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## Selfinjective algebras with hereditary stable slice<sup>☆</sup>

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### ABSTRACT

We determine the structure of all finite-dimensional selfinjective algebras over a field whose Auslander-Reiten quiver admits a hereditary stable slice.

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## 1. Introduction and the main result

In this paper, by an algebra we mean a basic, indecomposable, finite-dimensional associative  $K$ -algebra with identity over a field  $K$ . For an algebra  $A$ , we denote by  $\text{mod } A$  the category of finite-dimensional right  $A$ -modules, by  $\text{ind } A$  the full subcategory of  $\text{mod } A$  formed by a chosen set of representatives of the isomorphism classes of indecomposable modules, by  $D$  the standard duality  $\text{Hom}_K(-, K)$  on  $\text{mod } A$ , by  $\Gamma_A$  the Auslander-Reiten quiver of  $A$ , and by  $\tau_A$  and  $\tau_A^{-1}$  the Auslander-Reiten translations  $D \text{Tr}$  and  $\text{Tr } D$ , respectively. An algebra  $A$  is called *selfinjective* if  $A_A$  is injective, or equivalently, the projective modules in  $\text{mod } A$  are injective. If  $A$  is a selfinjective algebra, then the left socle of  $A$  and the right socle of  $A$  coincide, and we denote them by  $\text{soc}(A)$ . Two selfinjective algebras  $A$  and  $A'$  are said to be *socle equivalent* if the quotient algebras  $A/\text{soc}(A)$  and  $A'/\text{soc}(A')$  are isomorphic.

In the representation theory of selfinjective algebras a prominent role is played by the selfinjective algebras  $A$  which admit Galois coverings of the form  $\widehat{B} \rightarrow \widehat{B}/G = A$ , where  $\widehat{B}$  is the repetitive category of an algebra  $B$  of finite global dimension and  $G$  is an admissible group of automorphisms of  $\widehat{B}$ . Namely, frequently interesting selfinjective algebras are socle equivalent to such orbit algebras  $\widehat{B}/G$  and we may reduce their representation theory to that for the corresponding algebras of finite global dimension occurring in  $\widehat{B}$ . For example, for  $K$  algebraically closed, this is the case for selfinjective algebras of polynomial growth (see [31,32]), the restricted enveloping algebras [10], or more generally the tame Hopf algebras with infinitesimal group schemes [11], in odd characteristic, as well as for the special biserial algebras [7,25]. We also mention that for algebras  $B$  of finite global dimension the stable module category  $\underline{\text{mod}} \widehat{B}$  is equivalent (as a triangulated category) to the derived category  $D^b(\text{mod } B)$  of bounded complexes in  $\text{mod } B$  [13].

Among the algebras of finite global dimension a prominent role is played by the tilted algebras of hereditary algebras, for which the representation theory is rather well understood (see [1,3,6,14,18,20,21,23,27–29] and [43] for some basic results and characterizations). This made it possible to understand the representation theory of the orbit algebras  $\widehat{B}/G$  of tilted algebras  $B$ , called *selfinjective algebras of tilted type* (we refer to [1,4,5,8,9,15–17,19,22,31,32,35,36,40,42] for some general results and applications). In particular, it was shown that every admissible group  $G$  of the repetitive category  $\widehat{B}$  of a tilted algebra  $B$  is an infinite cyclic group generated by a strictly positive automorphism of  $\widehat{B}$ . In the series of articles [33–35,37–39] we developed the theory of selfinjective algebras with deforming ideals and established necessary and sufficient conditions for a selfinjective algebra  $A$  to be socle equivalent to an orbit algebra  $\widehat{B}/G$ , for an algebra  $B$  and an infinite cyclic group  $G$  generated by a strictly positive automorphism of  $\widehat{B}$  being the composition  $\varphi\nu_{\widehat{B}}$  of the Nakayama automorphism  $\nu_{\widehat{B}}$  of  $\widehat{B}$  and a positive automorphism  $\varphi$  of  $\widehat{B}$ .

In this paper we concentrate on the question of when a selfinjective algebra  $A$ , and its module category  $\text{mod } A$ , can be recovered from a finite collection of modules in  $\text{ind } A$  satisfying some homological conditions. We will show that this is possible when these

indecomposable modules form a hereditary stable slice in the Auslander-Reiten quiver  $\Gamma_A$  of  $A$ .

We shall describe the main result of the paper.

