Journal of Algebra 517 (2019) 439–456

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the S_n-invariant F-conjecture

Han-Bom Moon, David Swinarski *

Department of Mathematics, Fordham University, New York, NY 10023, United States of America

ARTICLE INFO

Article history:
Received 1 March 2017
Available online 11 September 2018
Communicated by Seth Sullivant

Keywords:
Moduli space
Rational curves
Birational geometry
Nef cone
Base point free divisor

ABSTRACT

By using classical invariant theory, we reduce the S_n-invariant F-conjecture to a feasibility problem in polyhedral geometry. We show by computer that for $n \leq 19$, every integral S_n-invariant F-nef divisor on the moduli space of genus zero stable n-pointed curves is semi-ample, over arbitrary characteristic. Furthermore, for $n \leq 16$, we show that for every integral S_n-invariant nef (resp. ample) divisor D on the moduli space, $2D$ is base-point-free (resp. very ample). As applications, we obtain the nef cone of the moduli space of stable curves without marked points, and the semi-ample cone of the moduli space of genus 0 stable maps to Grassmannian for small numerical values.

1. Introduction

When one studies birational geometric aspects of a projective variety X, the first step is to understand two cones of divisors in $\mathrm{N}^1(X)$: the effective cone $\mathrm{Eff}(X)$ and the nef cone $\mathrm{Nef}(X)$. The first cone contains information on rational contractions of X, and the second cone contains data on regular contractions of X.

* Corresponding author.
E-mail addresses: hmoon8@fordham.edu (H.-B. Moon), dswinarski@fordham.edu (D. Swinarski).

https://doi.org/10.1016/j.jalgebra.2018.08.030
0021-8693/

In this paper, we study the moduli space $\overline{\mathrm{M}}_{0,n}$ of genus 0 stable n-pointed curves. Its elementary construction in [20] suggests that its geometry is similar to that of toric varieties, and therefore many people conjectured that $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$ and $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ are polyhedral. However, the birational geometric properties of $\overline{\mathrm{M}}_{0,n}$ seem to be very complicated. The cone of effective divisors was conjectured to be generated by boundary divisors. However, now there are many known examples of non-boundary extremal effective divisors ([29,5,26]). Doran, Giansiracusa, and Jensen showed that the effectivity of a divisor class depends on the base ring, and there are generators of the Cox ring which do not lie on extremal rays of $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})$ ([7]). Furthermore, recently it was shown that $\overline{\mathrm{M}}_{0,n}$ is not a Mori dream space for $n \geq 10$ ([6,12,17]). On the other hand, the S_n-invariant part $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n}$ is simplicial and generated by symmetrized boundary divisors $\{B_i = \sum_{|I|=i} B_I\}$ ([23, Theorem 1.3]).

For $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$, there has been less progress, but there is an explicit conjectural description. From the analogy with toric varieties again, a natural candidate of the generating set of the Mori cone of $\overline{\mathrm{M}}_{0,n}$ is the set of one-dimensional boundary strata, called *F-curves*.

Definition 1.1. An effective divisor $D = \sum b_I B_I$ on $\overline{\mathrm{M}}_{0,n}$ is *F-nef* if for any F-curve F, $D \cdot F \geq 0$.

Although this definition uses intersection theory, we can explicitly state the set of linear inequalities with respect to the coefficients $\{b_I\}$ ([13, (0.14)]). Thus we can formally define F-nefness over Spec $\mathbb{Z}$, as well.

Fulton conjectured that the Mori cone of $\overline{\mathrm{M}}_{0,n}$ is generated by F-curves. Dually:

Conjecture 1.2 *(F-conjecture). A divisor on $\overline{\mathrm{M}}_{0,n}$ is nef if and only if it is F-nef.*

This conjecture was shown for $n \leq 7$ in [23] by Keel and McKernan in characteristic 0. However, as n grows, the Picard number of $\overline{\mathrm{M}}_{0,n}$ grows exponentially, so $\overline{\mathrm{M}}_{0,8}$ is already out of reach.

On the other hand, we may ask the same question for S_n-invariant divisors:

Conjecture 1.3 *(S_n-invariant F-conjecture). An S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$ is nef if and only if it is F-nef.*

In characteristic 0, Gibney proved Conjecture 1.3 for $n \leq 24$ ([11]). Recently, Fedorchuk showed that the S_n-invariant F-conjecture is true for $n \leq 16$ in arbitrary characteristic ([10]). In fact, he proved a stronger result: for $n \leq 16$, an S_n-invariant F-nef divisor is *boundary semi-ample* (Definition 5.3), which implies that it is nef. In Section 5, we show that the boundary semi-ampleness of Fedorchuk is equivalent to *G-semi-ampleness*, which is the key notion to our computational approach.

1.1. Results

In this paper, we translate Conjecture 1.3 into a feasibility problem in polyhedral geometry, namely, the nonemptiness of certain polytopes. We use this approach to show the following result.

Theorem 1.4 *(Theorem 4.1).*

(1) For $n \leq 19$, over $\operatorname{Spec} \mathbb{Z}$, *every S_n-invariant F-nef divisor on $\overline{\mathrm{M}}_{0,n}$ is semi-ample.*
(2) For $n \leq 16$, over $\operatorname{Spec} \mathbb{Z}$, *for every S_n-invariant F-nef divisor D on $\overline{\mathrm{M}}_{0,n}$, $2D$ is base-point-free.*

By using [24] or [28], we obtain the following consequence.

Theorem 1.5 *(Theorem 4.7). Suppose $n \leq 16$. Over any algebraically closed field, for every integral S_n-invariant ample divisor A on $\overline{\mathrm{M}}_{0,n}$, $2A$ is very ample.*

We immediately obtain the following two corollaries.

Corollary 1.6. *For $n \leq 19$, over any algebraically closed field,* $\operatorname{Nef}(\overline{\mathrm{M}}_{0,n}/S_n)$ *is equal to the F-nef cone and every nef divisor on $\overline{\mathrm{M}}_{0,n}/S_n$ is semi-ample.*

Corollary 1.7. *For $n \leq 19$, over any algebraically closed field, the Mori cone of $\overline{\mathrm{M}}_{0,n}/S_n$ is generated by F-curves.*

This result also yields the description of the nef cones of some other moduli spaces. By [13, Theorem 0.3], we obtain the nef cone of $\overline{\mathcal{M}}_g$, whose description was a question raised by Mumford.

Corollary 1.8. *Over any algebraically closed field, for $g \leq 19$, the nef cone of $\overline{\mathcal{M}}_g$, the moduli space of genus g stable curves, is equal to the F-nef cone.*

By [3, Theorem 1.1] and [4, Theorem 1.1], we obtain the nef cone of the moduli space of genus 0 stable maps to a Grassmannian, including the case of projective space.

