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Stability of depths of powers of edge ideals

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A R T I C L E I N F O

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Let G be a graph and let $I := I(G)$ be its edge ideal. In this paper, we provide an upper bound of n from which depth $R/I(G)^n$ is stationary, and compute this limit explicitly. This bound is always achieved if G has no cycles of length 4 and every its connected component is either a tree or a unicyclic graph.

Introduction

Let $R = K[x_1, \dots, x_r]$ be a polynomial ring over a field K and I be a homogeneous ideal in R. Brodmann [2] showed that depth R/I^n is a constant for sufficiently large n. Moreover

$$\lim_{n\to\infty} \operatorname{depth} R/I^n \leqslant \dim R - \ell(I),$$

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where $\ell(I)$ is the analytic spread of I. It was shown in [6, Proposition 3.3] that this is an equality when the associated graded ring of I is Cohen–Macaulay. We call the smallest number n_0 such that $\operatorname{depth} R/I^n = \operatorname{depth} R/I^{n_0}$ for all $n \geqslant n_0$, the *index of depth stability* of I, and denote this number by $\operatorname{dstab}(I)$. It is of natural interest to find a bound for $\operatorname{dstab}(I)$. As until now we only know effective bounds of $\operatorname{dstab}(I)$ for few special classes of ideals I, such as complete intersection ideals (see [5]), square-free Veronese ideals (see [8]), polymatroidal ideals (see [10]). In this paper we will study this problem for *edge ideals*.

From now on, every graph G is assumed to be simple (i.e., a finite, undirected, loopless and without multiple edges) without isolated vertices on the vertex set $V(G) = [r] := \{1, \ldots, r\}$ and the edge set $E(G)$ unless otherwise indicated. We associate to G the quadratic squarefree monomial ideal

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq R = K[x_1, \ldots, x_r]$$

which is called the edge ideal of G.

If I is a polymatroidal ideal in R, Herzog and Qureshi proved that $\operatorname{dstab}(I) < \dim R$ and they asked whether $\operatorname{dstab}(I) < \dim R$ for all Stanley–Reisner ideals I in R (see [10]). For a graph G, if every its connected component is nonbipartite, then we can see that $\operatorname{dstab}(I(G)) < \dim R$ from [4]. In general, there is no an absolute bound of $\operatorname{dstab}(I(G))$ even in the case G is a tree (see [20]). In this paper we will establish a bound of $\operatorname{dstab}(I(G))$ for any graph G. In particular, $\operatorname{dstab}(I(G)) < \dim R$.

The first main result of the paper shows that the limit of the sequence $\operatorname{depth} R/I(G)^n$ is the number s of connected bipartite components of G and $\operatorname{depth} R/I(G)^n$ immediately becomes constant once it reaches the value s. Moreover, $\operatorname{dstab}(I(G))$ can be obtained via its connected components.

Theorem 4.4. *Let G be a graph with p connected components $G_1, \ldots, G_p$. Let s be the number of connected bipartite components of G. Then*

(1) $\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = s$.
(2) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = s\}$.
(3) $\operatorname{dstab}(I(G)) = \sum_{i=1}^{p} \operatorname{dstab}(I(G_i)) - p + 1$.

The second one estimates an upper bound for $\operatorname{dstab}(I(G))$. Before stating our result, we recall some terminologies from graph theory. In a graph G, a *leaf* is a vertex of degree one and a *leaf edge* is an edge incident with a leaf. A connected graph is called a *tree* if it contains no cycles, and it is called a *unicyclic* graph if it contains exactly one cycle. We use the symbols $v(G)$, $\varepsilon(G)$ and $\varepsilon_0(G)$ to denote the number of vertices, edges and leaf edges of G, respectively.

Theorem 4.6. *Let G be a graph. Let $G_1, \ldots, G_s$ be all connected bipartite components of G and let $G_{s+1}, \ldots, G_{s+t}$ be all connected nonbipartite components of G. Let $2k_i$ be the* *maximum length of cycles of* G_i ($k_i := 1$ *if* G_i *is a tree) for all* $i = 1, \dots, s$*; and let* $2k_i - 1$ *be the maximum length of odd cycles of* G_i *for every* $i = s+1, \dots, s+t$*. Then*

$$\operatorname{dstab}(I(G)) \leqslant v(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1.$$

It is interesting that this bound is always achieved if G has no cycles of length 4 and every its connected component is either a tree or a unicyclic graph (see Theorem 5.1).

Our approach is based on a generalized Hochster formula for computing local cohomology modules of arbitrary monomial ideals formulated by Takayama [24]. The efficiency of this formula was shown in recent papers (see [7,12,17–19]). Using this formula and an explicit description of it for symbolic powers of Stanley–Reisner ideals given in [17], we are able to study the stability of depths of powers of edge ideals.

The paper is organized as follows. In Section 1, we give some useful formulas on $\operatorname{dstab}(I(G))$ for the case when all components of G are either nonbipartite or bipartite. We also recall the generalized Hochster formula to compute local cohomological modules of monomial ideals formulated by Takayama. In Section 2 and Section 3 we set up an upper bound of the index of depth stability for connected graphs which are either nonbipartite or bipartite, respectively. The core of the paper is Section 4. There we compute the limit of the sequence $\operatorname{depth} R/I(G)^n$. Then combining with results in Sections 2 and 3 on the index of depth stability of connected graphs we obtain a bound of $\operatorname{dstab}(I(G))$ for all any graph G. In the last section, we compute the index of depth stability of trees and unicyclic graphs.

1. Preliminary

We recall some standard notation and terminology from graph theory here. Let G be a graph. The ends of an edge of G are said to be incident with the edge, and vice versa. Two vertices which are incident with a common edge are adjacent, and two distinct adjacent vertices are neighbors. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ and the degree of a vertex v in G, denoted by $\deg_G(v)$, is the number of neighbors of v in G. If there is no ambiguity in the context, we write $\deg v$ instead of $\deg_G(v)$. The graph G is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a bipartition of G. It is well-known that G is bipartite if and only if G contains no odd cycle (see [1, Theorem 4.7]).

Let I be a homogeneous ideal in a polynomial ring $R = K[x_1, \dots, x_r]$ over the field K. As introduced in [9] we define *the index of depth stability* of I to be the number

$$\operatorname{dstab}(I) := \min\{n_0 \geqslant 1 \mid \operatorname{depth} S/I^n = \operatorname{depth} S/I^{n_0} \text{ for all } n \geqslant n_0\}.$$

In this paper we will establish a bound of $\operatorname{dstab}(I(G))$ for any graph G. First we have some information about $\operatorname{dstab}(I(G))$ when every component of G is nonbipartite.

Lemma 1.1. *Let G be a graph with connected components $G_1, \dots, G_t$. If all these components are nonbipartite, then*

(1) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = 0\}$;
(2) $\operatorname{dstab}(I(G)) = \sum_{i=1}^t \operatorname{dstab}(I(G_i)) - t + 1$.

Proof. (1) Let $\mathfrak{m}_i := (x_j \mid j \in V(G_i))$ and $R_i := K[x_j \mid j \in V(G_i)]$, i.e., $\mathfrak{m}_i$ is the maximal homogeneous ideal of R_i, for $i = 1, \dots, t$. Let $\mathfrak{m} := (x_j \mid j \in V(G))$ be the maximal homogeneous ideal of R, so that $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_t$.

By [4, Corollary 3.4] we have $\mathfrak{m}_i \in \operatorname{Ass}(R_i/I(G_i)^{n_i})$ for some integer $n_i \geqslant 1$. Let $n_0 := \sum_{i=1}^t (n_i - 1) + 1$. By [4, Corollary 2.2] we have $\mathfrak{m} \in \operatorname{Ass}(R/I(G)^n)$ for all $n \geqslant n_0$. On the other hand, the sequence $\{\operatorname{Ass}(R/I(G)^n)\}_{n \geqslant 1}$ is increasing by [15, Theorem 2.15] and note that $\operatorname{depth} R/I(G)^n = 0$ if and only if $\mathfrak{m} \in \operatorname{Ass}(R/I(G)^n)$, this implies $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = 0\}$.

(2) By Part 1 we also have $\operatorname{dstab}(I(G_i)) = \min\{n \geqslant 1 \mid \mathfrak{m}_i \in \operatorname{Ass}(R/I(G_i)^n)\}$ for each component G_i. On the other hand, by [4, Corollary 2.2] we have $\mathfrak{m} \in \operatorname{Ass}(R/I(G)^n)$ if and only if we can write $n = \sum_{i=1}^t (n_i - 1) + 1$ where the n_i are positive integers such that $\mathfrak{m}_i \in \operatorname{Ass}(R_i/I(G_i)^{n_i})$. Thus the statement follows. □

Next, we consider bipartite graphs. Note that all connected components of such graphs are bipartite as well. Bipartite graphs have a nice algebraic characterization.

Lemma 1.2. *(See [22].) A graph G is bipartite if and only if $I(G)^n = I(G)^{(n)}$ for all $n \geqslant 1$.*

Using this characterization we obtain.

Lemma 1.3. *Let G be a bipartite graph with s connected components. Then*

(1) $\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = s$, *and*
(2) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = s\}$.

Proof. Since G is bipartite, by Lemma 1.2 we have $I(G)$ is normally torsion-free, and so by [13] the Rees ring $\mathcal{R}[I(G)]$ of $I(G)$ is Cohen–Macaulay. Then by [14] the associated graded ring of $I(G)$ is Cohen–Macaulay as well. Hence, by [6, Proposition 3.3] we have

(1) $\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = r - \ell(I(G))$, and
(2) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = r - \ell(I(G))\}$.

On the other hand, $r - \ell(I(G)) = s$ (see [25, Page 50]). Thus the lemma follows. □

In the general case, our main tool to study $\operatorname{dstab}(I(G))$ is a generalized version of a Hochster's formula (see [23, Theorem 4.1 in Chapter II]) to compute local cohomology modules of monomial ideals given in [24].

Let $\mathfrak{m} := (x_1, \ldots, x_r)$ be the maximal homogeneous ideal of R and I a monomial ideal in R. Since R/I is an $\mathbb{N}^r$-graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over R/I. For every degree $\boldsymbol{\alpha} \in \mathbb{Z}^r$ we denote by $H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ the $\boldsymbol{\alpha}$-component of $H^i_{\mathfrak{m}}(R/I)$.

Let $\Delta(I)$ denote the simplicial complex corresponding to the Stanley–Reisner ideal $\sqrt{I}$. For every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ we set $G_{\boldsymbol{\alpha}} := \{i \mid \alpha_i < 0\}$ and we denote by $\Delta_{\boldsymbol{\alpha}}(I)$ the simplicial complex of all sets of the form $F \setminus G_{\boldsymbol{\alpha}}$, where F is a face of $\Delta(I)$ containing $G_{\boldsymbol{\alpha}}$ such that for every minimal generator $x^{\boldsymbol{\beta}}$ of I there exists an $i \notin F$ such that $\alpha_i < \beta_i$. To represent $\Delta_{\boldsymbol{\alpha}}(I)$ in a more compact way, for every subset F of $[r]$ let $R_F := R[x_i^{-1} \mid i \in F \cup G_{\boldsymbol{\alpha}}]$ and $I_F := IR_F$. This means that the ideal I_F of R_F is generated by all monomials of I by setting $x_i = 1$ for all $i \in F \cup G_{\boldsymbol{\alpha}}$. Then $x^{\boldsymbol{\alpha}} \in R_F$ and by [7, Lemma 1.1] we have

$$\Delta_{\boldsymbol{\alpha}}(I) = \{F \subseteq [r] \setminus G_{\boldsymbol{\alpha}} \mid x^{\boldsymbol{\alpha}} \notin I_F\}. \tag{1}$$

Lemma 1.4. *(See [24, Theorem 1].)* $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(I); K)$.

Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. The Stanley–Reisner ideal of Δ can be written as (see [16, Theorem 1.7]):

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where P_F is the prime ideal of R generated by variables x_i with $i \notin F$. For every integer $n \geqslant 1$, the n-th symbolic power of I_Δ is the monomial ideal

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^n.$$

Note that $\Delta(I_\Delta^{(n)}) = \Delta$. In [17, Lemma 1.3] there was given an useful formula for computing $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$. We apply it to edge ideals.

An independent set in a graph G is a set of vertices no two of which are adjacent to each other. An independent set S in G is maximal if the addition to S of any other vertex in the graph destroys the independence. Let $\Delta(G)$ be the set of independent sets of G. Then $\Delta(G)$ is a simplicial complex and this complex is the so-called independence complex of G; and facets of $\Delta(G)$ are just maximal independent sets of G. It is easy to see that $I(G) = I_{\Delta(G)}$.

Now we can compute $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ for bipartite graphs G.

Lemma 1.5. *Let G be a bipartite graph. Then, for all $\boldsymbol{\alpha} \in \mathbb{N}^r$ and $n \geqslant 1$, we have*

$$\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \left\langle F \in \mathcal{F}(\Delta(G)) \mid \sum_{i \notin F} \alpha_i \leqslant n-1 \right\rangle.$$

Proof. Let $\Delta := \Delta(G)$. Then, $I_\Delta = I(G)$. By Lemma 1.2, we have $I(G)^n = I(G)^{(n)}$. Therefore, $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$. The lemma now follows from [17, Lemma 1.3]. □

We conclude this section with some remarks about operations on monomial ideals. Let $A := K[x_1, \dots, x_s], B := K[y_1, \dots, y_t]$ and $R := K[x_1, \dots, x_s, y_1, \dots, y_t]$ be polynomial rings where $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ are two disjoint sets of variables. Then for monomial ideals I, I_1, I_2 of R we have

$$I \cap (I_1 + I_2) = I \cap I_1 + I \cap I_2. \tag{2}$$

Let I_1, I_2 be monomial ideals in A and let J_1, J_2 be monomial ideals in B. For simplicity, we denote $I_s R$ by I_s and $J_s R$ by J_s for $s = 1, 2$, then by [11, Lemma 1.1] we have

$$I_1 J_1 \cap I_2 J_2 = (I_1 \cap I_2)(J_1 \cap J_2). \tag{3}$$

Lemma 1.6. *Let I be a proper monomial ideal of A and J a proper monomial ideal of B. Then, for all $n \geqslant 1$ we have*

$$\operatorname{depth} R/(I+J)^n \geqslant \min\{\operatorname{depth} A/I^m \mid 1 \leqslant m \leqslant n\}.$$

Proof. Since the case $I = \mathbf{0}$ or $J = \mathbf{0}$ is obvious, so we may assume that I and J are nonzero ideals. For each $i = 0, \dots, n$, we put:

$$W_i := I^i J^{n-i} + \cdots + I^n J^0 \subseteq R,$$

where $I^0 = J^0 = R$. Since $W_0 = (I+J)^n$, in order to prove the lemma it suffices to show that

$$\operatorname{depth} R/W_i \geqslant \min\{\operatorname{depth} A/I^j \mid \max\{i, 1\} \leqslant j \leqslant n\} \text{ for all } i = 0, \dots, n. \tag{4}$$

Indeed, if $i = n$, then $\operatorname{depth} R/W_n = \operatorname{depth} R/I^n = \operatorname{depth} A/I^n + t \geqslant \operatorname{depth} A/I^n$. Next assume that the claim holds for $i+1$ with $0 \leqslant i < n$. By Equations (2) and (3) we have $I^i J^{n-i} \cap W_{i+1} = I^{i+1} J^{n-i}$. Since $W_i = I^i J^{n-i} + W_{i+1}$, we have an exact sequence

$$\mathbf{0} \longrightarrow R/I^{i+1}J^{n-i} \longrightarrow R/I^i J^{n-i} \oplus R/W_{i+1} \longrightarrow R/W_i \longrightarrow \mathbf{0}.$$

By Depth Lemma (see, e.g., [3, Proposition 1.2.9]), we have

$$\operatorname{depth} R/W_i \geqslant \min\{\operatorname{depth} R/I^{i+1}J^{n-i} - 1, \operatorname{depth} R/I^i J^{n-i}, \operatorname{depth} R/W_{i+1}\}.$$

On the other hand, by [11, Lemma 2.2] we have

$$\operatorname{depth} R/I^{i+1}J^{n-i} - 1 = \operatorname{depth} A/I^{i+1} + \operatorname{depth} B/J^{n-i} \geqslant \operatorname{depth} A/I^{i+1}.$$

Together with the induction hypothesis we then get

$$\operatorname{depth} R/W_i \geqslant \min\{\operatorname{depth} R/I^iJ^{n-i}, \operatorname{depth} A/I^j \mid j = i+1, \ldots, n\}.$$

If $i \geqslant 1$, by [11, Lemma 2.2] we have

$$\operatorname{depth} R/I^iJ^{n-i} = \operatorname{depth} A/I^i + \operatorname{depth} B/J^{n-i} + 1 \geqslant \operatorname{depth} A/I^i,$$

which yields the claim.

If $i = 0$, then $\operatorname{depth} R/W_0 \geqslant \min\{\operatorname{depth} R/J^n, \operatorname{depth} A/I^j \mid j = 1, \ldots, n\}$. Note that $\operatorname{depth} R/J^n = s + \operatorname{depth} B/J^n \geqslant s \geqslant \operatorname{depth} A/I$, hence the claim also holds. The proof now is complete. □

2. Depths of powers of edge ideals of connected nonbipartite graphs

Note that for a graph G we always assume that $V(G) = [r]$; $R = K[x_1, \ldots, x_r]$ is a polynomial ring over fields K and $\mathfrak{m} = (x_1, \ldots, m_r)$ is the maximal homogeneous ideal of R. In this section we always assume that G is a connected nonbipartite graph.

By Lemma 1.1 we have $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \mathfrak{m} \in \operatorname{Ass} R/I(G)^n\}$. Based on [4], we will determine explicitly when $\mathfrak{m} \in \operatorname{Ass} R/I(G)^n$ for a unicylic graph G.

Recall that a vertex cover (or a cover) of G is a subset S of $V(G)$ such that every edge of G has at least one endpoint in S. A cover is minimal if none of its proper subsets is itself a cover. It is well-known that $P = (x_{i_1}, \ldots, x_{i_t})$ is a minimal prime of the edge ideal $I(G)$ if and only if $\{i_1, \ldots, i_t\}$ is a minimal cover of G. For a subset U of $V(G)$, the neighbor set of U is the set

$$N(U) := \{v \in V(G) \mid v \text{ is adjacent to some vertex in } U\}.$$

We now describe the process that builds $\operatorname{Ass} R/I(G)^n$ for a unicylic graph G. Let C be a cycle of G of length $2k-1$. Let R_k be the set of vertices of C, $B_k := N(R_k) \setminus R_k$ and a monomial

$$d_k := \prod_{i \in R_k} x_i.$$

We now build recursively sets R_n, B_n and a monomial d_n for $n \geqslant k$. Suppose that $i \in R_s$ and $j \in R_s \cup B_s$ for some $s \geqslant k$ such that $\{i, j\}$ is an edge of G. Now if $j \in R_s$, then let $R_{s+1} := R_s$ and $B_{s+1} := B_s$. If $j \in B_s$, then let $R_{s+1} := R_s \cup \{j\}$ and $B_{s+1} := (B_s \cup N(j)) \setminus R_{s+1}$. In either case, let $d_{s+1} := d_s(x_i x_j)$.

Now for such a couple (R_n, B_n) with $n \geqslant k$, we take V to be any minimal subset of $V(G)$ such that $R_n \cup B_n \cup V$ is a cover of G. Then, $(R_n, B_n, V) := (x_i \mid i \in R_n \cup B_n \cup V)$ is an associated prime of $R/I(G)^n$ by [4, Theorem 3.3]. Let P_n be the set of such all prime ideals. Then, by [4, Theorem 5.6] we have

$$\operatorname{Ass} R/I(G)^n = \operatorname{Min}(R/I(G)) \cup P_n. \tag{5}$$

For unicyclic graphs, we have the following observation.

Remark 2.1. Assume that G is a unicyclic graph with a cycle C such that $G \neq C$. For any $v \in V(G) \setminus V(C)$, there is a unique simple path of the form: $v_0, v_1 \dots, v_d$, where $v_0 \in V(C)$, $v_1, \dots, v_d \notin V(C)$ and $v_d = v$. We say that this path connects C and v. Moreover,

(1) $d_G(v, C) = d$.
(2) This simple path can extend to a simple path connecting C to a leaf, i.e., there are vertices $u_1, \dots, u_t$ such that u_s is a leaf and $v_0, v_1 \dots, v_d, u_1, \dots, u_t$ is a simple path.
(3) If $d_G(v, C)$ is maximal, i.e., $d_G(v, C) \geqslant d_G(u, C)$ for any $u \in V(G)$, then v is a leaf. Assume further that $d \geqslant 2$, then $N_G(v_{d-1})$ contains only one non-leaf v_{d-2}.

We now can determine $\operatorname{dstab}(I(G))$ with unicyclic nonbipartite graphs G.

Lemma 2.2. *Let G be a unicyclic nonbipartite graph. If the length of the unique cycle is $2k - 1$, then* $\operatorname{dstab}(I(G)) = v(G) - \varepsilon_0(G) - k + 1$.

Proof. By [4, Corollaries 3.4 and 4.3] we have

$$\mathfrak{m} \in \operatorname{Ass} R/I(G)^n \text{ for all } n \geqslant v(G) - \varepsilon_0(G) - k + 1.$$

Therefore,

$$\operatorname{depth} R/I(G)^n = 0 \text{ for all } n \geqslant v(G) - \varepsilon_0(G) - k + 1,$$

so that $\operatorname{dstab}(I(G)) \leqslant v(G) - \varepsilon_0(G) - k + 1$.

We next prove the converse inequality. It suffices to show that if $\mathfrak{m} \in \operatorname{Ass} R/I(G)^n$, then $n \geqslant v(G) - \varepsilon_0(G) - k + 1$.

By Equation (5) we deduce that $\mathfrak{m} \in P_n$. Thus, $\mathfrak{m} = (R_n, B_n, V)$ where V is a minimal subset of $V(G)$ such that $R_n \cup B_n \cup V$ is a vertex cover of G. In particular, $V(G) = R_n \cup B_n \cup V$.

Claim 1. $V = \emptyset$. *Indeed, if V contains no leaves of G, then every leaf of G is in either R_n or B_n, and so $R_n \cup B_n = V(G)$ by Remark 2.1. This forces $V = \emptyset$.*

Suppose V contains a leaf, say i. Let j be the unique neighbor of i in G. Then, $j \in V(G) = R_n \cup B_n \cup V$. Therefore, $R_n \cup B_n \cup (V \setminus \{i\})$ is also a vertex cover of G. This contradicts the minimality of V. Hence, $V = \emptyset$, as claimed.

