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# On the completion of the mapping class group of genus two



Tatsunari Watanabe

Department of Mathematics, Purdue University, West Lafayette, IN, United States

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## ABSTRACT

In this paper, we will study the Lie algebra of the pronipotent radical of the relative completion of the mapping class group of genus two. In particular, we will partially determine a minimal presentation of the Lie algebra by determining the generators and bounding the degree of the relations of it.

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## 1. Introduction

Let  $X$  be a complex smooth algebraic variety. Fix a point  $x$  in  $X$ . The Lie algebra  $\mathfrak{g}$  of the unipotent (Malcev) completion over  $\mathbb{Q}$  of  $\pi_1(X, x)$  carries a natural  $\mathbb{Q}$ -MHS. A minimal presentation of  $\mathrm{Gr}_{\bullet}^W \mathfrak{g}$  was studied by Morgan in [14]. It is generated in weight  $-1$  and  $-2$  by  $H_1(X, \mathbb{Q})$  and has relations in weight  $-2$ ,  $-3$ , and  $-4$  coming from  $H_2(X, \mathbb{Q})$ . Denote by  $\mathcal{M}_g$  the moduli stack of smooth projective curves of genus  $g$  over  $\mathbb{C}$ .

E-mail address: [twatana@purdue.edu](mailto:twatana@purdue.edu).

URL: <http://www.math.purdue.edu/people/bio/twatana>.

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Assume that  $g \geq 2$ . It will be considered as an orbifold in this paper. It is a natural problem to look for a presentation of the Lie algebra of the unipotent completion of the orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{M}_g, x)$  of  $\mathcal{M}_g$ , since  $\mathcal{M}_g$  is smooth and has a finite cover by a smooth algebraic variety. The issue is that the rational homology  $H_1(\mathcal{M}_g, \mathbb{Q})$  vanishes and hence the unipotent completion is trivial. Instead, we will consider the relative completion of a discrete group due to Deligne, which generalizes the unipotent completion of a group. Associated to the universal family  $f : \mathcal{C}_g \rightarrow \mathcal{M}_g$ , there is a natural monodromy representation in the orbifold sense

$$\rho_x : \pi_1^{\text{orb}}(\mathcal{M}_g, x) \rightarrow \text{Sp}(H^1(C, \mathbb{Q})),$$

where  $C$  is the fiber of  $f$  over  $x$ . The orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{M}_g, x)$  is naturally isomorphic to the mapping class group  $\Gamma_g$ . It is the group of isotopy classes of orientation-preserving diffeomorphisms of a compact oriented surface  $S_g$  of genus  $g$ . The relative completion  $\mathcal{G}_g$  of  $\Gamma_g$  with respect to  $\rho_x$  is the inverse limit of all algebraic group  $G$  over  $\mathbb{Q}$  that is an extension of  $\text{Sp}_g$  by a unipotent group over  $\mathbb{Q}$  and such that there is a Zariski-dense representation  $\phi_G : \Gamma_g \rightarrow G(\mathbb{Q})$  making the diagram

$$\begin{array}{ccc} \Gamma_g & & \\ \downarrow \phi_G & \searrow \rho_x & \\ G(\mathbb{Q}) & \longrightarrow & \text{Sp}(H^1(C, \mathbb{Q})) \xrightarrow{\cong} \text{Sp}_g(\mathbb{Q}) \end{array}$$

commute, where the isomorphism  $\text{Sp}(H^1(C, \mathbb{Q})) \cong \text{Sp}_g(\mathbb{Q})$  is given by fixing a symplectic basis for  $H^1(C, \mathbb{Q})$ . The completion  $\mathcal{G}_g$  is a proalgebraic group over  $\mathbb{Q}$  that is an extension of  $\text{Sp}_g$  with a prounipotent group  $\mathcal{U}_g$  over  $\mathbb{Q}$ . In [8], Hain further developed the theory of relative completion and constructed a canonical MHS on the Lie algebra  $\mathfrak{u}_g$  of  $\mathcal{U}_g$ . In this paper, we will study the associated graded Lie algebra  $\text{Gr}_{\bullet}^W \mathfrak{u}_g$  for  $g = 2$ .

Hain proved in [6,9] that for  $g \geq 4$ , the Lie algebra  $\mathfrak{u}_g$  is generated in weight  $-1$  and quadratically presented and that for  $g = 3$ , it is generated in weight  $-1$  and admits quadratic and cubic relations. Furthermore, Hain used this result to prove that the Lie algebra  $\mathfrak{t}_g$  of the unipotent completion of the Torelli group  $T_g$  for  $g \geq 3$  is finitely presented with quadratic and possible cubic relations. This gave an important insight on the open problem of the finite presentation of  $T_g$  for  $g \geq 3$ . It is known that  $T_g$  is finitely generated for  $g \geq 3$ , but for any  $g \geq 3$ , it remains open whether or not  $T_g$  is finitely presented. For the case when  $g = 2$ , Mess showed in [13] that  $T_2$  is countably generated free group. Therefore, the Lie algebra  $\mathfrak{t}_2$  is simply the inverse limit of all nilpotent quotients of the free Lie algebra generated by  $H_1(T_2, \mathbb{Q})$ . However, our main result suggests that the Lie algebras  $\mathfrak{t}_2$  and  $\mathfrak{u}_2$  are essentially different, while for  $g \geq 3$ ,  $\mathfrak{t}_g$  is the central extension of  $\mathfrak{u}_g$  by  $\mathbb{G}_a$  (see [5]).

Hain used the Johnson's fundamental work [3] to determine the generators and Kanbanov's theorem [11] on  $H^2(\mathcal{M}_g, \mathbb{V})$ , where  $\mathbb{V}$  is a symplectic local system, to bound

the degree of relations, which then allowed him to determine explicitly the relations by producing certain commuting two elements in the Torelli group. In this paper, we will partially extend Hain's work to the case when  $g = 2$ . More precisely, we will determine the generators of the graded Lie algebra  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_2$  and bound the degree of its relations.

Let  $V_{a+b}$  denote the irreducible rational representation of  $\mathrm{Sp}_2$  corresponding to the partition  $n = a + b$  with  $a \geq b$  of a nonnegative integer  $n$ . Then we have the local system  $\mathbb{V}_{a+b}$  over  $\mathcal{M}_2$  that comes from the representation  $V_{a+b}$ . We can consider  $\mathbb{V}_{a+b}$  as a variation of Hodge structure of weight  $a + b$ . Fix a point  $x = [C]$  in  $\mathcal{M}_2$ . Then the fiber of  $\mathbb{V}_{a+b}$  over  $x$  is a Hodge structure of weight  $a + b$  and as an  $\mathrm{Sp}_2$  representation, it is isomorphic to  $V_{a+b}$ . We simply denote the fiber by  $V_{a+b}$ . Let  $V_{a+b}(m)$  denote the Tate twist of  $V_{a+b}$  by  $\mathbb{Q}(m)$ . It is a Hodge structure of weight  $a + b - 2m$ . For the chosen point  $x$  in  $\mathcal{M}_2$ , we have the Lie algebra  $\mathfrak{u}_2$  viewed as a mixed Hodge structure. Our main result is the following.

**Theorem 1.** *There is a Hodge weight preserving isomorphism of graded Lie algebras*

$$\mathbb{L}(V_{2+2}(3))/R \cong \mathrm{Gr}_{\bullet}^W \mathfrak{u}_2$$

*in the category of  $\mathrm{Sp}_2$ -representations. The Lie ideal  $R$  is generated by a subrepresentation of*

$$\bigoplus_{a+b} V_{a+b}(a+2)$$

*where the sum is taken over the partitions  $a + b$  of 2, 4, 6, 10 with  $a > b$ . In particular,  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_2$  is finitely presented with possibly the relations of cubic to septic degree.*

In Section 9, a table of the multiplicity in  $\mathbb{L}(V_{2+2}(3))$  of each  $V_{a+b}(a+2)$  appearing in the sum  $\bigoplus_{a+b} V_{a+b}(a+2)$  from Theorem 1 is given.

We can also consider the relative completion  $\mathcal{G}_{g,n}$  of the mapping class group of type  $(g, n)$ , denoted by  $\Gamma_{g,n}$ . For a fixed  $x = [C]$  in  $\mathcal{M}_{g,n}$ , denote the Lie algebra of  $\mathcal{U}_{g,n}$  by  $\mathfrak{u}_{g,n}$ . Let  $\pi_{g,n}$  denote the fundamental group of the configuration space of  $n$  points on  $C$  and  $\mathfrak{p}_{g,n}$  be the Lie algebra of the unipotent completion of  $\pi_{g,n}$ . The Lie algebras  $\mathfrak{p}_{g,n}$ ,  $\mathfrak{u}_{g,n}$ , and  $\mathfrak{u}_g$  admit canonical mixed Hodge structures and for  $g \geq 2$ , there is an exact sequence of graded Lie algebras

$$0 \rightarrow \mathrm{Gr}_{\bullet}^W \mathfrak{p}_{g,n} \rightarrow \mathrm{Gr}_{\bullet}^W \mathfrak{u}_{g,n} \rightarrow \mathrm{Gr}_{\bullet}^W \mathfrak{u}_g \rightarrow 0.$$

Since  $\mathrm{Gr}_{\bullet}^W \mathfrak{p}_{g,n}$  is finitely generated and presented, we have

**Corollary 2.** *For  $n \geq 0$ , the graded Lie algebra  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_{2,n}$  is finitely generated and presented.*

**Theorem 1** will follow immediately from [Corollary 6.5](#) and [Corollary 7.4](#). Our approach is fundamentally different from the higher genus case by Hain. Our main result is largely due to Petersen’s work [\[17\]](#) on the cohomology of symplectic local systems on  $\mathcal{A}_2$ . Together with Petersen’s nonvanishing theorem [\[18, Thm. 5.1\]](#) and an extension of it, we are able to determine the generators and bound the degree of the relations. While the computations for  $g \geq 3$  are topological in nature, our approach is arithmetic. This is because,  $\mathcal{M}_2$  being the complement of the locus  $\mathcal{D}_{1,1}$  of the products of two elliptic curves in  $\mathcal{A}_2$ , the cohomology of  $\mathcal{M}_2$  is determined by that of  $\mathcal{D}_{1,1}$  and  $\mathcal{A}_2$  by the Gysin sequence.

In [Sections 2, 3, and 4](#), we will review briefly the mapping class groups, the moduli stacks of smooth projective curves and principally polarized abelian varieties, and set notations for our basic objects for the case  $g = 2$ . In [Section 5](#), we will review the theory of relative completion along its Hodge theory aspect developed by Hain and then discuss a minimal presentation of  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_g$  for  $g \geq 3$ . In [Section 6](#), we will determine the generators of  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2$  and in [Section 7](#), we will prove a nonvanishing theorem for the Gysin sequence for the cohomology groups of  $\mathcal{M}_2$ ,  $\mathcal{A}_2$ , and  $\mathcal{D}_{1,1}$  to bound the degree of the relations of  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2$ . In [Section 8](#), we will prove [Corollary 2](#) and also describe the generators of  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_{2,n}$ .

