Journal of Algebra 496 (2018) 48–60

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Groups in which each subgroup is commensurable with a normal subgroup

Carlo Casolo [a], Ulderico Dardano [b,*], Silvana Rinauro [c]

[a] *Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67A, I-50134 Firenze, Italy*
[b] *Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", Via Cintia – Monte S. Angelo, I-80126 Napoli, Italy*
[c] *Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell'Ateneo Lucano 10 – Contrada Macchia Romana, I-85100 Potenza, Italy*

A R T I C L E   I N F O

Article history:
Received 4 April 2017
Available online 21 November 2017
Communicated by E.I. Khukhro

To the memory of Jim Wiegold

MSC:
primary 20F24
secondary 20F18, 20F50, 20E15

Keywords:
Locally finite
Core-finite
Subnormal
Nilpotent

A B S T R A C T

A group G is a CN-group if for each subgroup H of G there exists a normal subgroup N of G such that the index $|HN : (H \cap N)|$ is finite. The class of CN-groups contains properly the classes of core-finite groups and that of groups in which each subgroup has finite index in a normal subgroup.
In the present paper it is shown that a CN-group whose periodic images are locally finite is finite-by-abelian-by-finite. Such groups are then described into some details by considering automorphisms of abelian groups. Finally, it is shown that if G is a locally graded group with the property that the above index is bounded independently of H, then G is finite-by-abelian-by-finite.

* Corresponding author.
E-mail addresses: casolo@math.unifi.it (C. Casolo), dardano@unina.it (U. Dardano), silvana.rinauro@unibas.it (S. Rinauro).

https://doi.org/10.1016/j.jalgebra.2017.11.016

1. Introduction and main results

In a celebrated paper, B.H. Neumann [10] showed that for a group G the property that each subgroup H has finite index in a normal subgroup of G (i.e., $|H^G : H|$ is finite) is equivalent to the fact that G has finite derived subgroup (G is *finite-by-abelian*).

A class of groups with a dual property was considered in [1]. A group G is said to be a CF-group (*core-finite*) if each subgroup H contains a normal subgroup of G with finite index in H (i.e., $|H : H_G|$ is finite). As Tarski groups are CF, a complete classification of CF-groups seems to be rather difficult. However, in [1] and [12] it has been proved that *a* CF*-group G whose periodic quotients are locally finite is abelian-by-finite and, if G is periodic, there exists an integer n such that $|H : H_G| \leq n$ for all $H \leq G$* (say that *G is* BCF*, boundedly* CF) and that *a locally graded* BCF*-group is abelian-by-finite*. Furthermore, an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given. It seems to be a still open question whether every locally graded CF-group is abelian-by-finite. Recall that a group is said to be *abelian-by-finite* if it has an abelian subgroup with finite index and that a group is said to be *locally finite* (*locally graded*, respectively) if each non-trivial finitely generated subgroup is finite (has a proper subgroup with finite index, respectively).

With the aim of considering the above properties in a common framework, recall that two subgroups H and K of a group G are said to be *commensurable* if $H \cap K$ has finite index in both H and K. This is an equivalence relation and will be denoted by $\sim$. Clearly, if $H \sim K$, then $(H \cap L) \sim (K \cap L)$ and $HM \sim KM$ for each $L \leq G$ and $M \triangleleft G$.

Thus, in the present paper we consider the class of CN*-groups*, that is, *groups in which each subgroup is commensurable with a normal subgroup*. Into details, for a subgroup H of a group G define $\delta_G(H)$ to be the minimum index $|HN : (H \cap N)|$ with $N \triangleleft G$. Then G is a CN-group if and only if $\delta_G(H)$ is finite for all $H \leq G$. Clearly, subgroups and quotients of CN-groups are also CN-groups.

Note that if a subgroup H of a group G is commensurable with a normal subgroup N, then $S := (H \cap N)_N$ has finite index in H. Thus the class of CN-groups is contained in the class of *sbyf-groups*, that is, groups in which each subgroup H contains a subnormal subgroup S of G such that the index $|H : S|$ is finite (i.e., H is *subnormal-by-finite*). It is known that *locally finite sbyf-groups are (locally nilpotent)-by-finite* (see [7]) and *nilpotent-by-Chernikov* (see [3]).

The extension of a finite group by a CN-group is easily seen to be a CN-group, see Proposition 1.1 below. Moreover, from Proposition 9 in [4] it follows that *for an abelian-by-finite group properties* CN *and* CF *are equivalent*. However, for each prime p there is a nilpotent p-group with the property CN which is neither finite-by-abelian nor abelian-by-finite, see Proposition 1.2.

Our main result is the following.

Theorem A. *Let G be a* CN*-group such that every periodic image of G is locally finite. Then G is finite-by-abelian-by-finite.*

Here by a *finite-by-abelian-by-finite* group we mean a group which has a finite-by-abelian subgroup of finite index. The proof of Theorem A will be given in Section 3. The strategy of the proof will be to reduce to the case when G is nilpotent and then to apply techniques of nilpotent groups theory. To this end, in Section 2, we will study the action of a CN-group on its abelian sections.

