big|H^2\big(G, k^\times\big)\big| \cdot \big|\mathsf{Out}(G)\big| \cdot \big|\mathbb{L}_0(G)\big|.$$

Corollary 7.8. *Let G be a finite group without normal Abelian subgroups (e.g., a simple non-Abelian group). Then*

$$\mathfrak{A}(G) \cong H^2\big(G, k^\times\big) \rtimes \mathsf{Out}(G).$$

Corollary 7.9. *Let G be a finite non-Abelian simple group. Then the Brauer–Picard group of Vec_G is solvable.*

Proof. This is a consequence of the Schreier conjecture [9, p. 133] stating that $\mathsf{Out}(G)$ is solvable (this conjecture is verified using the classification of finite simple groups). □

Remark 7.10. It can happen that the group $\mathfrak{A}(G)$ is trivial. This is the case for every simple group G such that both $\mathsf{Out}(G)$ and $H^2(G, k^\times)$ are trivial. Among the groups that have these properties are the Mathieu group M_{11} and the Fischer–Griess Monster group.

On the other hand, if G is a p-group of order > 2 then $\mathfrak{A}(G)$ is non-trivial, since in this case the group $\mathsf{Out}(G)$ is non-trivial [13].

The next proposition describes the action of $\mathfrak{A}_0(G)$ on $\mathbb{L}(G)$.

By Lemma 6.10, elements of $\mathfrak{A}_0(G)$ are precisely braided autoequivalences induced from $\mathsf{Aut}(\mathsf{Vec}_G)$ and so are of the form $\Delta_{(a,\zeta)}$ for some $a \in \mathsf{Out}(G)$ and $\zeta \in H^2(G, k^\times)$, see (21).

Proposition 7.11. *Let $\mathcal{L}_{(N,\mu)}$ be a Lagrangian subcategory of $\mathcal{Z}(\mathsf{Vec}_G)$. Then*

$$\Delta_{(a,\zeta)}(\mathcal{L}_{(N,\mu)}) = \mathcal{L}_{(a(N),\mu^a\zeta^a)}.$$

Proof. Let us apply $\Delta_{(a,\zeta)}$ to an object $(\bigoplus_{g\in N} V_g, \{\gamma(x,g)\})$ of $\mathcal{L}_{(N,\mu)}$. Using Example 6.11, we obtain

$$\Delta_{(a,\zeta)}\left(\bigoplus_{g\in G} V_g, \{\gamma(x,g)\}_{x,g\in G}\right) = \left(\bigoplus_{g\in G} V_{a(g)}, \{\widetilde{\gamma(x,g)}\}_{x,g\in G}\right),$$

where

$$\widetilde{\gamma(x,g)} = \gamma\big(a^{-1}(x), a^{-1}(g)\big)\frac{\zeta^a(x,g)}{\zeta^a(g,x)} = \frac{\mu^a(x,g)}{\mu^a(g,x)}\frac{\zeta^a(x,g)}{\zeta^a(g,x)}\,\mathrm{id}_{V_g}, \tag{32}$$

for all $x, g \in N$, which implies the result. □

Proposition 7.12. *Let L be a subgroup of $G \times G^{\mathrm{op}}$ and let μ be a 2-cocycle in $Z^2(L, k^\times)$ satisfying conditions of Proposition 5.2. Let $\mathcal{M}(L,\mu)$ denote the corresponding element of $\mathsf{BrPic}(\mathsf{Vec}_G)$ and let $\alpha_{(L,\mu)}$ be the braided autoequivalence of $\mathcal{Z}(\mathsf{Vec}_G)$ corresponding to $\mathcal{M}(L,\mu)$ upon the isomorphism (14). Then*

$$\alpha_{(L,\mu)}(\mathcal{L}_{(1,1)}) = \mathcal{L}_{(L_1,\mu|_{L_1\times L_1})}, \tag{33}$$

where $L_1 = L \cap (G \times 1)$.