Corollary 1.9. *Let k, n, and d be positive integers such that $1 \leq k \leq n-1$ and $d \leq 19$. Let $\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,n), d)$ be the moduli space of genus 0 stable maps to the Grassmannian* $\mathrm{Gr}(k,n)$. *Over any algebraically closed field,* $\operatorname{Nef}(\overline{\mathrm{M}}_{0,0}(\mathrm{Gr}(k,n), d))$ *coincides with the semi-ample cone, and it is polyhedral with explicit generators.*

By [11], the S_n-invariant F-conjecture is known for $n \leq 24$ in characteristic zero. Thus the nefness parts of the above four corollaries are already known in the same characteristic. Fedorchuk's work shows that these results are independent of the characteristic of the base field and nefness can be strengthened to semi-ampleness for $n \leq 16$; we extend these statements to $n \leq 19$. Our new contributions are the base-point-freeness portion of Theorem 1.4 and a new derivation of the criterion in Corollary 3.16 via geometry and invariant theory. These results support the conjecture that the cone of S_n-invariant divisors of $\overline{\mathrm{M}}_{0,n}$ is in the Mori dream region.

1.2. *Idea of the proof*

The main ingredient of the proof of Theorem 1.4 is classical invariant theory, in particular *graphical algebras*.

A simple but crucial observation in this paper is that an S_n-invariant F-nef divisor D can be written as $\pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ where $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n//\mathrm{SL}_2$ is a regular contraction and D_2 is an ample divisor. Then the linear system $|D|$ can be identified with a sub linear system $|cD_2|_{\mathbf{a}}$ of $|cD_2|$ on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ (Proposition 3.7). Thus the study of $|D|$ can be reduced to the study of a non-complete linear system on $(\mathbb{P}^1)^n//\mathrm{SL}_2$. Furthermore, there is a sub linear system $|cD_2|_{\mathbf{a},G} \subset |cD_2|_{\mathbf{a}}$ which can be described combinatorially via the *graphical algebra*. The Cox ring of $(\mathbb{P}^1)^n//\mathrm{SL}_2$, which is the ring of SL_2-invariant divisors on $(\mathbb{P}^1)^n$, is classically known as the graphical algebra since the 19th century. Its generators can be described in terms of finite graphs, thus we may study it by using graph theory. By using the graphical algebra, we obtain a combinatorial description of $|cD_2|_{\mathbf{a},G}$ and its base locus in terms of polytopes (Corollary 3.16).

1.3. *Comparison with other cones*

By using various geometric and combinatorial methods, several authors previously described some polyhedral lower bounds of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$. There is a natural embedding $i : \overline{\mathrm{M}}_{0,n} \hookrightarrow X_\Delta$ into a non-proper toric variety. By taking the pull-back of the cone of semi-ample divisors, Gibney and Maclagan defined a polyhedral subcone $i^*(\mathcal{G}_\Delta)$ ([14]). On the other hand, by analyzing Keel's relations carefully, Fedorchuk described a cone of semi-ample divisors whose linear system intersects nicely with every boundary stratum and called it the cone of boundary semi-ample divisors ([10]). In Proposition 5.2, we show that for an S_n-invariant divisor, these two cones and our new cone of G-semi-ample divisors all coincide, even though these constructions look very different from each other. We regard this as weak evidence in support of the S_n-invariant F-conjecture.

1.4. *A remark on the computational complexity*

Gibney's proof in [11] of the S_n-invariant F-conjecture for $n \leq 24$ already uses a computer verification. It is fair to ask why our new approach cannot beat the record. The major reason why her calculations extend farther than ours is that for the same n, the size of the feasibility problem in her approach is smaller than the corresponding problem our approach. Her approach finds a point on approximately an $O(n)$-dimensional polyhedral cone (the dimension of the S_n-invariant Picard group of $\overline{\mathrm{M}}_{0,n}$) while our approach tries to find a point on an $O(n^2)$-dimensional cone (the number of edges of the complete graph K_n).

There is room to extend our computational result. To address questions of base-point-freeness, we computed integer valued solutions in polytopes; for semi-ampleness, rational valued solutions would suffice, and may be faster to compute. Moreover, an advantage of our method is that it is very easy to parallelize – once we have a list of S_n-orbits in the set of F-points, we can perform the computation for all F-points at the same time. With a high-performance computing environment, we expect that one could easily go farther and perhaps beyond the current record. Finally, we would like to point out that surprisingly, for most divisors, it is enough to solve only a few (in many cases only one) feasibility problems by taking the intersection of all the polytopes whose feasibility we want to test. It would be an interesting future project to refine and extend our computational approach.

1.5. Structure of the paper

This paper is organized as follows. In Section 2, we recall the definition of the graphical algebra. In Section 3, we translate the S_n-invariant F-conjecture into a polyhedral feasibility problem. Section 4 presents the proof of the main theorem, computational results, and some examples. We explain the comparison with other cones obtained by various authors in Section 5.

Acknowledgments

We would like to thank Maksym Fedorchuk, who kindly pointed out a crucial error in an earlier draft of this paper and gave us many valuable suggestions. We also thank Young-Hoon Kiem, Ian Morrison, and the anonymous referees for many helpful suggestions and comments. The first author was partially supported by the Minerva Research Foundation.

2. Graphical algebra

In this section, we recall the definition and basic properties of graphical algebras, which are key algebraic tools in this approach to the S_n-invariant F-conjecture. We work over $\mathbb{Z}$, but all of the results stated here are valid over any base ring.

The *graphical algebra* is a $\mathbb{Z}$-algebra which was introduced to describe a result in classical invariant theory. Consider $(\mathbb{P}^1)^n$, the space of n points on a projective line. There is a natural diagonal SL_2-action on this space, which is induced by a homomorphism $\mathrm{SL}_2 \to \mathrm{PGL}_2 \cong \mathrm{Aut}(\mathbb{P}^1)$. The graphical algebra is the ring of SL_2-invariant multi-homogeneous polynomials.

Let $[X_i : Y_i]$ be the homogeneous coordinates of i-th factor of $(\mathbb{P}^1)^n$. It is straightforward to check that $Z_{ij} := (X_iY_j - X_jY_i)$ is an SL_2-invariant polynomial, and their products are all invariant polynomials. We can index such polynomials by using finite digraphs. Let $\overrightarrow{\Gamma}$ be a finite directed graph on the vertex set $[n] := \{1, 2, \cdots, n\}$, and let Γ be its underlying undirected graph. We allow multiple edges, but a loop is not allowed. For a vertex i, the *degree* of i (denoted by d_i) is the number of edges incident to i regardless of their directions. The *multidegree* of $\overrightarrow{\Gamma}$ is defined by the degree sequence $(d_1, d_2, \cdots, d_n)$ and denoted by $\mathbf{deg}\ \overrightarrow{\Gamma}$. Let $V_{\overrightarrow{\Gamma}} = V_\Gamma$ be the set of vertices, and let $E_{\overrightarrow{\Gamma}}$ be the set of directed edges. We define $\mathbf{deg}\ \Gamma := \mathbf{deg}\ \overrightarrow{\Gamma}$ and E_Γ is the set of undirected edges in Γ. For any $I \subset [n]$, let w_I be the number of edges connecting vertices in I. For notational simplicity, we set $w_{ij} = w_{\{i,j\}}$, which is the number of edges connecting i and j. So $w_I = \sum_{i,j \in I} w_{ij}$. For $e \in E_{\overrightarrow{\Gamma}}$, $h(e) \in V_{\overrightarrow{\Gamma}}$ is the head and $t(e) \in V_{\overrightarrow{\Gamma}}$ is the tail.