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## 2. Mapping class groups and Torelli groups

Let  $S_g$  be a compact oriented surface of genus  $g$ . Let  $P$  be a subset of  $S_g$  consisting of  $n$  distinct points on  $S_g$ . The group of the orientation-preserving diffeomorphisms of  $S_g$  fixing  $P$  pointwise will be denoted by  $\mathrm{Diff}^+(S_g, P)$ , which is given open-compact topology. The mapping class group of type  $(g, n)$ , denoted by  $\Gamma_{g,n}$ , is defined to be the group of path-connected components of  $\mathrm{Diff}^+(S_g, P)$ :

$$\Gamma_{g,n} = \pi_0(\mathrm{Diff}^+(S_g, P)).$$

We easily see that the group  $\Gamma_{g,n}$  is independent of the choice of the reference surface  $S_g$  and  $P$ . In this paper, we will always assume that  $2g - 2 + n > 0$ . When  $n = 0$ ,  $\Gamma_{g,0}$  will be denoted by  $\Gamma_g$ .

The first integral cohomology group  $H^1(S_g, \mathbb{Z})$  is equipped with the cup product pairing

$$\langle \ , \ \rangle : H^1(S_g, \mathbb{Z}) \otimes H^1(S_g, \mathbb{Z}) \rightarrow H^2(S_g, \mathbb{Z}) \cong \mathbb{Z},$$

that is a nondegenerate, skew symmetric bilinear pairing. The mapping class group  $\Gamma_{g,n}$  acts on the first homology group  $H_1(S_g, \mathbb{Z})$  and hence on  $H^1(S_g, \mathbb{Z})$ , preserving the pairing  $\langle \cdot, \cdot \rangle$ . Therefore, there is a natural homomorphism

$$\rho : \Gamma_{g,n} \rightarrow \text{Aut}(H^1(S_g, \mathbb{Z}), \langle \cdot, \cdot \rangle) = \text{Sp}(H^1(S_g, \mathbb{Z})) = \text{Sp}(H_1(S_g, \mathbb{Z})).$$

It is well-known that if  $g \geq 1$ , then the homomorphism  $\rho$  is surjective. The kernel of  $\rho$  is called the Torelli group, denoted by  $T_{g,n}$ . The group  $T_1$  is trivial, Mess showed in [13] that  $T_2$  is an infinitely generated free group, and Johnson proved in [4] that  $T_g$  is finitely generated for  $g \geq 3$ . It is not known whether  $T_g$  is finitely presented or not for  $g \geq 3$ . Hain showed in [9] that the Lie algebra  $\mathfrak{t}_g$  of the Malcev completion of  $T_g$  is finitely presented.

### 3. Moduli stacks $\mathcal{M}_{g,n}$ and $\mathcal{A}_g$

By a compact complex curve  $C$  of genus  $g$ , we mean a compact Riemann surface of genus  $g$ , and a marking is an orientation-preserving diffeomorphism  $f : S \rightarrow C$  to a compact complex curve. The Teichmüller space of marked  $n$ -pointed compact curves of genus  $g$  will be denoted by  $\mathcal{X}_{g,n}$ . As a set, it consists of the isotopy classes of the markings. The mapping class group  $\Gamma_{g,n}$  acts on  $\mathcal{X}_{g,n}$  by  $[\phi] : [f] \mapsto [f \circ \phi^{-1}]$ . The fact that this action is properly discontinuous and virtually free imply that the quotient

$$\mathcal{M}_{g,n} = \mathcal{X}_{g,n} / \Gamma_{g,n}$$

is a complex orbifold and that it is covered by smooth complex varieties.

Let  $\mathfrak{h}_g$  be the Siegel upper half-space. It consists of the complex  $g \times g$  symmetric matrices  $Z$  such that  $\text{Im}(Z)$  is positive definite. It is the moduli space of principally polarized abelian varieties equipped with a choice of a symplectic basis and comes with an  $\text{Sp}_g(\mathbb{Z})$ -action:

$$\alpha \cdot Z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

The moduli stack  $\mathcal{A}_g$  of principally polarized abelian varieties is the quotient

$$\mathcal{A}_g = \mathfrak{h}_g / \text{Sp}_g(\mathbb{Z}).$$

It is considered as a complex orbifold. In Section 7, we will also view it as a locally symmetric space for  $\text{Sp}_g(\mathbb{R})$  and consider its Baily–Borel compactification and the boundary cohomology at the zero-dimensional stratum.

The Torelli map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  as stacks are a 2-to-1 map ramified along the hyperelliptic locus,  $\mathcal{H}_g$ , for  $g \geq 3$  and it is an open immersion for  $g = 2$ . For  $g = 1$ , the Torelli map

$\mathcal{M}_{1,1} \rightarrow \mathcal{A}_1$  is an isomorphism of stacks. In this paper, we especially pay our attention to the case  $g = 2$ . Via the Torelli map  $i : \mathcal{M}_2 \rightarrow \mathcal{A}_2$ , we may consider  $\mathcal{M}_2$  as an open substack of  $\mathcal{A}_2$ . The complement in  $\mathcal{A}_2$  of  $\mathcal{M}_2$  is a smooth divisor consisting of the unordered products of two elliptic curves, which we will denote by  $\mathcal{D}_{1,1}$ .

#### 4. Local systems over $\mathcal{M}_2$ , $\mathcal{A}_2$ , and $\mathcal{D}_{1,1}$

Let  $\pi : \mathcal{Y} \rightarrow \mathcal{A}_2$  be the universal family over  $\mathcal{A}_2$ . Let  $\mathbb{V}$  be the sheaf  $R^1\pi_*\mathbb{Q}$  over  $\mathcal{A}_2$ . That the morphism  $\pi$  is smooth and proper implies that  $\mathbb{V}$  is a polarized variation of Hodge structure of weight 1. Let  $x$  be in  $\mathcal{A}_2$  and  $A$  be the fiber of  $\pi$  over  $x$ . The monodromy action of  $\pi_1^{\text{orb}}(\mathcal{A}_2, x)$  on  $H^1(A, \mathbb{Q})$  corresponds to the inclusion

$$\rho_x : \pi_1^{\text{orb}}(\mathcal{A}_2, x) = \text{Sp}_2(\mathbb{Z}) \rightarrow \text{Sp}_2(\mathbb{Q}).$$

A finite dimensional  $\text{Sp}_2(\mathbb{Q})$ -representation  $M$  defines a local system  $\mathbb{M}$  over  $\mathcal{A}_2$  via the homomorphism  $\rho_x$ . In fact, the local system  $\mathbb{M}$  underlies a variation of Hodge structure over  $\mathcal{A}_2$ .

We will briefly review the  $\text{Sp}_2$ -representation theory. We consider  $\text{Sp}_2$  as a linear algebraic group over  $\mathbb{Q}$ . The maximal torus  $T$  of  $\text{Sp}_2$  consists of the diagonal matrices  $t$  of the form  $\text{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})$  and the fundamental dominant weights  $\lambda_j(t) : T \rightarrow \mathbb{R}$  are given by

$$\lambda_1(t) = t_1 \text{ and } \lambda_2(t) = t_1 t_2.$$

Then there is a bijection between the positive integral linear combinations  $\alpha = n_1\lambda_1 + n_2\lambda_2$  and the set of isomorphism classes of irreducible  $\text{Sp}_2$ -representations  $M_{[n_1, n_2]}$  with highest weight  $\alpha$ .

We may also produce irreducible representations by Weyl's construction (see [2]). Let  $V$  be a 4-dimensional vector space over  $\mathbb{Q}$  equipped with a nondegenerate, skew symmetric bilinear form  $V \otimes V \rightarrow \mathbb{Q}$ . For a partition  $\lambda : n = a + b$  of a nonnegative integer  $n$  with  $a \geq b \geq 0$ , there is an irreducible representation of  $\text{Sp}_2$ , which we denote by  $V_{a+b}$ . The representation  $V_{a+b}$  can be realized as a summand in  $V^{\otimes n}$  which is specified by contraction maps and the Schur functor corresponding to the partition  $\lambda$ . The irreducible representation  $V_{a+b}$  corresponds to  $M_{[a-b, b]}$ . In this paper, we will use the partition notation.

Note that there is an isomorphism of local systems  $\mathbb{V} \cong \mathbb{V}_1$ . Moreover, we obtain the local system  $\mathbb{V}_{a+b}$  corresponding to the irreducible representation  $V_{a+b}$ . As  $\mathbb{V}$  is a variation of Hodge structure of weight 1 over  $\mathcal{A}_2$ , Weyl's construction implies that  $\mathbb{V}_{a+b}$  carries a variation of Hodge structure of weight  $n = a + b$ . By pulling back, we also obtain variations of Hodge structures over  $\mathcal{M}_2$  and  $\mathcal{D}_{1,1}$ , which we will also denote by  $\mathbb{V}_{a+b}$ . Therefore, the cohomology  $H^\bullet(X, \mathbb{V}_{a+b})$  with  $X \in \{\mathcal{M}_2, \mathcal{A}_2, \mathcal{D}_{1,1}\}$  carry natural Mixed Hodge structures, and so do the compactly supported cohomology  $H_c^\bullet(X, \mathbb{V}_{a+b})$ .

We will abbreviate Hodge structure, Mixed Hodge structure, and variations of them by HS, MHS, VHS, and VMHS, respectively.

We will need to understand the restrictions of  $\mathbb{V}_{a+b}$  to  $\mathcal{D}_{1,1}$ . As an orbifold,  $\mathcal{D}_{1,1}$  is the quotient

$$\mathcal{D}_{1,1} = \mathfrak{h}_1 \times \mathfrak{h}_1 / (\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \rtimes S_2),$$

where  $S_2$  interchanges the components of  $\mathrm{SL}_2 \times \mathrm{SL}_2$ . The space  $\mathfrak{h}_1 \times \mathfrak{h}_1$  is considered to be the subspace of  $\mathfrak{h}_2$ :

$$\mathfrak{h}_1 \times \mathfrak{h}_1 = \left\{ \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \mid \mathrm{Im}(z_i) > 0, i = 1, 2 \right\}.$$

The wreath product  $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \rtimes S_2$  embeds into  $\mathrm{Sp}_2(\mathbb{Z})$ , consisting of the matrices

$$\begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & a_1 & 0 & b_1 \\ a_2 & 0 & b_2 & 0 \\ 0 & c_1 & 0 & d_1 \\ c_2 & 0 & d_2 & 0 \end{bmatrix}, \quad \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Thus the restriction of the local system  $\mathbb{V}_{a+b}$  to  $\mathcal{D}_{1,1}$  corresponds to the restriction of the  $\mathrm{Sp}_2$ -representation  $V_{a+b}$  to the subgroup  $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \rtimes S_2$  of  $\mathrm{Sp}_2$ . So we need to know how  $V_{a+b}$  decomposes as the sum of the irreducible  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rtimes S_2$ -representations. Petersen gives a branching formula in [16, Prop.3.4] for this inclusion.

