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Irreducible characters of wreath products in reality-based algebras and applications to association schemes

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Wreath products in reality-based algebras are generalizations of wreath products of table algebras and generalized Camina–Frobenius pairs of C-algebras. In this paper we present characterizations of the wreath product in a reality-based algebra by its irreducible characters and by the size of the zero submatrix of its character table. Applications to finite groups, table algebras, and association schemes are also discussed. In particular, we will show that the wreath product of one-class association schemes is characterized by the zeros in its first eigenmatrix.

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1. Introduction

The wreath product of association schemes provides a useful way to construct new association schemes from old ones (cf. [10,16], etc.). The wreath product of table algebras was first used by Arad and Muzychuk [3] for the classification of certain classes

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of standard integral table algebras. Since the Bose–Mesner algebra of an association scheme is a standard table algebra, and as a table algebra, the Bose–Mesner algebra of the wreath product of association schemes is exactly isomorphic to the wreath product of the Bose–Mesner algebras of those association schemes, the study of the wreath product of table algebras has natural applications to association schemes. Some basic properties of wreath products of C-algebras, table algebras, and association schemes are presented in [1,15].

Reality-based algebras are generalizations of table algebras. In this paper we study wreath products of standard reality-based algebras. Our main results (Theorems 1.6 and 1.8 below) characterize wreath products of these algebras in terms of their character tables. An immediate consequence of Theorem 1.6 is a result of Arad and Fisman [1, Lemma 2.11] on C-algebras that are wreath products (see Corollary 1.7 below). Applications to finite groups, table algebras, and association schemes are also discussed. In particular, Belonogov's result [7] for the Camina–Frobenius pairs in finite groups (Corollary 1.10 below) is a direct consequence of Theorem 1.8, and the first and second eigenmatrices of the wreath product of one-class association schemes can be easily obtained without calculations from Theorem 1.8 and its proof. Furthermore, by applying Theorem 1.8, we will show that the wreath product of one-class association schemes is characterized by the zeros in its first eigenmatrix.

In the rest of this introductory section, we state the main results of the paper explicitly. Let us start with some necessary definitions and notation.

Definition 1.1. (Cf. [5, Definition 1.16].) A *reality-based algebra* (RBA) $(A, \mathbf{B})$ is a finite dimensional associative algebra A over the complex numbers $\mathbb{C}$ with a distinguished basis $\mathbf{B} := \{b_i \mid 0 \le i \le k\}$, where $b_0 = 1_A$, the identity element of A, and the following three conditions hold.

(i) The structure constants for $\mathbf{B}$ are real numbers; that is, for all $b_i, b_j \in \mathbf{B}$,

$$b_i b_j = \sum_{m=0}^{k} \lambda_{ijm} b_m, \quad \text{for some } \lambda_{ijm} \in \mathbb{R}.$$

(ii) There is an algebra anti-automorphism (denoted by $*$) of A such that $(a^*)^* = a$ for all $a \in A$ and $b_i^* \in \mathbf{B}$ for all $b_i \in \mathbf{B}$. (Hence i^* is defined by $b_{i^*} = b_i^*$.)

(iii) For all $b_i, b_j \in \mathbf{B}$, $\lambda_{ij0} = 0$ if $j \ne i^*$; and $\lambda_{ii^*0} = \lambda_{i^*i0} > 0$.

Let $(A, \mathbf{B})$ be a RBA. If all structure constants λ_{ijm} are nonnegative, then $(A, \mathbf{B})$ is a *table algebra*. A *degree map* (cf. [5, Definition 1.1d]) for $(A, \mathbf{B})$ is an algebra homomorphism $f : A \to \mathbb{C}$ such that $f(b_i) \in \mathbb{R} \setminus \{0\}$ for all $b_i \in \mathbf{B}$. The values $f(b_i)$ are called the *degrees* of $(A, \mathbf{B}, f)$. If $(A, \mathbf{B})$ is a table algebra, then there always exists a (unique) degree map that is positive on $\mathbf{B}$; such a degree map is called the *positive* degree map. If f is a degree map of $(A, \mathbf{B})$ such that $f(b_i) = \lambda_{ii^*0}$ for all i, then $(A, \mathbf{B}, f)$ is called a *standard* RBA (SRBA), cf. [5, Definition 2.10]. A *standard table algebra* is a SRBA with nonnegative structure constants, and a *C-algebra* (in the sense of [8]) is a commutative SRBA. The basis $\mathbf{B}$ can be *rescaled* (cf. [5, Definition 2.11]), replacing each b_i by $\mu_i b_i$ for some $\mu_i \in \mathbb{R} \setminus \{0\}$ such that $\mu_i = \mu_{i^*}$ for all i and $\mu_0 = 1$. It is clear that the rescaled basis again yields a RBA. If $(A, \mathbf{B})$ has a degree map f, then there is a unique rescaling $\mathbf{B}'$ such that $(A, \mathbf{B}', f)$ is a SRBA (cf. [5, Proposition 2.20]).

Let $(A, \mathbf{B})$ be a RBA. Then by [5, Proposition 2.1], there exists a positive definite sesquilinear form $(\ ,\ )$ on A (that is, the form $(\ ,\ )$ is biadditive, and for all $a, b \in A$ and $\gamma \in \mathbb{C}$, $(\gamma a, b) = \gamma(a, b)$, $(b, a) = \overline{(a, b)}$ and $(a, a) > 0$ if $a \neq 0$) such that for all $b_i, b_j, b_m \in \mathbf{B}$,

$$(b_i, b_j) = \delta_{ij}\lambda_{ii^*0} \quad \text{and} \quad (b_i b_j, b_m) = \left(b_j, b_i^* b_m\right) = \left(b_i, b_m b_j^*\right). \tag{1.1}$$

Furthermore, A is a semisimple algebra (cf. [5, Proposition 2.3]). As usual, a *representation* of A is an algebra homomorphism $\phi : A \to M_n(\mathbb{C})$ such that $\phi(1_A) = I_n$, where n is a positive integer, $M_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices over $\mathbb{C}$, and I_n is the $n \times n$ identity matrix. Let $\phi : A \to M_n(\mathbb{C})$ be a representation of A. ϕ is called *irreducible* if $\phi(A)$ acts irreducibly on $\mathbb{C}^n$. The *character afforded by* ϕ is the linear map $\chi : A \to \mathbb{C}, a \mapsto \mathrm{Tr}(\phi(a))$, where $\mathrm{Tr}(\phi(a))$ is the trace of $\phi(a)$. χ is called *irreducible* if ϕ is irreducible. The set of all irreducible characters of $(A, \mathbf{B})$ is denoted by $\mathrm{Irr}(A)$ or $\mathrm{Irr}(\mathbf{B})$.

Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Then χ_0 is an irreducible character of A. Let $\mathrm{Irr}(\mathbf{B}) := \{\chi_0, \chi_1, \ldots, \chi_r\}$, and $\mathrm{Irr}(\mathbf{B})^\sharp := \mathrm{Irr}(\mathbf{B}) \setminus \{\chi_0\}$. For a nonempty subset $\mathbf{S}$ of $\mathbf{B}$, define

$$\mathbf{S}^+ := \sum_{b_i \in \mathbf{S}} b_i \quad \text{and} \quad o(\mathbf{S}) := \sum_{b_i \in \mathbf{S}} \chi_0(b_i) = \chi_0\left(\mathbf{S}^+\right).$$

A linear map $\sigma : A \to \mathbb{C}$ is called a *feasible trace* (cf. [12]) if for any $x, y \in A$, $\sigma(xy) = \sigma(yx)$. Define

$$\zeta : \ A \to \mathbb{C}, \qquad x \mapsto \alpha_0, \quad \text{if } x = \sum_{i=0}^{k} \alpha_i b_i. \tag{1.2}$$

Then it follows from $\lambda_{ii^*0} = \lambda_{i^*i0}$ for all i that ζ is a feasible trace of A. According to [12],

$$\zeta = \sum_{i=0}^{r} z_i \chi_i, \quad \text{where } z_i \in \mathbb{C} \text{ are } \textit{feasible multiplicities}. \tag{1.3}$$