For each $\overrightarrow{\Gamma}$, let

$$Z_{\overrightarrow{\Gamma}} := \prod_{e \in E(\Gamma)} (X_{t(e)}Y_{h(e)} - X_{h(e)}Y_{t(e)}).$$

Then

$$Z_{\overrightarrow{\Gamma}} \in \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(\mathbf{deg}\ \overrightarrow{\Gamma}))^{\mathrm{SL}_2}.$$

We define the multiplication $\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}$ of two graphs $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ by a graph with the vertex set $[n]$ and $E_{\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}} := E_{\overrightarrow{\Gamma_1}} \sqcup E_{\overrightarrow{\Gamma_2}}$, the disjoint union of the edge sets. Note that our graphs and directed graphs are not simple, so we allow several edges between two vertices. If two graphs $\overrightarrow{\Gamma_1}$ and $\overrightarrow{\Gamma_2}$ have common edges, then we retain all of them. Then $\mathbf{deg}\ \overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2} = \mathbf{deg}\ \overrightarrow{\Gamma_1} + \mathbf{deg}\ \overrightarrow{\Gamma_2}$. Furthermore,

$$Z_{\overrightarrow{\Gamma_1} \cdot \overrightarrow{\Gamma_2}} = Z_{\overrightarrow{\Gamma_1}} \cdot Z_{\overrightarrow{\Gamma_2}}.$$

Note that if we reverse the direction of an edge $e \in E_{\overrightarrow{\Gamma}}$ and make a new graph $\overrightarrow{\Gamma}'$, then $Z_{\overrightarrow{\Gamma}'} = -Z_{\overrightarrow{\Gamma}}$.

The first fundamental theorem of invariant theory ([9, Theorem 2.1]) says that the ring of SL_2-invariants of $(\mathbb{P}^1)^n$ is generated by the polynomials $Z_{\overrightarrow{\Gamma}}$.

Definition 2.1. The (total) *graphical algebra* R of order n is defined by

$$R := \bigoplus_{L \in \mathrm{Pic}((\mathbb{P}^1)^n)} \mathrm{H}^0((\mathbb{P}^1)^n, L)^{\mathrm{SL}_2} = \bigoplus_{(a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n_{\ge 0}} \mathrm{H}^0((\mathbb{P}^1)^n, \mathcal{O}(a_1, a_2, \cdots, a_n))^{\mathrm{SL}_2}.$$

The support of $Z_{\overrightarrow{\Gamma}}$ is independent of the direction of each edge. Thus we may denote $\mathrm{Supp}(Z_{\overrightarrow{\Gamma}})$ by D_Γ. This SL_2-invariant divisor on $(\mathbb{P}^1)^n$, or a Weil divisor on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$, is called a *graphical divisor*. The support of Z_{ij} is denoted by D_{ij}.

The homogeneous coordinate ring of the GIT quotient is a slice of R. Fix an effective linearization (a linearization with a nonempty semistable locus) $L \cong \mathcal{O}(a_1, a_2, \cdots, a_n)$. Then the homogeneous coordinate ring of $(\mathbb{P}^1)^n /\!/_L \mathrm{SL}_2$ is

$$R_L := \bigoplus_{d \geq 0} \mathrm{H}^0((\mathbb{P}^1)^n, L^d)^{\mathrm{SL}_2} \subset R.$$

The ideal of relations is explicitly described in [18,19].

From now on, we will use the symmetric linearization $\mathcal{O}(1, 1, \cdots, 1)$ only. In this case, the GIT quotient $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ is a projective variety with a natural S_n-action permuting the n factors. If n is odd, it is regular. If n is even, there are $\binom{n}{n/2}/2$ non-regular closed points which are associated to closed orbits of two distinct points with multiplicities $n/2$.

The generating set of R_L has been well-understood since the 19th century by Kempe ([21]). A combinatorial description, including the relation ideal, is given in [18]. We summarize the description here.

Theorem 2.2 *([21], [18, Theorem 2.3]). The homogeneous coordinate ring R_L is generated by $Z_{\overrightarrow{\Gamma}}$ for $\overrightarrow{\Gamma}$ with* $\mathbf{deg}\, \overrightarrow{\Gamma} = (\epsilon, \epsilon, \cdots, \epsilon)$ *where $\epsilon = 2$ if n is odd, and $\epsilon = 1$ if n is even.*

Let e_i be the i-th standard vector in $\mathbb{Z}^n \cong \mathrm{Pic}((\mathbb{P}^1)^n)$. Each D_{ij} on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ is the image of $V(Z_{ij})$ in $(\mathbb{P}^1)^n$ and $Z_{ij} \in \mathrm{H}^0(\mathcal{O}(e_i + e_j))$. Thus $\mathrm{Cl}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2)$ is identified with an index two sub-lattice of $\mathrm{Pic}((\mathbb{P}^1)^n)$, generated by $\deg D_{ij} = e_i + e_j$. Note that a generator D_{ij} of $\mathrm{Cl}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2)$ has a simple moduli theoretic interpretation. Indeed,

$$D_{ij} = \{(p_1, p_2, \cdots, p_n) \in (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2 \mid p_i = p_j\}.$$

Let $D_2 = \sum D_{ij}$.

Lemma 2.3. *The S_n-invariant Picard group $\mathrm{Pic}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2)^{S_n}$ is generated by $\frac{1}{n-1} D_2$ (resp. $\frac{2}{n-1} D_2$) when n is even (resp. odd).*

Proof. On $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$, we denote the descent of the line bundle $\mathcal{O}(a_1, a_2, \cdots, a_n)$ on $(\mathbb{P}^1)^n$ by $\overline{\mathcal{O}}(a_1, a_2, \cdots, a_n)$. By Kempf's descent lemma ([8, Theorem 2.3]), we are able to check when a line bundle on $(\mathbb{P}^1)^n$ descends to $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$. If n is odd, it descends if and only if $\sum a_i$ is even. If n is even, there is one extra constraint: For any $I \subset [n]$ with $|I| = n/2$, $\sum_{i \in I} a_i = \sum_{i \notin I} a_i$. The only line bundles satisfying this condition are the symmetric bundles $\mathcal{O}(a, a, \cdots, a)$. Therefore if n is odd, $\mathrm{Pic}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2)$ is isomorphic to an index two sub-lattice of $\mathrm{Pic}((\mathbb{P}^1)^n) \cong \mathbb{Z}^n$. If n is even, $\mathrm{Pic}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2) \cong \mathbb{Z}$. In particular, $\overline{\mathcal{O}}(1, 1, \cdots, 1)$ (resp. $\overline{\mathcal{O}}(2, 2, \cdots, 2)$) is an integral generator of $\mathrm{Pic}((\mathbb{P}^1)^n /\!/ \mathrm{SL}_2)^{S_n} \cong \mathbb{Z}$ when n is even (resp. odd). Because $\mathcal{O}(D_2) = \overline{\mathcal{O}}(n-1, n-1, \cdots, n-1)$, we obtain the desired result. $\square$

3. G-base-point-freeness

In this section, by using graphical algebra, we translate the S_n-invariant F-conjecture to a polyhedral feasibility problem. We work over Spec $\mathbb{Z}$, unless there is an explicit assumption on the base scheme.