Proof. Let $\alpha \in \mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$ be an autoequivalence such that $\alpha(\mathcal{L}_{(1,1)}) = \mathcal{L}_{(N,\nu)}$. Let $\mathcal{M}_\alpha \in \mathsf{BrPic}(\mathsf{Vec}_G)$ be the corresponding invertible Vec_G-bimodule category (see discussion after Theorem 6.2). By Remark 6.3 $\mathcal{M}_\alpha$ is identified, as a left Vec_G-module category, with the category of modules over the twisted subgroup algebra $(kN)_\nu$, i.e., $\mathcal{M}_\alpha \cong \mathcal{M}(N,\nu)$. Comparing this with Remark 5.4 we obtain the result. $\square$

Remark 7.13. Here is an alternative way to deduce Proposition 7.12. It was observed in [20] that there is a bijection between the set of Vec_G-module categories $\mathcal{M}$ such that the dual category $(\mathsf{Vec}_G)^*_{\mathcal{M}}$ is pointed and $\mathbb{L}(G)$. This bijection is equivariant with respect to the group isomorphism $\mathsf{BrPic}(\mathsf{Vec}_G) \xrightarrow{\sim} \mathsf{Aut}^{br}(\mathcal{Z}(\mathsf{Vec}_G))$. This implies (33).

8. Examples: symmetric and alternating groups

8.1. Symmetric group S_3

It is well known that $H^2(S_3, k^\times) = 0$, see [15, Theorem 2.12.3] and $\mathsf{Out}(S_3) = 1$, hence $\mathfrak{A}_0(S_3) = 1$. The only nontrivial normal Abelian subgroup of S_3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. It follows that $|\mathbb{L}_0(S_3)| = 2$. By Corollary 7.7 $|\mathfrak{A}(S_3)| = 2$, thus

$$\mathfrak{A}(S_3) \cong \mathbb{Z}/2\mathbb{Z}. \tag{34}$$

Example 8.1. By (34) we have $\mathsf{BrPic}(\mathsf{Rep}(S_3)) \cong \mathbb{Z}/2\mathbb{Z}$. There exists a non-trivial $\mathbb{Z}/2\mathbb{Z}$-extension of the fusion category $\mathsf{Rep}(S_3)$, namely the category $\mathcal{C}(sl(2), 4)$ of highest weight integrable modules over the affine Lie algebra $\widehat{sl}(2)$ of level 4. This is a weakly integral fusion category of dimension 12. Its simple objects lying in the non-trivial component have dimension $\sqrt{3}$.

8.2. Symmetric group S_4

It is known that $H^2(S_4, k^\times) \cong \mathbb{Z}/2\mathbb{Z}$, see [15, Theorem 2.12.3], and $\mathsf{Out}(S_4) = 1$. Hence, $\mathfrak{A}_0(S_4) \cong \mathbb{Z}/2\mathbb{Z}$.

The set $\mathbb{L}(S_4)$ consists of three subcategories (see [20, Example 5.2]):

$$\mathcal{L}_{(1,1)}, \quad \mathcal{L}_{(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},1)}, \quad \text{and} \quad \mathcal{L}_{(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\mu)},$$

where $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is identified with a normal subgroup of S_4 and μ is the non-trivial class in $H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k^\times)$.

We claim that $\mathfrak{A}(S_4)$ permutes the two later categories. To prove this we apply Proposition 7.11. It is enough to check that the restriction

$$H^2\left(S_4,k^\times\right) \to H^2\left(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},k^\times\right)$$

is injective. This follows from Theorem 3.3 since $H^1(S_3,\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})=1$.

Thus, the map $\mathfrak{A}(S_4)\to \mathsf{Sym}(\mathbb{L}_0(S_4))$ is injective and, hence

$$\mathfrak{A}(S_4)\cong S_3. \tag{35}$$

8.3. Alternating group A_4

It is known that $H^2(A_4,k^\times)\cong\mathbb{Z}/2\mathbb{Z}$, see [15], and $\mathsf{Out}(A_4)\cong\mathbb{Z}/2\mathbb{Z}$. The set $\mathbb{L}_0(A_4)$ consists of three Lagrangian subcategories (see [20, Example 5.2]):

$$\mathcal{L}_{(1,1)},\quad \mathcal{L}_{(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},1)},\quad \text{and}\quad \mathcal{L}_{(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\mu)},$$

where $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ is identified with the Sylow 2-subgroup of A_4 and μ is the non-trivial class in $H^2(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},k^\times)$. We claim that $H^2(A_4,k^\times)\subset\mathfrak{A}_0(A_4)$ permutes the two later categories. To prove this we apply Proposition 7.11. It is enough to check that the restriction

$$H^2\left(A_4,k^\times\right) \to H^2\left(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},k^\times\right)$$

is injective, which follows immediately from Theorem 3.4. Hence, the map

$$\pi:\mathfrak{A}(A_4)\to\mathsf{Sym}\left(\mathbb{L}_0(A_4)\right)\cong S_3$$

is surjective.

