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Lie algebroids arising from simple group schemes

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ABSTRACT

A classical construction of Atiyah assigns to any (real or complex) Lie group G, manifold M and principal homogeneous G-space P over M, a Lie algebroid over M ([1]). The spirit behind our work is to put this work within an algebraic context, replace M by a scheme X and G by a "simple" reductive group scheme $\mathcal{G}$ over X (in the sense of Demazure–Grothendieck) that arise naturally with an attached torsor (which plays the role of P) in the study of Extended Affine Lie Algebras (see [9] for an overview). Lie algebroids in an algebraic sense were also considered by Beilinson and Bernstein. We will explain how the present work relates to theirs.

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1. Introduction

To motivate our work we will quickly recall the main ingredients appearing in the definition of Lie algebroids within the "differential" framework. These are

i) A differentiable (real or complex) manifold M.
ii) A vector bundle $\mathcal{E}$ over M.
iii) A Lie algebra structure $[\,,\,]$ on the (real or complex) space $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$.
iv) A vector bundle morphism $\sharp$ (called the anchor map) from $\mathcal{E}$ to the tangent bundle TM of M satisfying

$$[X, fY] = f[X, Y] + \sharp(X)(f)Y \tag{1}$$

for all $X, Y \in \Gamma(\mathcal{E})$ and $f \in C^\infty(M)$. As usual the elements of TM are thought of as vector fields on M, and $\sharp(X)(f)$ is the derivative (with real or complex values) of f along the vector field $\sharp(X)$.

We now turn our attention to an a priori completely different set of algebraic objects that, as we shall see, bear a striking similarity to Lie algebroids.

Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra over a field k of characteristic 0. Let R be a ring extension of k, and let $\mathcal{L}$ be a Lie algebra over R (hence over k) with the property that $\mathcal{L} \otimes_R S \simeq \mathfrak{g} \otimes_k S$ as S-Lie algebras for some faithfully flat and finitely presented extension S/R. The centroid $\mathrm{Ctd}_k(\mathcal{L})$ of the k-Lie algebra $\mathcal{L}$ (see Section 2 for all relevant definitions) can be naturally identified with R. Every derivation δ of the k-Lie algebra $\mathcal{L}$ induces a derivation $\eta_{\mathcal{L}}(\delta)$ of its centroid, hence of R. This yields a k-Lie algebra homorphism $\eta_{\mathcal{L}} : \mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k(R)$ with the property

$$[\delta_1, r\delta_2] = r[\delta_1, \delta_2] + \eta_{\mathcal{L}}(\delta_1)(r)\delta_2 \tag{2}$$

for all $\delta_1, \delta_2 \in \mathrm{Der}_k(\mathcal{L})$ and $r \in R$.

It is inevitable to try to reconcile this last equation with (1). This can all be done in a natural way if R is *smooth* over k. Indeed $X = \mathrm{Spec}(R)$ plays the role of M while $\mathrm{Der}_k(R)$ is thought of as the sections of TM. By descent theory the R-module $\mathrm{Der}_k(\mathcal{L})$ is projective of finite type,[4] and therefore corresponds to the sections of an (algebraic) vector bundle $\mathcal{E}$ over X. The Lie algebra structure on $\Gamma(\mathcal{E}) = \mathrm{Der}_k(\mathcal{L})$ is the natural one (commutator of derivations).

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Notation Throughout this work k will denote a field of characteristic 0 and k-alg the category of associative commutative and unital k-algebras. If R is an object of k-alg the R-module of Kähler differentials of the k-algebra R and its universal derivation will be denoted by $d_{R/k} : R \to \Omega_{R/k}$.

[4] Here is where the smoothness of R/k is crucial.

We will denote by $\mathfrak{g}$ a finite dimensional split simple Lie algebra over k. The algebraic k-group of automorphisms of $\mathfrak{g}$ will be denoted by $\mathbf{Aut}(\mathfrak{g})$.

2. Lie algebras. Centroids and twisted forms

Fix an object R of k-alg. Let $\mathcal{L}$ be a Lie algebra over R (hence over k) with the property that $\mathcal{L} \otimes_R R' \simeq \mathfrak{g} \otimes_k R'$ as R'-Lie algebras for some faithfully flat and finitely presented extension R'/R. Such algebras are called *twisted forms* of $\mathfrak{g} \otimes_k R$. To single out a particular R' we speak of *twisted forms split by* R'. To $\mathcal{L}$ corresponds a torsor over R under $\mathbf{Aut}(\mathfrak{g})$, namely the affine R-scheme $\mathbf{Iso}_{R\text{-}Lie}(\mathcal{L}, \mathfrak{g} \otimes_k R)$, and the set of isomorphism classes of R-Lie algebras which are twisted forms of $\mathfrak{g} \otimes_k R$ is parametrized by the pointed set $H^1\big(R, \mathbf{Aut}(\mathfrak{g})\big)$. Details about torsors and non-abelian cohomology can be found in [4,8] and [12].[5]

Recall that the *centroid* $\mathrm{Ctd}_R(\mathcal{L})$ of the R-Lie algebra $\mathcal{L}$ consists of the endomorphisms of the R-module $\mathcal{L}$ that commute with left and right multiplication by elements of $\mathcal{L}$. That is,

$$\mathrm{Ctd}_R(\mathcal{L}) = \{\chi \in \mathrm{End}_R(\mathcal{L}) : \chi[x, y] = [\chi(x), y] = [x, \chi(y)]\, \forall x, y \in \mathcal{L}\}$$

for all $x, y \in \mathcal{L}$.[6] The centroid is a subalgebra of the (associative and unital) R-algebra $\mathrm{End}_R(\mathcal{L})$. For each $r \in R$ the homothety $\chi_r : x \mapsto rx$ belongs to $\mathrm{Ctd}_R(\mathcal{L})$. This yields an R-algebra homomorphism

$$\chi_{\mathcal{L},R} : R \to \mathrm{Ctd}_R(\mathcal{L}). \tag{3}$$

Recall that $\mathcal{L}$ is called *central* if the map $\chi_{\mathcal{L},R}$ is an isomorphism, and *perfect* if $\mathcal{L}$ is spanned as a k-module (in fact as an abelian group) by the set $\{[x, y] : x, y \in \mathcal{L}\}$.

By restriction of scalars we can view $\mathcal{L}$ also as a k-Lie algebra. At the centroid level, this yields the natural inclusion

$$\mathrm{Ctd}_R(\mathcal{L}) \subset \mathrm{Ctd}_k(\mathcal{L}). \tag{4}$$

Perfectness, on the other hand, is independent of whether we view $\mathcal{L}$ as an algebra over R or k.

For convenience we recall the following simple yet useful facts (see [6, lemma 4.6] or, more generally, [10, lemma 3.4] for details).

Lemma 2.1. *Let $\mathcal{L}$ be a twisted form of $\mathfrak{g} \otimes_k R$. Then*

i) *$\mathcal{L}$ is a projective R-module of finite type (hence finitely presented). Its rank is constant and it equals* $\dim_k(\mathfrak{g})$.

[5] Strictly speaking these are torsors under the R-group scheme $\mathbf{Aut}(\mathfrak{g})_R$. The abuse of notation is harmless. The relevant non-abelian H^1 are to be computed in the fppf topology. Since $\mathbf{Aut}(\mathfrak{g})$ is smooth the fppf topology can be replaced by the étale topology.

[6] The last equality is redundant because of the skew-symmetry of the Lie bracket. We kept it in the definition to remind the reader of the correct concept of centroid for arbitrary algebras.

ii) $\mathcal{L}$ *is perfect, the centroid* $\mathrm{Ctd}_R(\mathcal{L})$ *is commutative and the inclusion* $\mathrm{Ctd}_R(\mathcal{L}) \subset \mathrm{Ctd}_k(\mathcal{L})$ *is an equality.*

iii) $\mathcal{L}$ *is central. That is, the canonical map* $\chi_{\mathcal{L},R} : R \to \mathrm{Ctd}_R(\mathcal{L})$ *is an isomorphism of* k-alg. □

In what follows we will denote $\mathrm{Ctd}_R(\mathcal{L}) = \mathrm{Ctd}_k(\mathcal{L})$ simply by $\mathrm{Ctd}(\mathcal{L})$. We will also identify R with $\mathrm{Ctd}(\mathcal{L})$ via $r \mapsto \chi_r$.