Furthermore, each $z_i > 0$ by [5, Lemma 2.11(i)]. Let e_i be the central primitive idempotent of A corresponding to $\chi_i \in \mathrm{Irr}(\mathbf{B})$, $0 \leq i \leq k$. Then by [5, Proposition 2.14],

$$e_i = \sum_{l=0}^{k} \frac{z_i}{\chi_0(b_l)} \chi_i\left(b_l^*\right) b_l, \quad 0 \leq i \leq k.$$

In particular, $e_0 = z_0\mathbf{B}^+$. Thus, $z_0 o(\mathbf{B}) = \chi_0(z_0\mathbf{B}^+) = \chi_0(e_0) = 1$. So $z_0 = o(\mathbf{B})^{-1}$, and hence $e_0 = o(\mathbf{B})^{-1}\mathbf{B}^+$. e_0 will also be denoted by $e_{\mathbf{B}}$. Since $e_0 A$ has dimension $\chi_0(1) = 1$,

$$b_i e_{\mathbf{B}} = \chi_0(b_i)e_{\mathbf{B}}, \quad \text{for all } b_i \in \mathbf{B}.$$

Let $(A, \mathbf{B})$ be a RBA. For any $b_i, b_j \in \mathbf{B}$, define

$$\operatorname{Supp}(b_i b_j) := \bigl\{b_l \in \mathbf{B} \bigm| (b_i b_j, b_l) \neq 0\bigr\}.$$

Note that for any $b_i, b_j, b_l \in \mathbf{B}$, (1.1) implies that

$$b_l \in \operatorname{Supp}(b_i b_j) \quad \Leftrightarrow \quad b_i \in \operatorname{Supp}\bigl(b_l b_j^*\bigr) \quad \Leftrightarrow \quad b_j \in \operatorname{Supp}\bigl(b_i^* b_l\bigr). \tag{1.4}$$

For any nonempty subsets $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{B}$, define the set product $\mathbf{ST}$ by

$$\mathbf{ST} := \bigcup_{b_i \in \mathbf{S}, b_j \in \mathbf{T}} \operatorname{Supp}(b_i b_j).$$

Note that the set product need not be associative for RBAs. A nonempty subset $\mathbf{N}$ of $\mathbf{B}$ is called a *closed subset* if $\mathbf{N} = \mathbf{N}^*$ and $\mathbf{NN} \subseteq \mathbf{N}$, where $\mathbf{N}^* := \{b_i^* \mid b_i \in \mathbf{N}\}$. If $\mathbf{N}$ is a closed subset of $\mathbf{B}$, then $(\mathbb{C}\mathbf{N}, \mathbf{N})$ is also a RBA, where $\mathbb{C}\mathbf{N}$ is the $\mathbb{C}$-space spanned by $\mathbf{N}$. Furthermore, when $\mathbf{N}$ is a closed subset of $\mathbf{B}$, then (1.4) implies that

$$\mathbf{N}(\mathbf{B} \setminus \mathbf{N}) \cup (\mathbf{B} \setminus \mathbf{N})\mathbf{N} \subseteq \mathbf{B} \setminus \mathbf{N}. \tag{1.5}$$

Definition 1.2. Let $(A, \mathbf{B}, \chi_0)$ be a SRBA, and $\mathbf{N}$ a nonempty subset of $\mathbf{B}$. If

$$\operatorname{Supp}(b_i b_j) = \operatorname{Supp}(b_j b_i) = \{b_j\}, \quad \text{for all } b_i \in \mathbf{N}, b_j \in \mathbf{B} \setminus \mathbf{N},$$

then $(A, \mathbf{B}, \chi_0)$ is called a *wreath product* $(\mathbf{B}, \mathbf{N})$.

Remark 1.3. (i) The above definition implies that if $(A, \mathbf{B}, \chi_0)$ is a wreath product $(\mathbf{B}, \mathbf{N})$, then $\mathbf{N}$ is a closed subset. In fact, for any $b_i \in \mathbf{N}$ such that $b_i \neq b_0$, since $b_0 \in \operatorname{Supp}(b_i b_i^*)$, we see that $\operatorname{Supp}(b_i b_i^*) \neq \{b_i^*\}$, and hence $b_i^* \in \mathbf{N}$. Thus, $\mathbf{N}^* = \mathbf{N}$. For any $b_i, b_j \in \mathbf{N}$, if there is $b_l \in \mathbf{B} \setminus \mathbf{N}$ such that $b_l \in \operatorname{Supp}(b_i b_j)$, then by (1.4), $b_j \in \operatorname{Supp}(b_i^* b_l) = \{b_l\}$, a contradiction. Hence, $\mathbf{NN} \subseteq \mathbf{N}$. Thus, $\mathbf{N}$ is a closed subset.

(ii) In the above definition for any $b_i \in \mathbf{N}$ and $b_j \in \mathbf{B} \setminus \mathbf{N}$, $\operatorname{Supp}(b_i b_j) = \operatorname{Supp}(b_j b_i) = \{b_j\}$ is equivalent to $b_i b_j = b_j b_i = \chi_0(b_i) b_j$. Hence, if A is commutative, then Definition 1.2 is the same as the definition of a *generalized Camina–Frobenius pair* for a C-algebra in [1].

(iii) The definition of a generalized Camina–Frobenius pair for RBAs in [6, Definition 1.5] is a mild generalization of Definition 1.2 above. In particular, when there is no degree map, then the product $b_i b_j$ can be 0.

(iv) Let $(A, \mathbf{B})$ and $(C, \mathbf{D})$ be SRBAs, with $\mathbf{B} = \{b_0 = 1_A, b_1, \dots, b_k\}$ and $\mathbf{D} = \{d_0 = 1_C, d_1, \dots, d_h\}$. It is immediate that $(A \otimes C, \mathbf{B} \otimes \mathbf{D})$ is an SRBA, the same as for table algebras in [2] or [14]. Also as for table algebras, the (constructive) wreath product $(A \wr C, \mathbf{B} \wr \mathbf{D})$ is defined as

$$\mathbf{B} \wr \mathbf{D} := \{b_0 \otimes d_j \mid 0 \le j \le h\} \cup \left\{b_i \otimes \mathbf{D}^+ \,\middle|\, 1 \le i \le k\right\}.$$

Then $(A \wr C, \mathbf{B} \wr \mathbf{D})$ is a wreath product $(\mathbf{B} \wr \mathbf{D}, b_0 \otimes \mathbf{D})$ in the sense of Definition 1.2, where $b_0 \otimes \mathbf{D} := \{b_0 \otimes d_j \mid 0 \le j \le h\}$.

Definition 1.4. Let $(A, \mathbf{B}, \chi_0)$ be a SRBA, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. For each $\chi \in \mathrm{Irr}(\mathbf{B})$, define

$$\chi{\downarrow_{\mathbf{N}}} = \chi{\downarrow_{\mathbb{C}\mathbf{N}}} \quad \text{and} \quad \mathrm{Irr}(\chi{\downarrow_{\mathbf{N}}}) := \left\{\psi_t \in \mathrm{Irr}(\mathbb{C}\mathbf{N}) \,\middle|\, \chi{\downarrow_{\mathbb{C}\mathbf{N}}} = \sum_t m_t \psi_t \text{ with } m_t > 0\right\}.$$

Remark 1.5. (i) Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Let $\mathbf{N}$ be a closed subset of $\mathbf{B}$, and ψ_0 the degree map of $(\mathbb{C}\mathbf{N}, \mathbf{N})$. Then $\chi_0{\downarrow_{\mathbb{C}\mathbf{N}}} = \psi_0$.

(ii) Let ζ be the feasible trace of A defined by (1.2). Then $\zeta{\downarrow_{\mathbb{C}\mathbf{N}}}$ is the feasible trace of $\mathbb{C}\mathbf{N}$. Since the feasible multiplicities defined in (1.3) for ζ and $\zeta{\downarrow_{\mathbb{C}\mathbf{N}}}$ are positive, each $\psi \in \mathrm{Irr}(\mathbf{N})$ is in $\mathrm{Irr}(\chi{\downarrow_{\mathbf{N}}})$ for some $\chi \in \mathrm{Irr}(\mathbf{B})$.

Let $(A, \mathbf{B})$ be a RBA, and $\chi \in \mathrm{Irr}(\mathbf{B})$. Then for any nonempty subset $\mathbf{S}$ of $\mathbf{B}$, let $\chi(\mathbf{S}) := \{\chi(b_i) \mid b_i \in \mathbf{S}\}$. Now we are ready to state the main results of the paper.