The following result explains an explicit connection between $\overline{\mathrm{M}}_{0,n}$ and $(\mathbb{P}^1)^n//\mathrm{SL}_2$.

Theorem 3.1 *([20], [16, Theorem 4.1 and 8.3]). There is a birational contraction morphism*

$$\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n//\mathrm{SL}_2.$$

Remark 3.2. In [2], Alexeev and the second author studied nef divisors on $\overline{\mathrm{M}}_{0,n}$ obtained by pulling back nef divisors on $(\mathbb{P}^1)^n//\mathrm{SL}_2$. These "GIT divisors" were used to show that log canonical models of $\overline{\mathrm{M}}_{0,n}$ are Hassett's moduli spaces of weighted stable curves. Here, we examine certain linear systems for one of these divisors in much greater detail.

For any $I \subset [n]$ with $2 \le I \le n/2$, let $B_I \subset \overline{\mathrm{M}}_{0,n}$ be the associated boundary divisor, and let $B_i := \sum_{|I|=i} B_I$. The image of $B_{ij} := B_{\{i,j\}}$ is D_{ij}. For $I \subset [n]$ with $3 \le |I| < n/2$, B_I is contracted by π, and its image is

$$\pi(B_I) = \{(p_1, p_2, \cdots, p_n) \mid p_i = p_j \text{ for all } i, j \in I\}.$$

If $p : ((\mathbb{P}^1)^n)^{ss} \to (\mathbb{P}^1)^n//\mathrm{SL}_2$ is the GIT quotient map, then $\pi(B_I)$ is the image $p(W_I)$ of $W_I := V(Z_{ij})_{i,j\in I} \subset ((\mathbb{P}^1)^n)^{ss}$.

When n is even and $|I| = n/2$, $B_I = B_{I^c}$. Then $\pi(B_I) = \pi(B_{I^c})$ is an isolated singular point on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ and the associated closed orbit is

$$\{(p_1, p_2, \cdots, p_n) \mid p_i = p_j \text{ for all } i, j \in I \text{ or } i, j \in I^c\}.$$

Thus $\pi(B_I)$ is the image $p(W_I)$ of $W_I := V(Z_{ij})_{i,j\in I} \cap V(Z_{ij})_{i,j\in I^c}$. We denote $\pi(B_I) = p(W_I)$ by V_I for all I.

By Theorem 2.2, D_2 is very ample on $(\mathbb{P}^1)^n//\mathrm{SL}_2$ since $\mathcal{O}(D_2) = \overline{\mathcal{O}}(n-1, n-1, \cdots, n-1)$. We have

$$\pi^*(D_2) = \sum_{i\ge 2}^{\lfloor n/2 \rfloor} \binom{i}{2} B_i \tag{1}$$

([22, Lemma 5.3]). Since π is a regular contraction, the complete linear system $|\pi^*(D_2)|$ is base-point-free on $\overline{\mathrm{M}}_{0,n}$. Indeed, $\pi^*(D_2)$ is an extremal ray of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n}$ ([1, Proposition 6.8]).

The following is a very simple but important observation.

Lemma 3.3. *Every non-trivial S_n-invariant F-nef $\mathbb{Q}$-divisor on $\overline{\mathrm{M}}_{0,n}$ can be written uniquely as*

$$\pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$$

for some rational numbers $c > 0$ and $0 \leq a_i < c\binom{i}{2}$. Furthermore, it is integral if and only if $a_i \in \mathbb{Z}$ and $c \in \frac{1}{n-1}\mathbb{Z}$ (resp. $c \in \frac{2}{n-1}\mathbb{Z}$) when n is even (resp. n is odd).

For notational simplicity, we set $a_1 = a_2 = 0$.

Proof. Since F-nefness is defined formally, it is sufficient to prove the result over any algebraically closed field. In $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$, by [23, Theorem 1.3], $\{B_i\}_{2 \leq i \leq \lfloor n/2 \rfloor}$ forms a $\mathbb{Q}$-basis. By Equation (1), it is straightforward to check that $\{\pi^*(D_2), B_i\}_{3 \leq i \leq \lfloor n/2 \rfloor}$ is also a basis. Thus we have the existence and the uniqueness of the expression. Since $\mathrm{Eff}(\overline{\mathrm{M}}_{0,n})^{S_n}$ is generated by B_i for $2 \leq i \leq \lfloor n/2 \rfloor$ and every S_n-invariant F-nef divisor is big ([11, Proposition 4.5]), an S_n-invariant F-nef divisor is a strictly positive linear combination of B_i's. From (1), $a_i < c\binom{i}{2}$ and $c > 0$. Let F_j be any F-curve class whose associated partition has parts $\{1, 1, j, n-2-j\}$ for $1 \leq j \leq \lfloor n/2 \rfloor - 2$. Then since F_j is contracted by π,

$$0 \leq F_j \cdot (\pi^*(cD_2) - \sum_{i \geq 3} a_i B_i) = a_j + a_{j+2} - 2a_{j+1},$$

so the sequence $\{a_j\}$ is convex. From $a_1 = a_2 = 0$, inductively we obtain $a_i \geq 0$ for all i.

The last assertion follows from Lemma 2.3, since each B_i are all integral. □

Definition 3.4. For a non-trivial integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$, let $|cD_2|_{\mathbf{a}}$ be the sub linear system of $|cD_2|$ on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$ consisting of the divisors whose multiplicity along V_I is at least $a_{|I|}$.

Lemma 3.5. *Let $D = \pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. The complete linear system $|D|$ is identified with $|cD_2|_{\mathbf{a}}$ on $(\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$.*

Proof. Since D is non-trivial, $c > 0$ by Lemma 3.3. For any $E \in |cD_2|_{\mathbf{a}}$, $\pi^*E = E' + \sum_{i \geq 3} a_i B_i$ and $E' \in |D|$. Thus we have an injective map $|cD_2|_{\mathbf{a}} \to |D|$. Any divisor $F \in |D|$ can be written as $F + \sum_{i \geq 3} a_i B_i = \pi^* F'$ for some $F' \in |cD_2|$, and from the expression $F' \in |cD_2|_{\mathbf{a}}$. Thus we have a map $|D| \to |cD_2|_{\mathbf{a}}$. It is straightforward to see that they are inverses of each other. □

Definition 3.6. For a divisor $E \in |cD_2|_{\mathbf{a}}$, let $\widetilde{E} \in |D|$ be $\pi^*(E) - \sum_{i \geq 3} a_i B_i$, the divisor identified with E via the isomorphism $|D| \cong |cD_2|_{\mathbf{a}}$.