Denote the standard representation of  $\mathrm{SL}_2$  by  $H$ . The symmetric tensor power  $\mathrm{Sym}^m H$  will be denoted by  $H_m$ . These are the irreducible representations of  $\mathrm{SL}_2$ . The irreducible representations of  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rtimes S_2$  are given by the following three types:

- (i)  $U_{a,b} = H_a \otimes H_b \oplus H_b \otimes H_a$ ,  $\sigma : u \otimes v + v' \otimes u' \mapsto u' \otimes v' + v \otimes u$ , with  $a \neq b$ ,
- (ii)  $U_a^+ = H_a \otimes H_a$ ,  $\sigma : u \otimes v \mapsto v \otimes u$ ,
- (iii)  $U_a^- = H_a \otimes H_a$ ,  $\sigma : u \otimes v \mapsto -v \otimes u$ ,

where  $S_2 = \langle \sigma \rangle$ . Petersen's branching formula explicitly describes the restriction  $\mathrm{Res}_{\mathrm{SL}_2 \times \mathrm{SL}_2 \rtimes S_2}^{\mathrm{Sp}_2} V_{a+b}$  as the sum of the irreducible representations of the above three types.

**Example 4.1.** The key representations that we will consider are  $V_{2l+2l}$  with  $l \geq 1$  and  $V_{a+b}$  with  $a+b$  even and  $a > b$ . The restriction  $\mathrm{Res}_{\mathrm{SL}_2 \times \mathrm{SL}_2 \rtimes S_2}^{\mathrm{Sp}_2} V_{2l+2l}$  contains a single copy of the trivial representation  $U_0^+$  as a summand and  $\mathrm{Res}_{\mathrm{SL}_2 \times \mathrm{SL}_2 \rtimes S_2}^{\mathrm{Sp}_2} V_{a+b}$  contains a single copy of  $U_{a-b,0}$ . The corresponding local systems  $\mathbb{U}_{a,b}$ ,  $\mathbb{U}_a^+$ , and  $\mathbb{U}_a^-$  over  $\mathcal{D}_{1,1}$  also carry VHSs, and in order to be consistent with Hodge weights, we need to apply Tate twists. That is,  $U_0^+$  is twisted by  $-2l$ , i.e.,  $U_0^+(-2l)$  and  $U_{a-b,0}$  by  $-b$ ,  $U_{a-b,0}(-b)$ .

As an orbifold, the moduli stack  $\mathcal{M}_{1,1}$  is the quotient  $\mathcal{M}_{1,1} = \mathfrak{h}_1 / \mathrm{SL}_2(\mathbb{Z})$  with  $\pi_1^{\mathrm{orb}}(\mathcal{M}_{1,1}) = \mathrm{SL}_2(\mathbb{Z})$ . Each representation  $H_m$  defines a local system  $\mathbb{H}_m$  over  $\mathcal{M}_{1,1}$

underlying a VHS of weight  $m$ . Recall that the Torelli map  $\mathcal{M}_{1,1} \rightarrow \mathcal{A}_1$  is an isomorphism of stacks. The cohomology of the local systems over  $\mathcal{D}_{1,1}$  can be computed by pulling back them along the map  $\mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{D}_{1,1}$ .

**Remark 4.2.** There are isomorphisms

- (i)  $H^\bullet(\mathcal{D}_{1,1}, \mathbb{U}_{a,b}) \cong H^\bullet(\mathcal{M}_{1,1}, \mathbb{H}_a) \otimes H^\bullet(\mathcal{M}_{1,1}, \mathbb{H}_b)$ ,
- (ii)  $H^\bullet(\mathcal{D}_{1,1}, \mathbb{U}_a^+) \cong \mathrm{Sym}^2 H^\bullet(\mathcal{M}_{1,1}, \mathbb{H}_a)$ , and
- (iii)  $H^\bullet(\mathcal{D}_{1,1}, \mathbb{U}_a^-) \cong \Lambda^2 H^\bullet(\mathcal{M}_{1,1}, \mathbb{H}_a)$ .

## 5. Review of relative completion of $\Gamma_{g,n}$ and a minimal presentation of the Lie algebra $\mathrm{Gr}_\bullet^W \mathfrak{u}_g$

A detailed treatment of the theory of relative completion can be found in [8] and [10]. Here, we will summarize and state the properties of the theory needed for our main result.

Assume that  $F$  is a field of characteristic zero. By an algebraic group over  $F$ , we mean an affine group scheme  $G$  of finite type over  $F$ . A proalgebraic group over  $F$  is the projective limit of algebraic groups over  $F$ . The set of  $F$ -rational points of  $G$  is denoted by  $G(F)$ .

Consider the following data

- (i)  $R$  is a reductive group over  $F$ , and
- (ii)  $\Gamma$  is a discrete group with a Zariski-dense homomorphism  $\rho : \Gamma \rightarrow R(F)$ .

The relative completion of  $\Gamma$  with respect to  $\rho$  is a pair  $(\mathcal{G}, \tilde{\rho})$  of a proalgebraic group  $\mathcal{G}$  over  $F$  that is an extension of  $R$  by a prounipotent group  $\mathcal{U}$  over  $F$  and a natural map  $\tilde{\rho} : \Gamma \rightarrow \mathcal{G}(F)$ , satisfying the universal property: If  $G$  is a proalgebraic group over  $F$  that is also an extension of  $R$  by a prounipotent group  $U$  over  $F$  such that  $\rho$  factors through  $G(F) \rightarrow R(F)$  with Zariski-dense image in  $G(F)$ , then there exists a unique morphism  $\phi : \mathcal{G} \rightarrow G$  of proalgebraic groups over  $F$  such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\rho}} & \mathcal{G}(F) \\ \downarrow & \searrow \phi & \downarrow \\ G(F) & \longrightarrow & R(F) \end{array}$$

commutes. One of the key properties of relative completion we will need is the following. Denote the Lie algebra of  $\mathcal{U}$  by  $\mathfrak{u}$ . For our purpose, we take  $R = \mathrm{Sp}_g$  and  $F = \mathbb{Q}$ , and the cohomology we use here is the continuous cohomology of a discrete group and pro-Lie algebras.



**Proposition 5.1.** *For each partition  $\lambda$ , let  $V_\lambda$  be the corresponding irreducible representation of  $\mathrm{Sp}_g$ . Then*

(i) *there is a natural  $\mathrm{Sp}_g$ -invariant isomorphism*

$$H^1(\mathfrak{u}) \cong \bigoplus_{\lambda} H^1(\Gamma, V_\lambda) \otimes V_\lambda^*,$$

and

(ii) *there is a natural  $\mathrm{Sp}_g$ -invariant injection*

$$H^2(\mathfrak{u}) \rightarrow \bigoplus_{\lambda} H^2(\Gamma, V_\lambda) \otimes V_\lambda^*.$$

We take the relative completion of  $\Gamma_{g,n}$  with respect to the homomorphism  $\rho : \Gamma_{g,n} \rightarrow \mathrm{Sp}(H^1(S_g, \mathbb{Q}))$ . Since the image of  $\rho$  is  $\mathrm{Sp}(H^1(S_g, \mathbb{Z}))$ ,  $\rho$  has a Zariski-dense image. Denote the completion of  $\Gamma_{g,n}$  by  $\mathcal{G}_{g,n}$  and its pronilpotent radical by  $\mathcal{U}_{g,n}$ . The Lie algebras of  $\mathcal{G}_{g,n}$  and  $\mathcal{U}_{g,n}$  will be denoted by  $\mathfrak{g}_{g,n}$  and  $\mathfrak{u}_{g,n}$ .

For each  $x$  in  $\mathcal{M}_{g,n}$ , there is a monodromy representation

$$\rho_x : \pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}, x) \rightarrow \mathrm{Sp}(H^1(C, \mathbb{Q})),$$

where  $C$  is the fiber of the universal family over the point  $x$ . The natural isomorphism  $\Gamma_{g,n} \cong \pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}, x)$  identifies this monodromy action with the homomorphism  $\rho$ . Denote the completion of  $\rho_x$  by  $\mathcal{G}_{g,n}(x)$  and its pronilpotent radical by  $\mathcal{U}_{g,n}(x)$ . Their Lie algebras are denoted by  $\mathfrak{g}_{g,n}(x)$  and  $\mathfrak{u}_{g,n}(x)$ , respectively. The following result in [8] implies that for each choice of  $x$ , the Lie algebras  $\mathfrak{g}_{g,n}(x)$  and  $\mathfrak{u}_{g,n}(x)$  carry canonical  $\mathbb{R}$ -MHSs.

**Theorem 5.2** (Hain). *Suppose that  $X$  is a smooth complex variety and that  $\mathbb{V}$  is a polarized variation of  $\mathbb{Q}$ -HS over  $X$  of geometric origin. If the monodromy action*

$$\rho_x : \pi_1(X, x) \rightarrow \mathrm{Aut}(\mathbb{V}_x, \langle \ , \ \rangle)$$

*has a Zariski-dense image, then the Lie algebra  $\mathfrak{g}$  of the completion of  $\pi_1(X, x)$  and the pronilpotent Lie algebra  $\mathfrak{u}$  of the pronilpotent radical of the completion admit natural  $\mathbb{Q}$ -MHSs, where brackets are morphisms of MHSs.*

**Remark 5.3.** The original statement is stated over  $\mathbb{R}$ . However, if the Zariski-closure of the image of  $\rho_x$  is defined over  $\mathbb{Q}$ , the  $\mathbb{R}$ -structure canonically lifts to a  $\mathbb{Q}$ -structure. For a concrete explanation, see [7, 4.3].

Fix a base point  $x$  in  $\mathcal{M}_{g,n}$ . We will omit the reference to the base point  $x$  from now on. Now, we review the presentations of the pronilpotent Lie algebras  $\mathfrak{u}_g$  for  $g \geq 3$ . We will need the following fact (cf. [9, Prop. 5.6], [10, Prop. 3.1]).