It is straightforward to verify that if $\delta \in \mathrm{Der}_k(\mathcal{L})$ and $\chi \in \mathrm{Ctd}(\mathcal{L})$, then the commutator $[\delta, \chi] = \delta \circ \chi - \chi \circ \delta$ is an element of $\mathrm{Ctd}(\mathcal{L})$. This yields a natural k-Lie algebra homomorphism $\eta_{\mathcal{L}} : \mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k\big(\mathrm{Ctd}_k(\mathcal{L})\big)$ given by

$$\eta_{\mathcal{L}}(\delta)(\chi) = [\delta, \chi] = \delta \circ \chi - \chi \circ \delta \tag{5}$$

for all $\delta \in \mathrm{Der}_k(\mathcal{L})$ and $\chi \in \mathrm{Ctd}_k(\mathcal{L})$. Under our identification $R = \mathrm{Ctd}(\mathcal{L})$ this corresponds to the Lie algebra homomorphism (also denoted $\eta_{\mathcal{L}}$)

$$\eta_{\mathcal{L}} : \ \mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k(R) \tag{6}$$

given by

$$\eta_{\mathcal{L}}(\delta)(r) = t \iff [\delta, \chi_r] = \chi_t \text{ for all } r, t \in R. \tag{7}$$

The derivation $r\delta$ is nothing but $\chi_r \circ \delta$. From (5) and (6) we get the fundamental equation

$$[\delta_1, r\delta_2] = r[\delta_1, \delta_2] + \eta_{\mathcal{L}}(\delta_1)(r)\delta_2. \tag{8}$$

Note that the kernel of $\eta_{\mathcal{L}}$ consists of the derivations that commute with all χ_r. This is to say the elements of $\mathrm{Der}_R(\mathcal{L})$. The characterization of R-linear derivations of $\mathcal{L}$ is quite simple.

Lemma 2.2. *The adjoint representation* $\mathrm{ad}_{\mathcal{L}}$ *of* $\mathcal{L}$ *induces an* R*-Lie algebra isomorphism* $\mathcal{L} \simeq \mathrm{Der}_R(\mathcal{L})$.

Proof. If $x \in \mathcal{L}$ then the inner derivation $\mathrm{ad}_{\mathcal{L}}(x)$ is obviously R-linear. Thus $\mathrm{IDer}(\mathcal{L}) \subset \mathrm{Der}_R(\mathcal{L})$ where $\mathrm{IDer}(\mathcal{L})$ denotes the R-module of all inner derivations of $\mathcal{L}$. It is shown in [10, Ex. 4.9] that in fact $\mathrm{IDer}(\mathcal{L}) = \mathrm{Der}_R(\mathcal{L})$. To show that $\mathrm{ad}_{\mathcal{L}} : \mathcal{L} \to \mathrm{IDer}(\mathcal{L})$ is an isomorphism we must prove that $\mathcal{L}$ has trivial center. Let R'/R be a faithfully flat extension splitting $\mathcal{L}$. If x is in the center of $\mathcal{L}$ then $x \otimes 1$ is in the center of $\mathcal{L} \otimes_R R' \simeq \mathfrak{g} \otimes_k R'$. Since $\mathfrak{g}$ has trivial center it is easy to see (by using that R' is free as a k-module) that $\mathfrak{g} \otimes_k R'$ has trivial center. Thus $x \otimes 1 = 0$. Since R'/R is faithfully flat we conclude that $x = 0$ as desired. □

Our constructions have at their hearts descent theory. It is therefore important to have a complete understanding of the "split" case, that is when $\mathcal{L} = \mathfrak{g} \otimes_k R$. This is the content of the following.

Example 2.3. The k-Lie algebra homomorphism $\eta_{\mathfrak{g}\otimes_k R} : \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \to \mathrm{Der}_k(R)$ has a natural section: Indeed for $d \in \mathrm{Der}_k(R)$ it is clear that $\mathrm{id}_{\mathfrak{g}} \otimes d$ is an element of $\mathrm{Der}_k(\mathfrak{g} \otimes_k R)$ which is mapped to d under $\eta_{\mathfrak{g}\otimes_k R}$. As a consequence we have following split exact sequence of k-Lie algebras

$$0 \to \mathfrak{g} \otimes_k R \xrightarrow{\mathrm{ad}_{\mathfrak{g}\otimes_k R}} \mathrm{Der}_k(\mathfrak{g} \otimes_k R) \xrightarrow{\eta_{\mathfrak{g}\otimes_k R}} \mathrm{Der}_k(R) \to 0. \tag{9}$$

3. *k*-derivations and centroids of twisted forms

As we have seen the structure of the Lie algebra $\mathrm{Der}_R(\mathcal{L})$ is quite simple: It is isomorphic to $\mathcal{L}$ via the adjoint representation. By restriction of scalars we can view $\mathcal{L}$ as a Lie algebra over k and we next look at the much more delicate nature of $\mathrm{Der}_k(\mathcal{L})$, the set derivation of the k-Lie algebra $\mathcal{L}$. It is obvious that $\mathrm{Der}_k(\mathcal{L})$ has a natural R-module structure (see below).

The naive idea is to define an R-functor on Lie algebras that attaches to a given S/R the Lie algebra $\mathrm{Der}_k(\mathcal{L} \otimes_R S)$ of derivations of the k-Lie algebra $\mathcal{L} \otimes_R S$. But there is an immediate obstacle to this idea because given a morphism of R-algebras $f : S \to T$, there is no natural (or even reasonable) map from $\mathrm{Der}_k(\mathcal{L} \otimes_R S)$ to $\mathrm{Der}_k(\mathcal{L} \otimes_R T)$ that one can attach to f. Remarkably enough, as we will explain below, this functorial construction is possible if we limit ourselves to étale extensions of R (in which case our arrow $S \to T$ would out of necessity be an étale morphism, see [11, Exp. I Cor. 4.8]).

In [7] a theory of differentials for Lie algebras is developed that plays the same role that Kähler differentials play in commutative algebra. We will recall (without proofs) a suitable version of this theory that is sufficient for our present work.

In what follows by an R–$\mathcal{L}$-module we will understand a module of the R-Lie algebra $\mathcal{L}$ (in particular such modules have a natural R-module structure). If M is an $\mathcal{L}$-module we will let $\mathrm{Der}_k(\mathcal{L}, M)$ be the set of k-linear derivations of $\mathcal{L}$ with values in M. These are the k-linear maps $d : \mathcal{L} \to M$ with the property that $d([x, y]) = x.d(y) - y.d(x)$. Note that if M is an R–$\mathcal{L}$-module then $\mathrm{Der}_k(\mathcal{L}, M)$ has a natural R-module structure given by $(rd)(x) = r \cdot d(x)$.

The main result of [7] is the construction of an R–$\mathcal{L}$-module $\Omega_{R,\mathcal{L}/k}$ and a derivation (called universal) $d_{R,\mathcal{L},k} \in \mathrm{Der}_k(\mathcal{L}, \Omega_{R,\mathcal{L}/k})$ with the following property:

If $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/k}, M)$ denotes the R–$\mathcal{L}$-module homomorphisms from $\Omega_{R,\mathcal{L}/k}$ to M (namely the R-linear maps between the two modules that commute with the action of $\mathcal{L}$) we have a bijection

$$\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/k}, M) \to \mathrm{Der}_k(\mathcal{L}, M) \tag{10}$$

given by

$$\alpha \mapsto \alpha \circ d_{R,\mathcal{L},k}. \tag{11}$$

In particular if $\mathcal{L}$ is viewed as an R–$\mathcal{L}$-module via the adjoint representations we have a bijection

$$\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/k}, \mathcal{L}) \to \mathrm{Der}_k(\mathcal{L}). \tag{12}$$

Let S/R be an arbitrary morphism of k-alg. Consider the S-Lie algebra $\mathcal{L} \otimes_R S$ and its corresponding universal derivation $d_{S,\mathcal{L}\otimes_R S,k} : \mathcal{L} \otimes_R S \to \Omega_{S,\mathcal{L}\otimes_R S,k}$. The natural map $\mathcal{L} \to \mathcal{L} \otimes_R S$ followed by $d_{S,\mathcal{L}\otimes_R S,k}$ is an element of $\mathrm{Der}_k(\mathcal{L}, \Omega_{S,\mathcal{L}\otimes_R S/k})$. By (10) we obtain an R–$\mathcal{L}$-module homomorphism

$$\phi_{\mathcal{L}}^{S/R/k} : \Omega_{R,\mathcal{L}/k} \to \Omega_{S,\mathcal{L}\otimes_R S/k}. \tag{13}$$

Applying the base change S/R yields an S-$(\mathcal{L} \otimes_R S)$-module map

$$\phi_{\mathcal{L}}^{S/R/k} \otimes \mathrm{Id} : \Omega_{R,\mathcal{L}/k} \otimes_R S \to \Omega_{S,\mathcal{L}\otimes_R S/k}. \tag{14}$$

Since $\mathcal{L}$ is perfect, lemma 5.3 of [7] states that this last map is an isomorphism whenever S/R is étale. Furthermore, by *loc. cit.* theorem 6.4 and lemma 6.5 the canonical map

$$\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/k}, \mathcal{L}) \otimes_R S \to \mathrm{Hom}_{S,\mathcal{L}\otimes_R S}(\Omega_{R,\mathcal{L}/k} \otimes_R S, \mathcal{L} \otimes_R S) \tag{15}$$

is an S-module isomorphism. Combining (14) and (15) we obtain an S-module isomorphism[7]

$$\psi_{\mathcal{L}}^{S/R/k} : \mathrm{Der}_k(\mathcal{L}) \otimes_R S \to \mathrm{Der}_k(\mathcal{L} \otimes_R S). \tag{16}$$

This last isomorphism is the Lie algebra counterpart of a classical commutative algebra result on Kähler differentials that we now recall for future use.