Theorem 1.6. *Let $(A, \mathbf{B}, \chi_0)$ be a SRBA, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. Then the following are equivalent.*

(*i*) *$(A, \mathbf{B}, \chi_0)$ is a wreath product $(\mathbf{B}, \mathbf{N})$.*
(*ii*) *For any $\psi \in \mathrm{Irr}(\mathbf{N})^\sharp$, $\psi = \chi{\downarrow_{\mathbf{N}}}$ for some $\chi \in \mathrm{Irr}(\mathbf{B})$ such that $\chi(\mathbf{B} \setminus \mathbf{N}) = \{0\}$.*

Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Assume that A is commutative. That is, $(A, \mathbf{B}, \chi_0)$ is a C-algebra. Then up to exact isomorphism, $(A, \mathbf{B}, \chi_0)$ has a unique dual C-algebra $(A, \widehat{\mathbf{B}}_{\chi_0}, \chi_0)$. For any nonempty subset $\mathbf{N}$ of $\mathbf{B}$, the *kernel* of $\mathbf{N}$, $\ker_{\chi_0} \mathbf{N}$, is a subset of $\widehat{\mathbf{B}}_{\chi_0}$. The reader is referred to [4] for more details about dual C-algebras and kernels of subsets. The following result of Arad and Fisman (cf. [1, Lemma 2.11]) is an immediate consequence of Theorem 1.6 and (3.3) in its proof (see below).

Corollary 1.7. *(Cf. [1, Lemma 2.11].) Let $(A, \mathbf{B}, \chi_0)$ be a C-algebra that is a wreath product $(\mathbf{B}, \mathbf{N})$ for some closed subset $\mathbf{N}$. Then the dual C-algebra $(A, \widehat{\mathbf{B}}_{\chi_0}, \chi_0)$ is a wreath product $(\widehat{\mathbf{B}}_{\chi_0}, \ker_{\chi_0} \mathbf{N})$.*

Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. The character table of $(A, \mathbf{B}, \chi_0)$ is regarded as a matrix whose columns are indexed by the elements of $\mathbf{B}$ and whose rows are indexed by the irreducible characters of A. Assume that $\mathbf{B} = \{b_0 = 1_A, b_1, \dots, b_k\}$ and $\mathrm{Irr}(A) = \{\chi_0, \chi_1, \dots, \chi_r\}$. Then for any $0 \le i \le r$ and $0 \le j \le k$, the (χ_i, b_j)-entry of the character table is $\chi_i(b_j)$. If the character table of $(A, \mathbf{B}, \chi_0)$ has an $s \times t$ zero submatrix, then we will prove that $s + t \le |\mathbf{B}| - 1$ (see Proposition 3.1 below).

Theorem 1.8. *Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Then the following are equivalent.*

(*i*) *By permuting the rows and columns if necessary, the character table of $(A, \mathbf{B})$ has an $s \times t$ zero submatrix such that $s + t = |\mathbf{B}| - 1$, and the irreducible characters of A corresponding to the zero submatrix are linear.*
(*ii*) *There is a proper closed subset $\mathbf{N}$ of $\mathbf{B}$ such that $\mathbb{C}\mathbf{N}$ is commutative, $|\mathbf{N}| = s + 1$, and $(A, \mathbf{B}, \chi_0)$ is a wreath product $(\mathbf{B}, \mathbf{N})$.*

Remark 1.9. Let G be an extra-special p-group of order p^{2n+1}. That is, $|G| = p^{2n+1}$, $|Z(G)| = p$, and $G/Z(G)$ is an elementary abelian p-group, where $Z(G)$ is the center of G. Let $(A, \mathbf{B}) := (\mathbb{C}G, G)$, and $\mathbf{N} = Z(G)$. Then $(A, \mathbf{B})$ is a standard table algebra, and $\mathbf{N}$ is a closed subset of $\mathbf{B}$. It is well known that A has $p - 1$ irreducible characters χ_j of degree p^n such that $\chi_j(\mathbf{B} \setminus \mathbf{N}) = \{0\}$. Let $s = p - 1$, and $t = |\mathbf{B}| - p$. Then $s + t = |\mathbf{B}| - 1$, and by permuting the rows and columns if necessary, the character table of $(A, \mathbf{B})$ has an $s \times t$ zero submatrix. However, $(A, \mathbf{B})$ is not a wreath product $(\mathbf{B}, \mathbf{N})$. ($(A, \mathbf{B})$ is not a wreath product $(\mathbf{B}, \mathbf{M})$ for any proper closed subset $\mathbf{M}$ such that $|\mathbf{M}| > 1$, because $\mathbf{B}$ is a group.) So in the above theorem, if the assumption that the irreducible characters corresponding to the zero submatrix are linear is removed, then the implication (i) $\Rightarrow$ (ii) is false, even if $(A, \mathbf{B})$ is a standard table algebra. Also note that for group G as above, $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ is a wreath product $(\mathrm{Cla}(G), Z(G))$, where $Z(\mathbb{C}G)$ is the center of the group algebra $\mathbb{C}G$, and $\mathrm{Cla}(G)$ consists of class sums.

For a finite group G and a normal subgroup N of G, let $k(G)$ denote the number of conjugacy classes of G, and $k_G(N)$ the number of conjugacy classes of G contained in N. As a direct application of Proposition 3.1 and Theorem 1.8 to $(Z(\mathbb{C}G), \mathrm{Cla}(G))$ and its irreducible (central) characters, we have the following

Corollary 1.10. *(See [7, Corollary 1.2, Theorem 1.2].) Let G be a finite group. Then the following hold.*

(*i*) *If the character table of G has an $s \times t$ zero submatrix, then $s + t \le k(G) - 1$.*
(*ii*) *The following are equivalent.*
  (*a*) *By permuting the rows and columns if necessary, the character table of G has an $s \times t$ zero submatrix such that $s + t = k(G) - 1$.*
  (*b*) *G has a proper normal subgroup N such that $k_G(N) + k(G/N) = k(G) + 1$.*

Let G be a finite group. If G has a proper normal subgroup N such that $k_G(N) + k(G/N) = k(G)+1$, then (G, N) is called a *Camina–Frobenius pair*. The reader is referred to [7] for more details.

Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be an association scheme, and let $A_0, A_1, \ldots, A_d$ be its adjacency matrices. Let $\mathcal{A}$ be the Bose–Mesner algebra of $\mathcal{X}$. Then $(\mathcal{A}, \mathbf{B})$ is a standard table algebra, where $\mathbf{B} := \{A_0, A_1, \ldots, A_d\}$. A subset T of $\{R_i\}_{0 \le i \le d}$ is a *closed subset of* $\mathcal{X}$ if $\{A_i \mid R_i \in T\}$ is a closed subset of $\mathbf{B}$. Let T be a closed subset of $\mathcal{X}$, and $X/T := \{xT \mid x \in X\}$, where $xT := \{y \in X \mid (x, y) \in t$ for some $t \in T\}$. Then we have the *quotient scheme* $\mathcal{X}//T$ on the set X/T. Furthermore, fix $x \in X$, and let $t_{xT} := t \cap (xT \times xT)$, for any $t \in T$. Then we have the *subscheme* $T_{xT} := \{t_{xT} \mid t \in T\}$ on xT. The reader is referred to [20] for more details about subschemes and quotient schemes. Also see [17, 19]. It is well known that for a closed subset T of $\mathcal{X}$ and $x \in X$, the Bose–Mesner algebras of $\mathcal{X}//T$ and T_{xT} are exactly isomorphic to the *quotient table algebra* $(\mathcal{A}//\mathbf{N}, \mathbf{B}//\mathbf{N})$ (see Section 3) and *table subalgebra* $(\mathbb{C}\mathbf{N}, \mathbf{N})$, respectively, where $\mathbf{N} := \{A_i \mid R_i \in T\}$.

Since the first eigenmatrix of a commutative association scheme is the character table of the adjacency algebra, the next corollary is immediate from Proposition 3.1 and Theorem 1.8. More applications to commutative association schemes will be discussed in Section 4. For the definition of the wreath product of association schemes, see Section 4.