Note that $\widetilde{E}$ is *not* the proper transform of E in general.

The non complete sub linear system $|cD_2|_{\mathbf{a}}$ is key to understanding the base-point-freeness or the semi-ampleness of $|D|$; unfortunately, it is still difficult to analyze. We will define a sub linear system of $|cD_2|_{\mathbf{a}}$ which can be studied in purely combinatorial terms.

Recall that for any $I \subset [n]$, w_I is the number of edges connecting vertices in I. Recall also that for notational simplicity, we set $a_2 = 0$.

Proposition 3.7. *Let $D = \pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Let D_Γ be a graphical divisor associated to a graph Γ. Then $D_\Gamma \in |cD_2|_{\mathbf{a}}$ if and only if*

(1) $\mathbf{deg}\, \Gamma = (c(n-1), c(n-1), \cdots, c(n-1))$*;*
(2) For every $I \subset [n]$ with $2 \leq |I| \leq n/2$, $w_I \geq a_{|I|}$;

Proof. The condition (1) is exactly $D_\Gamma \in |cD_2|$, because $\mathcal{O}(cD_2) = \overline{\mathcal{O}}(c(n-1), c(n-1), \cdots, c(n-1))$. Note that for $3 \leq |I| < n/2$, a general point of V_I is smooth. Thus D_Γ vanishes on V_I with multiplicity at least $a_{|I|}$ if and only if $p^*(D_\Gamma)$ vanishes on $W_I = V(Z_{ij})_{i,j\in I}$ with multiplicity at least $a_{|I|}$ if and only if Γ has at least a_i edges connecting vertices in I.

When n is even and $|I| = n/2$, V_I is an isolated singular point and it is the image of $W_I = V(Z_{ij})_{i,j\in I} \cap V(Z_{ij})_{i,j\in I^c}$ on $((\mathbb{P}^1)^n)^{ss}$. Suppose that Z_Γ satisfies the following condition:

$$(*) \text{ For every } I \subset [n] \text{ with } |I| = n/2,\ w_I + w_{I^c} \geq 2a_{n/2}.$$

We claim that this condition is equivalent to the condition that multiplicity along $B_{n/2}$ is at least $a_{n/2}$.

Consider the following commutative diagram:

$$\begin{array}{ccccc} & & (\mathrm{Bl}_{W_I}(\mathbb{P}^1)^n)^s & \xrightarrow{q} & ((\mathbb{P}^1)^n)^{ss} \\ & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ \overline{\mathrm{M}}_{0,n} & \longrightarrow & \mathrm{Bl}_{W_I}(\mathbb{P}^1)^n/\!/\mathrm{SL}_2 & \xrightarrow{\bar{q}} & (\mathbb{P}^1)^n/\!/\mathrm{SL}_2 \end{array}$$

The vertical arrows are GIT quotients and q is the blow-up along W_I. The superscript ss (resp. s) denotes the semistable (resp. stable) locus. In [22, Theorem 1.1], it was shown that Hassett's contraction $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n/\!/\mathrm{SL}_2$ is decomposed into $\overline{\mathrm{M}}_{0,n} \to \mathrm{Bl}_{W_I}(\mathbb{P}^1)^n/\!/\mathrm{SL}_2 \xrightarrow{\bar{q}} (\mathbb{P}^1)^n/\!/\mathrm{SL}_2$ and $\bar{q}$ is Kirwan's partial desingularization.

For any $D_G \in |cD_2|$, $p^*(D_G)$ is an SL_2-invariant divisor on $((\mathbb{P}^1)^n)^{ss}$. Its multiplicity along W_I is at least $2a_{|I|} = 2a_{n/2}$. Thus $q^*p^*(D_G)$ has the multiplicity at least $2a_{n/2}$ along the exceptional divisor E_I. But since $-\mathrm{Id} \in \mathrm{SL}_2$ acts on E_I nontrivially, $2E_I$ descends to B_I. Therefore the multiplicity along B_I on $\mathrm{Bl}_{W_I}(\mathbb{P}^1)^n/\!/\mathrm{SL}_2$, which is equal to the multiplicity along B_I on $\overline{\mathrm{M}}_{0,n}$, is at least $a_{n/2}$. The converse is similar.

Since the degree of each vertex is $c(n-1)$, $w_I = cn(n-1)/2 - \sum_{i\in I, j\in I^c} w_{ij} = w_{I^c}$. Thus we may reduce Condition (*) to (2). $\square$

Definition 3.8. Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be a non-trivial S_n-invariant F-nef divisor with $c, a_i \in \mathbb{Z}$. Let $|cD_2|_{\mathbf{a},G} \subset |cD_2|_{\mathbf{a}}$ be the sub linear system generated by D_Γ satisfying two conditions in Proposition 3.7. Let $|D|_G \subset |D|$ be the sub linear system which is identified with $|cD_2|_{\mathbf{a},G}$ via the identification

$$|D| \cong |cD_2|_{\mathbf{a}}.$$

In other words, $|D|_G$ is generated by $\widetilde{D}_\Gamma$ for Γ in Proposition 3.7.

Remark 3.9. In general, $|D|_G$ is not equal to $|D|$. Equivalently, $|cD_2|_{\mathbf{a},G}$ is not equal to $|cD_2|_{\mathbf{a}}$. See Example 4.6.

The following lemma is straightforward. (It is also a special case of [10, Lemma 2.3.3]. We thank one of the anonymous referees for this observation.)

Lemma 3.10. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Let $B = \bigcap_{I\in T} B_I$ be a boundary stratum indexed by a nonempty subset $T \subset \{I \subset [n] \mid 2 \leq |I| \leq \lfloor n/2 \rfloor\}$. Then a general point in B is not in the support of $\widetilde{D}_\Gamma \in |D|_G$ if and only if for every $I \in T$ with $|I| \leq n/2$, $w_I = a_{|I|}$.*

Definition 3.11. An integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ is called *G-base-point-free* if $|D|_G$ is base-point-free. It is *G-semi-ample* if $|mD|_G$ is base-point-free for some $m \gg 0$.

This sub linear system is particularly nice because the base locus can be described in a combinatorial way.