Let S/R be an extension in k-alg. Assume that R/k is smooth. Then $\Omega_{R/k}$ is a projective R-module of finite type and as a consequence the canonical map $\mathrm{Hom}_R(\Omega_{R/k}, R) \otimes_R S \simeq \mathrm{Hom}_S(\Omega_{R/k} \otimes_R S, S)$ is an isomorphism. If S/R is étale, then the canonical map $\phi^{S/R/k} : \Omega_{R/k} \to \Omega_{S/k}$ induces an S-module isomorphism $\phi^{S/R/k} \otimes \mathrm{Id} : \Omega_{R/k} \otimes_R S \to \Omega_{S/k}$. Combining these two we get an S-module isomorphism

$$\psi^{S/R/k} : \mathrm{Der}_k(R) \otimes_R S = \mathrm{Hom}_R(\Omega_{R/k}, R) \otimes_R S \simeq \mathrm{Hom}_S(\Omega_{S/k}, S) = \mathrm{Der}_k(S). \tag{17}$$

Proposition 3.1. *Let the notation be as above, and suppose R/k is of finite type. Then*

[7] We will later see that this map is in fact a k-Lie algebra isomorphism.

i) *The map* $\eta_{\mathcal{L}} : \mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k(R)$ *gives rise to a split exact sequence of* R*-modules*

$$0 \to \mathcal{L} \xrightarrow{\mathrm{ad}_{\mathcal{L}}} \mathrm{Der}_k(\mathcal{L}) \xrightarrow{\eta_{\mathcal{L}}} \mathrm{Der}_k(R) \to 0. \tag{18}$$

ii) *This sequence is dual to the split exact sequence*

$$0 \to \Omega_{R/k} \otimes_R \mathcal{L} \to \Omega_{R,\mathcal{L}/k} \to \Omega_{R,\mathcal{L}/R} \to 0 \tag{19}$$

of R*–*$\mathcal{L}$*-modules established in corollary 6.2 of [7].*

iii) *If* R/k *is smooth* $\mathrm{Der}_k(\mathcal{L})$ *is a projective* R*-module of finite type.*

Proof. The main assertion is that the first part is the dual of the second, i.e. obtained from the second by applying $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\ \cdot\ ,\mathcal{L})$. First observe, that $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/k},\mathcal{L})$ by definition is $\mathrm{Der}_k(\mathcal{L})$. Similarly, $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R,\mathcal{L}/R},\mathcal{L})$ is by definition $\mathrm{Der}_R(\mathcal{L}) \simeq \mathcal{L}$. This isomorphisms are natural (and functorial in R). Next, $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R/k} \otimes_R \mathcal{L},\mathcal{L}) \simeq \mathrm{Der}_k(R, \mathrm{Ctd}(\mathcal{L}))$, by combining propositions 4.4, 4.6, and 4.8, and theorem 6.1 of [7], whenever $\mathcal{L}$ is a form of $\mathfrak{g} \otimes_k R$.[8] Indeed, as $\mathcal{L}$ is perfect, proposition 4.8 and lemma 4.5 loc. cit. establish that the bi-module derivations $R \to \mathrm{Ctd}(\mathcal{L})$ are canonically isomorphic to $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R/k} \otimes_R \mathcal{L},\mathcal{L})$. Again, as $\mathcal{L}$ is perfect, $\mathrm{Ctd}(\mathcal{L})$ is the trivial bi-module (that is, the left- and right-actions coincide), so the bi-module derivations are just regular derivations, and since in our situation R can be canonically identified with $\mathrm{Ctd}(\mathcal{L})$ by Lemma 2.1(iii), we have

$$\mathrm{Hom}_{R\text{–}\mathcal{L}}(\Omega_{R/k} \otimes_R \mathcal{L},\mathcal{L}) \simeq \mathrm{Der}_k(R). \tag{20}$$

Now recall from loc. cit. theorem 6.1 and its proof that the natural map $\Omega_{R/k} \otimes_R \mathcal{L} \to \Omega_{R,\mathcal{L}/k}$ is defined by $d_{R,k}r \otimes x \mapsto d_{R,\mathcal{L},k}(rx) - rd_{R,\mathcal{L},k}(x)$. Let $\alpha\colon \Omega_{R,\mathcal{L}/k} \to \mathcal{L}$ be an R–$\mathcal{L}$-homomorphism. Then $\delta := \alpha \circ d_{R,\mathcal{L},k}$ is the corresponding derivation. Note that α restricted to $\Omega_{R/k} \otimes_R \mathcal{L}$ is the map

$$d_{R,k}r \otimes x \mapsto \delta(rx) - r\delta(x)).$$

This is the map $d_{R,k}r\otimes x \mapsto \eta_{\mathcal{L}}(\delta)(\chi_r)(x)$. Under the identification of $\mathrm{Hom}_{R\text{-}\mathcal{L}}(\Omega_{R/k,\mathcal{L}}\otimes_R \mathcal{L},\mathcal{L})$ with the derivations of R described in (20) this becomes the map $r \mapsto \eta_{\mathcal{L}}(\delta)(r)$. In other words, the map $\mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k(R)$ obtained from applying $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\ \cdot\ ,\mathcal{L})$ to the exact sequence (19) is $\eta_{\mathcal{L}}$.

Now as $\mathcal{L}$ is a form of a perfect $\mathfrak{g}$, (19) splits as a sequence of R–$\mathcal{L}$-modules, by theorem 6.4 of [7]. Thus, as R–$\mathcal{L}$-modules, $\Omega_{R,\mathcal{L}/k} \simeq \Omega_{R/k} \otimes_R \mathcal{L} \oplus \Omega_{R,\mathcal{L}/R}$. It then follows that

$$\mathrm{Der}_k(\mathcal{L}) \simeq \mathrm{Der}_k(R) \oplus \mathrm{Der}_R(\mathcal{L}) \simeq \mathrm{Der}_k(R) \oplus \mathcal{L}$$

[8] The only assumption on $\mathfrak{g}$ needed for this result is that it be perfect.

and the projection along this direct sum decomposition is $\eta_{\mathcal{L}}$. In particular, $\eta_{\mathcal{L}}$ is surjective.

From this it follows that as an R-module $\mathrm{Der}_k(\mathcal{L})$ is isomorphic to $\mathcal{L} \oplus \Omega^*_{R/k}$. If R/k is smooth, then $\Omega^*_{R/k}$ is projective of finite type. So is $\mathcal{L}$. This completes the proof of the Proposition. □

Lemma 3.2. *Assume S/R is étale. Then the diagram of S-modules*

$$\begin{tikzcd} \mathrm{Der}_k(\mathcal{L}) \otimes_R S \arrow[r, "(\eta_{\mathcal{L}}\otimes \mathrm{Id})"] \arrow[d, "\psi_{\mathcal{L}}^{S/R/k}"'] & \mathrm{Der}_k(R) \otimes_R S \arrow[d, "\psi^{S/R/k}"] \\ \mathrm{Der}_k(\mathcal{L} \otimes_R S) \arrow[r, "\eta_{\mathcal{L}\otimes_R S}"'] & \mathrm{Der}_k(S) \end{tikzcd} \tag{21}$$

commutes.

Proof. The vertical maps are functorial in S as they are applications of the universal property of the modules of differentials together with the base change for étale morphisms and the discussion above regarding (15) and (16).