Claim 2. $|B_n| \leqslant \varepsilon_0(G)$. *Indeed, assume on the contrary that* $|B_n| > |\varepsilon_0(G)|$, *so that* B_n *contains a non-leaf of* G, *say* i. *Let* p *be a simple path connecting* C *and a leaf that passes through* i. *Let* j *be a vertex of* p *after* i. *Then, by Remark 2.1 and the construction of* R_n *and* B_n *we deduce that* $j \notin R_n \cup B_n$, *so* $j \notin V(G)$ *by Claim 1, a contradiction. Hence,* $|B_n| \leqslant \varepsilon_0(G)|$, *as claimed.*

We now prove the lemma. Since $|R_k| = 2k - 1$ and $|R_n| \leqslant |R_k| + (n - k)$, together with Claim 2 we obtain $v(G) = |R_n| + |B_n| \leqslant |R_k| + (n - k) + \varepsilon_0(G) = n + k - 1 + \varepsilon_0(G)$, so $n \geqslant v(G) - \varepsilon_0(G) - k + 1$, as required. □

Lemma 2.3. *Let* G *be a unicyclic nonbipartite graph. Assume that the unique odd cycle of* G *is of length* $2k - 1$. *Let* $n := v(G) - \varepsilon_0(G) - k + 1$. *Then, there is a monomial* f *of* R *such that* $\deg f = 2n - 1$ *and* $fx_i \in I(G)^n$ *for all* $i = 1, \dots, r$.

Proof. By Lemma 2.2 and Equation (5) we have $\mathfrak{m} \in P_n$. Thus, $\mathfrak{m} = (R_n, B_n, V)$ where V is a minimal subset of $V(G)$ such that $R_n \cup B_n \cup V$ is a vertex cover of G. In particular, $V(G) = R_n \cup B_n \cup V$. By the same way as in the proof of Claim 1 in Lemma 2.2 we have $V = \emptyset$. Hence, $R_n \cup B_n = \{1, \dots, r\}$.

Let $f := d_n$. Together with [4, Lemma 3.2] we imply that $\deg(f) = 2n - 1$ and $fx_i \in I(G)^n$ for all $i = 1, \dots, r$, as required. □

Let G be a connected nonbipartite graph and let $2l - 1$ be the minimum length of odd cycles of G. Then $\operatorname{dstab}(G) \leqslant v(G) - \varepsilon_0(G) - l + 1$ by [4, Corollaries 3.4 and 4.3]. The following result improves this bound a little bit.

Proposition 2.4. *Let* G *be a connected nonbipartite graph. Let* $2k - 1$ *be the maximum length of odd cycles of* G. *Then,* $\operatorname{dstab}(I(G)) \leqslant v(G) - \varepsilon_0(G) - k + 1$.

Proof. Let C be an odd cycle of G of length $2k - 1$. If C' is another cycle of G, then C' has an edge e not lying on the cycle C. Delete this edge from G, thereby obtaining a connected subgraph G' of G with $V(G') = V(G)$ and C is a cycle of G'. This process continues until we obtain a connected subgraph H of G such that $V(G) = V(H)$ and H has only one cycle, that is C. Let $n := v(H) - \varepsilon_0(H) - k + 1$. By Lemma 2.3, there is a monomial $f \in R$ such that $\deg f = 2n - 1$ and $x_i f \in I(H)^n$ for all $i = 1, \dots, r$. Since $I(H) \subseteq I(G)$, we have

$$x_i f \in I(G)^n \text{ for all } i = 1, \dots, r. \tag{6}$$

As $I(G)$ is generated by quadratic monomials and $\deg f = 2n-1$, so $f \notin I(G)^n$. Together with Equation (6) one has $I(G)^n : f = \mathfrak{m}$. Hence, $\operatorname{depth} R/I(G)^n = 0$, which implies $\operatorname{dstab}(I(G)) \leqslant n$ by Lemma 1.1. Since $\upsilon(G) = \upsilon(H)$ and $\varepsilon_0(G) \leqslant \varepsilon_0(H)$,

$$\operatorname{dstab}(I(G)) \leqslant n \leqslant \upsilon(G) - \varepsilon_0(G) - k + 1,$$

as required. □

3. Depths of powers of edge ideals of connected bipartite graphs

Let G be a biparite graph with bipartition (X, Y). Clearly, X and Y are then facets of $\Delta(G)$. Assume further that G is connected. By Lemma 1.3, one has $\operatorname{dstab}(I(G))$ is the smallest integer n such that $\operatorname{depth} R/I(G)^n = 1$. For such graphs we can find $\operatorname{dstab}(I(G))$ via integer linear programming.

Lemma 3.1. *Let G be a connected bipartite graph with bipartition (X, Y) and n a positive integer. Then,* $\operatorname{depth} R/I(G)^n = 1$ *if and only if $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y\rangle$ for some $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$. Moreover, if $n = \operatorname{dstab}(I(G))$, then such $\boldsymbol{\alpha}$ must satisfy*

$$\sum_{i\notin X} \alpha_i = \sum_{i\notin Y} \alpha_i = n - 1.$$

Proof. Since G is bipartite, by Lemma 1.2 one has $I(G)^n = I(G)^{(n)}$. Hence,

$$\operatorname{depth} R/I(G)^n = \operatorname{depth} R/I(G)^{(n)} \geqslant 1,$$

and hence $\operatorname{depth} R/I(G)^n = 1$ if and only if $H^1_{\mathfrak{m}}(R/I(G)^n) \neq \mathbf{0}$. By [17, Corollary 1.2] this is equivalent to the condition $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ being disconnected for some $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$.

Therefore, in order to prove the lemma it suffices to show that if $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ is disconnected, then $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y\rangle$. Indeed, since $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ is disconnected, there are two facets F and H of it such that $F \cap H = \emptyset$. Hence, $(V(G) \setminus F) \cup (V(G) \setminus H) = V(G)$. Together with the fact that $X \cap Y = \emptyset$ and $X \cup Y = V(G)$ we get

$$\sum_{i\notin X} \alpha_i + \sum_{i\notin Y} \alpha_i = \sum_{i\in V(G)} \alpha_i \leqslant \sum_{i\notin F} \alpha_i + \sum_{i\notin H} \alpha_i.$$

Since F and H are members of $\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G)^n))$, by Lemma 1.5 we have

$$\sum_{i\notin F} \alpha_i \leqslant n - 1, \quad \text{and} \quad \sum_{i\notin H} \alpha_i \leqslant n - 1.$$

Therefore,

$$\sum_{i\notin X} \alpha_i + \sum_{i\notin Y} \alpha_i \leqslant 2(n - 1),$$

which yields

$$\sum_{i \notin X} \alpha_i \leqslant n-1 \quad \text{or} \quad \sum_{i \notin Y} \alpha_i \leqslant n-1.$$

Thus we may assume that

$$\sum_{i \notin X} \alpha_i \leqslant n-1,$$

and thus $X \in \Delta_{\boldsymbol{\alpha}}(I(G)^n)$ by Lemma 1.5. As $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ is disconnected, there is a facet L of $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ such that $X \cap L = \emptyset$. We then have $L \subseteq V(G) \setminus X = Y$. The maximality of L forces $L = Y$, hence $Y \in \Delta_{\boldsymbol{\alpha}}(I(G)^n)$. If $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ has another facet, say T, that is different from X and Y, then neither X nor Y contains T, and then T meets both X and Y. This is impossible since $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$ is disconnected. Hence, $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$, as claimed.

Finally, assume that $n = \operatorname{dstab}(I(G))$. Then, by Lemma 1.3, n is the smallest positive integer such that $\operatorname{depth} R/I(G)^n = 1$.

Assume that $\sum_{i \notin X} \alpha_i < n-1$ and $\sum_{i \notin Y} \alpha_i < n-1$. Then, $n-1 \geqslant 1$ and

$$\sum_{i \notin X} \alpha_i \leqslant (n-1)-1 \text{ and } \sum_{i \notin Y} \alpha_i \leqslant (n-1)-1.$$

If F is a facet of $\Delta(G)$ that is different from X and Y, then $F \notin \mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G)))$, and then $\sum_{i \notin F} \alpha_i \geqslant n > n-1$ according to Lemma 1.5. From these equations and Lemma 1.5, we get $\Delta_{\boldsymbol{\alpha}}(I(G)^{n-1}) = \langle X, Y \rangle$. In particular, $\Delta_{\boldsymbol{\alpha}}(I(G)^{n-1})$ is disconnected, so $\operatorname{depth} R/I(G)^{n-1} = 1$. This contradicts to the minimality of n. Thus, we may assume that $\sum_{i \notin Y} \alpha_i = n-1$.

Assume now that $\sum_{i \notin X} \alpha_i < n-1$. Since

$$\sum_{i \in X} \alpha_i = \sum_{i \notin Y} \alpha_i = n-1 \geqslant 1,$$

$\alpha_i \geqslant 1$ for some $i \in X$. We may assume that $i = 1$. Let $\boldsymbol{\beta} = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_r)$, so that $\boldsymbol{\beta} \in \mathbb{N}^r$ as $\alpha_1 \geqslant 1$. By the same way as in the previous paragraph we get $\Delta_{\boldsymbol{\beta}}(I(G)^{n-1}) = \langle X, Y \rangle$, which yields $\operatorname{depth} R/I(G)^{n-1} = 1$. This also contradicts to the minimality of n. Hence,

$$\sum_{i \notin X} \alpha_i = \sum_{i \notin Y} \alpha_i = n-1,$$

as required. $\square$

We now give an explicit solution of the equation $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$. This solution turns out to be optimal for studying $\operatorname{dstab}(I(G))$.

Definition 3.2. Let G be a graph. We define:

(1) For each $i \in V(G)$, denote $\mu_G(i)$ to be the number of non-leaf edges of G that are incident with i,
(2) $\mu(G) := (\mu_G(1), \ldots, \mu_G(r)) \in \mathbb{N}^r$.

Lemma 3.3. *Let G be a connected bipartite graph with bipartition (X, Y). Let $\boldsymbol{\alpha} := \mu(G)$ and $n := \varepsilon(G) - \varepsilon_0(G) + 1$. Then,*

$$\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle, \quad \text{and} \sum_{i \notin X} \alpha_i = \sum_{i \notin Y} \alpha_i = \varepsilon(G) - \varepsilon_0(G).$$

Proof. Clearly, X and Y are facets of $\Delta(G)$. If $\upsilon(G) = 2$, i.e., G is exactly an edge $\{1, 2\}$, then $n = 1$ and $\boldsymbol{\alpha} = (0, 0)$. We may assume that $X = \{1\}$ and $Y = \{2\}$. Then, $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \Delta(I(G)) = \Delta(G) = \langle \{1\}, \{2\} \rangle$, so the lemma holds for this case.

Assume that $\upsilon(G) \geqslant 3$. Let $S := \{i \in X \mid \deg i = 1\}$ and $T := \{j \in Y \mid \deg j = 1\}$, so that

$$|S| + |T| = \varepsilon_0(G). \tag{7}$$

From [1, Theorem 1.1 and Exercise 1.1.9] we have

$$\sum_{i \in X} \deg i = \sum_{j \in Y} \deg j = \varepsilon(G). \tag{8}$$

Note that the unique neighbor of each leaf of G in X is a non-leaf of G in Y. Together with Formulas (7)–(8), this fact gives

$$\sum_{i \in X} \mu_G(i) = \sum_{i \in X} \deg i - |S| - |T| = \varepsilon(G) - \varepsilon_0(G) = n - 1.$$

Similarly,

$$\sum_{j \in Y} \mu_G(j) = \sum_{j \in Y} \deg j - |S| - |T| = \varepsilon(G) - \varepsilon_0(G) = n - 1.$$

Hence, $X, Y \in \mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G)^n))$ by Lemma 1.5. So in order to prove the lemma it remains to prove that $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$, or equivalently, if $F \in \mathcal{F}(\Delta(G)) \setminus \{X, Y\}$ then $F \notin F(\Delta_{\boldsymbol{\alpha}}(I(G)^n))$.