**Proposition 5.4.** *Let  $\mathfrak{n}$  be a negatively graded Lie algebra. Let  $\mathfrak{f}$  be the free Lie algebra generated by  $H_1(\mathfrak{n})$ , i.e.,  $\mathfrak{f} = \mathbb{L}(H_1(\mathfrak{n}))$ . Then there is an injection of graded vector spaces  $\gamma : H_2(\mathfrak{n}) \rightarrow [\mathfrak{f}, \mathfrak{f}]$  such that  $\mathfrak{n}$  admits a presentation*

$$\mathfrak{n} \cong \mathfrak{f} / \langle \mathrm{im} \gamma \rangle,$$

where  $\langle \mathrm{im} \gamma \rangle$  is the Lie ideal of  $\mathfrak{f}$  generated by the image of  $\gamma$ .  $\square$

The Lie algebra  $\mathfrak{u}_g$  is pronilpotent and hence graded by bracket. It is also graded by the weight filtration. For  $g \geq 3$ , it follows from the Johnson's work [3] that  $H_1(\mathfrak{u}_g)$  is a  $\mathbb{Q}$ -HS of weight  $-1$ , which together with the strictness of morphisms of MHSs implies that the weight filtration agrees with its lower central series (for a proof, see [9, Lem. 4.7]). The induced weight filtration on  $\mathbb{L}(H_1(\mathfrak{u}_g))$  is also the lower central series. The associated graded Lie algebra  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$  has a minimal presentation

$$\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g \cong \mathbb{L}(H_1(\mathfrak{u}_g)) / \langle \mathrm{Gr}_{\bullet}^W H_2(\mathfrak{u}_g) \rangle$$

as a graded Lie algebra in the category of the  $\mathrm{Sp}_g$ -representations. The key points for determining this presentation are the Johnson's work on the abelianization of the Torelli group  $T_{g,1}$  for determining  $H_1(\mathfrak{u}_g)$  and Kabanov's purity theorem [11] for bounding  $H_2(\mathfrak{u}_g)$ :

**Theorem 5.5** (Kabanov). *Let  $V_{\lambda}$  be an irreducible rational representation of  $\mathrm{Sp}_g$  and  $\mathbb{V}_{\lambda}$  be the corresponding VHS on  $\mathcal{M}_g$ . Then the weights of the MHS  $H^2(\mathcal{M}_g, \mathbb{V}_{\lambda})$  lie in  $\{2 + |\lambda|, 3 + |\lambda|\}$  for  $3 \leq g < 6$  and the weight is equal to  $2 + |\lambda|$  for  $g \geq 6$ , where  $|\lambda|$  denotes the weight of  $\mathbb{V}_{\lambda}$ .*

In Proposition 7.1, we will extend Kabanov's result to the case when  $g = 2$ . The relation between the MHSs  $H_{\bullet}(\mathfrak{u}_g)$  and  $H^{\bullet}(\mathcal{M}_g, \mathbb{V})$  for low degrees are given by

**Proposition 5.6** ([9, Prop. 7.1 & 7.3]). *Suppose that  $g \geq 1$ . If  $\mathbb{V}$  is the VHS over  $\mathcal{M}_{g,n}$  corresponding to an irreducible rational  $\mathrm{Sp}_g$ -representation  $V$ , then for each  $k$ , there is a morphism of MHSs*

$$\mathrm{Hom}_{\mathrm{Sp}_g}(H_k(\mathfrak{u}_{g,n}), V) \rightarrow H^k(\mathcal{M}_{g,n}, \mathbb{V}),$$

which is an isomorphism for  $k = 0, 1$  and an injection for  $k = 2$ .

**Remark 5.7.** In [9, Prop. 7.3], the result is stated for the case when  $g \geq 3$ , but since the canonical  $\mathbb{R}$ -MHS on  $\mathfrak{u}_{g,n}$  lifts to a  $\mathbb{Q}$ -MHS, the result can be stated for  $g \geq 1$  as well.

From this, it follows that  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$  has possibly quadratic and cubic relations for  $3 \leq g < 6$ , and only quadratic relations for  $g \geq 6$ . In [9], Hain determined all quadratic relations for  $g \geq 4$ , and in [6] showed that for  $g \geq 4$ ,  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_g$  is quadratically presented and determined the quadratic and cubic relations for  $g = 3$ . Hain used the fact that  $\mathrm{Gr}_{\bullet}^W \mathfrak{u}_3$  is not quadratically presented to show that  $\Gamma_3$  does not arise as a fundamental group of a compact Kähler manifold.

**Theorem 5.8** (*Hain*).

(i) If  $g \geq 3$ , then

$$H_1(\mathfrak{u}_g) \cong V_{1+1+1}.$$

(ii) There are isomorphisms

$$\mathrm{Gr}_{-2}^W H_2(\mathfrak{u}_g) \cong \begin{cases} V_0 & g = 3 \\ \text{the } \mathrm{Sp}_g\text{-complement of } V_{2+2} \text{ in } \mathbb{L}_2(V_{1+1+1}) & g \geq 4, \end{cases}$$

and

$$\mathrm{Gr}_{-3}^W H_2(\mathfrak{u}_g) \cong \begin{cases} \text{the } \mathrm{Sp}_g\text{-complement of } V_{3+1+1} \text{ in } \mathbb{L}_3(V_{1+1+1}) & g = 3 \\ 0 & g \geq 4, \end{cases}$$

where  $\mathbb{L}_k(V)$  denotes the  $k$ th component of the free Lie algebra  $\mathbb{L}(V)$ .

Note that Tate twists are omitted from the right-hand terms.

## 6. Generators of the Lie algebra $\mathfrak{u}_2$

By Proposition 5.1, the generators of  $\mathfrak{u}_2$  are determined by the cohomology groups  $H^1(\mathcal{M}_2, \mathbb{V}_{a+b})$ , which we will determine in this section.

Recall that the Torelli map  $i : \mathcal{M}_2 \rightarrow \mathcal{A}_2$  is an open immersion and the complement of the image is a divisor, denoted by  $\mathcal{D}_{1,1}$ , whose generic point is a product of two unordered elliptic curves. The product  $\mathcal{A}_1 \times \mathcal{A}_1$  is a two-sheeted cover of  $\mathcal{D}_{1,1}$ , where the symmetric group  $S_2$  simply interchanges the two elliptic components. Each irreducible rational representation  $V_{a+b}$  of  $\mathrm{Sp}_2$  defines a variation of Hodge structure  $\mathbb{V}_{a+b}$  of weight  $a + b$  on  $\mathcal{A}_2$ . The involution acts on  $\mathbb{V}_{a+b}$  by  $(-1)^{a+b} \mathrm{id}$ , while it acts trivially on the cohomology, and hence the cohomology of  $\mathbb{V}_{a+b}$  vanishes when  $a + b = \text{odd}$ . Thus we will only need to consider the case when  $a + b$  even. The restriction of  $\mathbb{V}_{a+b}$  to  $\mathcal{M}_2$  and  $\mathcal{D}_{1,1}$  are also denoted by  $\mathbb{V}_{a+b}$ . Associated to the open immersion  $i : \mathcal{M}_2 \rightarrow \mathcal{A}_2$ , there is a Gysin sequence

$$H^1(\mathcal{A}_2, \mathbb{V}_{a+b}) \rightarrow H^1(\mathcal{M}_2, \mathbb{V}_{a+b}) \rightarrow H^0(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) \rightarrow H^2(\mathcal{A}_2, \mathbb{V}_{a+b}).$$

The first key fact to remark is that the first term vanishes for  $a + b > 0$ . This follows from Raghunathan's vanishing theorem [20]:

**Theorem 6.1.** *Let  $V$  be a nontrivial irreducible rational representation of  $\mathrm{Sp}_g$  and  $\Gamma$  be a finite index subgroup of  $\mathrm{Sp}_g(\mathbb{Z})$ . If  $d < g$ , then  $H^d(\Gamma, V) = 0$ .*

The cohomology group  $H^1(\mathcal{A}_2, \mathbb{V}_{a+b}) \cong H^1(\mathrm{Sp}_2(\mathbb{Z}), V_{a+b})$  vanishes for  $a + b > 0$  by Theorem 6.1. Thus  $H^1(\mathcal{M}_2, \mathbb{V}_{a+b}) \cong H^1(\Gamma_2, V_{a+b})$  is the kernel of the Gysin map  $H^0(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) \rightarrow H^2(\mathcal{A}_2, \mathbb{V}_{a+b})$ . It follows immediately from [17, Thm. 2.1] that  $H^2(\mathcal{A}_2, \mathbb{V}_{a+b})$  vanishes except when  $a = b$  even and the dimension of the cusp forms for  $SL_2(\mathbb{Z})$  of weight  $a+b+4$  is positive, and that when  $a = b = 2l$  and  $l \geq 2$ ,  $H^2(\mathcal{A}_2, \mathbb{V}_{2l+2l})$  is the direct sum of copies of Tate HS  $\mathbb{Q}(-2l-1)$ , having dimension at least one. Firstly, we note that the branching formula [16, Prop. 3.4] implies that  $H^0(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}) = 0$  unless  $a = b$  even. Therefore, we will only need to consider the case when  $a = b = 2l$ . In this case, the restriction of  $\mathbb{V}_{2l+2l}$  to  $\mathcal{D}_{1,1}$  contains a single copy of the trivial local system  $\mathbb{U}_0^+(-2l)$  as a summand and thus  $H^0(\mathcal{D}_{1,1}, \mathbb{V}_{2l+2l})(-1) = \mathbb{Q}(-2l-1)$ . Secondly, we observe that for  $l = 1$ ,  $H^2(\mathcal{A}_2, \mathbb{V}_{2+2}) = 0$ , since there is no cusp form of weight 8, and hence  $H^1(\Gamma_2, V_{2+2}) = \mathbb{Q}(-3)$ . For  $l \geq 2$ , we need the following nonvanishing theorem by Petersen.

**Theorem 6.2** ([18, Thm. 5.1]). *For  $l \geq 2$ , the Gysin map*

$$H^0(\mathcal{D}_{1,1}, \mathbb{V}_{2l+2l})(-1) \rightarrow H^2(\mathcal{A}_2, \mathbb{V}_{2l+2l})$$

*is nontrivial.*

**Remark 6.3.** We will adopt Petersen's proof of this theorem to bound  $H_2(u_2)$ . The strategy and more details will be given in section 7.

Therefore, we have

**Corollary 6.4.** *There is an isomorphism of  $\mathbb{Q}$ -HSs*

$$H^1(\mathcal{M}_2, \mathbb{V}_{a+b}) \cong \begin{cases} \mathbb{Q}(-3) & \text{if } a = b = 2 \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Proposition 5.1 gives us

**Corollary 6.5.** *There is an isomorphism of  $\mathbb{Q}$ -HSs*

$$H_1(u_2) \cong V_{2+2}(3).$$

That  $H_1(u_2)$  is pure of weight  $-2$  and that the bracket is a morphism of MHSs give

**Corollary 6.6.** *For  $m \geq 1$ , we have*

$$W_{-2m+1}\mathfrak{u}_2 = W_{-2m}\mathfrak{u}_2 = L^m\mathfrak{u}_2,$$

where  $L^\bullet \mathfrak{n}$  denotes the lower central series of  $\mathfrak{n}$ .  $\square$

## 7. Bounds for the degree of the relations of the Lie algebra $\mathfrak{u}_2$

In this section, we will bound the cohomology  $H^2(\mathfrak{u}_2)$  by partially computing  $H^2(\mathcal{M}_2, \mathbb{V}_{a+b})$ . This will be done by considering the natural inclusion

$$\mathrm{Hom}_{\mathrm{Sp}}(H_2(\mathfrak{u}_2), V_{a+b}) \cong (H^2(\mathfrak{u}_2) \otimes V_{a+b})^{\mathrm{Sp}} \rightarrow H^2(\mathcal{M}_2, \mathbb{V}_{a+b})$$

that is a morphism of MHS. We will consider the part of the Gysin sequence

$$\begin{aligned} H^0(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) &\rightarrow H^2(\mathcal{A}_2, \mathbb{V}_{a+b}) \rightarrow H^2(\mathcal{M}_2, \mathbb{V}_{a+b}) \\ &\rightarrow H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) \rightarrow H^3(\mathcal{A}_2, \mathbb{V}_{a+b}). \end{aligned}$$

The following result is a key to bound  $H^2(\mathfrak{u}_2)$  and follows easily from Petersen's work [17].