By the previous proposition, the bottom horizontal map is obtained by applying $\mathrm{Hom}_{S\text{-}\mathcal{L}_S}(\ \cdot\ , \mathcal{L}_S)$ to the first part of (19) (in case $R = S$ and $\mathcal{L} = \mathcal{L}_S$). By lemmas 6.5 and 6.6 of [7] applying $\otimes_R S$ and taking $\mathrm{Hom}_{S\text{-}\mathcal{L}_S}(\ \cdot\ , \mathcal{L}_S)$ to

$$0 \to \Omega_{R/k} \otimes_R \mathcal{L} \to \Omega_{R,\mathcal{L}/k} \tag{22}$$

is the same as applying $\mathrm{Hom}_{R\text{–}\mathcal{L}}(\ \cdot\ , \mathcal{L}) \otimes_R S$, and the isomorphisms between the two sequences are given by $\psi_{\mathcal{L}}^{S/R/k}$ and $\psi_{\mathcal{L}}^{S/R/k}$. Moreover, applying the base change S/R to (22) and using the isomorphisms $\phi_{\mathcal{L}}^{S/R/k}$ (13) and $\phi^{S/R/k}$ results in the natural map

$$0 \to \Omega_{S/k} \otimes_S \mathcal{L}_S \to \Omega_{S,\mathcal{L}_S/k}.$$

Thus, the diagram commutes. □

4. $\mathfrak{X}/k$-schemes on Lie algebras

Let $\mathfrak{X}$ be a k-scheme. In what follows $\mathfrak{X}_{\text{ét}}$ will denote the small étale site of $\mathfrak{X}$. Recall that the objects are étale scheme morphisms $\mathfrak{Y} \to \mathfrak{X}$, and that the morphism between objects are just scheme morphism over $\mathfrak{X}$.[9]

Definition 4.1. By an $\mathfrak{X}/k$-scheme on Lie algebras in the étale sense we will understand the following:

[9] We remind the reader that any morphism between objects $\mathfrak{Y}$ and $\mathfrak{Z}$ of $\mathfrak{X}_{\text{ét}}$ is necessarily an étale morphism of schemes.

i) A scheme $\mathfrak{L}$ over $\mathfrak{X}$.[10]
ii) A k-Lie algebra structure $[-,-]_{\mathfrak{L}}$ on $\mathfrak{L}$. That is, a k-Lie algebra structure on each of the sets $\mathfrak{L}(\mathfrak{Y})$ for $\mathfrak{Y}/\mathfrak{X}$ étale.
iii) If $f:\mathfrak{Y}\to\mathfrak{Z}$ is a morphisms of étale extensions of $\mathfrak{X}$, the map $\mathfrak{L}(f):\mathfrak{L}(\mathfrak{Z})\to\mathfrak{L}(\mathfrak{Y})$ is a k-Lie algebra homomorphism.

We leave to the reader to generalize this concept at the level of sheaves (i.e., without the assumption that $\mathfrak{L}$ is a scheme), or Grothendieck topologies other than the étale. We will make no use of such generalization, nor do we know of any interesting examples. Unless specific mention to the contrary in what follows an $\mathfrak{X}/k$-scheme of Lie algebras will always be assumed to be *in the étale sense.*

4.1. Relevant examples of $\mathfrak{X}/k$-scheme on Lie algebras

All of the examples of $\mathfrak{X}/k$-schemes on Lie algebras that are of interest to us are when the base scheme $\mathfrak{X}=\operatorname{Spec}(R)$ is affine (we then talk about R/k-schemes on Lie algebras). We will henceforth give the relevant proofs in all detail under this assumption. We will in Remark 4.9 outline the constructions for $\mathfrak{X}$ arbitrary. As it is often the case this is done by reducing to the affine case (in which the proofs, as mentioned, will be given in full detail).

If M is an R-module we will denote by $\widetilde{M}$ the corresponding quasicoherent $\mathcal{O}_{\mathfrak{X}}$-module. If $\mathfrak{Y}$ is a scheme over $\mathfrak{X}$, we will denote the inverse image over $\mathfrak{Y}$ of $\widetilde{M}$ by $\widetilde{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{Y}}$.

Attached to M we have the corresponding "vector bundle" $\mathbb{V}(\widetilde{M})$.[11] Recall that by definition for any ring extension $\mathfrak{Y}/\mathfrak{X}$

$$\mathbb{V}(\widetilde{M})(\mathfrak{Y})=\operatorname{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(\widetilde{M}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{Y}},\mathcal{O}_{\mathfrak{Y}}). \tag{23}$$

In what follows we identify $\mathbb{V}(\widetilde{M})$ with the affine group scheme represented by the symmetric R-algebra $S(M)$ of M.

If $\mathfrak{Y}=\operatorname{Spec}(S)$ then

$$\mathbb{V}(\widetilde{M})(\mathfrak{Y})=\operatorname{Hom}_S(M\otimes_R S,S) \tag{24}$$

viewed as an S-module. By patching these we obtain the $\mathcal{O}_{\mathfrak{Y}}$-module structure on an arbitrary $\mathbb{V}(\widetilde{M})(\mathfrak{Y})$.

Vector bundles can be defined in a dual fashion. Let N be an R-module and let $\widetilde{N}$ be the corresponding quasicoherent $\mathcal{O}_{\mathfrak{X}}$-module. We have an $\mathfrak{X}$-group functor $\mathbb{W}(\widetilde{N})$ that assigns to $\mathfrak{Y}/\mathfrak{X}$ the abelian group of sections of the $\mathcal{O}_{\mathfrak{Y}}$-module $\widetilde{N}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{Y}}$. In symbols

$$\mathbb{W}(\widetilde{N})(\mathfrak{Y})=\Gamma(\mathfrak{Y},\widetilde{N}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{O}_{\mathfrak{Y}}). \tag{25}$$

[10] Note that $\mathfrak{L}/\mathfrak{X}$ is not assumed to be étale.
[11] "Fibration vectorielle définie par $\widetilde{M}$" in the terminology of [12].

If $\mathfrak{Y} = \mathrm{Spec}(S)$ then

$$\mathbb{W}(\widetilde{N})(\mathfrak{Y}) = N \otimes_R S. \tag{26}$$

It is clear that $\mathbb{W}(\widetilde{N})$ is always a sheaf in the fppf sense.

We recall that if N is projective of finite type, then $\mathbb{W}(\widetilde{N})$ is an affine scheme isomorphic to $\mathrm{Spec}(S(N^*))$ where $S(N^*)$ is the symmetric algebra of the dual R-module N^* of N. In particular if M is projective of finite type, then $\mathbb{V}(\widetilde{M}) \simeq \mathbb{W}(\widetilde{M^*})$ by means of the canonical isomorphism of M with its double dual. We will identify without any further reference

$$\mathbb{V}(\widetilde{M}) = \mathbb{W}(\widetilde{M^*}). \tag{27}$$

All of the above is based on the fact that

$$\mathbb{V}(\widetilde{M})(S) \simeq \mathrm{Hom}_{S\text{-}mod}(M \otimes_R S, S) \simeq \mathrm{Hom}(M, R) \otimes_R S = \mathbb{W}(\widetilde{M^*})(S).$$

For this reason we will think of $\mathbb{W}(\widetilde{M^*})$ as the scheme of sections of the R-scheme $\mathfrak{Y} \mapsto \mathbb{V}(\widetilde{M})(\mathfrak{Y})$ of section of $\mathbb{V}(\widetilde{M})$. This will be relevant in our definition of Lie algebroid as we try to preserve the analogy with the differential setup outlined in the Introduction.

Remark 4.2. Assume that $\mathfrak{Y}/\mathfrak{X}$ is étale. Since $\mathbb{V}(-)$ commutes with base change we have

$$\mathbb{V}(\Omega_{\mathfrak{X}/k})(\mathfrak{Y}) = \mathbb{V}(\Omega_{\mathfrak{X}/k})_{\mathfrak{Y}}(\mathfrak{Y}) = \mathbb{V}(\Omega_{\mathfrak{Y}/k})(\mathfrak{Y}) = \mathrm{Hom}_{\mathcal{O}_{\mathfrak{Y}}}(\Omega_{\mathfrak{Y}/k}, \mathcal{O}_{\mathfrak{Y}}).$$

By considering the universal derivation $\delta_{\mathfrak{Y}/k} : \mathcal{O}_{\mathfrak{Y}} \to \Omega_{\mathfrak{Y}/k}$ we obtain a natural map

$$\mathbb{V}(\Omega_{\mathfrak{X}/k})(\mathfrak{Y}) \to \mathrm{Der}_k(\mathcal{O}_{\mathfrak{Y}}).$$

Evaluating this last at $\mathfrak{Y}$ and identifying $\mathbb{V}(\Omega_{\mathfrak{X}/k}) = \mathbb{W}(\Omega^*_{\mathfrak{X}/k})$ yields a natural map

$$\mathbb{W}(\Omega^*_{\mathfrak{X}/k})(\mathfrak{Y}) \to \mathrm{Der}_k\big(\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})\big).$$

***

Let $\mathfrak{g}$ and $\mathcal{L}$ be as above. We henceforth assume that R/k is smooth (in particular of finite type). We will give now a list of $\mathfrak{X}/k$-schemes of Lie algebras that are relevant to the definition of Lie algebroids.