We will consider also BCN-*groups*, that is, groups G for which there is $n \in \mathbb{N}$ such that $\delta_G(H) \leq n$ for all $H \leq G$ and prove the following theorem.

Theorem B. *Let G be a finite-by-abelian-by-finite group.*

i) G is CN *if and only if it is finite-by-*CF*.*
ii) G is BCN *if and only if it is finite-by-*BCF*.*

It follows that if the group G is periodic and finite-by-abelian-by-finite, then G is BCN if and only if it is CN. Then we consider non-periodic finite-by-abelian-by-finite BCF-groups in Proposition 3.2.

The more restrictive property BCN remains treatable when we consider the wider class of locally graded groups.

Theorem C. *A locally graded* BCN*-group is finite-by-abelian-by-finite.*

Preliminaries

Our notation is mostly standard. For undefined terminology and basic facts we refer to [11]. If Γ is a group acting on a group G and $H \leq G$, we denote $H_\Gamma := \cap_{\gamma\in\Gamma} H^\gamma$ and $H^\Gamma := \langle H^\gamma \mid \gamma \in \Gamma\rangle$. We say that H is Γ-*invariant* (or a Γ-*subgroup*) if $H^\Gamma = H$.

We first point out a sufficient condition for a group to be CN (or even BCN) and give examples of non-trivial CN-groups.

Proposition 1.1. *Let G be a group with a normal series $G_0 \leq G_1 \leq G$, where G_0 and G/G_1 have finite order, m and n respectively.*

If $H \leq G$, then H is commensurable with $H_1 := (H \cap G_1)G_0 \leq G_1$ and $\delta_G(H) \leq mn \cdot \delta_{G/G_0}(H_1/G_0)$.

In particular, if each subgroup of G_1/G_0 is commensurable with a normal subgroup of G/G_0, then G is a CN*-group.* □

Proposition 1.2. *For each prime p there is a nilpotent p-group with the property* BCN*, which is neither abelian-by-finite nor finite-by-abelian.*

Proof. Consider a sequence P_n of isomorphic groups with order p^4 defined by $P_n := \langle x_n, y_n \mid x_n^{p^3} = y_n^p = 1,\ x_n^{y_n} = x_n^{1+p^2}\rangle = \langle x_n\rangle \rtimes \langle y_n\rangle$ where clearly $P_n' = \langle x_n^{p^2}\rangle$ has order p. Let $P := \mathrm{Dr}_{n\in\mathbb{N}} P_n$ and consider the automorphism γ of P such that $x_n^\gamma = x_n^{1+p}$ and

$y_n^\gamma = y_n$, for each $n \in \mathbb{N}$. Clearly, γ has order p^2 (2 resp.) if $p \neq 2$ (if $p = 2$ resp.), it acts as the automorphism $x \mapsto x^{1+p}$ on P/P' (which has exponent p^2) and acts trivially on P' (which is elementary abelian). Finally let $N := \langle x_0^{p^2} x_n^{p^2} \mid n \in \mathbb{N}\rangle$. Then N is a γ-invariant subgroup of P' with index p. Thus the p-group $G := (P \rtimes \langle\gamma\rangle)/N$ is a BCN-group by Proposition 1.1 applied to the series $P'/N \leq P/N \leq G$.

We have that G' is infinite, since for each n we have $x_n^p = [x_n, \gamma] \in [P_n, \gamma] > P_n'$. Moreover, we have that $gN \in Z(P/N)$ if and only if $\forall i\ [g, P_i] \leq N$, and $N \cap P_i = 1$. Thus $Z(P/N) = Z(P)/N$ where $Z(P) = \mathrm{Dr}_n\langle x_n^p\rangle$ has infinite index in P.

If, by contradiction, G is abelian-by-finite, then there is an abelian normal subgroup A/N of P/N with finite index. Then for some $m \in \mathbb{N}$ we have $P = AF$, where $F = \mathrm{Dr}_{n<m} P_n$ is a finite normal subgroup of P. Therefore P/N is center-by-finite, a contradiction. $\square$

2. Automorphisms of abelian groups

Recall that an automorphism γ of a group A is said to be a *power automorphism* if $H^\gamma = H$ for each subgroup $H \leq A$. It is well-known (see [11]) that, if A is an abelian p-group, then there exists a p-adic integer α such that $a^\gamma = a^\alpha$ for all $a \in A$. Here a^α stands for a^n, where n is any integer congruent to α modulo the order of a. On the other hand, a power automorphism of a non-periodic abelian group is either the identity or the inversion map.

As in [4], if Γ is a group acting on an abelian group A, we consider the following properties:

P) $\forall H \leq A\ H = H^\Gamma$;
AP) $\forall H \leq A\ |H : H_\Gamma| < \infty$;
BP) $\forall H \leq A\ |H^\Gamma : H| < \infty$;
CP) $\forall H \leq A\ \exists K = K^\Gamma \leq A$ *such that* $H \sim K$ (H, K *are commensurable*).

Obviously both AP and BP imply CP. Moreover, from Propositions 8 and 9 in [4] it follows that *these three properties are equivalent, provided A is abelian and Γ is finitely generated, while they are in fact different in the general case even when A and Γ are elementary abelian p-groups*. On the other hand, the properties AP and BP have been previously characterized in [6] and [2] respectively, as we are going to recall.