**Proposition 7.1.** *Assume that  $a + b$  is even.*

- (i)  $H^2(\mathcal{M}_2, \mathbb{V}_{a+b}) = 0$  if  $a = b$  odd.
- (ii) *The possible weights of  $H^2(\mathcal{M}_2, \mathbb{V}_{a+b})$  are given by*

$$\begin{cases} 2a + 2 & \text{if } a = b \text{ even} \\ a + b + 3, \quad 2a + 4 = a + b + a - b + 4 & \text{if } a > b. \end{cases}$$

**Proof.** For (i), first we note that the cohomology group  $H^2(\mathcal{A}_2, \mathbb{V}_{a+a})$  vanishes [17, Thm. 2.1] and that Petersen's branching formula implies that  $H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+a})$  vanishes. Thus the result follows.

For (ii), first assume that  $a = b$  even. Then similarly, we have  $H^1(\mathcal{D}_{1,1}, V_{a+a}) = 0$ . Thus the restriction map  $H^2(\mathcal{A}_2, V_{a+a}) \rightarrow H^2(\mathcal{M}_2, \mathbb{V}_{a+a})$  is surjective. By [17, Thm. 2.1],  $H^2(\mathcal{A}_2, \mathbb{V}_{a+a})$  is pure of weight  $2a + 2$  and hence the claim. Next, assume that  $a > b$ . The branching formula [16, Prop. 3.4] by Petersen shows that the restriction of  $\mathbb{V}_{a+b}$  to  $\mathcal{D}_{1,1}$  contains a single copy of the local system  $\mathbb{U}_{a-b,0}(-b)$  as a summand and that it is the only term contributing to cohomology, i.e., we have  $H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) = H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0})(-b - 1)$ . From Remark 4.2 We have isomorphisms

$$H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0}) = H^1(\mathcal{A}_1, \mathbb{H}_{a-b}) \otimes H^0(\mathcal{A}_1, \mathbb{Q}) \cong H^1(\mathrm{SL}_2(\mathbb{Z}), H_{a-b}).$$

The weight filtration on  $H^1(\mathrm{SL}_2(\mathbb{Z}), H_{a-b})$  splits ([1]) and hence we have

$$H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0})(-b-1) = (W_{a-b+1}H^1(\mathrm{SL}_2(\mathbb{Z}), H_{a-b}))(-b-1) \oplus \mathbb{Q}(-a-2).$$

The first summand is a  $\mathbb{Q}$ -HS of weight  $a+b+3$  and the second a Tate HS of weight  $2a+4$ . Since  $H^2(\mathcal{A}_2, \mathbb{V}_{a+b}) = 0$  in this case, our claim follows.  $\square$

**Corollary 7.2.** *The only possible weights of  $H_2(\mathbf{u}_2)$  are given by  $-(a-b)-4$  for  $a+b$  even and  $a > b \geq 0$ .*

**Proof.** From Proposition 7.1 and Proposition 5.1 (ii), it follows that the possible weights of  $H^2(\mathbf{u}_2)$  are 2, 3, and  $a-b+4$ , and thus those of  $H_2(\mathbf{u}_2)$  are  $-2$ ,  $-3$ , and  $-(a-b)-4$  with  $a > b$ . By Corollary 6.5,  $H_1(\mathbf{u}_2)$  is pure of weight  $-2$ , and hence the free Lie algebra  $\mathbb{L}(H_1(\mathbf{u}_2))$  has only even weights. By Proposition 5.4, there is an injection of graded vector spaces  $\gamma : H_2(\mathbf{u}_2) \rightarrow [\mathbb{L}(H_1(\mathbf{u}_2)), \mathbb{L}(H_1(\mathbf{u}_2))] \subset \mathbb{L}(H_1(\mathbf{u}_2))$  such that we have a presentation  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2 \cong \mathbb{L}(H_1(\mathbf{u}_2))/\langle \mathrm{im} \gamma \rangle$ . Since the image of  $\gamma$  is contained in  $[\mathbb{L}(H_1(\mathbf{u}_2)), \mathbb{L}(H_1(\mathbf{u}_2))]$ , it follows that  $H_2(\mathbf{u}_2)$  can have only even weights  $-2m$  with  $m \geq 2$ . Therefore, we can exclude the weights  $-2$  and  $-3$ , and thus the only possible weights of  $H_2(\mathbf{u}_2)$  are given by  $-(a-b)-4$  with  $a > b$ .  $\square$

Consequently, we will only need to consider the HS  $\mathbb{Q}(-a-2)$  appearing as a summand in  $H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1))$  with  $a > b$ . In order to determine whether the Tate term  $\mathbb{Q}(-a-2)$  appears in  $H^2(\mathcal{M}_2, \mathbb{V}_{a+b})$ , we will consider the Gysin map

$$\delta : H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) \rightarrow H^3(\mathcal{A}_2, \mathbb{V}_{a+b}).$$

We note from [17, Thm. 2.1] that  $H^3(\mathcal{A}_2, \mathbb{V}_{a+b})$  contains the direct sum of  $s_{a+b+4}$  copies of  $\mathbb{Q}(-a-2)$  as a summand, where  $s_m$  denotes the dimension of the cusp forms for  $\mathrm{SL}_2(\mathbb{Z})$  of weight  $m$ . Petersen kindly shared with me his unpublished note [19] in which he conjectures that whenever  $s_{a+b+4} > 0$ , the Gysin map is injective. We will prove the following partial analogue of Theorem 6.2 by adopting Petersen's approach.

**Theorem 7.3.** *Assume that  $a+b$  is even and  $a > b$ . If  $s_{a+b+4} > 0$ , the restriction of the Gysin map  $\delta : H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) \rightarrow H^3(\mathcal{A}_2, \mathbb{V}_{a+b})$  to the  $\mathbb{Q}$ -sub HS  $\mathbb{Q}(-a-2)$  of  $H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1))$  is nontrivial.*

**Corollary 7.4.** *If  $s_{a+b+4} > 0$ , then  $(H^2(\mathbf{u}_2) \otimes V_{a+b})^{\mathrm{Sp}_2} = 0$ . Furthermore,  $H_2(\mathbf{u}_2)$  is an  $\mathrm{Sp}_2$ -subrepresentation of the finite direct sum  $\bigoplus_{a+b} V_{a+b}(a+2)$ , where the sum is taken over the partitions  $a+b$  of 2, 4, 6, 10 with  $a > b$ .*

**Remark 7.5.** The weights of  $H_2(\mathbf{u}_2)$  are given by  $-a+b-4$  for  $a > b$ , and so we see that there are no quadratic relations in  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2$ . The range of the degrees of the relations on  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2$  is from the cubic to the septic. Thus  $\mathrm{Gr}_{\bullet}^W \mathbf{u}_2$  is finitely presented. However, the author is not able to determine  $H_2(\mathbf{u}_2)$  explicitly at this time. Since  $T_2$  is free, Hain's approach does not seem to extend to the case when  $g = 2$ .

### 7.1. An overview of Petersen’s approach on the nonvanishing of Gysin maps

We will briefly go over Petersen’s approach on the nonvanishing theorem [18, §5, Thm. 5.1]. Petersen’s idea is that since the cohomology classes of one’s interest are represented by Eisenstein series, one should be able to reduce the problem to computations at the boundary. Denote the Baily–Borel compactification of  $\mathcal{A}_g$  by  $\overline{\mathcal{A}}_g$ . Let  $j : \mathcal{A}_2 \rightarrow \overline{\mathcal{A}}_2$  and  $\tilde{j} : \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1$  be the inclusions into their Baily–Borel compactifications. We will consider the Gysin map between the stalks of the pushforwards  $(R\tilde{j}_*\mathbb{V}_{a+b})^{S_2}$  and  $Rj_*\mathbb{V}_{a+b}$ . Let  $i_0 : \mathcal{A}_0 \rightarrow \overline{\mathcal{A}}_2$  and  $\tilde{i}_0 : \mathcal{A}_0 \times \mathcal{A}_0 \rightarrow \overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1$  be the inclusions of their zero-dimensional strata into the respective Baily–Borel compactifications. There is a commutative diagram:

$$\begin{array}{ccc} H^{k-2}(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}(-1)) & \longrightarrow & (\tilde{i}_0^* R^{k-2} \tilde{j}_* \mathbb{V}_{a+b}(-1))^{S_2} \\ \downarrow & & \downarrow \\ H^k(\mathcal{A}_2, \mathbb{V}_{a+b}) & \longrightarrow & i_0^* R^k j_* \mathbb{V}_{a+b} \end{array}$$

The nonvanishing of the left-hand Gysin map will be shown if the right-hand Gysin map at stalks maps the class of the Eisenstein series of our interest to a nonzero class in  $i_0^* R^k j_* \mathbb{V}_{a+b}$ .

The first key point is that the stalks on the right are expressed as the cohomology groups of reductive groups with the Lie algebra cohomology of nilpotent Lie algebras as coefficient groups. Petersen calls this as Harder’s formula (see [12] for more details). Furthermore, these stalks carry natural MHS. This fact does not play a role for  $k = 2$ , but we will need this for  $k = 3$ . The zero-dimensional strata of  $\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1$  and  $\overline{\mathcal{A}}_2$  correspond to the parabolic subgroup  $P = B \times B$  of  $\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q})$  and the Siegel parabolic subgroup  $Q$  of  $\mathrm{Sp}_2(\mathbb{Q})$ , respectively, where  $B$  is the Borel subgroup of  $\mathrm{SL}_2(\mathbb{Q})$ . The natural map  $\mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{A}_2$  corresponds to an inclusion  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{Sp}_2$ , so we may consider  $\mathrm{SL}_2 \times \mathrm{SL}_2$  as a subgroup of  $\mathrm{Sp}_2$ . Then  $P$  is the intersection  $Q \cap \mathrm{SL}_2 \times \mathrm{SL}_2$  in  $\mathrm{Sp}_2$ . Let  $M_P$  and  $M_Q$  be the reductive quotients of  $P$  and  $Q$  by the unipotent radicals  $N_P$  and  $N_Q$ , respectively. Denote the Lie algebras of  $N_P$  and  $N_Q$  by  $\mathfrak{n}_P$  and  $\mathfrak{n}_Q$ , respectively. The idea behind the Harder’s formula is that the cohomology of the sufficiently small enough open punctured neighborhoods of the cusps  $i_0$  and  $\tilde{i}_0$  with the coefficient local system  $\mathbb{V}_{a+b}$  can be computed by identifying the neighborhoods as the fibrations over the locally symmetric spaces associated to  $M_P$  and  $M_Q$  whose fibers are the nilmanifolds associated to  $N_P$  and  $N_Q$ , respectively. The Leray spectral sequences of the fibrations degenerate, and hence the local cohomology we are after are given by  $H^\bullet(M_*(\mathbb{Z}), H^\bullet(\mathfrak{n}_*, V_{a+b}))$ , where  $*$   $\in$   $\{P, Q\}$ . The Gysin maps on the stalks are obtained by considering the Leray spectral sequences associated to the two fibrations and the Gysin sequence for the embedding of the neighborhood of  $\tilde{i}_0$  into that of  $i_0$ . This is nicely explained in [18, §5.3].