By Corollary 7.7 $\mathfrak{A}(A_4)$ is a non-Abelian group of order 12. Its Sylow 2-subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\cong\mathfrak{A}_0(A_4)$. Furthermore, $\mathfrak{A}(A_4)$ contains a normal subgroup of order 2 (the kernel of π). It is easy to check that the only group of order 12 with above properties is the dihedral group of order 12. Thus,

$$\mathfrak{A}(A_4)\cong D_{12}. \tag{36}$$

Remark 8.2. Let $G=A_n$ or S_n, where $n\geqslant 5$. Then G has no normal Abelian subgroups and $\mathfrak{A}(G)=H^2(G,k^\times)\rtimes\mathsf{Out}(G)$ by Corollary 7.8. The groups $H^2(G,k^\times)$ and $\mathsf{Out}(G)$ in this case are well known, see [15, Section 2.12]:

$$H^2\left(S_n,k^\times\right)=\mathbb{Z}/2\mathbb{Z},\qquad \mathsf{Out}(S_n)=\begin{cases}1 & n\neq 6,\\ \mathbb{Z}/2\mathbb{Z} & n=6,\end{cases}$$

$$H^2\left(A_n,k^\times\right)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & n\neq 6,7,\\ \mathbb{Z}/6\mathbb{Z} & n=6,7,\end{cases}\qquad \mathsf{Out}(A_n)=\begin{cases}\mathbb{Z}/2\mathbb{Z} & n\neq 6,\\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & n=6.\end{cases}$$

9. Examples: non-Abelian groups of order 8

9.1. Dihedral group D_8

The following is a standard representation of D_8 by generators and relations:

$$D_8 = \langle r, s \mid r^4 = s^2 = 1,\ sr = r^{-1}s \rangle.$$

We have $\mathsf{Out}(D_8) \cong \mathbb{Z}/2\mathbb{Z}$, where the nontrivial element is represented by the automorphism a given by

$$a(r) = r, \qquad a(s) = sr.$$

Also, $H^2(D_8, k^\times) \cong \mathbb{Z}/2\mathbb{Z}$, see [15, Theorem 2.11.4]. Thus, $\mathfrak{A}_0(D_8) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

By analyzing subgroup structure of D_8 we see that $\mathcal{Z}(\mathsf{Vec}_{D_8})$ has seven Lagrangian subcategories,

$$\begin{gathered}\mathcal{L}_{(1,1)}, \quad \mathcal{L}_{(\langle r\rangle,1)}, \quad \mathcal{L}_{(\langle s,r^2\rangle,1)}, \quad \mathcal{L}_{(\langle sr,r^2\rangle,1)}, \quad \mathcal{L}_{(\langle s,r^2\rangle,\mu_1)}, \\ \mathcal{L}_{(\langle sr,r^2\rangle,\mu_2)}, \quad \text{and} \quad \mathcal{L}_{(\langle r^2\rangle,1)},\end{gathered} \tag{37}$$

where μ_1, μ_2 denote nontrivial cohomology classes of the respective subgroups.

The following fact was established in [20, Example 5.1]. It is included here for the reader's convenience.

Lemma 9.1. *The Lagrangian subcategories in $\mathbb{L}_0(D_8)$ are precisely the following:*

$$\mathcal{L}_{(1,1)}, \quad \mathcal{L}_{(\langle r\rangle,1)}, \quad \mathcal{L}_{(\langle s,r^2\rangle,1)}, \quad \mathcal{L}_{(\langle sr,r^2\rangle,1)}, \quad \mathcal{L}_{(\langle s,r^2\rangle,\mu_1)}, \quad \textit{and} \quad \mathcal{L}_{(\langle sr,r^2\rangle,\mu_2)}. \tag{38}$$

Proof. Clearly, $\mathcal{L}_{(1,1)} \in \mathbb{L}_0(D_8)$. Using Remark 7.3 we see that Lagrangian subcategories $\mathcal{L}_{(\langle r\rangle,1)}$, $\mathcal{L}_{(\langle s,r^2\rangle,1)}$, and $\mathcal{L}_{(\langle sr,r^2\rangle,1)}$ are all equivalent to $\mathsf{Rep}(D_8)$, i.e., belong to $\mathbb{L}_0(D_8)$. To see that subcategories $\mathcal{L}_{(\langle s,r^2\rangle,\mu_1)}$ and $\mathcal{L}_{(\langle sr,r^2\rangle,\mu_2)}$ are in $\mathbb{L}_0(D_8)$ note that each of them is equivalent to a category $\mathsf{Rep}(G)$, where G is a non-Abelian group of order 8 having a normal subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The only group G with this property is D_8.