To shorten statements we define a further property:

$\tilde{\text{P}}$) Γ *has* P *on the factors of a* Γ-*series* $1 \leq V \leq D \leq A$ *where*
i) V *is free abelian of finite rank,*
ii) D/V *is divisible periodic with finite total rank,*
iii) A/D *is periodic and has finite* p-*exponent for each prime* $p \in \pi(D/V)$.

Theorem 2.1 *([6], [2]). Let Γ be group acting on an abelian group A. Then:*

a) Γ has AP *on A if and only if there is a Γ-subgroup A_1 such that A/A_1 is finite and Γ has either* P *or* $\tilde{\mathrm{P}}$ *on A_1.*
b) Γ has BP *on A if and only if there is a Γ-subgroup A_0 such that A_0 is finite and Γ has either* P *or* $\tilde{\mathrm{P}}$ *on A/A_0.*

In the next statement we give a characterization of the property CP along the same lines.

Theorem 2.2. *Let Γ be group acting on an abelian group A. Then:*

c) Γ has CP *on A if and only if there are Γ-subgroups $A_0 \leq A_1 \leq A$ such that A_0 and A/A_1 are finite and Γ has either* P *or* $\tilde{\mathrm{P}}$ *on A_1/A_0.*

The proof of Theorem 2.2 is at the end of this section. Here we deduce a corollary.

Corollary 2.3. *For a group Γ acting on an abelian group A, the following are equivalent:*

a) Γ has AP *on A/A_0 for a finite Γ-subgroup A_0 of A,*
b) Γ has BP *on a finite index Γ-subgroup A_1 of A,*
c) Γ has CP *on A.* □

Let us state a couple of elementary basic facts.

Proposition 2.4. *Let Γ be group acting on a locally nilpotent periodic group A. Then Γ has* AP, BP, CP *on A, respectively, if and only if Γ has* AP, BP, CP *on finitely many primary components of A, respectively, and* P *on all the other ones.*

Proof. This proof uses the same argument as in Proposition 4.1 in [5]. The sufficiency of the condition is clear once one notes that for each $H \leq A$ it follows that $H = \mathrm{Dr}_p(H \cap A_p)$, where A_p denotes the p-component of A.

Concerning necessity, suppose Γ does not have P on the primary p-component A_p of A for infinitely many primes p. Then for each such p there is $H_p \leq A_p$ which is not Γ-invariant. We have that the subgroup generated by the H_p's is not commensurable to any Γ-subgroup. □

Lemma 2.5. *Let Γ be a group acting on an abelian group A. If Γ has* CP *on A, then:*

i) Γ has P *on the largest periodic divisible subgroup of A;*
ii) if A is torsion-free, then each $\gamma \in \Gamma$ acts on A by either the identity or the inversion map.

Proof. Statement (i) follows from Lemma 4.3 in [5]. Concerning (ii), by Propositions 3.3 and 3.2 of [5] we have that there are coprime non-zero integers n, m such that $a^m = (a^\gamma)^n$ for each $a \in A$. Consider H such that $1 \neq H := \langle a_0 \rangle \leq A$. Then there is a Γ-invariant subgroup K of A which is commensurable with H. Thus there is $r \in \mathbb{N}$ such that K^r is a Γ-invariant nontrivial subgroup of H. This forces $mn = \pm 1$. $\square$

Now we prove some lemmas. In the first one we do not require that the group A is abelian.

Lemma 2.6. *Let* Γ *be a group acting on an* FC*-group* A*. If* Γ *has* CP *on* A*, then* Γ *has* BP *on the subgroup* $X := \{\, a \in A \,|\, \langle a \rangle^\Gamma$ *is finite*$\}$ *of* A*.*

Proof. Notice that X is the set of elements a of finite order of A such that $|\Gamma : C_\Gamma(a)|$ is finite, so X is a locally finite Γ-subgroup of A. For any $H \leq X$ there is $K \leq X$ such that $H \sim K = K^\Gamma \leq A$. Then there is a finite subgroup $F \leq X$ such that $H \leq KF$. Thus $H^\Gamma \leq KF^\Gamma$ and $|H^\Gamma : H| \leq |F^\Gamma| \cdot |HK : H|$ is finite. $\square$

Lemma 2.7. *Let* Γ *be a group acting on a* p*-group* A *which is the direct product of cyclic groups. If* Γ *has* CP *on* A*, then the subgroup* $X := \{\, a \in A \,|\, \langle a \rangle^\Gamma$ *is finite*$\}$ *has finite index in* A*.*

Proof. Assume by contradiction that A/X is infinite.

Let us see, by elementary facts, that there is a sequence (a_n) of elements of A such that

1) $\langle a_n | n \in \mathbb{N} \rangle = \mathrm{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$,
2) $A_I / A_I \cap X$ is infinite, for each infinite subset I of $\mathbb{N}$, where $A_I := \langle a_n | n \in I \rangle$.