Corollary 1.11. *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a commutative association scheme. Then the following hold.*

- (*i*) *If the first eigenmatrix P of $\mathcal{X}$ has an $s \times t$ zero submatrix, then $s + t \le d$.*
- (*ii*) *The following are equivalent.*
  - (*a*) *By permuting the rows and columns if necessary, the first eigenmatrix P of $\mathcal{X}$ has an $s \times t$ zero submatrix such that $s + t = d$.*
  - (*b*) *There is a proper closed subset T of $\mathcal{X}$ such that $|T| = s+1$ and the Bose–Mesner algebra of $\mathcal{X}$ is exactly isomorphic to the Bose–Mesner algebra of the wreath product $(\mathcal{X}//T) \wr T_{xT}$ as table algebras.*

The rest of the paper is organized as follows. In Section 2 we present some preliminary lemmas. Then in Section 3, we prove Theorems 1.6 and 1.8, and discuss applications to table algebras. Applications to association schemes are presented in Section 4.

2. Preliminary lemmas

In this section we present a few results that will be needed for the proofs of Theorems 1.6 and 1.8 in the next section. Let $(A, \mathbf{B}, \chi_0)$ be a SRBA, with $\mathbf{B} := \{b_i \mid 0 \le i \le k\}$. For any characters ϕ, ρ of A, define the form

$$\langle \phi, \rho \rangle_{\mathbf{B}} := \sum_{l=0}^{k} \frac{1}{\chi_0(b_l)} \phi\big(b_l^*\big) \rho(b_l).$$

Since $\phi(b_l^*)$ is the complex conjugate of $\phi(b_l)$ for any $b_l \in \mathbf{B}$, the form is Hermitian, and the following holds.

Lemma 2.1 *(Orthogonality Relations). (Cf. [5, Proposition 2.12], [2, Theorem 3.6].) Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Then for any $\chi_i, \chi_j \in \mathrm{Irr}(\mathbf{B})$,*

$$\langle \chi_i, \chi_j \rangle_{\mathbf{B}} = \frac{\chi_i(1)}{z_i} \delta_{ij},$$

where z_i is the feasible multiplicity.

The next lemma is a result for semisimple algebras over $\mathbb{C}$ in general.

Lemma 2.2. *Let A be a finite dimensional semisimple algebra over $\mathbb{C}$, and S a semisimple subalgebra of A with $1_A \in S$. Let $\Phi : A \to M_n(\mathbb{C})$ (for some n) be a representation of A such that $\Phi{\downarrow_S}$ is irreducible, and χ the character of A afforded by Φ. Let C be a $\mathbb{C}$-subspace of A that is a left (or similarly, a right) S-module. Then for $\chi(C) := \{\chi(c) \mid c \in C\}$,*

$$\chi(C) = \{0\} \quad \textit{if and only if } \Phi(C) = \{0\}.$$

Proof. It is obvious that if $\Phi(C) = \{0\}$, then $\chi(C) = \{0\}$. Now assume that $\chi(C) = \{0\}$. Toward a contradiction, suppose that $\Phi(c) \neq 0$ for some $c \in C$. Then for some $1 \leq i$, $j \leq n$, the (i, j)-entry $\gamma_{i,j}$ of $\Phi(c)$ is nonzero. Since S is semisimple and $\Phi{\downarrow_S}$ is irreducible, $\Phi(S) = M_n(\mathbb{C})$. So there is $s \in S$ such that $\Phi(s) = E_{ji}$, the matrix unit with 1 as the (j, i)-entry, and 0 elsewhere. Now $sc \in C$ by hypothesis, and

$$\Phi(sc) = \Phi(s)\Phi(c) = E_{ji}\Phi(c),$$

a matrix with γ_{ij} as its (j, j)-entry, and all other diagonal entries equal to zero. So $\chi(sc) = \gamma_{ij} \neq 0$, a contradiction. This proves that $\Phi(C) = \{0\}$. □

As a direct consequence of Lemma 2.2, we have the following

Corollary 2.3. *Let $(A, \mathbf{B}, \chi_0)$ be a SRBA, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. Let Φ be a representation of A such that $\Phi{\downarrow_{\mathbb{C}\mathbf{N}}}$ is irreducible, and χ the character of A afforded by Φ. Then*

$$\chi(\mathbf{B} \setminus \mathbf{N}) = \{0\} \quad \textit{if and only if } \Phi(\mathbf{B} \setminus \mathbf{N}) = \{0\}.$$

Proof. It is known that $\mathbb{C}\mathbf{N}$ is a semisimple subalgebra of A with $1_A \in \mathbb{C}\mathbf{N}$. Let $C := \mathbb{C}(\mathbf{B} \setminus \mathbf{N})$ be the $\mathbb{C}$-subspace of A spanned by $\mathbf{B} \setminus \mathbf{N}$. Then C is a left and right $(\mathbb{C}\mathbf{N})$-module by (1.5). So the corollary holds by Lemma 2.2. □

3. Proofs of Theorems 1.6 and 1.8

In this section we first prove Theorems 1.6 and 1.8. We will also present an upper bound for the size of a zero submatrix in the character table of a SRBA, and discuss applications to table algebras.

Proof of Theorem 1.6. (i) ⇒ (ii) Let $e_{\mathbf{N}} = o(\mathbf{N})^{-1}\mathbf{N}^+$, the central primitive idempotent of $\mathbb{C}\mathbf{N}$ that corresponds to the degree map ψ_0 of $(\mathbb{C}\mathbf{N}, \mathbf{N})$. If $\psi \in \mathrm{Irr}(\mathbf{N})^\sharp$ and M is an irreducible $(\mathbb{C}\mathbf{N})$-module that affords ψ, then $e_{\mathbf{N}}M = \{0\}$. Let $b_j \in \mathbf{B} \setminus \mathbf{N}$. Since $b_ib_j = b_jb_i = \psi_0(b_i)b_j$ for all $b_i \in \mathbf{N}$, it follows that $b_je_{\mathbf{N}} = e_{\mathbf{N}}b_j = b_j$, and $(b_jb_j^*, b_i) = (b_j, b_ib_j) = \psi_0(b_i)\chi_0(b_j)$ by (1.1). Hence,

$$b_jb_j^* = \chi_0(b_j)\mathbf{N}^+ + \sum_{b_m \in \mathbf{B}\setminus\mathbf{N}} \lambda_{jj^*m}b_m.$$

Therefore, M becomes an irreducible A-module with $b_jx = 0$ for all $b_j \in \mathbf{B} \setminus \mathbf{N}$ and $x \in M$. Let χ be the irreducible character of A afforded by M. Then $\psi = \chi{\downarrow_{\mathbf{N}}}$, and $\chi(\mathbf{B} \setminus \mathbf{N}) = \{0\}$. So (ii) holds.

(ii) ⇒ (i) Since A is semisimple, there is an algebra isomorphism

$$\Omega: \ A \to \bigoplus_{i=0}^{r} M_{n_i}(\mathbb{C}).$$

Let $\Pi_i : \bigoplus_{j=0}^{r} M_{n_j}(\mathbb{C}) \to M_{n_i}(\mathbb{C})$ be the projection, and $\Omega_i = \Pi_i\Omega$, $0 \le i \le r$. Then the irreducible representations of A, up to equivalence, are Ω_i, $i = 0, 1, \ldots, r$. For any $b_i \in \mathbf{N}$ and $b_j \in \mathbf{B} \setminus \mathbf{N}$, since $b_ib_j = b_jb_i = \chi_0(b_i)b_j$ if and only if $\Omega(b_ib_j) = \Omega(b_jb_i) = \Omega(\chi_0(b_i)b_j)$, it follows that (i) holds if and only if

$$\Omega_l(b_ib_j) = \Omega_l(b_jb_i) = \Omega_l\big(\chi_0(b_i)b_j\big), \quad \text{for all } b_i \in \mathbf{N},\ b_j \in \mathbf{B} \setminus \mathbf{N},\ 0 \le l \le r. \tag{3.1}$$

In the following we prove that (3.1) holds. Define

$$E(\mathbf{N}) := \big\{\chi \in \mathrm{Irr}(\mathbf{B}) \bigm| \chi{\downarrow_{\mathbf{N}}} \in \mathrm{Irr}(\mathbf{N})^\sharp \text{ and } \chi(\mathbf{B} \setminus \mathbf{N}) = \{0\}\big\}. \tag{3.2}$$