Finally, $\mathcal{L}_{(\langle r^2\rangle,1)}$ is equivalent to $\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})$ and so is not in $\mathbb{L}_0(D_8)$. $\square$

Lemma 9.2. *The restriction map $H^2(D_8, k^\times) \to H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k^\times)$ is an isomorphism.*

Proof. By Theorem 3.4 the restriction map $M(S_4) \to M(D_8)$ is injective. We saw in Section 8.2 that the restriction

$$H^2(S_4, k^\times) \to H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, k^\times)$$

is an isomorphism. This implies the claim. $\square$

Let us describe the action of $\mathfrak{A}_0(D_8)$ on $\mathsf{Sym}(\mathbb{L}_0(D_8))$.

Let $\mu \in H^2(D_8, k^\times) \subset \mathfrak{A}_0(D_8)$ and $a \in \mathsf{Out}(D_8) \subset \mathfrak{A}_0(D_8)$ be the generators of $\mathfrak{A}_0(D_8) \cong H^2(D_8, k^\times) \rtimes \mathsf{Out}(D_8) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By Lemma 9.2 μ maps $\mathcal{L}_{(\langle s,r^2\rangle,1)}$ to $\mathcal{L}_{(\langle s,r^2\rangle,\mu_1)}$ and $\mathcal{L}_{(\langle sr,r^2\rangle,1)}$ to $\mathcal{L}_{(\langle sr,r^2\rangle,\mu_2)}$. Also a maps $\mathcal{L}_{(\langle s,r^2\rangle,1)}$ to $\mathcal{L}_{(\langle sr,r^2\rangle,1)}$ and $\mathcal{L}_{(\langle s,r^2\rangle,\mu_1)}$ to $\mathcal{L}_{(\langle sr,r^2\rangle,\mu_2)}$.

Thus, $\pi : \mathfrak{A}(D_8) \to \mathsf{Sym}(\mathbb{L}_0(D_8)) \cong S_6$ is injective, i.e., $\mathfrak{A}(D_8)$ is a transitive subgroup of S_6. By Corollary 7.7 $|\mathfrak{A}(D_8)| = 24$.

Enumerating Lagrangian subcategories in the list (38) we have

$$\mathfrak{A}_0(D_8) = \{1, (35)(46), (34)(56), (36)(45)\}$$

as a subgroup of S_6. Other stabilizers of points in $\mathbb{L}_0(D_8)$ are the following conjugates of $\mathfrak{A}(D_8)$:

$$\{1, (12)(56), (15)(26), (16)(25)\} \quad \text{and} \quad \{1, (12)(34), (13)(24), (14)(23)\}. \tag{39}$$

Note that the elements

$$s_1 := (13)(24), \qquad s_2 := (15)(26), \qquad s_3 := (14)(23)$$

satisfy the usual symmetric group relations

$$(s_1 s_2)^3 = 1, \qquad (s_2 s_3)^3 = 1, \qquad s_1 s_3 = s_3 s_1, \qquad s_1^2 = s_2^2 = s_3^2 = 1,$$

and, hence, generate a subgroup isomorphic to S_4. Thus,

$$\mathfrak{A}(D_8) \cong S_4. \tag{40}$$

Example 9.3. Using Remark 4.2 we can construct a non-trivial $\mathbb{Z}/3\mathbb{Z}$-extension of $\mathsf{Rep}(D_8)$ (or Vec_{D_8}). Any such an extension is an integral fusion category of dimension 24 all whose non-invertible simple objects have dimension 2.

9.2. Quaternion group Q_8

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the quaternion group.

It is known that $H^2(Q_8, k^\times) = 1$, see [15], and $\mathsf{Out}(Q_8) = S_3$. Thus, $\mathfrak{A}_0(D_8) \cong S_3$.