Proposition 4.3. *There exists an of R/k-scheme on Lie algebras $\mathbf{Der}_k(\mathcal{L})$ in the étale sense such that $\mathbf{Der}_k(\mathcal{L})(S) = \mathrm{Der}_k(\mathcal{L} \otimes_R S)$ for all S/R étale.*

Proof. From Proposition 3.1 (ii) it follows that the R-functor $\mathbb{W}(\widetilde{\mathrm{Der}_k(\mathcal{L})})$ is in fact an affine R-scheme (it is represented by the symmetric algebra of the dual $\mathrm{Der}_k(\mathcal{L})^*$ of the R-module $\mathrm{Der}_k(\mathcal{L})$). We will denote this scheme by $\mathbf{Der}_k(\mathcal{L})$.

Assume that S/R is étale. Since $\mathbb{W}(\widetilde{\mathrm{Der}}_k(\mathcal{L}))(S) \simeq \mathrm{Der}_k(\mathcal{L} \otimes_R S)$ by (16), each of the S-modules $\mathbb{W}(\widetilde{\mathrm{Der}}_k(\mathcal{L}))(S)$ has a natural k-Lie algebra structure. We now address the functorial k-Lie algebra nature of this construction.

Let $f : S \to T$ be a morphism of étale extensions of R. Consider the diagram

$$\begin{tikzcd}
\mathrm{Der}_k(\mathcal{L}) \otimes_R S \arrow[r, "(\mathrm{Id} \otimes f)"] \arrow[d, "\psi_{\mathcal{L}}^{S/R/k}"'] & \mathrm{Der}_k(\mathcal{L}) \otimes_R T \arrow[d, "\psi_{\mathcal{L}}^{S/R/k}"] \\
\mathrm{Der}_k(\mathcal{L} \otimes_R S) \arrow[r] & \mathrm{Der}_k(\mathcal{L} \otimes_R T)
\end{tikzcd} \tag{28}$$

Both vertical maps are isomorphisms, and the top horizontal map is a k-Lie algebra homomorphism. All these maps are functorial in nature. The bottom arrow is hence a well-defined map of S-modules

$$\mathbf{Der}_k(\mathcal{L})(f) : \mathrm{Der}_k(\mathcal{L} \otimes_R S) \to \mathrm{Der}_k(\mathcal{L} \otimes_R T) \tag{29}$$

which is functorial.

We recall a particular case of Cor. 5.4 of [7].

Lemma 4.4. *Let $S \in k$-alg, and let $\mathcal{M}$ be a perfect Lie algebra over S. For any étale extension T/S the canonical map*

$$\mathrm{Der}_k(\mathcal{M} \otimes_S T) \to \mathrm{Der}_k(\mathcal{M}, \mathcal{M} \otimes_S T) \tag{30}$$

is an S-module isomorphism. $\square$

Of course in the above $\mathcal{M} \otimes_S T$ is viewed as an $\mathcal{M}$-S-module via the adjoint representation.

Consider now our étale extension $f : S \to T$. Given $d \in \mathrm{Der}_k(\mathcal{M})$ define $d^T \in \mathrm{Der}_k(\mathcal{M}, \mathcal{M} \otimes_S T)$ by $d^T(x) = d(x) \otimes 1$ for all $x \in \mathcal{M}$. By the previous Lemma d^T extends to a unique derivation $d_T \in \mathrm{Der}_k(\mathcal{M} \otimes_S T)$. Recall that by the meaning of "extension"

$$d_T(x \otimes 1) = d^T(x) = d(x) \otimes 1 \tag{31}$$

This procedure defines an S-linear map $D(f) : \mathrm{Der}_k(\mathcal{M}) \to \mathrm{Der}_k(\mathcal{M} \otimes_S T)$ given by $d \mapsto d_T$. We leave it to the reader to verify that $D(f) = \mathbf{Der}_k(\mathcal{L})(f)$.

It remains to show that each of the maps $D(f)$ is a k-Lie algebra homomorphism. This follows from (31). Indeed if $d, d' \in \mathrm{Der}_k(\mathcal{M})$, then from the definitions one easily sees that $[d, d']_T$ and $[d_T, d'_T]$ are two elements of $\mathrm{Der}_k(\mathcal{M} \otimes_S T)$ that have the same restriction to $\mathcal{M} \otimes 1$. By the uniqueness of the extension the two elements coincide.[12] This shows that our map $d \mapsto d_T$ is a k-Lie algebra homomorphism. In particular, the vertical maps (and consequently the bottom horizontal map) in (28) are k-Lie algebra homomorphisms.

The above provide all the ingredients to finish the proof of the R/k scheme on Lie algebras structure (in the étale sense) on $\mathbf{Der}_k(\mathcal{L})$. Indeed given an étale extension $f : S \to T$ of R, we set $\mathcal{M} = \mathcal{L} \otimes_R S$ and define

$$D(f) : \mathbf{Der}_k(\mathcal{L})(S) = \mathrm{Der}_k(\mathcal{M}) \to \mathrm{Der}_k(\mathcal{M} \otimes_S T) = \mathbf{Der}_k(\mathcal{L})(T)$$

with the aid of the canonical isomorphism $\mathcal{L} \otimes_R S \otimes_S T \simeq \mathcal{L} \otimes_R T$.

Let now $\mathfrak{Y}$ be a scheme étale over $\mathfrak{X}$, but not necessarily affine. Then $\mathbf{Der}_k(\mathcal{L})(\mathfrak{Y})$ still carries a natural structure of Lie algebra over k. Indeed, let $\coprod_i \mathrm{Spec}(S_i) \to \mathfrak{Y}$ be an open affine cover of $\mathfrak{Y}$, and let $f, g \colon \mathfrak{Y} \to \mathbf{Der}_k(\mathcal{L})$ be two morphisms. Let f_i, g_i be the respective restrictions to $\mathrm{Spec}(S_i)$. Then S_i/R is étale for all i, and by the above $[f_i, g_i] \colon \mathrm{Spec}(S_i) \to \mathbf{Der}_k(\mathcal{L})$ is well-defined. Moreover, if $\mathrm{Spec}(S_{ij}) \subset \mathrm{Spec}(S_i) \cap \mathrm{Spec}(S_j)$ is an open affine subset of the intersection, then

$$[f_i, g_i]|_{\mathrm{Spec}(S_{ij})} = [f_j, g_j]|_{\mathrm{Spec}(S_{ij})}$$

by (28) and the fact that $f_i = f_j$ (rest. $g_i = g_j$) on $\mathrm{Spec}(S_{ij})$. As $\mathbf{Der}_k(\mathcal{L})$ defines a sheaf on the étale site, there is a unique global morphism $[f, g] \colon \mathfrak{Y} \to \mathbf{Der}_k(\mathcal{L})$ restricting to $[f_i, g_i]$ on $\mathrm{Spec}(S_i)$. That this so defined bracket is a Lie-algebra structure follows by similar arguments relying on the fact that the elements of $\mathbf{Der}_k(\mathcal{L})(\mathfrak{Y})$ are determined by there restrictions to affine subsets. Finally, by the same token, if $f \colon \mathfrak{Y} \to \mathfrak{Z}$ is a morphism in the étale site over $\mathfrak{X}$, the induced morphism $\mathbf{Der}_k(\mathcal{L})(f) \colon \mathbf{Der}_k(\mathcal{L})(\mathfrak{Y}) \to \mathbf{Der}_k(\mathcal{L})(\mathfrak{Z})$ is a k-Lie algebra homomorphism, because applying $\mathbf{Der}_k(\mathcal{L})$ commutes with localizing to affine open subsets. □

The associative analogue of the above is:

Proposition 4.5. *There exists an R/k-scheme on Lie algebras $\mathbf{Der}_k(R)$ in the étale sense such that $\mathbf{Der}_k(R)(S) = \mathrm{Der}_k(S)$ whenever S/R is étale.*

Proof. Apply (17) to the projective R-module of finite type $\Omega_{R/k}$. The scheme under consideration is $\mathbf{Der}_k(R) = \mathbb{W}(\widetilde{\Omega^*_{R/k}})$. □

Proposition 4.6. *$\mathbb{W}(\widetilde{\mathcal{L}})$ has a natural structure of R/k-scheme on Lie algebras in the fppf sense. If S/R is any ring extension in k-alg, then $\mathbb{W}(\widetilde{\mathcal{L}})(S) = \mathcal{L} \otimes_R S$ viewed as a Lie algebra over k by restriction of scalars.*

[12] One cannot conclude anything about the original derivations. For example it is possible that $[d, d'] \neq 0$ but $[d, d']_T = 0$.