Lemma 3.12. *Let $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Then the base locus $\mathrm{Bs}(|D|_G)$ is a union of closures of boundary strata.*

Proof. For $\widetilde{D}_\Gamma \in |D|_G$,

$$\mathrm{Supp}(\widetilde{D}_\Gamma) = \bigcup_I B_I$$

where the sum is taken over all I where the number of edges connecting vertices in I is strictly larger than $a_{|I|}$. Since $|D|_G$ is generated by $\widetilde{D}_\Gamma$,

$$\mathrm{Bs}(|D|_G) = \bigcap_{\widetilde{D}_\Gamma \in |D|_G} \mathrm{Supp}(\widetilde{D}_\Gamma). \quad \square$$

Proposition 3.13. *Let $D = \pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$ be an integral S_n-invariant F-nef divisor. Then D is G-base-point-free if and only if for every F-point $F = \cap_{I \in T} B_I$, there is a graph Γ such that*

(1) $\mathbf{deg}\, \Gamma = (c(n-1), c(n-1), \cdots, c(n-1))$*;*
(2) For each $I \subset [n]$ *with* $2 \leq |I| \leq n/2$*,* $w_I \geq a_{|I|}$*;*
(3) For each $J \in T$ *with* $|J| \leq n/2$*,* $w_J = a_{|J|}$*;*

Proof. The above condition implies that the base locus of $|D|_G$ does not contain any F-point. Since $\mathrm{Bs}(|D|_G)$ is a union of boundary strata by Lemma 3.12, if it is non-empty, then there must be at least one F-point on it. Thus $|D|_G$ is base-point-free. □

Since G-base-point-freeness implies base-point-freeness, Proposition 3.13 provides a purely combinatorial sufficient condition for being a base-point-free divisor.

Note that sums and scalar multiples of graphical divisors are graphical divisors, too. So it is straightforward to see that for any two G-base-point-free (resp. G-semi-ample) divisors D and D', $D+D'$ and their nonnegative scalar multiples are all G-base-point-free (resp. G-semi-ample).

Definition 3.14. Let $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ be the convex cone generated by G-semi-ample divisors in $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$.

We have the following obvious implications for S_n-invariant divisors:

$$G\text{-base-point-free} \Rightarrow G\text{-semi-ample} \Rightarrow \text{semi-ample} \Rightarrow \text{nef} \Rightarrow F\text{-nef}. \tag{2}$$

Theorem 3.15. *The cone* $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ *of G-semi-ample divisors of* $\overline{\mathrm{M}}_{0,n}$ *is closed and polyhedral.*

Proof. We may consider a multigraph Γ as a graph weighting $w : E_{K_n} \to \mathbb{Q}$ where E_{K_n} is the set of edges on the complete graph K_n, by setting $w(\overline{ij}) = w_{ij}$, the number of edges between i and j. Consider $V := \mathbb{Q}^{\binom{n}{2}}$ with coordinates $\{w_{ij}\}_{1 \leq i < j \leq n}$. This space can be regarded as the space of graph weightings. By representing any non-trivial S_n-invariant F-nef divisor in the form $\pi^*(cD_2) - \sum_{i \geq 3} a_i B_i$, we may identify $\mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ with $\mathbb{Q}^{\lfloor n/2 \rfloor - 1}$ whose coordinates are $(c, a_i)_{3 \leq i \leq \lfloor n/2 \rfloor}$. For each F-point $F = \bigcap_{J \in T} B_J$, we can define a polyhedral cone $Q(n, F) \subset V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ by the following inequalities and equations:

(1) $c, a_i, w_{ij} \geq 0$;
(2) $\sum_{j \neq i} w_{ij} = c(n-1)$;
(3) $\sum_{i,j \in I} w_{ij} \geq a_{|I|}$ for each I with $3 \leq |I| \leq n/2$;
(4) $\sum_{i,j \in J} w_{ij} = a_{|J|}$ for each $J \in T$ with $|J| \leq n/2$.

Let $\rho : V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n} \to \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ be the projection defined by

$$\rho(w_{ij}, c, a_i) = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i = (c, a_i). \tag{3}$$

For an integral S_n-invariant F-nef divisor $D = \pi^*(cD_2) - \sum_{i\geq 3} a_i B_i$, $\mathrm{Bs}(|D|_G)$ does not contain an F-point $F := \bigcap_{J\in T} B_J$ if and only if $\rho^{-1}(D) \cap Q(n, F)$ has an integral point. So $\mathrm{Bs}(|mD|_G)$ does not contain F for some m if and only if $\rho^{-1}(D) \cap Q(n, F)$ has a rational point, if and only if $D \in \rho(Q(n, F))$.

Therefore $|mD|_G$ is base-point-free for some $m \gg 0$ if and only if $D \in \bigcap_F \rho(Q(n, F))$ where the intersection is taken over all F-points. Therefore it is polyhedral and closed. □

Thus we obtain a polyhedral lower bound of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})^{S_n}$.

By the proof of Theorem 3.15, we obtain the following corollary, which provides a computational approach to the S_n-invariant F-conjecture. For the reader's convenience, we state it in a self-contained form.

Corollary 3.16. *Let $\rho : V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n} \to \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ be a projection given by $\rho(w_{ij}, c, a_i) = (c, a_i)$. For each F-point $F = \bigcap_{J\in T} B_J$, let $Q(n, F)$ be a polyhedral cone in $V \times \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ defined by*

(1) $c, a_i, w_{ij} \geq 0$;
(2) $\sum_{j\neq i} w_{ij} = c(n-1)$;
(3) $\sum_{i,j\in I} w_{ij} \geq a_{|I|}$ *for each I with* $3 \leq |I| \leq n/2$;
(4) $\sum_{i,j\in I} w_{ij} = a_{|I|}$ *for each $J \in T$ with* $|J| \leq n/2$.

Let $D := \pi^*(cD_2) - \sum a_i B_i = (c, a_i) \in \mathrm{N}^1(\overline{\mathrm{M}}_{0,n})^{S_n}$ be an S_n-invariant F-nef divisor. Then D is G-semi-ample (hence semi-ample) if and only if $D \in \rho(Q(n, F))$ for every F-point F.

Remark 3.17. The polytopes defined by these sets of inequalities were previously obtained by Fedorchuk ([10, Lemma 2.3.3]). We will discuss the equivalence of our result and Fedorchuk's for S_n-invariant F-nef divisors in more detail in Proposition 5.4. Although the polytopes turn out to be the same, we obtain them by a completely different approach.

4. Computational results

In this section we list several computational results. The calculations can be found on the webpage [25].

Theorem 4.1. *For $n \leq 19$, over* Spec $\mathbb{Z}$*, the S_n-invariant F-nef cone coincides with the G-semi-ample cone.*

Proof. Since $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ is a convex subcone of the S_n-invariant F-nef cone, it is sufficient to show that every integral generator of an extremal ray of the S_n-invariant F-nef cone is G-semi-ample. By Corollary 3.16, it is sufficient to show the feasibility of the polytope $\rho^{-1}(D) \cap Q(n, F)$ for each integral generator D and an F-point F. By using Sage [27] and Gurobi [15], we checked that for $n \leq 19$, such a polytope is nonempty. □

Indeed a more refined result is true. Recall that the set of integral points on a strongly convex rational polyhedral cone forms a monoid and its generating set is called the Hilbert basis. By checking G-base-point-freeness of the Hilbert basis of the S_n-invariant F-nef cone, we obtain the following result.

Theorem 4.2.