Then by (ii), $\{\chi{\downarrow_{\mathbf{N}}} \mid \chi \in E(\mathbf{N})\} = \mathrm{Irr}(\mathbf{N})^\sharp$. Let $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$. Then for any $\chi \in E(\mathbf{N})$, $\chi(\mathbf{B} \setminus \mathbf{N}) = \{0\}$ and Lemma 2.1 imply that

$$0 = \langle \eta, \chi\rangle_{\mathbf{B}} = \langle \eta{\downarrow_{\mathbf{N}}}, \chi{\downarrow_{\mathbf{N}}}\rangle_{\mathbf{N}}.$$

Hence, $\mathrm{Irr}(\eta{\downarrow_{\mathbf{N}}}) \cap \mathrm{Irr}(\mathbf{N})^\sharp = \emptyset$. Let ψ_0 be the degree map of $(\mathbb{C}\mathbf{N}, \mathbf{N})$. It follows that

$$\eta{\downarrow_{\mathbf{N}}} = \eta(1)\psi_0, \quad \text{for all } \eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N}). \tag{3.3}$$

Let Ω_l be an irreducible representation of A that affords some $\chi \in E(\mathbf{N})$. Then $\Omega_l{\downarrow_{\mathbb{C}\mathbf{N}}}$ is irreducible, and hence $\Omega_l(\mathbf{B} \setminus \mathbf{N}) = \{0\}$ by Corollary 2.3. Thus, for all $b_i \in \mathbf{N}$ and $b_j \in \mathbf{B} \setminus \mathbf{N}$,

$$\Omega_l(b_i b_j) = \Omega_l(b_i)\Omega_l(b_j) = 0, \qquad \Omega_l(b_j b_i) = \Omega_l(b_j)\Omega_l(b_i) = 0,$$
$$\Omega_l\big(\chi_0(b_i)b_j\big) = \chi_0(b_i)\Omega_l(b_j) = 0.$$

Therefore,

$$\Omega_l(b_i b_j) = \Omega_l(b_j b_i) = \Omega_l\big(\chi_0(b_i)b_j\big), \quad \text{for all } b_i \in \mathbf{N}, b_j \in \mathbf{B} \setminus \mathbf{N}.$$

Now let Ω_m be an irreducible representation of A that affords some $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$. Then (3.3) implies that for all $b_i \in \mathbf{N}$, $\Omega_m(b_i) = \chi_0(b_i)I_{n_m}$, where $n_m = \eta(1)$. So for all $b_i \in \mathbf{N}$ and $b_j \in \mathbf{B} \setminus \mathbf{N}$,

$$\Omega_m(b_i b_j) = \Omega_m(b_i)\Omega_m(b_j) = \chi_0(b_i)\Omega_m(b_j) = \Omega_m\big(\chi_0(b_i)b_j\big),$$

and

$$\Omega_m(b_j b_i) = \Omega_m(b_j)\Omega_m(b_i) = \chi_0(b_i)\Omega_m(b_j) = \Omega_m\big(\chi_0(b_i)b_j\big).$$

Therefore, we have shown that (3.1) holds. □

The next proposition gives an upper bound for the size of a zero submatrix in the character table of a SRBA. We will need this result for the proof of Theorem 1.8.

Proposition 3.1. *Let $(A, \mathbf{B}, \chi_0)$ be a SRBA. Assume that the character table of $(A, \mathbf{B})$ has an $s \times t$ zero submatrix. Then $s + t \le |\mathbf{B}| - 1$.*

Proof. By permuting the rows and columns if necessary, we may assume that the character table of $(A, \mathbf{B})$ is of the form

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & O \end{pmatrix}, \quad \text{where } O \text{ is an } s \times t \text{ zero submatrix.} \tag{3.4}$$

Note that the rows of the character table are linearly independent by Lemma 2.1. Hence, the rows of C_{21} are also linearly independent. Thus, $\mathrm{rank}(C_{21}) = s$. But for any $\chi \in \mathrm{Irr}(A)^\sharp$, $\chi(e_\mathbf{B}) = 0$, i.e. $\sum_{b_i \in \mathbf{B}} \chi(b_i) = 0$. So the sum of any row of C_{21} is zero. Thus, the columns of C_{21} are linearly dependent, and hence the number of columns of C_{21} is at least $\mathrm{rank}(C_{21}) + 1$. But the number of columns of C_{21} is $|\mathbf{B}| - t$. Hence $|\mathbf{B}| - t \ge s + 1$, and the proposition holds. □

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8. (ii) $\Rightarrow$ (i) Let $E(\mathbf{N})$ be the same as in (3.2). Then by Theorem 1.6, there is a bijection between $E(\mathbf{N})$ and $\mathrm{Irr}(\mathbf{N})^\sharp$, via $\chi_j \leftrightarrow \chi_j{\downarrow_{\mathbf{N}}}$. Since $\mathbb{C}\mathbf{N}$ is commutative, $|E(\mathbf{N})| = |\mathrm{Irr}(\mathbf{N})^\sharp| = |\mathbf{N}| - 1 = s$, and every $\chi_j \in E(\mathbf{N})$ is linear. Furthermore, all $\chi_j \in E(\mathbf{N})$ vanish on the columns of the character table indexed by the $b_l \in \mathbf{B} \setminus \mathbf{N}$. There are $t = |\mathbf{B}| - |\mathbf{N}| = |\mathbf{B}| - (s+1)$ of these columns. Hence, (i) holds.

(i) $\Rightarrow$ (ii) Assume that $\mathbf{B} = \{b_0 = 1_A, b_1, \dots, b_k\}$ and $\mathrm{Irr}(\mathbf{B}) = \{\chi_0, \chi_1, \dots, \chi_r\}$ are indexed in such a way that the character table of $(A, \mathbf{B})$ is of the form (3.4). Hence,

$$\chi_j(b_l) = 0 \quad \text{for all } j > r - s \text{ and } l > s.$$

Let $\mathbf{N} := \{b_0, b_1, \dots, b_s\}$. Since $s + t = |\mathbf{B}| - 1$ is maximal for the size of a zero submatrix in the character table by Proposition 3.1, and $\chi_j(b_l^*)$ is the complex conjugate of $\chi_j(b_l)$ for all j, l, it follows that $(\mathbf{B} \setminus \mathbf{N})^* = \mathbf{B} \setminus \mathbf{N}$, and hence $\mathbf{N}^* = \mathbf{N}$. For each $j > r - s$, Lemma 2.1 and $\chi_j(b_l) = 0$ for all $l > s$ imply that

$$0 = \langle \chi_0, \chi_j \rangle_{\mathbf{B}} = \sum_{l=0}^{k} \frac{1}{\chi_0(b_l)} \chi_0(b_l^*) \chi_j(b_l) = \sum_{l=0}^{s} \chi_j(b_l), \quad \text{for any } j > r - s.$$

Hence, the $(s+1)$-tuple $(1, 1, \dots, 1)^T$ is in the null space of C_{21}. Since C_{21} has rank s and has $s+1$ columns, $(1, 1, \dots, 1)^T$ spans the null space of C_{21}. Let $x \in \mathbb{C}\mathbf{N}$ such that $\chi_j(x) = 0$ for all $j > r - s$. Assume that $x = \sum_{l=0}^{s} \beta_l b_l$ for some $\beta_l \in \mathbb{C}$. Then $0 = \sum_{l=0}^{s} \beta_l \chi_j(b_l)$ for all $j > r - s$ implies that $(\beta_0, \beta_1, \dots, \beta_s)^T$ is in the null space of C_{21}. Hence, $(\beta_0, \beta_1, \dots, \beta_s)^T = \gamma(1, 1, \dots, 1)^T$ for some $\gamma \in \mathbb{C}$, and

$$x \in \mathbb{C}\mathbf{N} \text{ and } \chi_j(x) = 0 \text{ for all } j > r - s \quad \Leftrightarrow \quad x = \gamma \mathbf{N}^+ \text{ for some } \gamma \in \mathbb{C}. \tag{3.5}$$

Let $b_v \in \mathbf{N}$, $b_u \in \mathbf{B} \setminus \mathbf{N}$, and set $b_u b_v = x + y$, where $x \in \mathbb{R}\mathbf{N}$ and $y \in \mathbb{R}(\mathbf{B} \setminus \mathbf{N})$. For each $j > r - s$, χ_j is linear by the assumption of (i), and hence χ_j is an algebra homomorphism from A to $\mathbb{C}$. Thus, and since $\chi_j(b_u) = 0$,

$$0 = \chi_j(b_u)\chi_j(b_v) = \chi_j(b_u b_v) = \chi_j(x) + \chi_j(y) = \chi_j(x), \quad \text{for all } j > r - s.$$