In fact, if A/X has finite rank, it has a Prüfer subgroup Q/X. Let Y be a countable subgroup of A such that $Q = YX$. By Kulikov's Theorem (see [11]) Y is the direct product of cyclic groups, so that we may choose elements $a_n \in Y$ such that $\langle a_n | n \in \mathbb{N} \rangle = \mathrm{Dr}_{n \in \mathbb{N}} \langle a_n \rangle \leq Y$ and $|a_n X| < |a_{n+1} X|$. The claim holds. Similarly, if A/X has infinite rank, consider a countably infinite subgroup Q/X of the socle of A/X. As above, let Y be a countable subgroup of A such that $Q = YX$. Then we may choose elements $a_n \in Y$ which are independent mod X and generate their direct product as claimed.

We claim now that *there are sequences of infinite subsets* I_n*,* J_n *of* $\mathbb{N}$ *and* Γ*-subgroups* $K_n \leq A$ *such that for each* $n \in \mathbb{N}$*:*

3) $I_n \cap J_n = \emptyset$ *and* $I_{n+1} \subseteq J_n$
4) $K_n \sim A_{I_n}$
5) $(K_1 \dots K_i) \cap (A_{I_1} \dots A_{I_n}) \leq (A_{I_1} \dots A_{I_i})$, $\forall i \leq n$.

To prove the claim, proceed by induction on n. Choose an infinite subset I_1 of $\mathbb{N}$ such that $J_1 := \mathbb{N} \setminus I_1$ is infinite. By the CP-property there exists $K_1 = K_1^\Gamma$ commensurable with A_{I_1}.

Suppose we have defined I_j, J_j and K_j for $1 \leq j \leq n$ such that (3–5) hold. Since $(K_1 \dots K_n) \sim (A_{I_1} \dots A_{I_n})$, there is $m \in \mathbb{N}$ such that

6) $(K_1 K_2 \dots K_n) \cap A_{\mathbb{N}} \leq (A_{I_1} A_{I_2} \dots A_{I_n})\langle a_1, \dots, a_m\rangle$.

Let I_{n+1} and J_{n+1} be disjoint infinite subsets of $J_n \setminus \{1, \dots, m\}$. By CP-property there exists $K_{n+1} = K_{n+1}^\Gamma$ commensurable with $A_{I_{n+1}}$. By the choice of I_{n+1} it follows that

7) $(K_1 \dots K_i) \cap (A_{I_1} \dots A_{I_{n+1}}) \leq (K_1 \dots K_i) \cap (A_{I_1} \dots A_{I_n})\ \forall i \leq n$

and so (5) holds for $n + 1$, as required. The claim is now proved.

Note that by (2) and (5) it follows that $A_{I_n}/A_{I_n} \cap X$ is infinite for each $n \in \mathbb{N}$ and that also the following property holds

8) $(K_1 K_2 \dots K_n) \cap \bar{A} \leq (A_{I_1} A_{I_2} \dots A_{I_n})\ \forall n$, where $\bar{A} := \mathrm{Dr}_{n\in\mathbb{N}} A_{I_n}$.

Now for each $n \in \mathbb{N}$, choose an element $b_n \in (A_{I_n} \cap K_n) \setminus X$. Then we have $B := \langle b_n \mid n \in \mathbb{N}\rangle = \mathrm{Dr}_n\langle b_n\rangle$, where $\langle b_n\rangle^\Gamma$ is infinite and $\langle b_n\rangle^\Gamma \leq K_n \sim A_{I_n}$, so that

9) $\langle b_n\rangle^\Gamma \cap A_{I_n}$ is infinite for each n.

Since there exists $B_0 = B_0^\Gamma \sim B$, we may take

$$B_* := (B_0 \cap B)^\Gamma = (B_* \cap B)^\Gamma \leq B^\Gamma \text{ where } B_* \sim B.$$

Now $B_*/(B_* \cap B)$ and $B/(B_* \cap B)$ are both finite and there is $n \in \mathbb{N}$ such that if $B_n := \langle b_1, \dots, b_n\rangle$ we have

$$(B_* \cap B)^\Gamma = B_* \leq (B_* \cap B)B_n^\Gamma \text{ and}$$
$$B = (B_* \cap B)B_n.$$

Since $b_n \in K_n$ for each n, we have $B_n \leq \bar{K}_n := K_1 K_2 \dots K_n$ and $B^\Gamma = (B_* \cap B)^\Gamma B_n^\Gamma \leq (B_* \cap B)B_n^\Gamma \leq (B_* \cap B)\bar{K}_n \leq B\bar{K}_n$, so that $B^\Gamma \cap \bar{A} \leq B\bar{K}_n \cap \bar{A} = B(\bar{K}_n \cap \bar{A}) \leq BA_{I_1}A_{I_2} \dots A_{I_n}$ by (8) above.

Thus $\langle b_{n+1}\rangle^\Gamma \cap A_{I_{n+1}} \leq B^\Gamma \cap A_{I_{n+1}} \leq (BA_{I_1}A_{I_2} \dots A_{I_n}) \cap A_{I_{n+1}} = \langle b_{n+1}\rangle$ is finite, a contradiction with (9). $\square$

Lemma 2.8. *Let Γ be a group acting on an abelian periodic reduced group A. If Γ has* CP *on A, then there are Γ-subgroups $A_0 \leq A_1 \leq A$ such that A_0 and A/A_1 are finite and Γ has* P *on A_1/A_0.*

Proof. By Proposition 2.4 it is enough to consider the case when A is a p-group. If A is the direct product of cyclic groups, by Lemma 2.7 we have that $A_1 := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite}\}$ has finite index in A. Further, by Lemma 2.6, Γ has BP on A_1. Then the statement follows from Theorem 2.1.