Indeed, by the maximality of F, we can partition F into $F = U \cup V$, where U and V are nonempty proper subsets of X and Y, respectively, such that every vertex in $X \setminus U$ (resp. in $Y \setminus V$) is adjacent to at least one vertex in V (resp. in U), and no vertex in U is adjacent to a vertex in V. Then, we have

$$\sum_{i \in X \setminus U} \mu_G(i) = \sum_{i \in X \setminus U} \deg i - |S \cap (X \setminus U)| - |T \cap V|,$$
$$\sum_{j \in Y \setminus V} \mu_G(j) = \sum_{j \in Y \setminus V} \deg j - |T \cap (Y \setminus V)| - |S \cap U|,$$

and

$$\sum_{j \in Y \setminus V} \deg j = \sum_{i \in U} \deg i + \sum_{j \in Y \setminus V} |N_G(j) \cap (X \setminus U)|.$$

Combining these Equations with Formulas (7)–(8) we obtain

$$\begin{aligned}
&\sum_{u \notin F} \mu_G(u) = \sum_{i \in X \setminus U} \mu_G(i) + \sum_{j \in Y \setminus V} \mu_G(j) \\
&= \sum_{i \in X \setminus U} \deg i - |S \cap (X \setminus U)| - |T \cap V| + \sum_{j \in Y \setminus V} \deg j - |T \cap (Y \setminus V)| - |S \cap U| \\
&= \sum_{i \in X \setminus U} \deg i + \sum_{j \in Y \setminus V} \deg j - (|S \cap U| + |S \cap (X \setminus U)| + |T \cap V| + |T \cap (Y \setminus V)|) \\
&= \sum_{i \in X \setminus U} \deg i + \sum_{j \in Y \setminus V} \deg j - (|S| + |T|) \\
&= \sum_{i \in X \setminus U} \deg i + \sum_{i \in U} \deg i + \sum_{j \in Y \setminus V} |N_G(j) \cap (X \setminus U)| - \varepsilon_0(G) \\
&= \sum_{i \in X} \deg i - \varepsilon_0(G) + \sum_{j \in Y \setminus V} |N_G(j) \cap (X \setminus U)| \\
&= \varepsilon(G) - \varepsilon_0(G) + \sum_{j \in Y \setminus V} |N_G(j) \cap (X \setminus U)|,
\end{aligned}$$

or equivalently,

$$\sum_{u \notin F} \mu_G(u) = \varepsilon(G) - \varepsilon_0(G) + |P| = n - 1 + |P|,$$

where $P = \{(a, b) \mid a \in X \setminus U, b \in Y \setminus V \text{ and } ab \in E(G)\}$. Therefore, by Lemma 1.5 we have $F \notin \Delta_{\boldsymbol{\alpha}}(I(G)^n)$ whenever $|P| \geqslant 1$, i.e., $P \neq \emptyset$.

In order to prove $P \neq \emptyset$, let $\ell := \min\{d_G(i, j) \mid i \in U \text{ and } j \in V\}$. Then, ℓ is finite because G is connected. Let $a \in U$ and $b \in V$ such that there is a path of length ℓ connects a and b. Suppose

$$a = a_1, b_1, a_2, b_2, \dots, a_s, b_s = b$$

is such a path, where $a_1, \dots, a_s \in X$ and $b_1, \dots, b_s \in Y$. Then, $b_1 \in Y \setminus V$ because $a_1 = a \in U$. Now if $a_2 \in U$, then we would have the path $a_2, b_2, \dots, a_s, b_s = b$ that connects $a_2 \in U$ and $b \in V$ of length $\ell - 2$. This contradicts to the minimality of ℓ. Thus, $a_2 \in X \setminus U$. This implies $(a_2, b_1) \in P$, so $P \neq \emptyset$, as required. $\square$

Let G be a graph and C be a cycle of G. For any vertex v of G, we define the distance from v to C to be:

$$d_G(v, C) = \{d_G(v, u) \mid u \in V(C)\}.$$

Proposition 3.4. *Let G be a connected bipartite graph and let $2k$ be the maximum length of cycle of G ($k := 1$ if G is a tree). Then,* $\operatorname{dstab}(I(G)) \leqslant v(G) - \varepsilon_0(G) - k + 1$.

Proof. Let (X, Y) be a bipartition of G.

If G is a tree, then $\varepsilon(G) = v(G) - 1$ by [1, Theorem 4.3]. Let $\boldsymbol{\alpha} := \mu(G)$ and $n := \varepsilon(G) - \varepsilon_0(G) + 1$. Then, $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$ by Lemma 3.3. Hence, by Lemma 3.1, we have

$$\operatorname{dstab}(I(G)) \leqslant n = \varepsilon(G) - \varepsilon_0(G) + 1 = v(G) - \varepsilon_0(G),$$

and the proposition follows.

Assume that G has a cycle, say C_{2k}, of length $2k$ where $k \geqslant 2$. If C is another cycle of G, then C has an edge e not lying in the cycle C_{2k}. Delete this edge from G, thereby obtaining a connected subgraph G' of G with $V(G') = V(G)$ and C_{2k} is a cycle of G'. This process continues until we obtain a connected subgraph H of G such that $V(G) = V(H)$ and H has only one cycle, that is C_{2k}. Note that H is also a bipartite graph with bipartition (X, Y). Assume that the cycle C_{2k} is:

$$1, 2, \ldots, 2k - 1, 2k, 1.$$

Let $n := v(H) - \varepsilon_0(H) - k + 1$ and define $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ by

$$\alpha_j := \begin{cases} \mu_H(j) - 1 & \text{if } 1 \leqslant j \leqslant 2k + 2, \\ \mu_H(j) & \text{otherwise.} \end{cases}$$

Claim 1.

$$\Delta_{\boldsymbol{\alpha}}(I(H)^n) = \langle X, Y \rangle \text{ and } \sum_{i \notin X} \alpha_i = \sum_{i \notin Y} \alpha_i = n - 1.$$

Proof. We will prove this claim by induction on $v(H)$. If $v(H) = 2k$, then $H = C_{2k}, r = 2k$ and $n = k+1$. We may assume also that $X = \{1, 3, \ldots, 2k-1\}$ and $Y = \{2, 4, \ldots, 2k\}$. By noticing that $\boldsymbol{\alpha} = (1, 1, \ldots, 1) \in \mathbb{N}^r$, we have

$$\sum_{i \notin X} \alpha_i = \sum_{i \notin Y} \alpha_i = k = n - 1,$$

and therefore X and Y are facets of $\Delta_{\boldsymbol{\alpha}}(I(H)^n)$. Hence, it remains to show that $\Delta_{\boldsymbol{\alpha}}(I(H)^n) = \langle X, Y\rangle$. Let F be a facet of $\Delta(H)$ that is different from X and Y. Since all facets of $\Delta(C_{2k})$ have at most k elements; and only X and Y have exactly k elements, we must have $|F| < k$. Hence,

$$\sum_{i\notin F} \alpha_i \geqslant k+1 = n,$$

and hence $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$. Therefore, $\Delta_{\boldsymbol{\alpha}}(I(H)^n) = \langle X, Y\rangle$, and the claim follows.

Assume that $v(H) > 2k$. Clearly, r is not in C_{2k}, so we may assume that $d_G(r, C_{2k}) \geqslant d_G(v, C_{2v})$ for any vertex v of G. Then, r is a leaf by Remark 2.1. Let t be the unique neighbor of r in G.

Let $T := H \setminus \{r\}$. Then, T is also a connected bipartite graph with only cycle C_{2k} and $v(T) = v(H) - 1$. We may assume that $r \in X$, so that $(X \setminus \{r\}, Y)$ is a bipartition of T. Let $s := v(T) - \varepsilon_0(T) - k + 1$ and define $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{r-1}) \in \mathbb{N}^{r-1}$ by

$$\beta_j := \begin{cases} \mu_T(j) - 1 & \text{if } 1 \leqslant j \leqslant 2k, \\ \mu_T(j) & \text{otherwise.} \end{cases}$$

We now distinguish two cases:

Case 1. $d_G(r, C_{2k}) = 1$. In this case $V(G) \setminus V(C_{2k})$ is the set of all leaves of G. Thus, $\boldsymbol{\beta} = (\alpha_1, \dots, \alpha_{r-1})$ and $\varepsilon_0(T) = \varepsilon_0(H) - 1$, and thus $s = n$.

Since $v(T) = v(H) - 1$ and $\alpha_r = 0$, by the induction hypothesis we have $\Delta_{\boldsymbol{\beta}}(I(T)^n) = \langle X \setminus \{r\}, Y\rangle$, and

$$\sum_{i\notin X} \alpha_i = \sum_{i\notin X\setminus\{r\}} \beta_i = n-1, \text{ and } \sum_{i\notin Y} \alpha_i = \sum_{i\notin Y} \beta_i + \alpha_r = n-1. \tag{9}$$

In particular, $X \in \Delta_{\boldsymbol{\alpha}}(I(H)^n)$ and $Y \in \Delta_{\boldsymbol{\alpha}}(I(H)^n)$. Thus it remains to show that $\Delta_{\boldsymbol{\alpha}}(I(H)^n) = \langle X, Y\rangle$. Let F be any facet of $\Delta(H)$ that is different from X and Y.

Assume that $t \in F$. Then, F is also a facet of $\Delta(T)$ that is different from $X \setminus \{r\}$ and Y. Therefore,

$$\sum_{i\notin F} \alpha_i = \sum_{i\notin F} \beta_i + \alpha_r = \sum_{i\notin F} \beta_i \geqslant n.$$

Therefore, $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$.

Assume that $t \notin F$. Then, $r \in F$ and $F \setminus \{r\}$ is a subset of neither $X \setminus \{r\}$ nor Y. Since $F \setminus \{r\} \in \Delta(T)$, there is a facet F' of $\Delta(T)$ containing F and being different from $X \setminus \{r\}$ and Y. Therefore,

$$\sum_{i\notin F} \alpha_i = \sum_{i\notin F\setminus\{r\}} \beta_i \geqslant \sum_{i\notin F'} \beta_i \geqslant n.$$

Which implies $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$. The claim holds for this case.

Case 2. $d_G(r, C_{2k}) \geqslant 2$. By Remark 2.1 we can assume that $N_G(t) = \{t-1, t+1, \ldots, r\}$ where $t-1$ is a non-leaf and $t+1, \ldots, r$ are leaves. We now distinguish two subcases:

Case 2a. $t+1 = r$. Then, $\varepsilon_0(T) = \varepsilon_0(H)$ and $s = n-1$. Since $\upsilon(T) = \upsilon(H) - 1, \alpha_r = 0$ and

$$\beta_j = \begin{cases} \alpha_j - 1 & \text{if } j = t-1 \text{ or } j = t, \\ \alpha_j & \text{otherwise,} \end{cases}$$

by the induction hypothesis we have $\Delta_{\boldsymbol{\beta}}(I(T)^{n-1}) = \langle X \setminus \{r\}, Y\rangle$, and

$$\sum_{i \notin X} \alpha_i = \sum_{i \notin X \setminus \{r\}} \beta_i + 1 = n-1, \text{ and } \sum_{i \notin Y} \alpha_i = \sum_{i \notin Y} \beta_i + \alpha_r + 1 = n-1. \tag{10}$$

In particular, $X \in \Delta_{\boldsymbol{\alpha}}(I(H)^n)$ and $Y \in \Delta_{\boldsymbol{\alpha}}(I(H)^n)$. Thus it remains to show that $\Delta_{\boldsymbol{\alpha}}(I(H)^n) = \langle X, Y\rangle$. Let F be any facet of $\Delta(H)$ that is different from X and Y.