Proof. By Lemma 2.1(i) $\mathcal{L}$ is a projective R-module of finite type. The Proposition now follows from the definition of $\mathbb{W}(\widetilde{\mathcal{L}})$. $\square$

In all of the above examples the $\mathfrak{X}/k$-scheme on Lie algebras have a natural $\mathcal{O}_{\mathfrak{X}}$-module structure in the sense of [12, Exp. I]. The following example does not.[13]

Example 4.7. Constant $\mathfrak{X}/k$-schemes on Lie algebras. Let $\mathfrak{l}$ be a Lie algebra over k. The corresponding constant scheme $\mathfrak{l}_{\mathfrak{X}}$ has a natural $\mathfrak{X}/k$-scheme on Lie algebra structure (for any Grothendieck topology).

Remark 4.8. The concept of morphisms of R/k-schemes on Lie algebras is the obvious one. An important example of a morphism of R/k-schemes on Lie algebras is the functor

$$\eta_{\mathcal{L}} : \mathbf{Der}_k(\mathcal{L}) \to \mathbf{Der}_k(R)$$

defined by means of Lemma 3.2.

What is the connection between $\mathbf{Der}_k(\mathcal{L})$ and $\mathbb{W}(\widetilde{\mathcal{L}})$? By the very definition, for each étale extension S/R,

$$\mathbf{Der}_k(\mathcal{L})(S) = \mathrm{Der}_k(\mathcal{L} \otimes_R S).$$

Thus, $\mathbf{Der}_k(\mathcal{L})$ is the sheaf associated to the pre-sheaf $\mathfrak{Y} \mapsto \mathrm{Der}_k(\mathbb{W}(\widetilde{\mathcal{L}})(\mathfrak{Y}))$. Here by pre-sheaf we mean a contra-variant functor on the affine members of the étale site. For general $\mathfrak{Y}$ étale over $\mathfrak{X}$, we still obtain a map $\mathbf{Der}_k(\widetilde{\mathcal{L}})(\mathfrak{Y}) \to \mathrm{Der}_k(\mathbb{W}(\widetilde{\mathcal{L}})(\mathfrak{Y}))$, but this need not be an isomorphism.

Remark 4.9. Let $\mathfrak{X}$ be a scheme over k, locally of finite type, and let $\mathcal{L}$ be a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-Lie algebras on $\mathfrak{X}$. To be precise, $\mathcal{L}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$ modules, such that $\mathcal{L}(U)$ is a $\mathcal{O}_{\mathfrak{X}}(U)$-Lie algebra, compatible with restriction maps. So if $U \subset \mathfrak{X}$ is open affine, then $\mathcal{L}|_U \simeq \mathbb{W}(\tilde{L})$ where L is a Lie-algebra over $\mathcal{O}_{\mathfrak{X}}(U)$, and the Lie-algebra structure is the one coming from L. Let $\mathbf{Hom}_k(\mathcal{L}, \mathcal{L})$ be the sheaf of k-linear sheaf-homomorphisms $\mathcal{L} \to \mathcal{L}$. Then $\mathbf{Der}_k(\mathcal{L})$ is the subsheaf of $\mathbf{Hom}_k(\mathcal{L}, \mathcal{L})$ defined as

$$\mathbf{Der}_k(\mathcal{L})(U) = \{\delta \in \mathrm{Hom}_k(\mathcal{L}|_U, \mathcal{L}|_U) \mid \delta \text{ is a derivation}\}.$$

Suppose $\mathcal{L}$ is perfect, that is, $\mathcal{L}(U)$ is a perfect Lie algebra for every $U \subset \mathfrak{X}$ open affine.[14] Then $\mathbf{Der}_k(\mathcal{L})$ is quasi-coherent. Indeed, fix some open affine subset $U \subset \mathfrak{X}$. Let $\mathcal{D}$ be the sheaf associated to the $\mathcal{O}_{\mathfrak{X}}(U)$-module $\mathrm{Der}_k(\mathcal{L}(U))$. To define a morphism $\mathcal{D} \to \mathbf{Hom}_k(\mathcal{L}, \mathcal{L})|_U$ is equivalent to defining a $\mathcal{O}_{\mathfrak{X}}(U)$-linear map $\mathrm{Der}_k(\mathcal{L}(U)) \to$

13 This will not be used in what follows. It is only given for illustrative purposes.

14 If $\mathcal{L}$ is a quasi-coherent sheaf of Lie-algebras, one may define $[\mathcal{L}, \mathcal{L}]$ as the sheaf associated to the presheaf $U \mapsto [\mathcal{L}(U), \mathcal{L}(U)]$. The condition then precisely states that $\mathcal{L} = [\mathcal{L}, \mathcal{L}]$.

$\mathrm{Hom}_k(\mathcal{L}|_U, \mathcal{L}|_U)$. Let $\delta \in \mathrm{Der}_k(\mathcal{L}(U))$. Then for every $V \subset U$ affine, δ defines a unique k-derivation $\delta_V : \mathcal{L}(V) \to \mathcal{L}(V)$ by the base-change formula (16). Covering an arbitrary open subset $V \subset U$ by open affine subsets V_i, the δ_{V_i} glue to give a unique derivation $\delta_V : \mathcal{L}(V) \to \mathcal{L}(V)$ and this does not depend on the choice of V_i (again using (16)). This results in a homomorphism of Lie-algebras over k, $\mathcal{D} \to \mathbf{Hom}_k(\mathcal{L}, \mathcal{L})|_U$. By (16), it is also clear, that this is injective. It clearly has image contained in $\mathbf{Der}_k(\mathcal{L})|_U$. If on the other hand, $V \subset U$ is affine, and $\delta : \mathcal{L}|_V \to \mathcal{L}|_V$ is any derivation, then the fact that (16) is an isomorphism shows that for any affine open $Z \subset V$, δ_Z must be the derivation induced by δ_V according to (16). But that means δ is the image of δ_V (as an element of $\mathcal{D}(V)$) in $\mathrm{Hom}_k(\mathcal{L}, \mathcal{L})|_V$. It follows that $\mathcal{D}$ and $\mathbf{Der}_k(\mathcal{L})$ are isomorphic over U, and $\mathbf{Der}_k(\mathcal{L})$ is quasi-coherent.

Finally, suppose $\mathfrak{X}$ is smooth over k, and that $\mathcal{L}$ is a form of a perfect finite dimensional Lie algebra $\mathfrak{g}$ over k (which means there is an étale cover $f : \mathfrak{Y} \to \mathfrak{X}$ such that $\mathcal{L} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{Y} \simeq \mathfrak{g} \otimes_k \mathcal{O}_\mathfrak{Y}$). Then $\mathcal{L}$ is locally free (as a sheaf of $\mathcal{O}_\mathfrak{X}$-modules) of finite type, and so is $\mathbf{Der}_k(\mathcal{L})$. In particular, $\mathbb{W}(\mathbf{Der}_k(\mathcal{L}))$ is a scheme, isomorphic to $\mathbb{V}(\mathbf{Der}_k(\mathcal{L})^*)$.