Hence by (3.5), $x = \gamma \mathbf{N}^+$ for some $\gamma \in \mathbb{R}$. If $x \neq 0$, then $b_0 \in \mathrm{Supp}(b_u b_v)$, and hence $b_v = b_u^* \in \mathbf{B} \setminus \mathbf{N}$, a contradiction. Thus, $x = 0$, and

$$\mathrm{Supp}(b_u b_v) \subseteq \mathbf{B} \setminus \mathbf{N}, \quad \text{for all } b_u \in \mathbf{B} \setminus \mathbf{N}, b_v \in \mathbf{N}. \tag{3.6}$$

If for some $b_{v_1}, b_{v_2} \in \mathbf{N}$, $\mathrm{Supp}(b_{v_1} b_{v_2})$ contains some $b_u \in \mathbf{B} \setminus \mathbf{N}$, then $b_{v_1} \in \mathrm{Supp}(b_u b_{v_2}^*) \subseteq \mathbf{B} \setminus \mathbf{N}$ by (1.4) and (3.6), a contradiction. Therefore, $\mathbf{N}\mathbf{N} \subseteq \mathbf{N}$, and $\mathbf{N}$ is a closed subset of $\mathbf{B}$. Furthermore, the s linear characters χ_j, $j > r - s$, yield s distinct irreducible characters $\chi_j{\downarrow_{\mathbf{N}}}$ of $\mathrm{Irr}(\mathbf{N})^\sharp$. Since $s = |\mathbf{N}| - 1$, $\mathbb{C}\mathbf{N}$ is commutative, and $\{\chi_j{\downarrow_{\mathbf{N}}} \mid j > r - s\} = \mathrm{Irr}(\mathbf{N})^\sharp$, with $\chi_j(\mathbf{B} \setminus \mathbf{N}) = \{0\}$ for all $j > r - s$. Thus, $(A, \mathbf{B})$ is a wreath product $(\mathbf{B}, \mathbf{N})$ by Theorem 1.6, and (ii) holds. $\square$

When $(A, \mathbf{B})$ is a table algebra, several equivalent variations on the statements of Theorem 1.6 arise. We present them in Corollary 3.4 below. Let $(A, \mathbf{B})$ be a standard table algebra, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. Then for any $b_i \in \mathbf{B}$, the positive degree of b_i is denoted by $o(b_i)$, and the set product $\mathbf{N}\{b_i\}\mathbf{N}$ is denoted by $\mathbf{N}b_i\mathbf{N}$. The *quotient table algebra* $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$ is also a standard table algebra (cf. [2, Theorem 4.9]), where

$$\mathbf{B}//\mathbf{N} := \{b_i//\mathbf{N} \mid b_i \in \mathbf{B}\} \quad \text{and} \quad b_i//\mathbf{N} := o(\mathbf{N})^{-1}(\mathbf{N}b_i\mathbf{N})^+.$$

Furthermore, the positive degree of $b_i//\mathbf{N}$ is $o(b_i//\mathbf{N}) = o(\mathbf{N})^{-1}o(\mathbf{N}b_i\mathbf{N})$, and by [2, Proposition 4.8],

$$b_i//\mathbf{N} = \frac{o(b_i//\mathbf{N})}{o(b_i)}(e_\mathbf{N} b_i e_\mathbf{N}).$$

Also each $b_i//\mathbf{N}$ is an element of A, and $A//\mathbf{N}$ is a subalgebra of A; but $1_A \notin A//\mathbf{N}$ if $\mathbf{N} \neq \{1_A\}$. For any $\eta \in \mathrm{Irr}(\mathbf{B})$ such that $\eta\downarrow_{A//\mathbf{N}} \neq 0$, $\eta\downarrow_{A//\mathbf{N}} \in \mathrm{Irr}(A//\mathbf{N})$ by [18, Theorem 3.2].

Proposition 3.2. *Let $(A, \mathbf{B})$ be a standard table algebra, and $\mathbf{N}$ a closed subset of $\mathbf{B}$. Let $\eta \in \mathrm{Irr}(\mathbf{B})$. Then the following are equivalent.*

(i) $\eta\downarrow_\mathbf{N} = \eta(1)\psi_0$, *where ψ_0 is the positive degree map of $(\mathbb{C}\mathbf{N}, \mathbf{N})$.*

(ii) *For all $b_i \in \mathbf{B}$,*

$$\eta(b_i) = \frac{o(b_i)}{o(b_i//\mathbf{N})}\eta(b_i//\mathbf{N}).$$

(iii) *For some character ξ of $A//\mathbf{N}$ and all $b_i \in \mathbf{B}$,*

$$\eta(b_i) = \frac{o(b_i)}{o(b_i//\mathbf{N})}\xi(b_i//\mathbf{N}).$$

Proof. Assume (i). Then for any $a \in A$, $\eta(e_\mathbf{N} a e_\mathbf{N}) = \eta(a)$ by [18, Lemma 3.4]. Hence,

$$\eta(b_i//\mathbf{N}) = \frac{o(b_i//\mathbf{N})}{o(b_i)}\eta(e_\mathbf{N} b_i e_\mathbf{N}) = \frac{o(b_i//\mathbf{N})}{o(b_i)}\eta(b_i),$$

and (ii) holds.

Assume (ii). Then (iii) follows, with $\xi = \eta\downarrow_{A//\mathbf{N}}$.

Assume (iii). Then for all $b_i \in \mathbf{N}$,

$$\eta(b_i) = \frac{o(b_i)}{o(1_A//\mathbf{N})}\xi(1_A//\mathbf{N}) = o(b_i)\xi(1_A//\mathbf{N}).$$

In particular, $\eta(1_A) = o(1_A)\xi(1_A//\mathbf{N}) = \xi(1_A//\mathbf{N})$. So $\eta(b_i) = o(b_i)\eta(1)$ for all $b_i \in \mathbf{N}$, and (i) holds. $\square$

Remark 3.3. If $(A, \mathbf{B})$ is a standard table algebra, then it is immediate from Remark 1.3(iv) and the definition of $\mathbf{B}//\mathbf{N}$ that $(A, \mathbf{B})$ is a wreath product $(\mathbf{B}, \mathbf{N})$ if and only if $(A, \mathbf{B}) \cong_x (A//\mathbf{N}, \mathbf{B}//\mathbf{N}) \wr (\mathbb{C}\mathbf{N}, \mathbf{N})$. Also see [15, Lemma 3.1].

The implication (i) ⇒ (v) in the next corollary was proved for association schemes in [11].

Corollary 3.4. *Let $(A, \mathbf{B})$ be a standard table algebra, $\mathbf{N}$ a closed subset of $\mathbf{B}$, and $E(\mathbf{N})$ the same as in (3.2). Then the following are equivalent.*

(*i*) $(A, \mathbf{B}) \cong_x (A//\mathbf{N}, \mathbf{B}//\mathbf{N}) \wr (\mathbb{C}\mathbf{N}, \mathbf{N})$.
(*ii*) *If $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$, then $\eta{\downarrow_{\mathbf{N}}} = \eta(1)\psi_0$, where ψ_0 is the positive degree map of $(\mathbb{C}\mathbf{N}, \mathbf{N})$.*
(*iii*) *If $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$, then for some character ξ of $A//\mathbf{N}$ and all $b_i \in \mathbf{B}$,*

$$\eta(b_i) = \frac{o(b_i)}{o(b_i//\mathbf{N})}\xi(b_i//\mathbf{N}).$$

(*iv*) *If $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$, then*

$$\eta(b_i) = \begin{cases} o(b_i)\eta(1_A//\mathbf{N}), & \text{if } b_i \in \mathbf{N}, \\ o(\mathbf{N})\eta(b_i//\mathbf{N}), & \text{if } b_i \in \mathbf{B} \setminus \mathbf{N}. \end{cases}$$

(*v*) *For any $\xi \in \mathrm{Irr}(A//\mathbf{N})$, $\widehat{\xi}$ is a character of A, where*

$$\widehat{\xi}(b_i) = \begin{cases} o(b_i)\xi(1_A//\mathbf{N}), & \text{if } b_i \in \mathbf{N}, \\ o(\mathbf{N})\xi(b_i//\mathbf{N}), & \text{if } b_i \in \mathbf{B} \setminus \mathbf{N}. \end{cases}$$

Proof. We may assume that $\mathbf{N} \neq \mathbf{B}$. Then the equivalence of (i) and (ii) follows from Theorem 1.6, (3.3), and Remark 3.3; and the equivalence of (ii) and (iii) follows from Proposition 3.2.