Assume that $t \in F$. Then, F is also a facet of $\Delta(T)$ that is different from $X \setminus \{r\}$ and Y. Since $t-1 \notin F$ and $\alpha_{t-1} = \beta_{t-1} + 1$, we have

$$\sum_{i \notin F} \alpha_i = \sum_{i \notin F} \beta_i + 1 + \alpha_r = \sum_{i \notin F} \beta_i + 1 \geqslant s + 1 = n.$$

Therefore, $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$.

Assume that $t \notin F$. Then, $r \in F$. If $t-1 \in F$, then $F \setminus \{r\}$ is a subset of neither $X \setminus \{r\}$ nor Y. Hence, there is a facet F' of $\Delta(T)$ containing F and being different from $X \setminus \{r\}$ and Y. Therefore,

$$\sum_{i \notin F} \alpha_i \geqslant \sum_{i \notin F} \beta_i + 1 \geqslant \sum_{i \notin F'} \beta_i + 1 \geqslant s + 1 = n.$$

Which implies $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$.

If $t-1 \notin F$, then $(F \cup \{t\}) \setminus \{r\}$ is a facet of $\Delta(T)$. Noticing that $\alpha_{t-1} = \beta_{t-1} + 1$ and $\alpha_t = 1$, we get

$$\sum_{i \notin F} \alpha_i = \sum_{i \notin (F \cup \{t\}) \setminus \{r\}} \beta_i + 1 + \alpha_t \geqslant (s-1) + 2 = n.$$

Which again implies $F \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$.

Case 2. $t+1 < r$. Thus $\boldsymbol{\beta} = (\alpha_1, \ldots, \alpha_{r-1})$, and thus $s = n$. Now we can proceed as in Case 1. This completes the proof of Claim 1.

Claim 2. $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y\rangle$.

Proof. by Claim 1 and Lemma 1.5, X and Y are facets of $\Delta_{\boldsymbol{\alpha}}(I(G)^n)$. It remains to show that for any facet F of $\Delta(G)$ being different from X and Y, then $F \notin \Delta_{\boldsymbol{\alpha}}(I(G)^n)$. Since F is a face of H, we have $F \subseteq F'$ for some facet F' of $\Delta(H)$. Then, F' is different from X and Y, and then $F' \notin \Delta_{\boldsymbol{\alpha}}(I(H)^n)$. Thus, by Lemma 1.5 we have

$$\sum_{i \notin F} \alpha_i \geqslant \sum_{i \notin F'} \alpha_i \geqslant n$$

and thus $F \notin \Delta_{\boldsymbol{\alpha}}(I(G)^n)$, as claimed.

Now we return to the proof of the proposition. Claim 2 and Lemma 3.1 give $\operatorname{dstab}(I(G)) \leqslant n$, or equivalently

$$\operatorname{dstab}(I(G)) \leqslant \varepsilon(H) - \varepsilon_0(H) - k + 1.$$

Let e be an edge of the cycle C_{2k}. Then, $H \setminus e$ is a tree. Hence, by [1, Theorem 4.3] we have $\varepsilon(H) = \varepsilon(H\backslash e) + 1 = (v(H\backslash e) - 1) + 1 = v(H\backslash e) = v(H) = v(G)$. Clearly, $\varepsilon_0(G) \leqslant \varepsilon_0(H)$. Therefore,

$$\operatorname{dstab}(I(G)) \leqslant \varepsilon(H) - \varepsilon_0(H) - k + 1 \leqslant v(G) - \varepsilon_0(G) - k + 1,$$

as required. $\square$

4. Depths of powers of edge ideals

In this section we study the stability of depth $R/I(G)^n$ for any graph G. First we need some basic facts of homological modules of simplicial complexes.

A tool which will be of much use is the Mayer–Vietoris sequence, see [21, Theorem 25.1] or [23, in Page 21]. For two simplicial complexes Δ_1 and Δ_2, we have the long exact sequence of reduced homology modules

$$\cdots \to \widetilde{H}_i(\Delta_1) \oplus \widetilde{H}_i(\Delta_2) \to \widetilde{H}_i(\Delta) \to \widetilde{H}_{i-1}(\Delta_1 \cap \Delta_2) \to \widetilde{H}_{i-1}(\Delta_1) \oplus \widetilde{H}_{i-1}(\Delta_2) \to \cdots$$

where $\Delta = \Delta_1 \cup \Delta_2$.

A simplicial complex Δ is a cone if there is a vertex v such that $\{v\} \cup F \in \Delta$ for every $F \in \Delta$. If Δ is a cone, then it is acyclic (see [21, Theorem 8.2]), i.e.,

$$\widetilde{H}_i(\Delta; K) = \mathbf{0} \text{ for every } i \in \mathbb{Z}.$$

Finally, for two simplicial complexes Δ and Γ over two disjoint vertex sets, the join of Δ and Γ, denoted by $\Delta * \Gamma$, is defined by

$$\Delta * \Gamma := \{F \cup G \mid F \in \Delta \text{ and } G \in \Gamma\}.$$

Lemma 4.1. *Let G be a bipartite graph with connected components $G_1, \dots, G_s$ and let $n := \sum_{i=1}^{s} \operatorname{dstab}(I(G_i)) - s + 1$. Then there is $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_r) \in \mathbb{N}^r$ such that*

$$\sum_{i \notin F} \alpha_i = n - 1 \quad \textit{for all } F \in \mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G)^n)) \quad \textit{and } \widetilde{H}_{s-1}(\Delta_{\boldsymbol{\alpha}}(I(G)^n); K) \neq \mathbf{0}.$$

Proof. For each i, let (X_i, Y_i) be a bipartition of G_i and $n_i := \operatorname{dstab}(I(G_i))$, so that

$$n = \sum_{i=1}^{s} n_s - s + 1.$$

Since the vertex sets of $G_1, \dots, G_s$ are mutually disjoint, by Lemma 3.1 there is $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ such that

$$\sum_{j \in V(G_i) \setminus X_i} \alpha_j = \sum_{j \in V(G_i) \setminus Y_i} \alpha_j = n_i - 1, \tag{11}$$

and

$$\sum_{j \in V(G_i) \setminus F_i} \alpha_j \geqslant n_i \quad \text{for all } F_i \in \mathcal{F}(\Delta(G_i)) \setminus \{X_i, Y_i\}. \tag{12}$$

For any $F \in \mathcal{F}(\Delta(G))$, we can partition F into $F = \bigcup_{i=1}^{s} F_i$ where $F_i \in \mathcal{F}(\Delta(G_i))$ for $i = 1, \dots, s$. By Equation (11) and Inequality (12) we get

$$\sum_{j \notin F} \alpha_j = \sum_{i=1}^{s} \sum_{j \in V(G_i) \setminus F_i} \alpha_j \geqslant \sum_{i=1}^{s} (n_i - 1) = n - 1$$

and the equality occurs if and only if

$$\sum_{j \in V(G_i) \setminus F_i} \alpha_j = n_i - 1 \quad \text{for all } i = 1, \dots, s,$$

or equivalently, either $F_i = X_i$ or $F_i = Y_i$ for all $i = 1, \dots, s$. Together with Lemma 1.5 we have

$$\sum_{j \notin F} \alpha_j = n - 1 \quad \text{for all } F \in \mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G)^n)),$$

and

$$\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X_1, Y_1 \rangle * \cdots * \langle X_s, Y_s \rangle.$$

So it remains to prove that $\widetilde{H}_{s-1}(\langle X_1, Y_1 \rangle * \cdots * \langle X_s, Y_s \rangle; K) \neq \mathbf{0}$. In order to prove this, let $\Delta_i := \langle X_1, Y_1 \rangle * \cdots * \langle X_i, Y_i \rangle$ for $i = 1, \dots, s$ and $\Delta_0 := \{\emptyset\}$. Then, for all $i = 1, \dots, s$

we have

$$\Delta_i = \langle X_i\rangle * \Delta_{i-1} \cup \langle Y_i\rangle * \Delta_{i-1} \ \text{ and } \Delta_{i-1} = \langle X_i\rangle * \Delta_{i-1} \cap \langle Y_i\rangle * \Delta_{i-1}.$$

Since $\langle X_i\rangle * \Delta_{i-1}$ and $\langle X_i\rangle * \Delta_{i-1}$ are cones, by using Mayer–Vietoris sequence, we get an exact sequence $\mathbf{0} \to \widetilde{H}_{s-1}(\Delta_s; K) \to \widetilde{H}_{s-2}(\Delta_{s-1}; K) \to \mathbf{0}$. Thus,

$$\widetilde{H}_{s-1}(\Delta_s; K) \cong \widetilde{H}_{s-2}(\Delta_{s-1}; K).$$

By repeating this way we obtain

$$\widetilde{H}_{s-1}(\Delta_s; K) \cong \widetilde{H}_{s-2}(\Delta_{s-1}; K) \cong \cdots \cong \widetilde{H}_{-1}(\Delta_0; K) \cong K,$$

and so $\widetilde{H}_{s-1}(\Delta_s; K) \neq \mathbf{0}$, as required. $\square$

The next lemma gives the limit of the sequence depth $R/I(G)^n$.

Lemma 4.2. *Let G be a graph. Assume that $G_1, \dots, G_s$ are all connected bipartite components of G and $G_{s+1}, \dots, G_{s+t}$ are all connected nonbipartite components of G. Then*

$$\operatorname{depth} R/I(G)^n = s \ \ \textit{for all } n \geqslant \sum_{i=1}^{s+t} \operatorname{dstab}(I(G_i)) - (s+t) + 1.$$

Proof. Let $n_i := \operatorname{dstab}(I(G_i))$ for $i = 1, \dots, s+t$. We divide the proof into three cases:

Case 1. $s = 0$, i.e., every component of G is nonbipartite. This case follows from Lemma 1.1.

Case 2. $t = 0$, i.e., G is bipartite. Let $m := \sum_{i=1}^{s} n_i - s + 1$. By Lemmas 1.4 and 4.1, there is $\boldsymbol{\alpha} \in \mathbb{N}^r$ such that

$$\dim_K H^s_{\mathfrak{m}}(R/I(G)^m)_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{s-1}(\Delta_{\boldsymbol{\alpha}}(I(G)^m); K) \neq 0.$$

Hence, $H^s_{\mathfrak{m}}(R/I(G)^m) \neq \mathbf{0}$, which yields depth $R/I(G)^m \leqslant s$. On the other hand, by Lemma 1.3 we have depth $R/I(G)^m \geqslant s$. Thus, depth $R/I(G)^m = s$. The lemma now follows from Lemma 1.3.