The second important point of Petersen’s method is that the nonvanishing of the Gysin maps on the stalks is related to the nontriviality of the modular symbols associated to

eigen cusp forms. It comes down to compute the restriction map  $H_c^1(\mathcal{A}_1, \mathbb{H}_m \otimes \mathbb{C}) \rightarrow H_c^1(\mathbb{R}_{>0}, \mathbb{H}_m \otimes \mathbb{C}) = H_m \otimes \mathbb{C}$ , which is induced by the inclusion of symmetric spaces  $\mathbb{R}_{>0} \rightarrow \mathfrak{h}$ . The  $\mathbb{G}_m$  action decomposes  $H_m \otimes \mathbb{C}$  into  $m + 1$  one-dimensional eigenspaces  $\mathbb{R}_{>0} \rightarrow \mathfrak{h}$ . The  $\mathbb{G}_m$  action decomposes  $H_m \otimes \mathbb{C}$  into  $m + 1$  one-dimensional eigenspaces  $H_m \otimes \mathbb{C} = \bigoplus_{l=0}^m E_{m-2l}$ . For a cusp form  $f$  of weight  $m + 2$ , we have a class  $[f]$  in  $H_c^1(\mathcal{A}_1, \mathbb{H}_m \otimes \mathbb{C})$  and we are interested in the image of  $[f]$  in  $\bigoplus_{l=0}^m E_{m-2l}$ . For  $k = 2$ , the nonvanishing of the Gysin map at the stalks is equivalent to the nontriviality of the image  $[f]$  in the eigenspace  $E_0$ , which is in fact related to the central value of the  $L$ -function attached to  $f$ , where  $f$  here is a normalized Hecke eigenform. For  $k = 3$ , we will consider the image of  $[f]$  in  $E_{-a+b-2}$ , and the nontriviality will follow from the fact that  $L(f, n + 1)$  is nontrivial for  $n \geq \frac{a+b+4}{2}$ , which is a basic fact about the convergence of the corresponding Euler product.

## 7.2. Proof of Theorem 7.3

As stated above, we will follow Petersen's approach with some modification to our case. We will use the same notation as above. The natural map  $\mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathcal{A}_2$  corresponds to the inclusion  $\mathrm{SL}_2 \times \mathrm{SL}_2 \rightarrow \mathrm{Sp}_2$  given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{bmatrix}.$$

The Siegel parabolic subgroup  $Q$  of  $\mathrm{Sp}_2$  consists of the matrices whose lower left quadrant is all zero. The parabolic subgroup  $P$  of  $\mathrm{SL}_2 \times \mathrm{SL}_2$  is the intersection of  $Q$  with  $\mathrm{SL}_2 \times \mathrm{SL}_2$  in  $\mathrm{Sp}_2$ . The reductive quotients  $M_P$  and  $M_Q$  of  $P$  and  $Q$ , respectively, consist of matrices of the form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a' & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a'^{-1} \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & A^{-T} \end{bmatrix},$$

respectively, where  $A$  in  $\mathrm{GL}_2$ . Therefore, we have  $M_Q \cong \mathrm{GL}_2$ . Let  $B$  be the standard Borel subgroup of  $\mathrm{SL}_2$  consisting of upper triangular matrices, and  $M_B$  the reductive quotient of  $B$  by the unipotent radical  $N_B$ , whose elements are of the form

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

Let  $T$  be the maximal  $\mathbb{Q}$ -split torus of  $\mathrm{SL}_2$ . The isomorphism  $\mathbb{G}_m \cong T$  is given by

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$



Thus the action of  $T$  on  $N_B$  has the weight  $t^2$ . Denote the Lie algebra of  $N_B$  by  $\mathfrak{n}_B$ . Then  $M_p = M_B \times M_B \cong \mathbb{G}_m \times \mathbb{G}_m$  and  $\mathfrak{n}_P = \mathfrak{n}_B \times \mathfrak{n}_B$ . The group  $\mathbb{G}_m(\mathbb{Z})$  is cyclic of order 2, which we denote by  $\langle \sigma \rangle$ .

Assume that  $a + b$  is even and  $a > b$ . The restriction of  $\mathbb{V}_{a+b}$  to  $\mathcal{D}_{1,1}$  contains a single copy of  $\mathbb{U}_{a-b,0}(-b)$  as a summand, and we have seen that  $H^1(\mathcal{D}_{1,1}, \mathbb{V}_{a+b}) = H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0}(-b))$ . Thus, we will consider the following commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0}(-b-1)) & \longrightarrow & \tilde{i}_0^* \mathbf{R}^1 \tilde{j}_* \mathbb{U}_{a-b,0}(-b-1) \\ \downarrow & & \downarrow \\ H^3(\mathcal{A}_2, \mathbb{V}_{a+b}) & \longrightarrow & i_0^* \mathbf{R}^3 j_* \mathbb{V}_{a+b} \end{array}$$

We have seen that  $H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0}(-b-1)) = W_{a+b+3} \oplus \mathbb{Q}(-a-2)$ . Harder's formula and a Künneth formula give the isomorphisms

$$\begin{aligned} \tilde{i}_0^* \mathbf{R}^1 \tilde{j}_* \mathbb{U}_{a-b,0} &\cong \bigoplus_{l=0}^1 H^l(M_P(\mathbb{Z}), H^{1-l}(\mathfrak{n}_P, U_{a-b,0})) \\ &\cong H^0(\langle \sigma \rangle \times \langle \sigma \rangle, H^1(\mathfrak{n}_P, U_{a-b,0})) \\ &\cong H^0(\langle \sigma \rangle \times \langle \sigma \rangle, H^\bullet(\mathfrak{n}_B, H_{a-b}) \otimes H^\bullet(\mathfrak{n}_B, \mathbb{Q})) \\ &\cong (H^1(\mathfrak{n}_B, H_{a-b}) \otimes H^0(\mathfrak{n}_B, \mathbb{Q})) \oplus (H^0(\mathfrak{n}_B, H_{a-b}) \otimes H^1(\mathfrak{n}_B, \mathbb{Q})). \end{aligned}$$

Note that the action of  $\sigma$  on the Lie algebra cohomology of  $\mathfrak{n}_B$  is trivial when  $a + b$  is even. We will consider  $H_m \otimes K$  as a  $K$ -vector space of homogeneous polynomials of degree  $m$  in two variable  $X$  and  $Y$ . In this paper,  $K$  is either  $\mathbb{Q}$  or  $\mathbb{C}$ . The action of  $\mathrm{SL}_2(K)$  on  $H_m \otimes K = K[X, Y]_m$  is given by

$$\alpha \cdot P(X, Y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} P(X, Y) = P(aX + cY, bX + dY).$$

Under this action of  $\mathrm{SL}_2(\mathbb{Q})$  on  $H_m$ , we have

$$H^0(\mathfrak{n}_B, H_m) \cong \mathbb{Q}X^m \text{ and } H^1(\mathfrak{n}_B, H_m) \cong \mathbb{Q}Y^m \otimes \eta,$$

where the element  $\eta$  is the dual of the generator of  $\mathfrak{n}_B$ . Note that  $T$  acts on  $\eta$  by  $t^{-2}$ . Thus there is an isomorphism of MHSs

$$\tilde{i}_0^* \mathbf{R}^1 \tilde{j}_* \mathbb{U}_{a-b,0} \cong \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta,$$

where the left-hand term is pure of weight  $2(a-b) + 2$  and the right-hand term pure of weight 2.

**Proposition 7.6.** *The generator of the term  $\mathbb{Q}(-(a-b)-1)$  in  $H^1(\mathcal{D}_{1,1}, \mathbb{U}_{a-b,0})$  maps nontrivially into the term  $\mathbb{Q}Y^{a-b} \otimes \eta$ .*

**Proof.** Let  $m = a - b$ . Let  $\Delta : \mathcal{A}_1 \rightarrow \mathcal{A}_1 \times \mathcal{A}_1$  and  $\overline{\Delta} : \overline{\mathcal{A}}_1 \rightarrow \overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1$  be the diagonal maps. Consider the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{A}_1, \mathbb{H}_m) & \xrightarrow{r} & H^1(\partial\mathcal{A}_1, \mathbb{H}_m) \\ \uparrow \Delta^* & & \uparrow \overline{\Delta}^* \\ H^1(\mathcal{A}_1 \times \mathcal{A}_1, \mathbb{U}_{m,0}) & \longrightarrow & \tilde{i}_0^* R^1 \tilde{j}_* \mathbb{U}_{m,0} \end{array}$$

where  $H^1(\partial\mathcal{A}_1, \mathbb{H}_m)$  is the boundary cohomology of  $\mathcal{A}_1$  with the local coefficient system  $\mathbb{H}_m$  and the upper horizontal map  $r$  is the restriction map. It is well-known that the map  $r$  takes the generator of  $\mathbb{Q}(-m-1) \subset H^1(\mathcal{A}_1, \mathbb{H}_m)$  to the generator of  $H^1(\partial\mathcal{A}_1, \mathbb{H}_m) \cong \mathbb{Q}Y^m \otimes \eta$ . Thus the image of the term  $\mathbb{Q}(-m-1)$  in the stalk  $\tilde{i}_0^* R^1 \tilde{j}_* \mathbb{U}_{m,0}$  is nontrivial. Furthermore, the weight of the term  $\mathbb{Q}Y^m \otimes \eta$  is given by  $m + m + 2 = 2m + 2$ , which implies that the image of  $\mathbb{Q}(-m-1)$  in  $\tilde{i}_0^* R^1 \tilde{j}_* \mathbb{U}_{m,0}$  is equal to  $\mathbb{Q}Y^m \otimes \eta$ .  $\square$

Now, we will consider the Gysin map between the stalks. Harder's formula applied to the stalk  $i_0^* R^3 j_* \mathbb{V}_{a+b}$  gives an isomorphism

$$i_0^* R^3 j_* \mathbb{V}_{a+b} \cong \bigoplus_{l=0}^3 H^l(M_Q(\mathbb{Z}), H^{3-l}(\mathfrak{n}_Q, V_{a+b})).$$

The coefficient group  $H^\bullet(\mathfrak{n}_Q, V_{a+b})$  are  $M_Q \cong \mathrm{GL}_2$  representations, and Kostants theorem [21, Table 2.3.4] gives us the corresponding  $M_Q$  representations. As in the case  $k = 2$  (Petersen's proof), we will only need to consider them as the representations of the derived subgroup of  $M_Q$ , which is isomorphic to  $\mathrm{SL}_2$ . These are

$$H^l(\mathfrak{n}_Q, V_{a+b}) \cong \begin{cases} H_{a-b} & \text{for } l = 0, 3 \\ H_{a+b+2} & \text{for } l = 1, 2. \end{cases}$$

Note that  $H^\bullet(\mathrm{GL}_2(\mathbb{Z}), H_m) \cong H^\bullet(\mathrm{SL}_2(\mathbb{Z}), H_m)^{(\sigma)}$ , and hence the stalk above is isomorphic to  $H^1(\mathrm{SL}_2(\mathbb{Z}), H_{a+b+2})^{(\sigma)} \cong H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})^{(\sigma)}$ .