It is easy to find Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_{Q_8})$. There are five normal Abelian subgroups of Q_8:

$$1, \quad \langle -1\rangle, \quad \langle i\rangle, \quad \langle j\rangle, \quad \text{and} \quad \langle k\rangle.$$

Since $H^2(N, k^\times) = 1$ for each of these subgroups there are precisely five Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_{Q_8})$. By Remark 7.3 the Lagrangian subcategory $\mathcal{L}_{(N,1)}$ is equivalent to $\mathsf{Rep}(\widehat{N} \rtimes (Q_8/N))$. But Q_8 is not isomorphic to any non-trivial semidirect product. Thus, $\mathsf{Rep}(\widehat{N} \rtimes (Q_8/N))$ is equivalent to $\mathsf{Rep}(Q_8)$ if and only if N is trivial. It follows that there is precisely one Lagrangian subcategory in $\mathbb{L}_0(Q_8)$. Thus,

$$\mathfrak{A}(Q_8) \cong S_3. \tag{41}$$

Remark 9.4. Since $\mathsf{BrPic}(\mathsf{Rep}(Q_8)) \cong \mathsf{BrPic}(\mathsf{Vec}_{Q_8}) = \mathsf{Out}(Q_8)$ we see that categories $\mathsf{Rep}(Q_8)$ and Vec_{Q_8} have no non-trivial extensions.

10. Examples: groups of order *pq*

Let p, q be prime numbers such that $q \equiv 1 \pmod p$. It is well known that there is a unique (up to an isomorphism) finite group G of order pq, namely $G = \mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$.

We will need the following presentation of G by generators and relations:

$$G = \langle x, y \mid x^q = y^p = 1 \text{ and } yxy^{-1} = x^a \rangle, \tag{42}$$

for a fixed a such that $a^p \equiv (1 \bmod q)$.

Lemma 10.1. *The group* $\mathsf{Out}(G)$ *is isomorphic to* $\mathbb{Z}/\frac{q-1}{p}\mathbb{Z}$.

Proof. It is clear that any automorphism α of G maps $\langle x \rangle$ to itself. Since all subgroups of G of order p are conjugate to each other it follows that the composition of α with some inner automorphism of G maps $\langle y \rangle$ to itself. Thus, modulo an inner automorphism, α is given by

$$\alpha(x) = x^m, \qquad \alpha(y) = y^n \tag{43}$$

for some m, n such that $1 \leqslant m < q$ and $1 \leqslant n < p$. It is straightforward to check that in order to preserve defining relations (42) of G we must have $n = 1$. Also, automorphisms (43) with $m = a^i$, $i = 0, \ldots, p-1$ are inner. Thus,

$$\mathsf{Out}(G) \cong (\mathbb{Z}/q\mathbb{Z})^\times/(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/\tfrac{q-1}{p}\mathbb{Z},$$

as required. $\square$

It is known that $H^2(G, k^\times) = 1$, see [15, Corollary 2.1.3]. The only normal Abelian subgroups of G are 1 and $\mathbb{Z}/q\mathbb{Z}$. Hence, $\mathcal{Z}(\mathsf{Vec}_G)$ contains precisely two Lagrangian subcategories.

$$\mathcal{L}_{(1,1)} \quad \text{and} \quad \mathcal{L}_{(\mathbb{Z}/q\mathbb{Z},1)}.$$

This implies that $|\mathfrak{A}(G)| = \frac{2(q-1)}{p}$.

Lemma 10.2. *Any $\alpha \in \mathfrak{A}(G)$ such that $\alpha \notin \mathfrak{A}_0(G)$ has order 2.*

Proof. The condition $\alpha \notin \mathfrak{A}_0(G)$ means that α permutes the pair of Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_G)$.

Consider the subgroup $L \subset G \times G^{\mathrm{op}}$ generated by (y, y^{-1}) and $\langle x\rangle \times \langle x\rangle$. We have

$$L \cong (\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}.$$

By Theorem 3.3 the restriction map

$$H^2(L, k^\times) \to H^2(\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}, k^\times) \cong \mathbb{Z}/q\mathbb{Z}$$

is an isomorphism. By Corollary 5.6 the braided auto-equivalence $\alpha_{(L,\mu)}$ corresponding to the Vec_G-bimodule category $\mathcal{M}(L,\mu)$ has order 2 in $\mathfrak{A}(G)$ for any non-trivial $\mu \in H^2(L, k^\times)$.

It is straightforward to check that every $\alpha \notin \mathfrak{A}_0(G)$ is isomorphic to some $\alpha_{(L,\mu)}$ (there are $\frac{q-1}{p}$ isomorphism classes of such equivalences). □

Lemma 10.2 implies that

$$\mathfrak{A}(G) \cong D_{\frac{2(q-1)}{p}}. \tag{44}$$

Example 10.3. Suppose that p, q are odd. Using Remark 4.2 we conclude that any "reflection" element of $\mathsf{BrPic}(\mathsf{Vec}_G) \cong D_{\frac{2(q-1)}{p}}$ gives rise to a non-trivial $\mathbb{Z}/2\mathbb{Z}$-extension of Vec_G. Any such an extension is a weakly integral fusion category of dimension $2pq$. The non-trivial component of this extension contains p classes of simple objects of dimension $\sqrt{q}$.