Let A be any reduced p-group and B_* be a basic subgroup of A. Then there is $B = B^\Gamma \sim B_*$. Since A/B_* is divisible, then the divisible radical of A/B has finite index. Thus we may assume that A/B is divisible. By Kulikov's Theorem (see [11]), also B is a direct product of cyclic groups, therefore by the above there are Γ-subgroups $B_0 \leq B_1 \leq B$ such that B_0 and B/B_1 are finite and Γ has P on B_1/B_0. We may assume $B_0 = 1$. Also, since A/B_1 is finite-by divisible, it is divisible-by-finite and we may assume it is divisible.

Let $\gamma \in \Gamma$ and α be a p-adic integer such that $x^\gamma = x^\alpha$ for all $x \in B_1$. Consider the endomorphism $\gamma - \alpha$ of A and note that $B_1 \leq \ker(\gamma - \alpha)$. Thus $A/\ker(\gamma-\alpha) \simeq \mathrm{im}(\gamma-\alpha)$ is both divisible and reduced, hence trivial. It follows $\gamma = \alpha$ on the whole of A. □

Proof of Theorem 2.2. For the sufficiency of the condition note that for any subgroup $H \leq A$ we have $H \sim H \cap A_1$ and the latter is in turn commensurable with a Γ-subgroup since Γ has BP on A_1 by Theorem 2.1.

Concerning necessity, we first prove the statement when A is periodic. Let $A = D \times R_1$, where D is divisible and R_1 is reduced. Then there is a subgroup $R = R^\Gamma \sim R_1$. Thus DR and $D \cap R$ are Γ-subgroups of A with finite index and order respectively. Then we can assume $A = D \times R$. Let $X := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite}\}$. Clearly $D \leq X$, as Γ has P on D by Lemma 2.5. On the other hand, $X \cap R$ has finite index in R by Lemma 2.8. It follows A/X is finite and by Lemma 2.6 and Theorem 2.1 the statement holds.

In the non-periodic case, note that if V_0 is a free subgroup of A such that A/V_0 is periodic, then there is $V_1 = V_1^\Gamma \sim V_0$. Let $n := |V_1/(V_0 \cap V_1)|$. Thus by applying Lemma 2.5 to the Γ-subgroup $V := V_1^n$ we have that *there is a free abelian Γ-subgroup V such that A/V is periodic and each $\gamma \in \Gamma$ acts on V by either the identity or the inversion map.*

Suppose that V has finite rank. Consider now the action of Γ on the periodic group A/V and apply the above. Then there is a series $V \leq A_0 \leq A_1 \leq A$ such that A_0/V and A/A_1 are finite and Γ has either P or $\tilde{\text{P}}$ on A_1/A_0. Since A_0 has finite torsion subgroup T we can factor out T and assume $A_0 = V$. Then Γ has either P or $\tilde{\text{P}}$ on A_1 as straightforward verification shows.

Suppose finally that V has infinite rank. Let $V_2 \leq V$ be such that V/V_2 is divisible periodic and its p-component has infinite rank for each prime p. We may assume $V := V_2$. By the above case when A is periodic, there is a Γ-series $V \leq A_0 \leq A_1 \leq A$ such that

A_0/V and A/A_1 are finite and Γ has P on A_1/A_0. We may factor out the torsion subgroup of A_0, as it is finite, and assume $A_0 = V$.

Again let $V_2 \leq V$ be such that V/V_2 is divisible periodic and its p-component has infinite rank for each prime p. Let $\gamma \in \Gamma$ and, for each prime p, let α_p be a p-adic integer such that $x^\gamma = x^{\alpha_p}$ for all x in the p-component of A_1/V. Let $\epsilon = \pm 1$ be such that $x^\gamma = x^\epsilon$ for all $x \in V$. By Lemma 2.5, γ has P on the maximum divisible subgroup D_p/V_2 of the p-component of A_1/V_2. Thus $\alpha_p = \epsilon$ on D_p/V_2. Therefore $x^\gamma = x^\epsilon$ for all $x \in V$ and for all $x \in A_1/V$. We claim that $a^\gamma = a^\epsilon$ for each $a \in A_1$. To see this, for any $a \in A_1$ consider $n \in \mathbb{N}$ such that $a^n \in V$. Then there is $v \in V$ such that $a^\gamma = a^\epsilon v$. Hence $a^{n\epsilon} = (a^n)^\gamma = (a^\gamma)^n = (a^\epsilon v)^n = a^{n\epsilon}v^n$. Thus $v^n = 1$. Therefore, as V is torsion-free, we have $v = 1$, as required. □

3. Proofs of the theorems

Recall that locally finite CF-groups are abelian-by-finite and BCF (see [1]).