(1) For $n \leq 16$, over Spec $\mathbb{Z}$, *for any integral S_n-invariant F-nef divisor D, $2D$ is G-base-point-free.*
(2) For $n \leq 11$ or 13, over Spec $\mathbb{Z}$, *for any integral S_n-invariant F-nef divisor D, D is G-base-point-free.*

Remark 4.3. This computation was faster than we anticipated. To check the G-base-point-freeness of a divisor D we need to do the following computation.

(1) Take a representative of an F-point F from each S_n-orbit. Let P be the set of the representatives.
(2) For each $F \in P$, compute the non-emptiness of $\rho^{-1}(D) \cap Q(n, F)$.

But when n is small, for many of the divisors D that we tested, $\rho^{-1}(D) \cap \cap_{F \in P} Q(n, F)$ is nonempty. Thus, for these divisors, it was sufficient to solve just one feasibility problem, instead of one feasibility problem for each $F \in P$. For such divisors D, in the linear system $|D|$, we found an effective sum of boundaries E whose support does not contain at least one F-point in each S_n-orbit in the set of all F-points. By S_n-symmetry, this is sufficient for the G-base-point-freeness.

Further tricks for speeding up the calculation are described at [25].

Conjecture 4.4. *For any integral S_n-invariant F-nef divisor D, $2D$ is G-base-point-free. In particular, $2D$ is base-point-free.*

Remark 4.5. For $n \leq 15$, most of the integral divisors are G-base-point-free. More precisely, for $n = 12$, there are only two integral S_n-invariant F-nef divisors which are not base-point-free. For $n = 14, 15$, there is only one for each n.

Example 4.6. Let $n = 12$. Consider the divisor class

$$D = \frac{1}{11}(4B_2 + 12B_3 + 13B_4 + 18B_5 + 16B_6) = \pi^*(\frac{4}{11}D_2) - B_4 - 2B_5 - 4B_6.$$

Here we give an example of an integral S_n-invariant base-point-free divisor which is not G-base-point-free. Then the base locus $\mathrm{Bs}(|D|_G)$ contains the S_{12}-orbit of an F-point

$$\begin{aligned} F =& B_{\{1,2\}} \cap B_{\{1,2,3\}} \cap B_{\{4,5\}} \cap B_{\{4,5,6\}} \cap B_{\{1,2,3,4,5,6\}} \cap B_{\{7,8\}} \cap B_{\{7,8,9\}} \\ & \cap B_{\{10,11\}} \cap B_{\{10,11,12\}}. \end{aligned}$$

One can check that the locus of curves having four tails with three marked points is the base locus of $|D|_G$. The base locus is a disjoint union of 15400 loci, each of which is isomorphic to $(\overline{\mathrm{M}}_{0,4})^5$.

On the other hand, by using a computer and Kapranov's model, we constructed a divisor $E \in |D|$ such that $F \notin E$. Therefore $|D|_G \neq |D|$. See [25].

Now the very ampleness of S_n-invariant ample divisors is an immediate consequence of the result of Keel and Tevelev.

Theorem 4.7. *Let $n \leq 16$. Over any algebraically closed field, for every integral S_n-invariant ample divisor A on $\overline{\mathrm{M}}_{0,n}$, $2A$ is very ample.*

Proof. By [28, Theorem 1.5] or [24, Theorem 1.1], the log canonical divisor $K_{\overline{\mathrm{M}}_{0,n}} + B$ is very ample. Note that for every F-curve F, $(K_{\overline{\mathrm{M}}_{0,n}} + B) \cdot F = 1$. So if A is an S_n-invariant integral ample divisor, then $A - (K_{\overline{\mathrm{M}}_{0,n}} + B)$ is an S_n-invariant nef divisor, so $2(A - (K_{\overline{\mathrm{M}}_{0,n}} + B))$ is base-point-free by Theorem 4.2. Therefore

$$2A = 2(A - (K_{\overline{\mathrm{M}}_{0,n}} + B)) + 2(K_{\overline{\mathrm{M}}_{0,n}} + B)$$

is a sum of a base-point-free divisor and a very ample divisor, which is very ample. □

Remark 4.8. By Remark 4.5, when $n \leq 11$ or $n = 13$, every integral S_n-invariant nef divisor is base-point-free. So for those n, every integral S_n-invariant ample divisor is very ample.

5. Comparison to other cones

Several lower bounds of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ have been previously described in the literature. In this section we compare the S_n-invariant part of these cones with the cone $\mathrm{GS}(\overline{\mathrm{M}}_{0,n})$ of G-semi-ample divisors introduced in this paper.

The first lower bound is due to Gibney and Maclagan. In [14], they define a lower bound of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$, by using an embedding of $\overline{\mathrm{M}}_{0,n}$ into a non-proper toric variety. Let X_Δ be a toric variety whose associated fan is $\Delta \subset \mathbb{R}^n$. Suppose that there is a projective toric variety X_Σ with $\Delta \subset \Sigma$.

Definition 5.1. The cone $\mathcal{G}_\Delta \subset \mathrm{Pic}(X_\Delta)_\mathbb{Q}$ is the semi-ample cone of X_Δ.

Let Y be a projective variety with an embedding $i : Y \hookrightarrow X_\Delta$. Then we obtain a lower bound

$$i^*(\mathcal{G}_\Delta) \subset \mathrm{Nef}(Y).$$

The cone $i^*(\mathcal{G}_\Delta)$ is polyhedral and can be described combinatorially. These properties follow from the corresponding properties of $\mathcal{G}_\Delta$:

Proposition 5.2 *([14, Proposition 2.3]). Let $D = \sum_{i\in\Delta(1)} a_i D_i \in \mathrm{Pic}(X_\Delta)$. Then the following are equivalent:*

(1) $D \in \mathcal{G}_\Delta$;
(2) $D \in \bigcap_{\sigma\in\Delta} \mathrm{pos}(D_i \mid i \notin \sigma)$;
(3) there is a piecewise linear convex function $\psi : N_\mathbb{R} \to \mathbb{R}$ and $\psi(v_i) = a_i$ where v_i is the first integral vector in the ray i.
(4) $D \in \bigcup_\Sigma i^*_\Sigma(\mathrm{Nef}(X_\Sigma))$ where the union is over all projective toric varieties X_Σ with $\Delta \subset \Sigma$ and $i_\Sigma : X_\Delta \to X_\Sigma$ is the inclusion. Furthermore, we may assume that $\Delta(1) = \Sigma(1)$.

It is well-known that $\overline{\mathrm{M}}_{0,n}$ can be embedded into a non-proper toric variety X_Δ where Δ is the space of phylogenetic trees. Thus we obtain a lower bound $i^*(\mathcal{G}_\Delta)$ of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$.

The second lower bound is due to Fedorchuk. In [10], he introduces a combinatorial notion called *boundary semi-ampleness*.

Definition 5.3. Let D be a divisor on $\overline{\mathrm{M}}_{0,n}$. D is *boundary semi-ample* if for every $x \in \overline{\mathrm{M}}_{0,n}$, there exists an effective boundary $\mathbb{Q}$-divisor $E \in |D|$ such that $x \notin \mathrm{Supp}(E)$.