Case 3. $s \neq 0$ and $t \neq 0$. Let G' and G'' be induced subgraphs of G defined by

$$G' := \bigcup_{i=1}^{s} G_i \ \text{ and } G'' := \bigcup_{i=1}^{t} G_{s+i}.$$

We may assume that $V(G') = [p]$ and $V(G'') = \{p+1, \dots, p+q\}$, where $p + q = r$. For simplicity, we set $y_1 := x_{p+1}, \dots, y_q := x_{p+q}$. Then $R = K[x_1, \dots, x_p, y_1, \dots, y_q]$. Let

$R' := K[x_1, \ldots, x_p], R'' := K[y_1, \ldots, y_q]$, $m := \sum_{i=1}^{s} n_i - s + 1$ and $n_0 := n - m + 1$. Note that $n_0 \geqslant \sum_{i=1}^{t} n_{s+i} - t + 1$, so $(y_1, \ldots, y_q) \in \operatorname{Ass}(R''/I(G'')^{n_0})$ by Lemma 1.1. Accordingly, there exists $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q) \in \mathbb{N}^q$ such that $(y_1, \ldots, y_q) = I(G'')^{n_0} : \mathbf{y}^{\boldsymbol{\beta}}$. This implies

$$\mathbf{y}^{\boldsymbol{\beta}} \in I(G'')^{n_0-1},\ \mathbf{y}^{\boldsymbol{\beta}} \notin I(G'')^{n_0} \text{ and } \mathbf{y}^{\boldsymbol{\beta}} \in I(G'')^{n_0}_F \text{ whenever } \emptyset \neq F \in \Delta(G''). \tag{13}$$

Next, by Lemma 4.1 there is $\boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_p) \in \mathbb{N}^p$ such that

$$\widetilde{H}_{s-1}(\Delta_{\boldsymbol{\alpha}}(I(G')^m); K) \neq \mathbf{0}, \text{ and } \sum_{i \notin V} \alpha_i = m - 1 \text{ for all } V \in \mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I(G')^m)). \tag{14}$$

Let $\boldsymbol{\gamma} := (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q) \in \mathbb{N}^r$. Note that $\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{y}^{\boldsymbol{\beta}} \in R$. We claim that

$$\Delta_{\boldsymbol{\gamma}}(I(G)^n) = \Delta_{\boldsymbol{\alpha}}(I(G')^m). \tag{15}$$

Indeed, for all $H \in \Delta_{\boldsymbol{\gamma}}(I(G)^n)$ we can partition H into $H = H_1 \cup H_2$ where $H_1 \in \Delta(G')$ and $H_2 \in \Delta(G'')$. By Equation (1) we have

$$\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{y}^{\boldsymbol{\beta}} \notin I(G)^n_H = (I(G')_{H_1} + I(G'')_{H_2})^n = \sum_{i=0}^{n} I(G')^i_{H_1} I(G'')^{n-i}_{H_2}. \tag{16}$$

Now, if $H_2 \neq \emptyset$, then by Formula (13) we would have $\mathbf{y}^{\boldsymbol{\beta}} \in I(G'')^{n_0}_{H_2}$. Then, Formula (16) forces $\mathbf{x}^{\boldsymbol{\alpha}} \notin I(G')^{n-n_0}_{H_1} = I(G')^{m-1}_{H_1}$, thus $H_1 \in \Delta_{\boldsymbol{\alpha}}(I(G')^{m-1})$. In particular, $\Delta_{\boldsymbol{\alpha}}(I(G')^{m-1}) \neq \emptyset$. Let us take arbitrary facet V of $\Delta_{\boldsymbol{\alpha}}(I(G')^{m-1})$. By Lemma 1.5 we then have $\sum_{i \notin V} \alpha_i \leqslant m - 2$. By Lemma 1.5 again, V is a facet of $\Delta_{\boldsymbol{\alpha}}(I(G')^m)$, which contradicts (14). Thus, $H_2 = \emptyset$ and $H = H_1$. Formula (16) now becomes

$$\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{y}^{\boldsymbol{\beta}} \notin (I(G')_H + I(G''))^n = \sum_{i=0}^{n} I(G')^i_H I(G'')^{n-i}.$$

Together with Formula (13), this fact implies $\mathbf{x}^{\boldsymbol{\alpha}} \notin I(G')^{n-n_0+1}_H = I(G')^m_H$, or equivalently, $H \in \Delta_{\boldsymbol{\alpha}}(I(G')^m)$, so $\Delta_{\boldsymbol{\gamma}}(I(G)^n) \subseteq \Delta_{\boldsymbol{\alpha}}(I(G')^m)$.

In order to prove the reverse inclusion, suppose that $H \in \Delta_{\boldsymbol{\alpha}}(I(G')^m)$. Then, $\mathbf{x}^{\boldsymbol{\alpha}} \notin I(G')^m_H$ by Equation (1). If $\mathbf{x}^{\boldsymbol{\gamma}} \in I(G)^n_H$, then

$$\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{x}^{\boldsymbol{\alpha}}\mathbf{y}^{\boldsymbol{\beta}} \in I(G)^n_H = (I(G')_H + I(G''))^n = \sum_{i=0}^{n} I(G')^i_H I(G'')^{n-i}.$$

Hence, $\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{y}^{\boldsymbol{\beta}} \in I(G')^{\nu}_H I(G'')^{n-\nu}$ for some nonnegative integer ν. Since $V(G') \cap V(G'') = \emptyset$, it yields $\mathbf{x}^{\boldsymbol{\alpha}} \in I(G')^{\nu}_H$ and $\mathbf{y}^{\boldsymbol{\beta}} \in I(G'')^{n-\nu}$. By Formula (13) we deduce that $n - \nu \leqslant n_0 - 1$, and so $\nu \geqslant n - n_0 + 1 = m$. But then $\mathbf{x}^{\boldsymbol{\alpha}} \in I(G')^m_H$, a contradiction.

Hence, $\mathbf{x}^{\boldsymbol{\gamma}} \notin I(G)^n_H$, i.e., $H \in \Delta_{\boldsymbol{\gamma}}(I(G)^n)$, and hence $\Delta_{\boldsymbol{\alpha}}(I(G')^m) \subseteq \Delta_{\boldsymbol{\gamma}}(I(G)^n)$, as claimed.

Combining Formulas (14) and (15) with Lemma 1.4, we get

$$\begin{aligned}\dim_K H^s_{\mathfrak{m}}(R/I(G)^n)_{\boldsymbol{\gamma}} &= \dim_K \widetilde{H}_{s-1}(\Delta_{\boldsymbol{\gamma}}(I(G)^n);K)\\ &= \dim_K \widetilde{H}_{s-1}(\Delta_{\boldsymbol{\alpha}}(I(G')^m);K) \neq 0.\end{aligned}$$

Therefore, $H^s_{\mathfrak{m}}(R/I(G)^n) \neq \mathbf{0}$, so

$$\operatorname{depth} R/I(G)^n \leqslant s. \tag{17}$$

On the other hand, since G' is bipartite, by Lemmas 1.3 and 1.6 we get

$$\operatorname{depth} R/I(G)^n = \operatorname{depth} R/(I(G') + I(G''))^n \geqslant \min_{\nu \geqslant 1} \operatorname{depth} R'/I(G')^\nu = s.$$

Together with Inequality (17), we obtain $\operatorname{depth} R/I(G)^n = s$, as required. □

Corollary 4.3. *For all graphs G we have $\lim_{n\to\infty} \operatorname{depth} R/I(G)^n = \dim R - \ell(I(G))$.*

Proof. Let s be the number of bipartite components of G. Then $s = \dim R - \ell(I(G))$ (see [25, Page 50]), so the corollary immediately follows from Lemma 4.2. □

We are now ready to prove the first main result of the paper.

Theorem 4.4. *Let G be a graph with p connected components $G_1, \ldots, G_p$. Let s be the number of connected bipartite components of G. Then*

(1) $\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = s$.
(2) $\operatorname{dstab}(I(G)) = \min\{n \geqslant 1 \mid \operatorname{depth} R/I(G)^n = s\}$.
(3) $\operatorname{dstab}(I(G)) = \sum_{i=1}^p \operatorname{dstab}(I(G_i)) - p + 1$.

Proof. We may assume that $G_1, \ldots, G_s$ are bipartite.

(1) If $s = 0$ (resp. $s = p$), then the first statement follows from Lemma 1.1 (resp. Lemma 1.3). Assume that $1 \leqslant s < p$. Let G' be the induced subgraph of G consisting of $G_1, \ldots, G_s$ and G'' the induced subgraph of G consisting of $G_{s+1}, \ldots, G_p$. Then, $I(G) = I(G') + I(G'')$. Let $R' := K[x_i \mid i \in V(G')]$. For all $n \geqslant 1$, since G' is bipartite, by Lemmas 1.3 and 1.6 we have

$$\operatorname{depth} R/I(G)^n \geqslant \min\{\operatorname{depth} R'/I(G')^m \mid m \geqslant 1\} = s.$$

Together with Lemma 4.2 we conclude that

$$\min\{\operatorname{depth} R/I(G)^n \mid n \geqslant 1\} = s,$$

and (1) follows.

We next prove (2) and (3) simultaneously by induction on p. If $p = 1$, then the theorem follows from Lemmas 1.1 and 1.3.

Assume that $p \geqslant 2$. If $s = 0$, our claim follows from Lemma 1.1. So we may assume that $s \geqslant 1$. Let H be the induced subgraph of G consisting of components $G_2, \dots, G_p$. Then, H has $p-1$ connected components and $s-1$ connected bipartite components. By Lemma 4.2 we have

$$\operatorname{depth} R/I(G)^n = s \text{ for all } n \geqslant \sum_{i=1}^{p} \operatorname{dstab}(I(G_i)) - p + 1.$$

Hence, in order to prove the theorem it suffices to show that if

$$\operatorname{depth} R/I(G)^n = s \tag{18}$$

for a given positive integer n, then $n \geqslant \sum_{i=1}^{p} \operatorname{dstab}(I(G_i)) - p + 1$.

In order to prove this assertion let $A := K[x_j \mid j \in V(G_1)]$ and $B := K[x_j \mid j \in V(H)]$. Then, we have $\dim A \geqslant 2$ and $\dim B \geqslant s$. For simplicity, we set $I := I(G_1)$ and $J := I(H)$. We now claim that

$$\operatorname{depth} R/I^i J^{n-i} \geqslant s + 1 \text{ for } i = 0, \dots, n. \tag{19}$$

Indeed, if $i = n$, since $\operatorname{depth} A/I^n \geqslant 1$ and $\dim B \geqslant s$, we have

$$\operatorname{depth} R/I^n J^0 = \operatorname{depth} R/I^n = \operatorname{depth} A/I^n + \dim B \geqslant 1 + s.$$

Since $\operatorname{depth} B/J^n \geqslant s - 1$ by Part 1, a similar proof also holds for $i = 0$. For all $i = 1, \dots, n-1$, by [11, Lemma 2.2] we have $\operatorname{depth} R/I^i J^{n-i} = \operatorname{depth} A/I^i + \operatorname{depth} B/J^{n-i} + 1$. Hence, $\operatorname{depth} R/I^i J^{n-i} \geqslant 1 + (s-1) + 1 = s + 1$, as claimed.

Let $n_1 := \operatorname{dstab}(G_1)$ and $n_2 := \operatorname{dstab}(H)$. We will prove that $n \geqslant n_1 + n_2 - 1$. Assume on the contrary that $n \leqslant n_1 + n_2 - 2$. For each $i = 0, \dots, n$, we put

$$W_i := I^i J^{n-i} + \cdots + I^n J^0,$$

where $I^0 = J^0 = R$. We next claim that

$$\operatorname{depth} R/W_i \geqslant s + 1 \text{ for all } i = 0, \dots, n. \tag{20}$$

Indeed, we prove this by induction on i. If $i = n$, then by Inequality (19) we have

$$\operatorname{depth} R/W_n = \operatorname{depth} R/I^n \geqslant s + 1.$$

Assume that $\operatorname{depth} R/W_{i+1} \geqslant s + 1$ for some $0 \leqslant i < n$. By Equations (2) and (3), we have $I^i J^{n-i} \cap W_{i+1} = I^{i+1} J^{n-i}$. Since $W_i = I^i J^{n-i} + W_{i+1}$, we have an exact sequence

$$\mathbf{0} \longrightarrow R/I^{i+1} J^{n-i} \longrightarrow R/I^i J^{n-i} \oplus R/W_{i+1} \longrightarrow R/W_i \longrightarrow \mathbf{0}.$$

By Depth Lemma, we have

$$\operatorname{depth} R/W_i \geqslant \min\{\operatorname{depth} R/I^{i+1}J^{n-i} - 1, \operatorname{depth} R/I^iJ^{n-i}, \operatorname{depth} R/W_{i+1}\}.$$

Together with Inequality (19) and the induction hypothesis, this fact yields

$$\operatorname{depth} R/W_i \geqslant \min\{\operatorname{depth} R/I^{i+1}J^{n-i} - 1, s+1\}.$$

Therefore, the inequality (20) will follows if $\operatorname{depth} R/I^{i+1}J^{n-i} \geqslant s+2$. In order to prove this inequality, note that $(i+1)+(n-i) = n+1 \leqslant n_1+n_2-1$. Hence, either $i+1 < n_1$ or $n-i < n_2$. Note that $n-i \geqslant 1$.