Proof of Theorem B. It follows from Proposition 1.1 and Proposition 3.1 below. □

Proposition 3.1. *Let G be an abelian-by-finite group.*

i) if G is CN, *then G is* CF;
ii) if G is BCN, *then G is* BCF.

Proof. Let A be a normal abelian subgroup with finite index r. Then each $H \leq A$ has at most r conjugates in G. If $\delta_G(H) \leq n < \infty$ then for each $g \in G$ we have $|H : (H \cap H^g)| \leq 2\delta_G(H) \leq 2n$ hence $|H/H_G| \leq (2n)^r$. More generally, if H is any subgroup of G, then $|H/H_G| \leq r(2n)^r$. □

Let us characterize BCF-groups among abelian-by-finite CF-groups.

Proposition 3.2. *Let G be a non-periodic group with an abelian normal subgroup A with finite index. Then the following are equivalent:*

i) G is a BCF*-group;*
ii) G is a CF*-group and there is $B \leq A$ such that B has finite exponent, $B \lhd G$ and each $g \in G$ acts by conjugation on A/B by either the identity or the inversion map.*

Proof. Let T be the torsion subgroup of A. By Lemma 2.5, for each $g \in G$ there exists $\epsilon_g = \pm 1$ such that γ acts on A/T as the automorphism $x \mapsto x^{\epsilon_g}$. Then the equivalence of (i) and (ii) holds with $B := \langle A^{g-\epsilon_g} \mid g \in G \rangle$, by Theorem 3 of [4]. □

To prove Theorem A, our first step is a reduction to nilpotent groups.

Lemma 3.3. *A soluble p-group G with the property* CN *is nilpotent-by-finite.*

Proof. By Theorem 2.2, one may refine the derived series of G to a finite G-series $\mathbf{S}$ such that G has P on each infinite factor of $\mathbf{S}$. Recall that a p-group of power automorphisms of an abelian p-group is finite (see [11]). Then the stability group $S \leq G$ of the series $\mathbf{S}$, that is, the intersection of the centralizers in G of the factors of the series, has finite index in G. On the other hand, by a theorem of Ph. Hall, S is nilpotent. □

We recall now an elementary property of nilpotent groups.

Lemma 3.4. *Let G be a nilpotent group with class c. If G' has finite exponent e, then $G/Z(G)$ has finite exponent dividing e^c.*

Proof. Argue by induction on c, the statement being clear for $c = 1$. Assume $c > 1$ and that G/Z has exponent dividing e^{c-1}, where $Z/\gamma_c(G) := Z(G/\gamma_c(G))$. Then for all $g, x \in G$ we have $[g^{e^{c-1}}, x] \in \gamma_c(G) \leq G' \cap Z(G)$. Therefore $1 = [g^{e^{c-1}}, x]^e = [g^{e^c}, x]$, and $g^{e^c} \in Z(G)$, as claimed. □

The next lemma follows easily from Lemma 6 in [9].

Lemma 3.5. *Let G be a nilpotent p-group and N a normal subgroup such that G/N is an infinite elementary abelian group. If H and U are finite subgroup of G such that $H \cap U = 1$, there exists a subgroup V of G such that $U \leq V$, $H \cap V = 1$ and VN/N is infinite.* □

We deduce a technical lemma which is a tool for our purpose.

Lemma 3.6. *Let G be a nilpotent p-group and N be a normal subgroup such that G/N is an infinite elementary abelian group. If N contains the* FC*-center of G and G' is abelian with finite exponent, then there are subgroups H, U of G such that $H \cap U = 1$, with injective maps $n \mapsto h_n \in H$ and $(i, n) \mapsto u_{i,n} \in [G, h_i^{-1}h_n] \cap U$, where $i, n \in \mathbb{N}$, $i < n$.*

Proof. Let us show that for each $n \in \mathbb{N}$ there is an $(n+1)$-uple $v_n := (h_n, u_{0,n}, u_{1,n}, \ldots, u_{n-1,n})$ of elements of G such that:

1) $\{h_1, \ldots, h_n\}$ is linearly independent modulo N;
2) $u_{i,n} \in [G, h_i^{-1}h_n]$ $\forall i \in \{0, \ldots, n-1\}$;
3) $\{u_{j,h} \mid 0 \leq j < k \leq n\}$ is $\mathbb{Z}$-independent in G';
4) $H_n \cap U_n = 1$, where $H_n := \langle h_1, \ldots, h_n \rangle$ and $U_n := \langle u_{j,h} \mid 0 \leq j < k \leq n \rangle$.

Then the statement is true for $H := \bigcup_{n\in\mathbb{N}} H_n$ and $U := \bigcup_{n\in\mathbb{N}} U_n$.

Let $h_0 := 1$ and choose $h_1 \in G \setminus N$. Since N contains the FC-center F of G, we have that h_1 has an infinite numbers of conjugates in G, hence $[G, h_1]$ is infinite and residually finite. Thus we may choose $u_{0,1} \in [G, h_1]$ such that $\langle u_{0,1} \rangle \cap \langle h_1 \rangle = 1$.