Assume (iii). Then for all $\eta \in \mathrm{Irr}(\mathbf{B}) \setminus E(\mathbf{N})$ and all $b_i \in \mathbf{B}$, Proposition 3.2 implies that

$$\eta(b_i) = \begin{cases} o(b_i)\eta(1_A//\mathbf{N}), & \text{if } b_i \in \mathbf{N}, \\ \frac{o(b_i)}{o(b_i//\mathbf{N})}\eta(b_i//\mathbf{N}), & \text{if } b_i \in \mathbf{B} \setminus \mathbf{N}. \end{cases}$$

Then $\eta{\downarrow_{A//\mathbf{N}}} \in \mathrm{Irr}(A//\mathbf{N})$ by [18, Theorem 3.2], and (i) implies that for all $b_i \in \mathbf{B} \setminus \mathbf{N}$, $b_i//\mathbf{N} = o(\mathbf{N})^{-1}b_i$. Thus, $\eta(b_i) = o(\mathbf{N})\eta(b_i//\mathbf{N})$ for all $b_i \in \mathbf{B} \setminus \mathbf{N}$, and (iv) holds.

Assume (iv). By [18, Theorem 3.2], there is $\eta \in \mathrm{Irr}(\mathbf{B})$ such that $\xi = \eta{\downarrow_{A//\mathbf{N}}}$. So $\eta \notin E(\mathbf{N})$. Hence $\widehat{\xi} = \eta$, and (v) holds.

Assume (v). Let $\xi_0 = \chi_0{\downarrow_{A//\mathbf{N}}}$, the positive degree map of $(A//\mathbf{N}, \mathbf{B}//\mathbf{N})$. Then for all $b_i \in \mathbf{N}$, $\widehat{\xi_0}(b_i) = o(b_i)\xi_0(1_A//\mathbf{N}) = o(b_i)$; hence $\widehat{\xi_0}(1_A) = 1$, and for all $b_i \in \mathbf{B} \setminus \mathbf{N}$,

$\widehat{\xi_0}(b_i) = o(\mathbf{N})\xi_0(b_i//\mathbf{N}) = o(\mathbf{N}b_i\mathbf{N})$. Thus, $\widehat{\xi_0} \in \mathrm{Irr}(\mathbf{B})$, and is positive-real-valued on $\mathbf{B}$. By Lemma 2.1, $\widehat{\xi_0} = \chi_0$. Therefore,

$$o(b_i) = \chi_0(b_i) = \widehat{\xi_0}(b_i) = o(\mathbf{N}b_i\mathbf{N}),$$

whence $\mathbf{N}b_i\mathbf{N} = \{b_i\}$ for all $b_i \in \mathbf{B} \setminus \mathbf{N}$. It follows that $(A, \mathbf{B})$ is the wreath product $(\mathbf{B}, \mathbf{N})$, and (i) holds. $\square$

4. Applications to commutative association schemes

In this section we discuss some applications to the wreath products of association schemes. In particular, we will show that the wreath product of one-class association schemes is characterized by the zeros in its first eigenmatrix.

Let $\mathcal{X} := (X, \{R_i\}_{0\le i\le d})$ be a *d-class association scheme of order* $|X|$. For any $0 \le i \le d$, the valency and the adjacency matrix of R_i are denoted by k_i and A_i, respectively. Let $\mathcal{A}$ be the Bose–Mesner algebra of $\mathcal{X}$, and let $\mathbf{B} := \{A_0, A_1, ..., A_d\}$. For the rest of this section, we will always assume that $\mathcal{X}$ is commutative (that is, $\mathcal{A}$ is commutative). Let $E_0, E_1, ..., E_d$ be the primitive idempotents of $\mathcal{A}$, with $E_0 = \frac{1}{|X|}J$, where J is the matrix whose entries are all 1. Then $\{E_0, E_1, ..., E_d\}$ is another basis of $\mathcal{A}$. Following the notation in [8, Section II.3], we assume that

$$A_i = \sum_{j=0}^{d} p_i(j)E_j \quad \text{and} \quad E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_i(j)A_j, \quad 0 \le i \le d. \tag{4.1}$$

Let P and Q be the $(d + 1) \times (d + 1)$ matrices whose (i, j)-entries are $p_j(i)$ and $q_j(i)$, respectively. Then P and Q are called the *first* and *second eigenmatrices* of $\mathcal{X}$, respectively. Let $\mathrm{Irr}(\mathcal{A}) := \{\chi_0, \chi_1, ..., \chi_d\}$ be the set of irreducible characters of $\mathcal{A}$, where χ_0 is the degree map of $\mathcal{A}$. Since P is also the character table of the standard table algebra $(\mathcal{A}, \mathbf{B})$, by renumbering χ_1, ..., χ_d if necessary, we may assume that

$$\chi_j(A_i) = p_i(j), \quad 0 \le i, j \le d. \tag{4.2}$$

Furthermore, let $\mathbf{E} := \{|X|E_0, |X|E_1, ..., |X|E_d\}$, and let $\circ$ denote the Hadamard product of matrices. Then $(\mathcal{A}, \mathbf{E}, \circ)$ is a standard table algebra, called the *dual* of $(\mathcal{A}, \mathbf{B})$. The primitive idempotents of $(\mathcal{A}, \mathbf{E}, \circ)$ are $A_0, A_1, ..., A_d$, and by (4.1),

$$|X|E_i = \sum_{j=0}^{d} q_i(j)A_j, \quad \text{and} \quad A_i = \frac{1}{|X|}\sum_{j=0}^{d} p_i(j)|X|E_j, \quad 0 \le i \le d.$$

Therefore, the first and second eigenmatrices of $(\mathcal{A}, \mathbf{E}, \circ)$ are Q and P, respectively. Note that $(\mathcal{A}, \mathbf{B}, \chi_0)$ is a C-algebra, and its dual C-algebra

$$(\mathcal{A}, \widehat{\mathbf{B}}_{\chi_0}, \chi_0) \cong_x (\mathcal{A}, \mathbf{E}, \circ) \quad \text{as table algebras.} \tag{4.3}$$

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0\le j\le e})$ be association schemes, and let $A_0, A_1, \dots, A_d$ and $B_0, B_1, \dots, B_e$ be the adjacency matrices of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then the adjacency matrices C_l, $0 \le l \le d+e$, of the *wreath product* $\mathcal{X} \wr \mathcal{Y}$ are given by

$$C_0 = A_0 \otimes B_0, \qquad C_1 = A_0 \otimes B_1, \dots, \qquad C_e = A_0 \otimes B_e,$$
$$C_{e+1} = A_1 \otimes J_m, \dots, \quad C_{e+d} = A_d \otimes J_m,$$

where $m = |Y|$ and J_m is the $m \times m$ matrix whose entries are all 1. Therefore, the Bose–Mesner algebra of $\mathcal{X} \wr \mathcal{Y}$ is exactly isomorphic to the wreath product of the Bose–Mesner algebras of $\mathcal{X}$ and $\mathcal{Y}$ as table algebras. Note that the wreath product $\mathcal{X} \wr \mathcal{Y}$ in this paper is the wreath product $\mathcal{Y} \wr \mathcal{X}$ in [13] and [16].

For any positive integer $n \ge 2$, let K_n denote the one-class association scheme of order n. It is clear that the Bose–Mesner algebra of K_n and its dual are exactly isomorphic as table algebras. Note that for C-algebras $(A, \mathbf{B}, f)$, $(C, \mathbf{D}, g)$, and $(X, \mathbf{Y}, h)$ such that $(A, \mathbf{B}, f) \cong (C, \mathbf{D}, g) \wr (X, \mathbf{Y}, h)$, by [1, Theorem 2.9] or [15, Theorem 2.9] we have $(A, \widehat{\mathbf{B}}_f, f) \cong (X, \widehat{\mathbf{Y}}_h, h) \wr (C, \widehat{\mathbf{D}}_g, g)$. The next lemma is immediate from this result and (4.3).