The Gysin map  $\tilde{i}_0^* R^1 \tilde{j}_* \mathbb{U}_{a-b,0} \rightarrow i_0^* R^3 j_* \mathbb{V}_{a+b}$  can be obtained by considering the embedding of the sufficiently small open neighborhood  $W_P$  of  $\tilde{i}_0$  in  $\mathcal{A}_1 \times \mathcal{A}_1$  into that  $W_Q$  of  $i$  in  $\mathcal{A}_2$ . Let  $* \in \{P, Q\}$ . The open neighborhood  $W_*$  can be identified as a fibration over the locally symmetric space associated to  $M_*$  with the fiber  $N_*(\mathbb{Z}) \backslash N_*(\mathbb{R})$ . The embedding yields the commutative diagram:

$$\begin{array}{ccc}
N_P(\mathbb{Z}) \backslash N_P(\mathbb{R}) & \longrightarrow & N_Q(\mathbb{Z}) \backslash N_Q(\mathbb{R}) \\
\downarrow & & \downarrow \\
W_P & \longrightarrow & W_Q \\
\downarrow & & \downarrow \\
M_P(\mathbb{Z}) \backslash M_P(\mathbb{R}) / K_{M_P} & \longrightarrow & M_Q(\mathbb{Z}) \backslash M_Q(\mathbb{R}) / K_{M_Q}.
\end{array}$$

The fact that symmetric spaces  $M_*(\mathbb{R})/K_{M_*}$  are contractible implies that the cohomology of  $M_*(\mathbb{Z}) \backslash M_*(\mathbb{R})/K_{M_*}$  agrees with the cohomology of  $M_*(\mathbb{Z})$ . The van Est isomorphism describes the cohomology of  $N_*(\mathbb{Z}) \backslash N_*(\mathbb{R})$  as that of the Lie algebra  $\mathfrak{n}_*$ . Then the corresponding map between the cohomology groups  $H^\bullet(M_*(\mathbb{Z}), H^\bullet(\mathfrak{n}_*, \mathbb{V}))$  with  $* \in \{P, Q\}$  arises as part of the Gysin map induced between the Leray spectral sequences associated to these fibrations. Thus we obtain the commutative diagram

$$\begin{array}{ccc}
\tilde{i}_0^* \mathbf{R}^1 \tilde{j}_* \mathbb{U}_{a-b,0} & \longrightarrow & i_0^* \mathbf{R}^3 j_* \mathbb{V}_{a+b} \\
\downarrow \cong & & \downarrow \cong \\
H^0(M_P(\mathbb{Z}), H^1(\mathfrak{n}_P, U_{a-b,0})) & \xrightarrow{g} & H^1(M_Q(\mathbb{Z}), H^2(\mathfrak{n}_Q, V_{a+b}))
\end{array}$$

Since the Hodge weights play no role for the rest of the proof, we will omit Tate twists.

We are led to understand the bottom horizontal map  $g$ . Petersen's idea is that this map is induced by the restriction of a local system over  $\mathcal{A}_1$  along the inclusion of the locally symmetric space  $\mathbb{R}_{>0} \times B\langle\sigma\rangle$  of  $\mathbb{G}_m$  into  $\mathcal{A}_1$  of  $\mathrm{SL}_2$ , where  $B\langle\sigma\rangle$  is the classifying space for the group  $\langle\sigma\rangle$ , and that it can be considered as the evaluations of the modular symbols associated to normalized eigenforms of a certain weight. Consider the commutative diagram of groups

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & M_P & \xrightarrow{\det_P} & \mathbb{G}_m \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \parallel \\
1 & \longrightarrow & \mathrm{SL}_2 & \longrightarrow & M_Q & \xrightarrow{\det_Q} & \mathbb{G}_m \longrightarrow 1,
\end{array}$$

where the map  $\det_P$  associates  $aa'$  to a matrix  $\mathrm{diag}(a, a', a^{-1}, a'^{-1})$  in  $M_P$  and the map  $\det_Q$   $\det(A)$  to an matrix  $A$  in  $M_Q$ , and the left-hand vertical map is induced by the middle inclusion. Assigning to each group in the diagram the corresponding locally symmetric space, we obtain an inclusion of fibrations. It then follows from the Leray spectral sequences of the two fibrations that there are isomorphisms

$$H^0(M_P(\mathbb{Z}), H^1(\mathfrak{n}_P, U_{a-b,0})) \cong H^0(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta)^{(\sigma)}$$

and

$$H^1(M_Q(\mathbb{Z}), H^2(\mathfrak{n}_Q, V_{a+b})) \cong H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})^{(\sigma)}.$$

For a rational representation  $V$  of  $M_Q$ , there is the restriction

$$H_c^\bullet(M_Q(\mathbb{Z}) \backslash M_Q(\mathbb{R}) / K_{M_Q}, \mathbb{V}) \rightarrow H_c^\bullet(M_P(\mathbb{Z}) \backslash M_P(\mathbb{R}) / K_{M_P}, \mathbb{V}),$$

where  $\mathbb{V}$  is the local system corresponding to  $V$ . Then for  $V = H^1(\mathfrak{n}_Q, V_{a+b})$ , we observe that the dual of the restriction contains the map  $g$ . Thus there is the commutative diagram

$$\begin{array}{ccc} H^0(M_P(\mathbb{Z}), H^1(\mathfrak{n}_P, U_{a-b,0})) & \xrightarrow{\cong} & H^0(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta)^{(\sigma)} \\ g \downarrow & & \downarrow \\ H^1(M_Q(\mathbb{Z}), H^2(\mathfrak{n}_Q, V_{a+b})) & \xrightarrow{\cong} & H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})^{(\sigma)}. \end{array}$$

We also denote by  $g$  the right-hand vertical map of the diagram. The map  $g$  is the restriction to the  $\langle \sigma \rangle$ -invariants of the map

$$\tilde{g} : H^0(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta) \rightarrow H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2}),$$

which is induced by the inclusion  $\mathbb{G}_m \rightarrow \mathrm{SL}_2$ . We have seen that

$$H^0(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta)^{(\sigma)} \cong \mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta,$$

and so  $\tilde{g}$  factors as

$$\mathbb{Q}Y^{a-b} \otimes \eta \oplus \mathbb{Q}X^{a-b} \otimes \eta \rightarrow H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})^{(\sigma)} \rightarrow H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2}).$$

Recall that  $s_{a+b+4}$  denotes the dimension of the space of the cusp forms of weight  $a+b+4$  for  $\mathrm{SL}_2(\mathbb{Z})$ .

**Proposition 7.7.** *If  $s_{a+b+4} > 0$ , then the generator  $Y^{a-b} \otimes \eta$  maps nontrivially into  $H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})$ .*

Before getting to the proof of this claim, we will briefly review the periods and modular symbols of a cusp form, and the  $L$ -function attached to a Heck eigenform. Let  $f$  be a cusp form of weight  $m+2$ . The  $n$ th period of  $f$  is defined to be

$$r_n(f) = \int_0^{i\infty} f(z) z^n dz$$

for  $0 \leq n \leq m$ . By identifying  $H_m \otimes \mathbb{C} = \mathbb{C}[X, Y]_m$ , the *period polynomial*  $r(f) \in H_m \otimes \mathbb{C}$  of  $f$  is defined to be

$$r(f)(X, Y) = \int_0^{i\infty} f(z)(zX + Y)^m dz = \sum_{n=0}^m \binom{m}{n} r_n(f) X^n Y^{m-n}.$$

The differential form  $f(z)(zX + Y)^m dz$  defines a class  $[f]$  in  $H_c^1(\mathcal{A}_1, \mathbb{H}_m)$  and integrating along the imaginary axis corresponds to the restriction map

$$H_c^1(\mathcal{A}_1, \mathbb{H}_m \otimes \mathbb{C}) \rightarrow H_c^1(\mathbb{R}_{>0}, \mathbb{H}_m \otimes \mathbb{C}) = H_m \otimes \mathbb{C},$$

where  $\mathbb{R}_{>0}$  is embedded into  $\mathfrak{h}$  by  $a \mapsto ia$ . Thus the class  $[f]$  maps to the period polynomial  $r(f)(X, Y)$ . The  $\mathbb{G}_m$ -action decomposes  $H_m$  into  $m+1$  one-dimensional eigenspaces

$$H_m = \bigoplus_{n=0}^m E_{-m+2n}.$$

We may identify  $X^n Y^{m-n}$  as a generator of  $E_{-m+2n}$ , and so  $E_{-m+2n} = \mathbb{Q} X^n Y^{m-n}$ . Composing the restriction map with the projection

$$p_n : H_c^1(\mathbb{R}_{>0}, \mathbb{H}_m \otimes \mathbb{C}) \rightarrow H_c^1(\mathbb{R}_{>0}, E_{-m+2n} \otimes \mathbb{C}) = \mathbb{C} X^n Y^{m-n} = \mathbb{C}$$

gives an evaluation, called a *modular symbol*, associating the value  $\binom{m}{n} r_n(f)$  to a cusp form  $f$ .

Let  $q = e^{2\pi iz}$  for  $z \in \mathfrak{h}$ . Let  $f(q) = \sum a(n)q^n$  be the expansion of  $f$  on the  $q$ -disk. If  $f$  is a Hecke eigenform and normalized, i.e.,  $a(1) = 1$ , then the  $L$ -function of  $f$  is defined to be the Dirichlet series

$$L(f, s) = \sum_{n>0} a(n)n^{-s},$$

which is known to be convergent for  $\operatorname{Re}(s) > \frac{m}{2} + 2$ .