Assume then that we have defined v_i for $i \leq n$, that is, we have elements $h_0, \ldots, h_n$, $u_{j,k}$, with $0 \leq j < k \leq n$ such that conditions (1–4) hold. To define an adequate v_{n+1}, note that by Lemma 3.5 we have that there exists $V_n \leq G$ such that $H_n \leq V_n$, $U_n \cap V_n = 1$ and $V_n N/N$ is infinite. Then choose

i) $h_{n+1} \in V_n \setminus NU_nH_n$.

Note that $h_{n+1} \notin FH_n \leq NH_n$, so that $\{h_1, \ldots, h_{n+1}\}$ is independent mod F. In particular $\forall i \in \{0, \ldots, n\}$, $h_i^{-1}h_{n+1} \notin F$, hence also $[G, h_i^{-1}h_{n+1}]$ is infinite. Since G' is residually finite, we may recursively choose $u_{0,n+1}, \ldots, u_{n,n+1}$ such that $\forall i \in \{0, \ldots, n\}$

ii) $u_{i,n} \in [G, h_i^{-1}h_n]$;
iii) $\langle u_{i,n+1} \rangle \cap U_n \langle u_{h,n+1} \mid 0 \leq h < i \rangle H_{n+1} = 1$.

Then properties (1–3) hold for v_{n+1}. Finally suppose there are $h \in H_n$, $u \in U_n$, $s, t_0, \ldots, t_n \in \mathbb{Z}$ such that

iv) $a = hh_{n+1}^s = uu_{0,n+1}^{t_1} \cdots u_{n,n+1}^{t_n} \in H_{n+1} \cap U_{n+1}$.

Then from (*iii*) it follows $u_{n,n+1}^{t_n} = \ldots = u_{0,n+1}^{t_1} = 1$. Hence $a = hh_{n+1}^s = u \in V_n \cap U_n = 1$ and 4 holds. □

Lemma 3.7. *Let G be a nilpotent p-group. If G is* CN*, then G' has finite exponent.*

Proof. If, by contradiction, G' has infinite exponent, then the same happens to the abelian group $G'/\gamma_3(G)$ and there is N such that $G' \geq N \geq \gamma_3(G)$ and G'/N is a Prüfer group. We may assume $N = 1$, that is, G' itself is a Prüfer group and $G' \leq Z(G)$. Let us show that for any $H \leq G$ we have $|H^G : H| < \infty$, hence G' is finite, a contradiction. In fact we have that, by the CN-property, there is $K \lhd G$ such that $K \sim H$. Thus H has finite index in HK and we can also assume $H = HK$, that is, H/H_G is finite. Thus, we can assume $H_G = 1$ and $H \cap G' = 1$, that is, H is finite with order p^n and HG' is an abelian Chernikov group. It follows that H is contained in the n-th socle S of $HG' \lhd G$, where S is finite and normal in G, as required. □

Lemma 3.8. *Let G be a nilpotent p-group. If G is* CN*, then G is finite-by-abelian-by-finite.*

Proof. Let G be a counterexample. Then both G' and $G/Z(G)$ are infinite. However, they have finite exponent by Lemmas 3.7 and 3.4. Moreover, by Lemma 2.6, the FC-center

F of G is finite-by-abelian. Thus F has infinite index in G. On the other hand, G/F has finite exponent, since $F \geq Z(G)$.

Then $N := FG^pG'$ has infinite index in G, otherwise the abelian group G/FG' has finite rank and finite exponent, hence it is finite. This implies that the nilpotent group G/F is finite, a contradiction.

If G' is abelian we are in a position to apply Lemma 3.6 and obtain infinitely many elements and subgroups $h_n \in H$, $u_{i,n} \in U$ as in that statement. By CN-property there is K such that $H \sim K \lhd G$. So that the set $\{h_n(H \cap K) \ / \ n \in \mathbb{N}\}$ is finite. Hence there is $i \in \mathbb{N}$ and an infinite set $I \subseteq \mathbb{N} \setminus \{1, \dots, i\}$ such that for each $n \in I$ we have $h_i^{-1}h_n \in H \cap K$ and $u_{i,n} \in U \cap [G, H \cap K] \leq U \cap K$. Therefore $U \cap K$ is infinite, in contradiction with $U \cap K \sim U \cap H = 1$.

For the general case, proceed by induction on the nilpotency class $c > 1$ of G and assume that the statement is true for $G/Z(G)$ and even that this is finite-by-abelian. Then there is a subgroup $L \leq G$ such that G/L is abelian and $L/Z(G)$ is finite. Thus L' is finite and, by the above, G/L' is finite-by-abelian-by-finite, a contradiction. □

Let us consider now non-periodic CN-groups.

Lemma 3.9. *Let G be a CN-group and $A = A(G)$ its subgroup generated by all infinite cyclic normal subgroups. Then G/A is periodic, A is abelian and each $g \in G$ acts on A by either the identity or the inversion map. In particular, $|G/C_G(A)| \leq 2$.*

Proof. For any $x \in G$ there is $N \lhd G$ which is commensurable with $\langle x \rangle$. Then $n := |N : (N \cap \langle x \rangle)|$ is finite. Thus $N^{n!} \leq \langle x \rangle$ where $N^{n!} \lhd G$. Hence G/A is periodic.