Lemma 4.1. *Let $\mathcal{X} = K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ be the wreath product of one-class association schemes $K_{n_1}, K_{n_2}, \dots, K_{n_d}$, where $d, n_1, n_2, \dots, n_d$ are positive integers greater than or equal to 2. Let $(\mathcal{A}, \mathbf{E}, \circ)$ be the dual of the Bose–Mesner algebra of $\mathcal{X}$. Then as a table algebra, $(\mathcal{A}, \mathbf{E}, \circ)$ is exactly isomorphic to the Bose–Mesner algebra of the wreath product $K_{n_d} \wr \cdots \wr K_{n_2} \wr K_{n_1}$.*

The next lemma describes the structures of Bose–Mesner algebras of wreath products of one-class association schemes.

Lemma 4.2. *(See [13, Theorem 2.2].) Let $\mathcal{X} = K_{n_d} \wr \cdots \wr K_{n_2} \wr K_{n_1}$ be the wreath product of one-class association schemes $K_{n_d} \dots, K_{n_2}, K_{n_1}$, where $d, n_1, n_2, \dots, n_d$ are positive integers greater than or equal to 2. Let $A_0, A_1, \dots, A_d$ be the adjacency matrices of $\mathcal{X}$, and let $k_0, k_1, \dots, k_d$ be the valencies of $\mathcal{X}$. Assume that $k_0 \le k_1 \le \cdots \le k_d$. Then $\mathcal{X}$ has the following properties.*

(i) *The valencies are $k_1 = n_1 - 1$ and*

$$k_i = (k_0 + k_1 + \cdots + k_{i-1})(n_i - 1) = n_1 \cdots n_{i-1}(n_i - 1), \quad \textit{for } i = 2, 3, \dots, d.$$

(ii) *$A_iA_j = k_iA_j$, $0 \le i < j \le d$, and*

$$(A_i)^2 = k_iA_0 + k_iA_1 + \cdots + k_iA_{i-1} + \frac{k_i(n_i - 2)}{n_i - 1}A_i, \quad 1 \le i \le d.$$

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. Assume that its first eigenmatrix P is of the form (3.4) such that $s+t = d$. Then by Theorem 1.8, Theorem 1.6, and (3.3), all the rows of C_{11} are equal.

The next result was first proved by S.-Y. Song (cf. [9,10]). It is clear that Corollary 4.3(i), (ii) follow directly from Lemmas 4.1 and 4.2, and Corollary 4.3(iii), (iv) directly from Corollary 1.11, Lemma 4.1, and the remark in the above paragraph. That is, this corollary can be easily obtained from the results proved in this paper without calculations.

Corollary 4.3. *Let $\mathcal{X} = K_{n_d} \wr \cdots \wr K_{n_2} \wr K_{n_1}$ be the wreath product of one-class association schemes $K_{n_d} \ldots, K_{n_2}, K_{n_1}$, where $d, n_1, n_2, \ldots, n_d$ are positive integers greater than or equal to 2. Let $k_0, k_1, \ldots, k_d$ be the valencies of $\mathcal{X}$ such that $k_0 \le k_1 \le \cdots \le k_d$. Let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of the Bose–Mesner algebra of $\mathcal{X}$, and let $m_0, m_1, \ldots, m_d$ be the multiplicities of $\mathcal{X}$. Assume that $m_0 \le m_1 \le \cdots \le m_d$. Then the following hold.*

(i) *The multiplicities are $m_1 = n_d - 1$ and*

$$m_i = (m_0 + m_1 + \cdots + m_{i-1})(n_{d+1-i} - 1) = n_d \cdots n_{d+2-i}(n_{d+1-i} - 1), \quad 2 \le i \le d.$$

(ii) $E_i \circ E_j = \frac{1}{|X|} m_i E_j$, $0 \le i < j \le d$, *and*

$$E_i \circ E_i = \frac{1}{|X|}\left(m_i E_0 + m_i E_1 + \cdots + m_i E_{i-1} + \frac{m_i(n_{d+1-i} - 2)}{n_{d+1-i} - 1} E_i\right), \quad 1 \le i \le d.$$

(iii) *The first eigenmatrix of $\mathcal{X}$ is*

$$P = \begin{pmatrix}
1 & k_1 & k_2 & k_3 & \cdots & k_{d-2} & k_{d-1} & k_d \\
1 & k_1 & k_2 & k_3 & \cdots & k_{d-2} & k_{d-1} & -\sum_{i=0}^{d-1} k_i \\
1 & k_1 & k_2 & k_3 & \cdots & k_{d-2} & -\sum_{i=0}^{d-2} k_i & 0 \\
1 & k_1 & k_2 & k_3 & \cdots & -\sum_{i=0}^{d-3} k_i & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\
1 & k_1 & k_2 & -\sum_{i=0}^{2} k_i & \cdots & 0 & 0 & 0 \\
1 & k_1 & -\sum_{i=0}^{1} k_i & 0 & \cdots & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.$$

(iv) *Replacing k_i by m_i in the first eigenmatrix P, $1 \le i \le d$, we obtain the second eigenmatrix Q of $\mathcal{X}$.*

(v) $P = Q$ *if and only if $m_i = k_i$, $i = 0, 1, 2, ..., d$, if and only if $n_{d+1-i} = n_i$, $i = 1, 2, ..., d$.*

Two association schemes $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0\le j\le e})$ are said to be *isomorphic*, if there is a bijection $\varphi: \ X \cup \{R_i\}_{0\le i\le d} \to Y \cup \{S_j\}_{0\le j\le e}$ such that

(i) $\varphi(X) = Y$ and $\varphi(\{R_i\}_{0\le i\le d}) = \{S_j\}_{0\le j\le e}$, and
(ii) for any $x_1, x_2 \in X$ such that $(x_1, x_2) \in R_i$ for some $0 \le i \le d$, $(\varphi(x_1), \varphi(x_2)) \in \varphi(R_i)$.

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme, with adjacency matrices A_0, A_1, ..., A_d and valencies k_0, k_1, ..., k_d such that $k_0 \le k_1 \le \cdots \le k_d$. It is proved in [16, Theorem 1.1] that $\mathcal{X}$ is isomorphic to the wreath product of one-class association schemes if and only if $A_iA_j = k_iA_j$, for any $0 \le i < j \le d$. Let $E_0, E_1, ..., E_d$ be the primitive idempotents of the Bose–Mesner algebra of $\mathcal{X}$, and let $m_0, m_1, ..., m_d$ be the multiplicities of $\mathcal{X}$ such that $m_0 \le m_1 \le \cdots \le m_d$. Then from Corollary 4.3 and [16, Theorem 1.1], $\mathcal{X}$ is isomorphic to the wreath product of one-class association schemes if and only if $E_i \circ E_j = \frac{1}{|X|} m_i E_j$, for any $0 \le i < j \le d$. Furthermore, we have the following

Proposition 4.4. *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. Then the following are equivalent.*

(i) $\mathcal{X}$ is isomorphic to the wreath product of one-class association schemes.
(ii) By permuting the rows and columns if necessary, the first eigenmatrix P of $\mathcal{X}$ is of the form

$$\begin{pmatrix} * & * & * & * & \cdots & * & * \\ * & * & * & * & \cdots & * & * \\ * & * & * & * & \cdots & * & 0 \\ * & * & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ * & * & * & 0 & \cdots & 0 & 0 \\ * & * & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.4}$$

(iii) By permuting the rows and columns if necessary, the second eigenmatrix Q of $\mathcal{X}$ is of the form (4.4).

Proof. Since (ii) and (iii) are equivalent by [8, Theorem 3.5(i), p. 63], and (i) implies (ii) by Corollary 4.3, we only need to prove that (ii) implies (i). Without loss of generality, we may assume that the eigenmatrix P of $\mathcal{X}$ is of the form (4.4). Then for any $s \in \{1, 2, \ldots, d-1\}$, P has an $s \times t$ zero submatrix such that $s + t = d$. Thus, it follows from the proof of Theorem 1.8 that $\{R_0, R_1, \ldots, R_s\}$ is a closed subset of $\mathcal{X}$, for any $1 \le s \le d-1$. Hence, by [16, Theorem 1.1], $\mathcal{X}$ is isomorphic to the wreath product of one-class association schemes, and (i) holds. □

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