If $i+1 < n_1$, by Part 1 we get $\operatorname{depth} A/I^{i+1} \geqslant 2$ and $\operatorname{depth} B/J^{n-i} \geqslant s-1$. Together with [11, Lemma 2.2] we obtain

$$\operatorname{depth} R/I^{i+1}J^{n-i} = \operatorname{depth} A/I^{i+1} + \operatorname{depth} B/J^{n-i} + 1 \geqslant 2 + (s-1) + 1 = s+2,$$

as claimed.

If $n-i < n_2$, the proof is similar. Thus, the claim (20) is proved.

Notice that $W_0 = (I+J)^n = (I(G_1)+I(H))^n = I(G)^n$. By (20) we have $\operatorname{depth} R/I(G)^n \geqslant s+1$. This contradicts (18). Therefore, we must have $n \geqslant n_1+n_2-1$.

Finally, by the induction hypothesis we have

$$n_2 = \operatorname{dstab}(I(H)) = \sum_{i=2}^{p} \operatorname{dstab}(I(G_i)) - (p-1) + 1.$$

Together with $n_1 = \operatorname{dstab}(I(G_1))$, we have

$$n \geqslant n_1 + n_2 - 1 = \sum_{i=1}^{p} \operatorname{dstab}(G_i) - p + 1,$$

as required. $\square$

Remark 4.5. From Theorem 4.4 and Lemmas 1.1 and 3.1 we see that $\operatorname{dstab}(I(G))$ is independent from the characteristic of the base field K, so it depends purely on the structure of G.

We next combine Theorem 4.4 and Propositions 2.4 and 3.4 to get the second main result of the paper, which sets up an upper bound for $\operatorname{dstab}(I(G))$.

Theorem 4.6. *Let G be a graph. Let $G_1, \ldots, G_s$ be all connected bipartite components of G and let $G_{s+1}, \ldots, G_{s+t}$ be all connected nonbipartite components of G. Let $2k_i$ be the* *maximum length of cycles of G_i ($k_i := 1$ if G_i is a tree) for all $i = 1, \dots, s$; and let $2k_i - 1$ be the maximum length of odd cycles of G_i for every $i = s+1, \dots, s+t$. Then*

$$\operatorname{dstab}(I(G)) \leqslant \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1.$$

Proof. Since

$$\upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1 = \sum_{i=1}^{s+t} (\upsilon(G_i) - \varepsilon_0(G_i) - k_i + 1) - (s+t) + 1,$$

by Propositions 2.4 and 3.4 we get

$$\upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1 \geqslant \sum_{i=1}^{s+t} \operatorname{dstab}(I(G_i)) - (s+t) + 1.$$

Together with Theorem 4.4 we obtain

$$\operatorname{dstab}(I(G)) = \sum_{i=1}^{s+t} \operatorname{dstab}(I(G_i)) - (s+t) + 1 \leqslant \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{s+t} k_i + 1,$$

as required. $\square$

5. The index of depth stability of trees and unicyclic graphs

The aim of this section is to prove that the upper bound of $\operatorname{dstab}(I(G))$ given in Theorem 4.6 is always achieved if G has no cycles of length 4 and every component of G is either a tree or a unicyclic graph. Recall that a connected graph G is a tree if it contains no cycles; and G is a unicyclic graph if it contains exactly one cycle.

If G is a unicyclic graph and C is the unique cycle of G, then for every vertex v of G not lying in C, there is a unique simple path of minimal distance from v to a vertex in C.

Theorem 5.1. *Let G be a graph with p connected components $G_1, \dots, G_p$ such that each G_i is either a tree or a unicyclic graph. For each i, if G_i is bipartite, let $2k_i$ be the length of its unique cycle ($k_i := 1$ if G_i is a tree); and if G_i is nonbipartite, let $2k_i - 1$ be the length of its unique cycle. If G has no cycles of length 4, then*

$$\operatorname{dstab}(I(G)) = \upsilon(G) - \varepsilon_0(G) - \sum_{i=1}^{p} k_i + 1.$$

By Theorem 4.6, it suffices to show that $\operatorname{dstab}(G_i) = v(G_i) - \varepsilon_0(G_i) - k_i + 1$ for each $i = 1, \dots, p$. If G_i is nonbipartite, the equality follows from Lemma 2.2. Thus, it remains to prove this equality for the case G_i is bipartite.

We divide the proof into two lemmas. The first lemma deals with unicyclic bipartite graphs and the second one deals with trees.

For a vertex x of G, we denote $L_G(x)$ to be the set of leaves of G that are adjacent to x. We start with the following observation.

Lemma 5.2. *Let G be a graph with $r = v(G)$. Let p be a leaf of G and q the unique neighbor of p in G. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ and we define $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r)$ by*

$$\beta_i := \begin{cases} \alpha_i + 1 & \text{if } i = p \text{ or } i = q, \\ \alpha_i & \text{otherwise.} \end{cases}$$

Then $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \Delta_{\boldsymbol{\beta}}(I(G)^{n+1})$ for all $n \geqslant 1$.

Proof. Let F be a facet of $\Delta(G)$. By the maximality of F, it must contain either p or q but not both, so

$$\sum_{i \notin F} \beta_i = \sum_{i \notin F} \alpha_i + 1.$$

Thus, by Lemma 1.5 we get $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \Delta_{\boldsymbol{\beta}}(I(G)^{n+1})$ for all $n \geqslant 1$. $\square$

Lemma 5.3. *Let G be a unicyclic bipartite graph. Assume that the unique cycle of G is C_{2k} of length $2k$ with $k \geqslant 3$. Then, $\operatorname{dstab}(I(G)) = v(G) - \varepsilon_0(G) - k + 1$.*

Proof. Let $n := \operatorname{dstab}(I(G))$. By Theorem 4.6 we have $n \leqslant v(G) - \varepsilon_0(G) - k + 1$. Thus, in order to prove the theorem it suffices to show $n \geqslant v(G) - \varepsilon_0(G) - k + 1$.

Let (X, Y) be a bipartition of G. Then, by Lemma 3.1 there is $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ such that

$$\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle \text{ and } \sum_{j \in X} \alpha_j = \sum_{j \in Y} \alpha_j = n - 1. \tag{21}$$

Observe that for any face F of $\Delta(G)$ with $F \cap X \neq \emptyset$ and $F \cap Y \neq \emptyset$, we have

$$\sum_{i \notin F} \alpha_i \geqslant n. \tag{22}$$

Indeed, let L be a facet of $\Delta(G)$ which contains F, so that L meets both X and Y. Since $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$, $L \notin \Delta_{\boldsymbol{\alpha}}(I(G)^n)$. By Lemma 1.5 we get

$$\sum_{i \notin F} \alpha_i \geqslant \sum_{i \notin L} \alpha_i \geqslant n,$$

and the formula (22) follows.

We now prove $n \geqslant \upsilon(G) - \varepsilon_0(G) - k + 1$ by induction on $\upsilon(G)$.

If $\upsilon(G) = 2k$, i.e., $G = C_{2k}$, then $\upsilon(G) - \varepsilon_0(G) - k + 1 = k + 1$. For each $i \in X$, let $N_G(i) = \{u_i, v_i\}$ and $F_i := \{i\} \cup (Y \setminus \{u_i, v_i\})$. Then, $F_i \in \Delta(G)$. Since $|X| = |Y| = k \geqslant 3$, $F_i \cap X \neq \emptyset$ and $F_i \cap Y \neq \emptyset$. Together with Formulas (21) and (22), this fact gives

$$n \leqslant \sum_{j \notin F_i} \alpha_j = \sum_{j \in X} \alpha_j + \alpha_{u_i} + \alpha_{v_i} - \alpha_i = n - 1 + \alpha_{u_i} + \alpha_{v_i} - \alpha_i,$$

whence $\alpha_i + 1 \leqslant \alpha_{u_i} + \alpha_{v_i}$. Hence,

$$\sum_{i \in X} \alpha_i + k = \sum_{i \in X} (\alpha_i + 1) \leqslant \sum_{i \in X} (\alpha_{u_i} + \alpha_{v_i}) = 2 \sum_{j \in Y} \alpha_j.$$

Together with Formula (21), this gives $(n-1) + k \leqslant 2(n-1)$. Thus, $n \geqslant k+1$, and thus the lemma holds for this case.

Assume that $\upsilon(G) > 2k$. We distinguish two cases:

Case 1. $G \setminus V(C_{2k})$ is totally disconnected. For any vertex u lying in C_{2k} with $L_G(u) \neq \emptyset$, we claim that

$$\alpha_u \geqslant 1, \text{ and } \alpha_i = 0 \text{ for every } i \in L_G(u). \tag{23}$$

Indeed, without loss of generality we may assume that $u \in Y$, so that $L_G(u) \subseteq X$. Let $F := (Y \setminus \{u\}) \cup L_G(u)$. Then, $F \in \Delta(G)$. Since the length of C_{2k} is at least 6, we have $F \cap Y \neq \emptyset$. Notice that $\emptyset \neq L_G(u) \subseteq F \cap X$. Therefore, $F \cap X \neq \emptyset$ and $F \cap Y \neq \emptyset$. By Formula (22) we have

$$\sum_{i \in X} \alpha_i + \alpha_u - \sum_{i \in L_G(u)} \alpha_i = \sum_{i \notin F} \alpha_i \geqslant n.$$

By (21), this gives

$$n - 1 + \alpha_u - \sum_{i \in L_G(u)} \alpha_i \geqslant n,$$

so

$$\alpha_u \geqslant \sum_{i \in L_G(u)} \alpha_i + 1 \geqslant 1.$$

Hence, it remains to prove that $\alpha_i = 0$ for all $i \in L_G(u)$. Assume that $\alpha_i \geqslant 1$ for some $i \in L_G(u)$. Define $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$ by

$$\beta_j := \begin{cases} \alpha_j - 1 & \text{if } j = u \text{ or } j = i, \\ \alpha_j & \text{otherwise.} \end{cases}$$

Then, $\boldsymbol{\beta} \in \mathbb{N}^r$. Since $u \in Y$ and $\alpha_u \geqslant 1$, by (21) we have

$$n - 1 = \sum_{j \in Y} \alpha_j \geqslant \alpha_u \geqslant 1.$$

By Lemma 5.2 we have $\Delta_{\boldsymbol{\beta}}(I(G)^{n-1}) = \Delta_{\boldsymbol{\alpha}}(I(G)^n)$. Consequently, $\Delta_{\boldsymbol{\beta}}(I(G)^{n-1}) = \langle X, Y \rangle$, which implies depth $R/I(G)^{n-1} = 1$ by Lemma 3.1, and so $\operatorname{dstab}(I(G)) \leqslant n - 1$ by Theorem 4.4. This contradicts to $n = \operatorname{dstab}(I(G))$. Thus, $\alpha_i = 0$, as claimed.