It is clear that A is abelian. Let $g \in G$. If $\langle a \rangle \lhd G$ and a has infinite order, then there is $\epsilon_a = \pm 1$ such that $a^g = a^{\epsilon_a}$. On the other hand, by Lemma 2.5, there is $\epsilon = \pm 1$ such that for each $a \in A$ there is a periodic element $t_a \in A$ such that $a^g = a^{\epsilon} t_a$. It follows $a^{\epsilon_a - \epsilon} = t_a$. Therefore $\epsilon_a = \epsilon$ is independent of a, as required. □

Proof of Theorem A. Recall from the Introduction that all subgroups of G are subnormal-by-finite. If G is periodic, then, by the above quoted results in [7] and [3] respectively, we may assume that G is locally nilpotent and soluble. Then, by Proposition 2.4, only finitely many primary components are non-abelian. Thus we may assume G is a p-group and apply Lemma 3.3 and Lemma 3.8. It follows that G is finite-by-abelian-by-finite.

To treat the general case, consider $A = A(G)$ using the notation of Lemma 3.9. We may assume A is central in G. Let V be a torsion-free subgroup of A such that A/V is periodic. Then G/V is locally finite and we may apply the above. Thus there is a series $V \leq G_0 \leq G_1 \leq G$ such that G acts trivially on V, G_1/G_0 is abelian, while G_0/V and G/G_1 are finite. Then we can assume $G = G_1$ and note that the stabilizer S of the series has finite index. Since S is nilpotent (by Ph. Hall Theorem) we can assume that $G = S$ is nilpotent. If T is the torsion subgroup of G, then VT/T is contained in the center of

G/T. Since all factors of the upper central series of G/T are torsion-free we have G/T is abelian. Thus $G' \leq T \cap G_0$ is finite. $\square$

Proof of Theorem C. If the statement is false, by Theorem A we may assume that there is a counterexample G that is periodic and not locally finite. Also we may assume G is finitely generated and infinite. Let R be the locally finite radical of G. By Theorem A again, R is finite-by-abelian-by-finite. By Theorem B(i), there is a finite subgroup $G_0 \triangleleft G$ such that R/G_0 is abelian-by-finite. We may assume $G_0 = 1$, so that R is abelian-by-finite.

We claim that $\bar{G} := G/R$ has finite exponent at most $(n+1)!$ where n is such that $n \geq \delta_G(H)$ for each $H \leq G$. In fact, for each $x \in \bar{G}$, there is $\bar{N} \triangleleft \bar{G}$ such that $|\bar{N} : (\bar{N} \cap \langle x \rangle)| \leq n$. Thus $\bar{N}^{n!} \leq \langle x \rangle$ and $\bar{N}^{n!} \triangleleft G$. Hence $\bar{N}^{n!} = 1$ and $x^{n \cdot n!} = 1$.

By the positive answer (for all exponents) to the Restricted Burnside Problem, there is a positive integer k such that every finite image of $\bar{G}$ has order at most k. Since $\bar{G}$ is finitely generated, this means that the finite residual $\bar{K}$ of $\bar{G}$ has finite index and is finitely generated as well. Since also $\bar{G}$ is locally graded (see [8]), we have $\bar{K} = 1$ and $\bar{G}$ is finite. Therefore G is abelian-by-finite, a contradiction. $\square$

References

[1] J.T. Buckley, J.C. Lennox, B.H. Neumann, H. Smith, J. Wiegold, Groups with all subgroups normal-by-finite, J. Aust. Math. Soc. A 59 (3) (1995) 384–398.

[2] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, Rend. Semin. Mat. Univ. Padova 81 (1989) 107–149.

[3] C. Casolo, Groups in which all subgroups are subnormal-by-finite, Adv. Group Theory Appl. 1 (2016) 33–45, https://doi.org/10.4399/97888548908173.

[4] U. Dardano, S. Rinauro, Inertial automorphisms of an abelian group, Rend. Semin. Mat. Univ. Padova 127 (2012) 213–233, https://doi.org/10.4171/RSMUP/127-11.

[5] U. Dardano, S. Rinauro, Inertial endomorphisms of an abelian group, Ann. Mat. Pura Appl. 195 (1) (2016) 219–234, https://doi.org/10.1007/s10231-014-0459-6.

[6] S. Franciosi, F. de Giovanni, M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, Comm. Algebra 23 (14) (1995) 5483–5497.

[7] H. Heineken, Groups with neighbourhood conditions for certain lattices, Note Mat. 1 (1996) 131–143.

[8] P. Longobardi, M. Maj, H. Smith, A note on locally graded groups, Rend. Semin. Mat. Univ. Padova 94 (1995) 275–277.

[9] W. Möhres, Torsionsgruppen, deren Untergruppen alle subnormal sind, Geom. Dedicata 31 (1989) 237–244.

[10] B.H. Neumann, Groups with finite classes of conjugate subgroups, Math. Z. 63 (1955) 76–96.

[11] D.J.S. Robinson, A Course in the Theory of Groups, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.

[12] H. Smith, J. Wiegold, Locally graded groups with all subgroups normal-by-finite, J. Aust. Math. Soc. A 60 (2) (1996) 222–227.