To establish that the above reasoning results in a scheme on Lie algebras, it remains to show that the Lie algebra structure on $\mathbf{Der}_k(\mathcal{L})$ transfers to the étale site over $\mathfrak{X}$. Now note that by general principles, $\mathbf{Der}_k(\mathcal{L}) \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{Y}$ is quasi-coherent, locally isomorphic to the sheaf associated to a module of the form $\mathrm{Der}_k(\mathcal{L}(U)) \otimes_S S' = \mathrm{Der}_k(\mathcal{L}(U) \otimes_S S')$, where $\mathcal{O}_\mathfrak{X}(U) = S$ and S' is the coordinate ring of an open affine subset V of $f^{-1}(U)$. Over V, it then follows that $\mathbf{Der}_k(\mathcal{L}) \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{Y}|_V$ embeds into $\mathbf{Der}_k(\mathcal{L}_\mathfrak{Y})|_V$, where $\mathcal{L}_\mathfrak{Y} = \mathcal{L} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{Y}$. These embeddings glue (thanks again to (16) and (28)). In other words, $\mathbf{Der}_k(\mathcal{L}) \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_\mathfrak{Y} \simeq \mathbf{Der}_k(\mathcal{L}_\mathfrak{Y})$, giving $\mathbb{W}(\mathbf{Der}_k(\mathcal{L}))$ the structure of a scheme on Lie-algebras on the étale site of $\mathfrak{X}$. By abuse of notation, we denote this scheme again $\mathbf{Der}_k(\mathcal{L})$. We then have $\mathbf{Der}_k(\mathcal{L})(\mathfrak{Y})$ is a sub-Lie algebra of $\mathrm{Hom}_k(\mathcal{L}_\mathfrak{Y}, \mathcal{L}_\mathfrak{Y})$.

5. Lie algebroids

Let $R \in k$-alg and $\mathfrak{X} = \mathrm{Spec}(R)$. We maintain the notation and terminology of the previous sections.

Definition 5.1. An R/k *Lie algebroid* is given by a triple $(M, [-,-]_{\mathbb{W}(\widetilde{M^*})}, \alpha)$ where

i) M is a projective R-module of finite type.
ii) $[-,-]_{\mathbb{W}(\widetilde{M^*})}$ is an R/k-scheme on Lie algebras on the scheme $\mathbb{W}(\widetilde{M^*})$ of sections of the vector bundle $\mathbb{V}(\widetilde{M})$. Furthermore, the k-vector space structure on each Lie algebra $\mathbb{W}(\widetilde{M^*})(\mathfrak{Y})$ is compatible with the way that k acts on this set via its $\mathcal{O}_\mathfrak{Y}(Y)$-module structure.
iii) $\alpha : \mathbb{V}(\widetilde{M}) \to \mathbb{V}(\widetilde{\Omega_{R/k}})$ is a morphism of vector bundles such that the induced morphism on sections $\alpha^* : \mathbb{W}(\widetilde{M^*}) \to \mathbb{W}(\widetilde{\Omega^*_{R/k}})$ has the property that for each étale extension $\mathfrak{Y}/X$, sections $\xi,\ \eta$ of $\mathbb{W}(\widetilde{M^*})(\mathfrak{Y})$ and element f of $\mathcal{O}_\mathfrak{Y}(\mathfrak{Y})$ we have

$$[\xi, f\eta] = \alpha^*(\xi)(f)\eta + f[\xi, \eta],$$

where $\alpha^*(\xi)(f)$ is the action of $\alpha^*(\xi)$ on f as a derivation of $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$ (recall that by Remark 4.2 we have a natural map $\mathbb{W}(\widetilde{\Omega^*_{R/k}})(\mathfrak{Y}) \to \mathrm{Der}_k\big(\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})\big)$.

Remark 5.2. Lie algebroids, from an algebraic point of view, were defined in [2]. Their definition is quite general and all of our examples fall within their description. That said, our definition of R/k-étale scheme on Lie algebras has no analogue in [2], and that of Lie algebroids mimic in spirit much closely the differential construction outlined in the Introduction. The étale nature of our construction arises naturally and is part of the definition.

Our work complements [2] in as much as it gives a very precise description of Lie algebroids based (for a lack of a better expression) on a finite dimensional split simple Lie algebra $\mathfrak{g}$ (in [2] the base Lie algebra is general or, in some cases, semisimple). Our anchor map could be defined in general, but it will not lead to Lie algebroids with $\mathrm{Spec}(R)$ as base unless $\mathfrak{g}$ is central simple. The reason is that the centroid of $\mathfrak{g} \otimes_k R$ can be identified with R only for $\mathfrak{g}$ central simple. It would be interesting to see what extra information our R/k-scheme on Lie algebras and Lie algebroids arising from simple group schemes shed into the formidable D-universe developed by Beilinson and Berstein. We intend to do so in future work.

Remark 5.3. We have again focused our attention in the case where the base scheme $\mathfrak{X}$ is affine. For an arbitrary base $\mathfrak{X}$ the definition is analogous but with $\widetilde{M}$ replaced by a locally free coherent sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathfrak{X}}$-modules, and $\widetilde{\Omega^*_{R/k}}$ replaced by $\Omega^*_{\mathfrak{X}/k}$.

Remark 5.4. Just as in the differential case, the often seen assumption that the anchor map be a Lie algebra homomorphism is superfluous. We have by the property of the anchor map that

$$[\xi, [\eta, f\rho]] = [\xi, f[\eta, \rho]] + [\xi, \alpha^*(\eta)(f)\rho].$$

Applying the anchor map property to both summands of the right hand side of this equation, and appealing to the Jacobi identity, easily leads to

$$\alpha^*\big([\xi, \eta]\big)(f)\rho = \big(\alpha^*(\xi) \circ \alpha^*(\eta) - \alpha^*(\eta) \circ \alpha^*(\xi)\big)(f)\rho$$

for all sections ξ, η, ρ of $\mathbb{W}(\widetilde{M^*})(\mathfrak{Y})$ and any element f of $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. Thus

$$\alpha^*\big([\xi, \eta]\big) = [\alpha^*(\xi), \alpha^*(\eta)]$$

as desired.

6. Lie algebroids attached to simple group schemes

Let R/k and $\mathfrak{g}$ be as above. We will assume throughout that $X = \mathrm{Spec}(R)$ is smooth over k.

Let $\mathcal{G}$ be a reductive group scheme over X, in the sense of [12]. We will say that $\mathcal{G}$ is *simple of type* $\mathfrak{g}$ if the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$ is a twisted form of $\mathfrak{g} \otimes_k R$.[15]

We will describe explicitly how to attach a Lie algebroid to a simple group scheme $\mathcal{G}$ of type $\mathfrak{g}$ in two different ways. The first one, a differential approach, will make use of the Lie algebra of $\mathcal{G}$ only. The second approach, a global one, will be based on $\mathcal{G}$ itself.

Remark 6.1. If $\mathcal{L}$ is a twisted form of $\mathfrak{g} \otimes_k R$, there always exist a simple group scheme $\mathcal{G}$ of type $\mathfrak{g}$ whose Lie algebra is $\mathcal{L}$. Indeed let S/R be a faithfully flat étale extension trivializing $\mathcal{L}$. Let $u \in \mathbf{Aut}(\mathfrak{g})(S \otimes_R S) = \mathbf{Aut}(\mathfrak{g} \otimes_k R)(S \otimes_R S)$ be a one cocycle defining $\mathcal{L}$. Let $\mathbf{G}$ be the simple simply connected split algebraic k-group of type $\mathfrak{g}$. Since R is a scheme of characteristic 0 the canonical R-group scheme homomorphism $\mathbf{Aut}(\mathbf{G} \times_k R) = \mathbf{Aut}(\mathfrak{g} \otimes_k R)$ is an isomorphism [12, Exp. XXVI]. By viewing u as a cocycle in $\mathbf{Aut}(\mathbf{G} \times_k R)(S \otimes_R S)$ we obtain an R-group scheme $\mathcal{G}$ such that $\mathcal{G} \times_R S \simeq \mathbf{G}_S = (\mathbf{G}_R)_S$. Since the computation of Lie algebras of group schemes commutes with twisting we have an R-Lie algebra isomorphism $\mathrm{Lie}(\mathcal{G}) \simeq \mathcal{L}$ as desired

Henceforth $\mathcal{G}$ will denote a simple reductive group scheme over R of type $\mathfrak{g}$, and $\mathcal{L}$ its Lie algebra.

6.1. *Differential construction of the Lie algebroid of $\mathcal{G}$*

Recall the R-module morphism

$$\eta_{\mathcal{L}} : \mathrm{Der}_k(\mathcal{L}) \to \mathrm{Der}_k(R) = \mathrm{Hom}(\Omega_{R/k}, R) = \Omega^*_{R/k}.$$

Let $M = \mathrm{Der}_k(\mathcal{L})^*$. By taking duals and identifying M with an R-submodule of its symmetric algebra $S(M)$ we obtain

$$\eta^*_{\mathcal{L}} \in \mathrm{Hom}_{R\text{-}mod}(\Omega_{R/k}, S(M)) = \mathrm{Hom}_{R\text{-}alg}(S(\Omega_{R/k}), S(M)).^{16}$$

By Yoneda considerations this corresponds to an element

$$\alpha \in \mathrm{Hom}_{R\text{-}sch}(\mathbb{V}(\widetilde{M}), \mathbb{V}(\widetilde{\Omega_{R/k}})).$$

[15] The concept of simple reductive group scheme is not defined in [12]. This is not surprising since, with the natural definition of such concept, the property of being simple is not stable under base change. Being simple of type $\mathfrak{g}$ corresponds in [12] to semisimple groups schemes whose type is constant (of type $\mathfrak{g}$). This concept is stable by base change.