We may assume that $V(H) = \{1, \ldots, 2k\}$. Let $\boldsymbol{\beta} := (\alpha_1, \ldots, \alpha_{2k}) \in \mathbb{N}^{2k}$, $X_0 := X \cap V(C_{2k})$ and $Y_0 := Y \cap V(C_{2k})$. Then, (X_0, Y_0) is a bipartition of C_{2k}. Clearly,

$$X = X_0 \cup \bigcup_{i \in Y_0} L_G(i) \text{ and } Y = Y_0 \bigcup_{i \in X_0} L_G(i).$$

Together with Claim (23) we have

$$\sum_{i \notin X_0} \beta_i = \sum_{i \notin X} \alpha_i = n - 1.$$

Similarly, $\sum_{i \notin Y_0} \beta_i = n - 1$. Therefore, $X_0, Y_0 \in \Delta_{\boldsymbol{\beta}}(I(C_{2k})^n)$.

For any facet F of $\Delta(C_{2k})$ which is different from X_0 and Y_0, let

$$F' := F \cup \bigcup_{i \in V(C) \setminus F} L_G(i).$$

Then, F' is a facet of $\Delta(G)$ which is different from X and Y. Together Claim (23) with Lemma 1.5, we have

$$\sum_{i \notin F} \beta_i = \sum_{i \notin F'} \alpha_i \geqslant n$$

so that $F \notin \Delta_{\boldsymbol{\beta}}(I(C_{2k})^n)$. Thus, $\Delta_{\boldsymbol{\beta}}(I(C_{2k})^n) = \langle X_0, Y_0 \rangle$.

This gives depth $S/I(C_{2k})^n = 1$ where $S = K[x_1, \ldots, x_{2k}]$. From the case $v(G) = 2k$ above, we imply that

$$n \geqslant k + 1 = v(G) - \varepsilon_0(G) - k + 1,$$

and the lemma holds in this case.

Case 2. $G \setminus V(C_{2k})$ is not totally disconnected. Let v be a leaf of G such that $d_G(v, C_{2k})$ is maximal. By Remark 2.1, we deduce that $N_G(v)$ has only one non-leaf, say u, and $N_G(u)$ also has only one non-leaf, say w. Note that $L_G(u) \neq \emptyset$ since $v \in L_G(u)$. We may assume that $u \in Y$, so that $v \in X$. We first claim that

$$\alpha_u \geqslant 1, \text{ and } \alpha_i = 0 \text{ for every } i \in L_G(u). \tag{24}$$

Indeed, let $F := (Y \setminus \{u\}) \cup L_G(u)$. Then, $F \in \Delta(G)$. Since $|N_G(w)| \geqslant 2$ and $N_G(w) \subseteq Y$, we have $\emptyset \neq N_G(w) \setminus \{u\} \subseteq Y \setminus \{u\} \subseteq F \cap Y$. Notice that $\emptyset \neq L_G(u) \subseteq F \cap X$. Therefore, $F \cap X \neq \emptyset$ and $F \cap Y \neq \emptyset$. The proof of claim now carries out the same as in Claim (23).

We next claim that

$$\alpha_w \geqslant 1. \tag{25}$$

Indeed, assume on the contrary that $\alpha_w = 0$. Note that $w \in X$ and $N_G(u) = L_G(u) \cup \{w\}$. Let $F := (X \cup \{u\}) \setminus N_G(u)$. Then, $F \in \Delta(G)$ and $u \in F \cap Y$. Since $N_G(u) \neq X$, $F \cap X \neq \emptyset$. By Formulas (21)–(24) and the assumption $\alpha_w = 0$, these facts give

$$n \leqslant \sum_{i \notin F} \alpha_i = \sum_{i \in Y} \alpha_i - \alpha_u + \alpha_w + \sum_{i \in L_G(u)} \alpha_i = n - 1 - \alpha_u,$$

and so $\alpha_u < 0$, a contradiction. Thus, $\alpha_w \geqslant 1$, as claimed.

Let $H := G \setminus L_G(u)$. Clearly, H is a connected bipartite graph with bipartition $(X \setminus L_G(u), Y)$. Moreover, H has only cycle C_{2k} as well. We may assume that $V(H) = \{1, \ldots, s\}$. Then $s \geqslant 2k$ and $L_G(u) = \{s+1, \ldots, r\}$. Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s) := (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$. We now prove that

$$\Delta_{\boldsymbol{\theta}}(I(H)^n) = \langle X \setminus L_G(u), Y \rangle. \tag{26}$$

Indeed, by (24) we get $\sum_{\alpha_i \in L_G(u)} \alpha_i = 0$. Together with Formula (21), this fact gives

$$\sum_{i \in V(H), i \notin Y} \theta_i = \sum_{i \in V(H), i \notin Y} \alpha_i = \sum_{i \in X \setminus L_G(u)} \alpha_i + \sum_{i \in L_G(u)} \alpha_i = \sum_{i \in X} \alpha_i = n - 1.$$

Hence, by Lemma 1.5, $Y \in \Delta_{\boldsymbol{\theta}}(I(H)^n)$. Similarly, $X \setminus L_G(u) \in \Delta_{\boldsymbol{\theta}}(I(H)^n)$. Now let F' be any facet of $\Delta(H)$ which is different from $X \setminus L_G(u)$ and Y.

If $u \in F'$ then F' is also a facet of $\Delta(G)$. By noticing that F' is different from X and Y and $\sum_{i \in L_G(u)} \alpha_i = 0$, so by (22) we have

$$\sum_{i \in V(H), i \notin F'} \theta_i = \sum_{i \in V(H), i \notin F'} \alpha_i + \sum_{i \in L_G(u)} \alpha_i = \sum_{i \notin F'} \alpha_i \geqslant n,$$

and so $F' \notin \Delta_{\boldsymbol{\theta}}(I(H)^n)$.

If $u \notin F'$, then $w \in F'$ since u is a leaf of H, hence $F' \cup L_G(u)$ is a facet of $\Delta(G)$. Similarly, we have $F' \notin \Delta_{\boldsymbol{\theta}}(I(H)^n)$, and the formula (26) follows.

Define $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_s) \in \mathbb{Z}^s$ by

$$\gamma_j := \begin{cases} \theta_j - 1 & \text{if } j = u \text{ or } j = w, \\ \theta_j & \text{otherwise.} \end{cases}$$

From Inequalities (24) and (25), we have $\gamma_u = \theta_u - 1 = \alpha_u - 1 \geqslant 0$ and $\gamma_w = \theta_w - 1 = \alpha_w - 1 \geqslant 0$, so $\boldsymbol{\gamma} \in \mathbb{N}^s$. Note that

$$n - 1 = \sum_{i \in X} \alpha_i \geqslant \alpha_u \geqslant 1.$$

Therefore, by Lemma 5.2 we have $\Delta_{\boldsymbol{\gamma}}(I(H)^{n-1}) = \Delta_{\boldsymbol{\theta}}(I(H)^n)$. Together with (26) we get

$$\Delta_{\boldsymbol{\gamma}}(I(H)^{n-1}) = \langle X \setminus L_G(1), Y \rangle.$$

Hence, by Lemma 3.1 we have $\operatorname{depth} S/I(H)^{n-1} = 1$, where $S = K[x_1, \ldots, x_s]$. By Theorem 4.4 we have $\operatorname{dstab}(I(H)) \leqslant n - 1$. On the other hand, since $v(H) = v(G) - |L_G(u)| < v(G)$, by the induction hypothesis we have $\operatorname{dstab}(H) \leqslant v(H) - \varepsilon_0(H) - k + 1$. As $\{w, u\}$ is not a leaf edge of G and recall that $H = G \setminus L_G(u)$, we conclude that $\varepsilon_0(G) = \varepsilon_0(H) + |L_G(u)| - 1$. Thus,

$$v(G) - \varepsilon_0(G) - k + 1 = v(H) + |L_G(u)| - (\varepsilon_0(H) + |L_G(u)| - 1) - k + 1 = v(H) - \varepsilon_0(H) - k.$$

Hence, $n - 1 \geqslant \operatorname{dstab}(I(H)) \geqslant v(G) - \varepsilon_0(G) - k$, and hence $n \geqslant v(G) - \varepsilon_0(G) - k + 1$. Thus, the proof now is complete. $\square$

Finally, we compute $\operatorname{dstab}(I(G))$ for trees G. If a tree G has a vertex x being adjacent to every other vertex, then G is called a star with a center x. Note that G is a star if and only if $\operatorname{diam}(G) \leqslant 2$ where $\operatorname{diam}(G)$ stands for the diameter of G. If $\operatorname{diam}(G) = d$, then there is a path $x_1 x_2 \ldots x_d x_{d+1}$ of length d in G. Such a path will be referred to as a path *realizing the diameter* of G.

Lemma 5.4. $\operatorname{dstab}(I(G)) = v(G) - \varepsilon_0(G)$ *for all trees* G.

Proof. Let $n := \operatorname{dstab}(I(G))$. By Theorem 4.6 we have $n \leqslant v(G) - \varepsilon_0(G)$. So it remains to show $n \geqslant v(G) - \varepsilon_0(G)$.

If G is a star, then $\varepsilon_0(G) = \varepsilon(G) = v(G) - 1$, and then $v(G) - \varepsilon_0(G) = 1 \leqslant n$. Thus, the lemma holds for this case.

We will prove by induction on $v(G) = r$. If $v(G) = 2$, then G is one edge, and then it is a star. This case is already proved.

If $v(G) \geqslant 3$. We may assume that G is not a star so that $\operatorname{diam} G \geqslant 3$. Since $\operatorname{depth} R/I(G)^n = 1$, there is $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ such that $\Delta_{\boldsymbol{\alpha}}(I(G)^n) = \langle X, Y \rangle$ where (X, Y) is a bipartition of G.

Let $vuw \ldots z$ be a path realizing the diameter of G. Then v is a leaf, u and w both are not leaves. By [20, Lemma 3.3] we have $N_G(u) = \{w\} \cup L_G(u)$. And now we prove $n \geqslant v(G) - \varepsilon_0(G)$ by the same way as in Case 2 in the proof of Lemma 5.3. Thus we only sketch the proof here:

First, we show that $\alpha_u \geqslant 1, \alpha_w \geqslant 1$ and $\alpha_i = 0$ for every $i \in L_G(u)$. Then, let $T := G \setminus L_G(u)$. Note that T is also a tree and $\upsilon(G) - \varepsilon_0(G) = \upsilon(T) - \varepsilon_0(T) + 1$. We may assume that $u \in Y$, $w = s-1$, $u = s$ and $L_G(u) = \{s+1, \dots, r\}$. Let $\boldsymbol{\theta} := (\alpha_1, \dots, \alpha_{s-2}, \alpha_{s-1} - 1, \alpha_s - 1) \in \mathbb{N}^s$. Then, we show that

$$\Delta_{\boldsymbol{\theta}}(I(T)^{n-1}) = \langle X \setminus L_G(u), Y \rangle .$$

This gives depth $S/I(T)^{n-1} = 1$ where $S = K[x_1, \dots, x_s]$. By the induction hypothesis we have $n - 1 \geqslant \upsilon(T) - \varepsilon_0(T)$. From that we obtain $n \geqslant \upsilon(G) - \varepsilon_0(G)$. $\square$

Remark 5.5. Let G be a unicyclic bipartite graph. If the unique circle of G is C_4 of length 4, by the same argument as in the proof of Lemma 5.3 we have the following situations:

(1) If $G = C_4$, then $\operatorname{dstab}(I(G)) = 1$.
(2) If $G \neq C_4$ and C_4 has at least two adjacent vertices of degree 2 in G, then $\operatorname{dstab}(I(G)) = \upsilon(G) - \varepsilon_0(G) - 2$.
(3) In the remaining cases, $\operatorname{dstab}(I(G)) = \upsilon(G) - \varepsilon_0(G) - 1$.

Thus if every connected component of G is either a tree or a unicyclic graph, then we can compute $\operatorname{dstab}(I(G))$ by using Theorem 4.4, Lemmas 2.2, 5.3, 5.4 and Remark 5.5.

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