The following result was pointed out to us by Fedorchuk.

Proposition 5.4. *Suppose that D is an S_n-invariant divisor on $\overline{\mathrm{M}}_{0,n}$. Then the following three conditions are equivalent:*

(1) $D \in \mathrm{GS}(\overline{\mathrm{M}}_{0,n})$;
(2) D is boundary semi-ample;
(3) $D \in i^*(\mathcal{G}_\Delta)$.

Proof. For an F-nef divisor $D = \pi^* cD_2 - \sum a_i B_i$, recall that $|D|_G$ is the sub linear system of $|D|$ generated by graphical divisors. Let $|D|_B$ be the sub linear system generated by effective sums of boundary divisors. Since any graphical divisor is an effective sum of boundaries, $|D|_G \subset |D|_B$. Conversely, if $E = \sum_I c_I B_I \in |D|_B$ is an effective sum of boundaries, then for $\pi : \overline{\mathrm{M}}_{0,n} \to (\mathbb{P}^1)^n /\!/ \mathrm{SL}_2$, $\pi_*(E) = \sum_{|I|=2} c_I D_I$, so this is a graphical divisor D_Γ for some graph Γ. Then $\widetilde{D}_\Gamma = \pi^* D_\Gamma - \sum_{i\geq 3} a_i B_i = \sum_{|I|=2} c_I B_I +$

$\sum_{|I|\geq 3} d_I B_I$ for some d_I. Then $d_I = c_I$ and $E = \widetilde{D}_\Gamma$ because all divisors B_I with $|I| \geq 3$ are independent. Thus $E \in |D|_G$ and $|D|_G = |D|_B$. Therefore, G-semi-ampleness, which is the base-point-freeness of $|mD|_G$ for some $m > 0$, is equivalent to boundary semi-ampleness, which is precisely the base-point-freeness of $|mD|_B$ for some $m > 0$.

Because $\mathrm{Pic}(X_\Delta) \cong \mathrm{Pic}(\overline{\mathrm{M}}_{0,n})$ and each boundary divisor B_I is a restriction of a toric boundary, from item (2) of Proposition 5.2, it is straightforward to see that D is boundary semi-ample if and only if $D \in i^*(\mathcal{G}_\Delta)$. $\square$

References

[1] Valery Alexeev, Angela Gibney, David Swinarski, Higher-level $\mathfrak{sl}_2$ conformal blocks divisors on $\bar{M}_{0,n}$, Proc. Edinb. Math. Soc. (2) 57 (1) (2014) 7–30.

[2] Valery Alexeev, David Swinarski, Nef divisors on $\overline{M}_{0,n}$ from GIT, in: Geometry and Arithmetic, in: EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 1–21.

[3] Izzet Coskun, Joe Harris, Jason Starr, The ample cone of the Kontsevich moduli space, Canad. J. Math. 61 (1) (2009) 109–123.

[4] Izzet Coskun, Jason Starr, Divisors on the space of maps to Grassmannians, Int. Math. Res. Not. 2006 (2006) 35273.

[5] Ana-Maria Castravet, Jenia Tevelev, Hypertrees, projections, and moduli of stable rational curves, J. Reine Angew. Math. 675 (2013) 121–180.

[6] Ana-Maria Castravet, Jenia Tevelev, $\overline{M}_{0,n}$ is not a Mori dream space, Duke Math. J. 164 (8) (2015) 1641–1667.

[7] Brent Doran, Noah Giansiracusa, David Jensen, A simplicial approach to effective divisors in $\overline{M}_{0,n}$, Int. Math. Res. Not. IMRN 2 (2017) 529–565.

[8] J.-M. Drezet, M.S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1) (1989) 53–94.

[9] Igor Dolgachev, Lectures on Invariant Theory, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003.

[10] Maksym Fedorchuk, Semiampleness criteria for divisors on $\overline{M}_{0,n}$, arXiv:1407.7839, 2014.

[11] Angela Gibney, Numerical criteria for divisors on $\overline{M}_g$ to be ample, Compos. Math. 145 (5) (2009) 1227–1248.

[12] José Luis González, Kalle Karu, Some non-finitely generated Cox rings, Compos. Math. 152 (5) (2016) 984–996.

[13] Angela Gibney, Sean Keel, Ian Morrison, Towards the ample cone of $\overline{M}_{g,n}$, J. Amer. Math. Soc. 15 (2) (2002) 273–294.

[14] Angela Gibney, Diane Maclagan, Lower and upper bounds for nef cones, Int. Math. Res. Not. IMRN 2012 (14) (2012) 3224–3255.

[15] Gurobi Optimization Inc., Gurobi optimizer reference manual, http://www.gurobi.com, 2016.

[16] Brendan Hassett, Moduli spaces of weighted pointed stable curves, Adv. Math. 173 (2) (2003) 316–352.

[17] Juergen Hausen, Simon Keicher, Antonio Laface, On blowing up the weighted projective space, arXiv:1608.04542, 2016.

[18] Benjamin Howard, John Millson, Andrew Snowden, Ravi Vakil, The equations for the moduli space of n points on the line, Duke Math. J. 146 (2) (2009) 175–226.

[19] Benjamin Howard, John Millson, Andrew Snowden, Ravi Vakil, The ideal of relations for the ring of invariants of n points on the line, J. Eur. Math. Soc. (JEMS) 14 (1) (2012) 1–60.

[20] M.M. Kapranov, Chow quotients of Grassmannians. I, in: I. M. Gelfand Seminar, in: Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 29–110.

[21] A.B. Kempe, On regular difference terms, Proc. Lond. Math. Soc. S1-25 (1) (1894) 343.

[22] Young-Hoon Kiem, Han-Bom Moon, Moduli spaces of weighted pointed stable rational curves via GIT, Osaka J. Math. 48 (4) (2011) 1115–1140.

[23] Seán Keel, James McKernan, Contractible extremal rays on $\overline{M}_{0,n}$, in: Handbook of Moduli. Vol. II, in: Adv. Lect. Math. (ALM), vol. 25, Int. Press, Somerville, MA, 2013, pp. 115–130.

[24] Sean Keel, Jenia Tevelev, Equations for $\overline{M}_{0,n}$, Internat. J. Math. 20 (9) (2009) 1159–1184.

[25] Han-Bom Moon, David Swinarski, Supplementary files for "On the S_n-invariant F Conjecture", http://faculty.fordham.edu/dswinarski/G-semiampleness, 2016.
[26] Morgan Opie, Extremal divisors on moduli spaces of rational curves with marked points, Michigan Math. J. 65 (2) (2016) 251–285.
[27] W.A. Stein, et al., Sage. The Sage Development Team, Sage Mathematics Software (Version 6.5).
[28] Jenia Tevelev, Compactifications of subvarieties of tori, Amer. J. Math. 129 (4) (2007) 1087–1104.
[29] Peter Vermeire, A counterexample to Fulton's conjecture on $\overline{M}_{0,n}$, J. Algebra 248 (2) (2002) 780–784.