11. Examples: dihedral groups D_{2n} where n is odd

Recall that for any integer $n \geqslant 3$ we denote by D_{2n} the dihedral group on n vertices. That is,

$$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ \text{and } (sr)^2 = 1\rangle. \tag{45}$$

Let ℓ be the number of distinct prime divisors of n.

Lemma 11.1. $\mathsf{Out}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$.

Proof. For any i relatively prime to n consider $a_i \in \mathsf{Aut}(D_{2n})$ given by

$$a_i(s) = s, \qquad a_i(r) = r^i.$$

It is straightforward to check that the automorphisms a_i and a_j are congruent modulo an inner automorphism of D_{2n} if and only if $i \equiv -j \pmod{n}$ and that every outer automorphism of D_{2n} is congruent to some a_i. This implies the result. $\square$

When n is odd, $H^2(D_{2n}, k^\times) = 0$ [15, Proposition 2.11.4].

Corollary 11.2. $\mathsf{Stab}(\mathsf{Rep}(D_{2n})) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$.

Proposition 11.3. *The set $\mathbb{L}_0(D_{2n})$ consists of subcategories $\mathcal{L}_{(\langle r^b\rangle,1)}$ where b divides n and $\frac{n}{b}$ and b are relatively prime.*

Proof. Lagrangian subcategories of $\mathcal{Z}(\mathsf{Vec}_{D_{2n}})$ are all of the form $\mathcal{L}_{(\langle r^b\rangle,1)}$, where b divides n. By Remark 7.3 the category $\mathcal{L}_{(\langle r\rangle,1)}$ is equivalent to

$$\mathsf{Rep}\left(\left(\mathbb{Z}/\frac{n}{b}\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}\right) \rtimes \mathbb{Z}/2\mathbb{Z}\right).$$

The latter category is equivalent to $\mathsf{Rep}(D_{2n})$ if and only if the group $\mathbb{Z}/\frac{n}{b}\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ has an element of order n, which is the case precisely when $\frac{n}{b}$ and b are relatively prime. $\square$

Remark 11.4. Note that $\mathfrak{A}_0(D_{2n})$ stabilizes *all* Lagrangian subcategories in $\mathbb{L}_0(D_{2n})$ and, hence, it is a normal subgroup of $\mathfrak{A}(D_{2n})$.

Lemma 11.5. *The image of $\mathfrak{A}(D_{2n})$ in $\mathsf{Sym}(\mathbb{L}_0(D_{2n}))$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\ell$.*

Proof. Let $\alpha \in \mathfrak{A}(D_{2n})$ be a braided autoequivalence such that $\alpha \notin \mathsf{Stab}(\mathsf{Rep}(D_{2n}))$. It suffices to show that the image of α in $\mathsf{Sym}(\mathbb{L}_0(D_{2n}))$ has order 2. For this end, observe that for any fixed $\mathcal{L}_0 \in \mathbb{L}_0(D_{2n})$ all the dimensions $\dim(\mathcal{L}_0 \cap \mathcal{L})$, $\mathcal{L} \in \mathbb{L}_0(D_{2n})$ are distinct (they correspond to subsets of the set of prime divisors of n). Since α preserves dimensions of subcategories, we have

$$\dim\bigl(\mathcal{L} \cap \alpha(\mathcal{L})\bigr) = \dim\bigl(\alpha(\mathcal{L}) \cap \alpha^2(\mathcal{L})\bigr),$$

and, hence, $\alpha^2(\mathcal{L}) = \mathcal{L}$ for any $\mathcal{L} \in \mathbb{L}_0(D_{2n})$. $\square$

Corollary 11.6. *There is a short exact sequence*

$$1 \to (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\} \to \mathfrak{A}(D_{2n}) \to (\mathbb{Z}/2\mathbb{Z})^\ell \to 1. \tag{46}$$

Remark 11.7. When $\ell = 1$, i.e., when n is a prime power, an argument similar to that in Section 10 shows that $\mathfrak{A}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\} \rtimes \mathbb{Z}/2\mathbb{Z}$, i.e., $\mathfrak{A}(D_{2n})$ is a generalized dihedral group. We conjecture that in general the sequence (46) splits, i.e.,

$$\mathfrak{A}(D_{2n}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\} \rtimes (\mathbb{Z}/2\mathbb{Z})^\ell.$$

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