Furthermore, as  $f$  is a normalized eigenform, the  $L$ -function has an Euler product expansion

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{k-1-2s})^{-1}.$$

In our case, this is a key point and for  $s = n + 1$  with  $2n > m + 2$ , the Euler product converges and hence  $L(f, n + 1)$  is nontrivial. A key relation between the  $L$ -function and the period of a normalized eigenform  $f$  is given by the Mellin transform

$$L(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z}.$$

For  $s = n + 1$ , we have

$$r_n(f) = n!(-2\pi i)^{-n-1}L(f, n+1).$$

Another key property of the  $L$ -function  $L(f, s)$  is that there is a functional equation relating  $L(f, s)$  and  $L(f, m+2-s)$ . Now, we will prove [Proposition 7.7](#).

**Proof.** We note that the map  $H^0(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta) \rightarrow H^1(\mathcal{A}_1, \mathbb{H}_{a+b+2})$  is the dual of the restriction map

$$H_c^1(\mathcal{A}_1, \mathbb{H}_{a+b+2}) \rightarrow H_c^1(\mathbb{R}_{>0}, \mathbb{Q}Y^{a-b} \otimes \eta).$$

Here we are identifying  $H_m \cong H_m^*$  induced by the polarization, and moreover there is an isomorphism  $\mathbb{Q}Y^{a-b} \otimes \eta \cong E_{-(a-b)-2}$  as  $\mathbb{G}_m$ -representations. Let  $f$  be a normalized eigenform of weight  $a+b+4$ . Let  $[f]$  be the class in  $H_c^1(\mathcal{A}_1, \mathbb{H}_{a+b+2} \otimes \mathbb{C})$  corresponding to the 1-form  $f(z)(zX+Y)^{a+b+2}dz$ . Then the image of the class  $[f]$  in

$$H_c^1(\mathbb{R}_{>0}, \mathbb{C}Y^{a-b} \otimes \eta) \cong E_{-(a-b)-2} \otimes \mathbb{C} \cong \mathbb{C}X^bY^{a+2}$$

is given by  $\binom{a+b+2}{b}r_b(f)X^bY^{a+2}$ . Thus we consider the value  $L(f, b+1)$ . By the functional equation mentioned above, the value  $L(f, b+1)$  is nontrivial if and only if  $L(f, a+3)$ . The assumption that  $a > b$  implies that  $2(a+2) > a+b+4$ , and hence the Euler product for  $L(f, a+3)$  converges and so  $L(f, a+3)$  is nontrivial.  $\square$

This completes the proof of [Theorem 7.3](#).

## 8. The pointed case $\mathfrak{u}_{2,n}$

In this section, we extend our main result to the Lie algebra  $\mathfrak{u}_{2,n}$ . Suppose that  $S$  is a compact oriented surface of genus  $g$ . The *configuration space*  $F^n(S)$  of  $n \geq 1$  points on  $S$  is defined to be

$$F^n(S) = S^n - \Delta,$$

where  $\Delta = \cup_{i,j} \Delta_{ij}$  and  $\Delta_{ij} = \{(\dots, x_i, \dots, x_j, \dots) \in S^n | x_i = x_j\}$ . Fix a point  $x = [C]$  in  $\mathcal{M}_g$ . Then for a fixed base point  $p$  in  $F^n(C)$ , we denote the fundamental group of  $F^n(C)$  by  $\pi_{g,n}$ . Suppose that  $g \geq 2$  and  $n \geq 1$ . We have the following exact sequence

$$1 \rightarrow \pi_{g,n} \rightarrow \Gamma_{g,n} \rightarrow \Gamma_g \rightarrow 1$$

coming from the projection  $\mathcal{M}_{g,n} \rightarrow \mathcal{M}_g$  that forgets the  $n$ -marked points. Denote the unipotent completion of  $\pi_{g,n}$  over  $\mathbb{Q}$  by  $\pi_{g,n}^{\text{un}}$ . Then the naturality and right exactness of

relative completion [10, Prop. 3.6 & 3.7] imply that the sequence of proalgebraic groups over  $\mathbb{Q}$

$$\pi_{g,n}^{\text{un}} \rightarrow \mathcal{G}_{g,n} \rightarrow \mathcal{G}_g \rightarrow 1$$

is exact. In fact, the sequence is exact at left as well. Denote the Lie algebra of  $\pi_{g,n}^{\text{un}}$  by  $\mathfrak{p}_{g,n}$ . By universal property of relative completion, the conjugation action of  $\Gamma_{g,n}$  on  $\pi_{g,n}$  induces a representation  $\mathcal{G}_{g,n} \rightarrow \text{Aut } \mathfrak{p}_{g,n}$  by [9, Lemma 3.8].

**Proposition 8.1.** *If  $g \geq 2$  and  $n \geq 1$ , then the sequences*

$$1 \rightarrow \pi_{g,n}^{\text{un}} \rightarrow \mathcal{G}_{g,n} \rightarrow \mathcal{G}_g \rightarrow 1$$

*and*

$$1 \rightarrow \pi_{g,n}^{\text{un}} \rightarrow \mathcal{U}_{g,n} \rightarrow \mathcal{U}_g \rightarrow 1$$

*of proalgebraic groups over  $\mathbb{Q}$  are exact.*

**Proof.** We note that the composition map  $\pi_{g,n}^{\text{un}} \rightarrow \mathcal{G}_{g,n} \rightarrow \text{Aut } \mathfrak{p}_{g,n}$  is the adjoint action of  $\pi_{g,n}^{\text{un}}$  on  $\mathfrak{p}_{g,n}$ . Since the center of  $\pi_{g,n}^{\text{un}}$  is trivial (cf. [15]), it follows that the adjoint action  $\pi_{g,n}^{\text{un}} \rightarrow \text{Aut } \mathfrak{p}_{g,n}$  is injective, and so is the map  $\pi_{g,n}^{\text{un}} \rightarrow \mathcal{G}_{g,n}$ . Since  $\mathcal{G}_{g,n}$  is an extension of  $\text{Sp}_g$  by  $\mathcal{U}_{g,n}$  and the image of  $\pi_{g,n}^{\text{un}}$  in  $\mathcal{G}_{g,n}$  is contained in  $\mathcal{U}_{g,n}$ , it follows that the sequence  $1 \rightarrow \pi_{g,n}^{\text{un}} \rightarrow \mathcal{U}_{g,n} \rightarrow \mathcal{U}_g \rightarrow 1$  is exact.  $\square$

Consequently, we have the exact sequence of MHSs

$$0 \rightarrow \mathfrak{p}_{g,n} \rightarrow \mathfrak{u}_{g,n} \rightarrow \mathfrak{u}_g \rightarrow 0$$

and the exactness of the functor  $\text{Gr}_{\bullet}^W$  on the category of MHS implies that the sequence of graded Lie algebras

$$0 \rightarrow \text{Gr}_{\bullet}^W \mathfrak{p}_{g,n} \rightarrow \text{Gr}_{\bullet}^W \mathfrak{u}_{g,n} \rightarrow \text{Gr}_{\bullet}^W \mathfrak{u}_g \rightarrow 0$$

is exact. The Lie algebra  $\mathfrak{p}_{g,n}$  is finitely generated and finitely presented by Morgan's theorem [14] (see [9, Thm. 1.3] for an explicit presentation), and therefore, we obtain Corollary 2. Furthermore, we can easily determine the generators of  $\mathfrak{u}_{2,n}$  by a basic spectral sequence argument applied to the exact sequence  $1 \rightarrow \pi_{g,n} \rightarrow \Gamma_{g,n} \rightarrow \Gamma_g \rightarrow 1$ .

**Proposition 8.2.** *Suppose  $n \geq 0$ . Then we have*

$$\text{Gr}_k^W H_1(\mathfrak{u}_{2,n}) = \begin{cases} V_{1+0}(1)^{\oplus n} & \text{for } k = -1 \\ V_{2+2}(3) & \text{for } k = -2 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let  $V$  be a finite dimensional rational  $\mathrm{Sp}_2$  representation. Then  $\pi_{2,n}$  acts trivially on  $V$  via the representation  $\Gamma_{2,n} \rightarrow \mathrm{Sp}_2$  and hence there is the exact sequence

$$0 \rightarrow H^1(\Gamma_2, V) \rightarrow H^1(\Gamma_{2,n}, V) \rightarrow \mathrm{Hom}_{\mathrm{Sp}_2}(H_1(\pi_{2,n}), V) \rightarrow H^2(\Gamma_2, V).$$

The group  $\Gamma_2$  acts on  $H_1(\pi_{2,n}) = \bigoplus_{i=1}^n H_1(C)_i$  diagonally via  $\mathrm{Sp}_2$ . For  $V = V_{1+0}$ , we have  $H^k(\Gamma_2, V_{1+0}) = 0$  for  $k = 1, 2$ , and hence there is an isomorphism of MHS  $H^1(\Gamma_{2,n}, V_{1+0}) \cong \mathbb{Q}(-1)^{\oplus n}$ . For  $V = V_{2+2}$ , the third term is trivial, and hence  $H^1(\Gamma_{2,n}, V_{2+2}) = \mathbb{Q}(-3)$ . For  $V \neq V_{1+0}$  or  $V_{2+2}$ , the first and third terms are trivial, and hence  $H^1(\Gamma_{2,n}, V) = 0$ . Then the result follows from [Proposition 5.1](#).  $\square$

## 9. Appendix: A table of multiplicities of $V_{a+b}$ in [Theorem 1](#)

In this Appendix, we give a table of the multiplicities in  $\mathbb{L}(V_{2+2}(3))$  of the irreducible  $\mathrm{Sp}_2$  representations  $V_{a+b}$  appearing in the sum  $\bigoplus_{a+b} V_{a+b}(a+2)$  from [Theorem 1](#), using the computer program LiE developed in the University of Amsterdam.

Denote the component of bracket length  $k$  of  $\mathbb{L}(V_{2+2}(3))$  by  $\mathbb{L}_k$ . The Hodge weight  $-(a-b)-4$  of  $V_{a+b}(a+2)$  determines the value  $k$  such that  $V_{a+b}(a+2) \subset \mathbb{L}_k$ . Since the free Lie algebra is generated by  $V_{2+2}(3)$  that is of weight  $-2$ , we have the relation  $k = \frac{-(a-b)-4}{-2}$ . Denote  $V_{a+b}$  by  $[a+b]$ . We have the following list of the multiplicities:

$k$ , weight $\backslash a+b$	2	4	6	10
3, $-6$	$1 \times [2+0]$	$1 \times [3+1]$	$2 \times [4+2]$	$1 \times [6+4]$
4, $-8$		$4 \times [4+0]$	$5 \times [5+1]$	$2 \times [7+3]$
5, $-10$			$22 \times [6+0]$	$18 \times [8+2]$
6, $-12$				$85 \times [9+1]$
7, $-14$				$420 \times [10+0]$

Since each  $V_{a+b}$  appears at least once in  $\mathbb{L}_k$  for an appropriate  $k$ , we cannot exclude any of the  $V_{a+b}$ 's from the possible generating representations for  $R$  in [Theorem 1](#).

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