[16] Throughout and without reference we canonically identify a projective R-module of finite type with its double dual.

It is evident by construction that at the level of sections α induces the scheme morphism $\eta_{\mathcal{L}}$ of Remark 4.8. In view of 8 and Proposition 3.1 we obtain

Theorem 6.2. *Let $\mathcal{L}$ be a twisted form of $\mathfrak{g}\otimes_k R$ where $\mathfrak{g}$ is a finite dimensional split simple Lie algebra over k. Assume that R/k is smooth. Let $M = \mathrm{Der}_k(\mathcal{L})^*$, and $[-,-]_{\mathbb{W}(\widetilde{M^*})}$ be the Lie algebra structure on $\mathbf{Der}_k(\mathcal{L})$.*

i) *$(M, [-,-]_{\mathbb{W}(\widetilde{M^*})}, \eta_{\mathcal{L}})$ is an R/k-Lie algebroid.*
ii) *$0 \to \mathbb{W}(\widetilde{\mathcal{L}}) \to \mathbf{Der}_k(\mathcal{L}) \to \mathbf{Der}_k(R) \to 0$ is an exact sequence of R/k-schemes on Lie algebras.*

Proof. Here $\mathbb{W}(\widetilde{M^*}) = \mathbb{V}(\widetilde{M}) = \mathbf{Der}_k(\mathcal{L})$. And $\eta_{\mathcal{L}}\colon \mathbf{Der}_k(\mathcal{L}) \to \mathbf{Der}_k(R)$ is a Lie algebra homomorphism. Let $\mathfrak{Y}/\mathfrak{X}$ be an étale extension, and $\xi, \zeta \in \mathbf{Der}_k(\mathcal{L})(\mathfrak{Y})$ and $f \in \mathcal{O}_{\mathfrak{Y}}(\mathfrak{Y})$. We may view $\mathbf{Der}_k(\mathcal{L})(\mathfrak{Y})$ as a sub-Lie algebra of $\mathrm{Hom}_k(\widetilde{\mathcal{L}}_{\mathfrak{Y}}, \widetilde{\mathcal{L}}_{\mathfrak{Y}})$. With this identification,

$$[\xi, f\zeta] = \xi\chi_f\zeta - \chi_f\zeta\xi = (\xi\chi_f - \chi_f\xi)\zeta + \chi_f(\xi\zeta - \chi_f\zeta\xi).$$

It follows that $[\xi, f\zeta] = \eta_{\mathcal{L}}(\xi)(f)\zeta + f[\xi,\zeta]$, as needed.

Regarding the second part, all members of the sequence are vector bundles over $\mathfrak{X}$, and that the sequence is exact as a sequence of vector bundles is the content of Proposition 3.1. Restricted to the étale site, the maps are Lie-algebra homomorphisms, and the claim follows. □

6.2. Global construction of the Lie algebroid attached to $\mathcal{G}$

For any ring extension T/S we let $\Omega_{T/S}$ is the S-module of Kähler differentials of a given S-algebra T.

By definition of reductive group scheme $\mathcal{G}$ is affine and smooth over $\mathfrak{X}$. Thus $\mathcal{G} = \mathrm{Spec}(A)$ where A is a smooth Hopf algebra over R (in particular flat and finitely presented). The smoothness of A over R implies (see [5] Theo. 20.5.7) that we have a split exact sequence of A-modules

$$0 \to A \otimes_R \Omega_{R/k} \to \Omega_{A/k} \to \Omega_{A/R} \to 0.$$

Consider the base change $\epsilon : A \to R$ attached to the unit of $\mathcal{G}$, and consider the corresponding R-modules $\omega_{A/R} := R \otimes_A \Omega_{A/R}$ and $\omega_{A/k} := R \otimes_A \Omega_{A/k}$. We then have the split exact sequence of R-modules

$$0 \to \Omega_{R/k} \to \omega_{A/k} \to \omega_{A/R} \to 0.$$

From this it follows (see [3, Ch. II §2 no. 1 Prop. 1]) that we have a split exact sequence of R-vector bundles

$$0 \to \mathbb{V}(\widetilde{\omega_{A/R}}) \to \mathbb{V}(\widetilde{\omega_{A/k}}) \to \mathbb{V}(\widetilde{\Omega_{R/k}}) \to 0. \tag{32}$$

Theorem 6.3. *Let $\mathcal{G}$ be a simple group scheme over R of type $\mathfrak{g}$, and let $\mathcal{L} = \mathrm{Lie}(\mathcal{G})$ be its Lie algebra (which is thus a twisted form of $\mathfrak{g} \otimes_k R$). Let the notation be as above.*

i) *There exists a canonical isomorphism $\mathbb{W}(\widetilde{\mathcal{L}}) \simeq \mathfrak{Lie}(\mathcal{G})$ of R/k-schemes on Lie algebras.*
ii) *As an R-scheme $\mathbb{W}(\widetilde{\omega^*_{A/k}})$ is isomorphic to $\mathbf{Der}_k(\mathcal{L})$. There exists a unique Lie algebroid $(\omega_{A/k}, [\,,\,]_{\mathbb{W}(\widetilde{\omega^*_{A/k}})}, \alpha)$ that when taking sections on (32) and under the identifications of (i) leads to the split exact sequence of R/k-scheme on Lie algebras of Theorem 6.2.*

In particular one can in a canonical fashion attach to $\mathcal{G}$ a Lie algebroid of the form $(\omega_{A/k}, [\,,\,]_{\mathbb{W}(\widetilde{\omega^*_{A/k}})}, \alpha)$ where the anchor map α corresponds to $\eta_{\mathcal{L}}$ and its kernel to $\mathbb{W}(\mathcal{L})$.

Proof. We have seen that the scheme of sections of $\mathbb{V}(\widetilde{\Omega_{R/k}})$ is $\mathbb{W}(\widetilde{\Omega^*_{R/k}}) = \mathbf{Der}_k(R)$. By [4, II.4, no. 3.4] we have $\mathfrak{Lie}(\mathcal{G}) \simeq \mathbb{V}(\widetilde{\omega_{A/R}})$. In particular $\mathrm{Lie}(\mathcal{G}) = \mathrm{Hom}_R(\omega_{A/R}, R)$. Since $\mathcal{G}$ is smooth over R the canonical map $\mathfrak{Lie}(\mathcal{G}) \to \mathbb{W}\big(\widetilde{\mathrm{Lie}(\mathcal{G})}\big)$ is an isomorphism. This provides the canonical isomorphism

$$\mathbb{W}\big(\widetilde{\mathrm{Lie}(\mathcal{G})}\big) \simeq \mathbb{W}(\widetilde{\omega^*_{A/R}}) \simeq \mathbb{V}(\widetilde{\omega_{A/R}}).$$

By taking these identifications into considerations, the rest of the proof follows by applying Theorem 6.2. □

Remark 6.4. Consider the algebraic k-group $\mathbf{Aut}(\mathfrak{g})$. By assumption the R-linear Lie algebra $\mathcal{L} := \mathrm{Lie}(\mathcal{G})$ is such that $\mathcal{L} \otimes_R S \simeq \mathfrak{g} \otimes_k S$ as Lie algebras over S for some faithfully flat étale S/R. In other words $\mathcal{L}$ is an S/R-form of $\mathfrak{g} \otimes_k S$, so that there is a cocycle $u \in \mathbf{Aut}(\mathfrak{g})(S \otimes_R S)$ such that

$$\mathcal{L} = \{x \in \mathfrak{g} \otimes_k S \,:\, up_1(x) = p_2(x)\}\,,$$

here $p_i : S \to S \otimes_R S$ $(i = 1, 2)$ are the inclusions $p_1(x) = x \otimes 1$, $p_2(x) = 1 \otimes x$. It can be shown that u leads to a descent data on the S-module $\mathrm{Der}_k(\mathfrak{g} \otimes_k S)$, that this descent data preserves the k-Lie algebra structure of $\mathrm{Der}_k(\mathfrak{g} \otimes_k S)$, and that the descended R-module is precisely the k-Lie algebra $\mathrm{Der}_k(\mathcal{L})$.

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