Let  $A$  be a selfinjective algebra and  $\Gamma_A^s$  the stable Auslander-Reiten quiver of  $A$ , obtained from  $\Gamma_A$  by removing the projective modules and the arrows attached to them. Following [42], a full valued subquiver  $\Delta$  of  $\Gamma_A$  is said to be a *stable slice* if the following conditions are satisfied:

- (1)  $\Delta$  is connected, acyclic, and without projective modules.
- (2) For any valued arrow  $V \xrightarrow{(a,a')} U$  in  $\Gamma_A$  with  $U$  in  $\Delta$  and  $V$  non-projective,  $V$  belongs to  $\Delta$  or to  $\tau_A \Delta$ .
- (3) For any valued arrow  $U \xrightarrow{(b,b')} V$  in  $\Gamma_A$  with  $U$  in  $\Delta$  and  $V$  non-projective,  $V$  belongs to  $\Delta$  or to  $\tau_A^{-1} \Delta$ .

Assume now that  $\Delta$  is a finite stable slice of  $\Gamma_A$ . Then  $\Delta$  is said to be *right regular* (respectively, *left regular*) if  $\Delta$  does not contain the radical  $\text{rad } P$  (respectively, socle factor  $P/\text{soc } P$ ) of an indecomposable projective module  $P$  in  $\text{mod } A$ . More generally,  $\Delta$  is said to be *almost right regular* if for any indecomposable projective module  $P$  from  $\text{mod } A$  with  $\text{rad } P$  lying on  $\Delta$ ,  $\text{rad } P$  is a sink of  $\Delta$ . Dually,  $\Delta$  is said to be *almost left regular* if for any indecomposable projective module  $P$  from  $\text{mod } A$  with  $P/\text{soc } P$  lying on  $\Delta$ ,  $P/\text{soc } P$  is a source of  $\Delta$ . Finally,  $\Delta$  is said to be *hereditary* if the endomorphism algebra  $H(\Delta) = \text{End}_A(M(\Delta))$  of the direct sum  $M(\Delta)$  of all modules lying on  $\Delta$  is a hereditary algebra and its valued quiver  $Q_{H(\Delta)}$  is the opposite quiver  $\Delta^{\text{op}}$  of  $\Delta$ .

The following theorem is the main result of this paper and extends results established in [35,42] to a general case.

**Theorem 1.1.** *Let  $A$  be a selfinjective algebra over a field  $K$ . The following statements are equivalent.*

- (i)  $\Gamma_A$  admits a hereditary almost right regular stable slice.
- (ii)  $\Gamma_A$  admits a hereditary almost left regular stable slice.
- (iii)  $A$  is socle equivalent to an orbit algebra  $\widehat{B}/(\varphi\nu_{\widehat{B}})$ , where  $B$  is a tilted algebra and  $\varphi$  is a positive automorphism of  $\widehat{B}$ .

Moreover, if  $K$  is algebraically closed, we may replace in (iii) “socle equivalent” by “isomorphic”.

We would like to stress that in general we cannot replace in (iii) “socle equivalent” by “isomorphic”. Namely, there exist fields  $K$ , with non-zero second Hochschild cohomology group  $H^2(K, K)$ , and non-splittable Hochschild extensions

$$0 \rightarrow D(H) \rightarrow \widetilde{H} \rightarrow H \rightarrow 0$$

of hereditary algebras  $H$  over  $K$  such that  $\widetilde{H}$  is a selfinjective algebra socle equivalent but non-isomorphic to the trivial extension algebra  $T(H) = H \ltimes D(H) = \widehat{H}/(\nu_{\widehat{H}})$  (see [33, Corollary 4.2 and Proposition 6.1]).

We also mention that the class of selfinjective algebras occurring in the statement (iii) is closed under stable, and hence derived, equivalences (see [24,34,39] and [26]).

We note that, for a selfinjective algebra  $A$ ,  $A$  satisfies (i) if and only if  $A^{\text{op}}$  satisfies (ii), and  $A$  satisfies (iii) if and only if  $A^{\text{op}}$  satisfies (iii). Therefore, in order to prove Theorem 1.1, it is sufficient to show that (i) is equivalent to (iii).

## 2. Orbit algebras of repetitive categories

Let  $B$  be an algebra and  $1_B = e_1 + \cdots + e_n$  a decomposition of the identity of  $B$  into a sum of pairwise orthogonal primitive idempotents. We associate to  $B$  a selfinjective locally bounded  $K$ -category  $\widehat{B}$ , called the *repetitive category* of  $B$  (see [16]). The objects of  $\widehat{B}$  are  $e_m, i$ ,  $m \in \mathbb{Z}$ ,  $i \in \{1, \dots, n\}$ , and the morphism spaces are defined as follows

$$\widehat{B}(e_m, i, e_r, j) = \begin{cases} e_j B e_i, & r = m, \\ D(e_i B e_j), & r = m + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that  $e_j B e_i = \text{Hom}_B(e_i B, e_j B)$ ,  $D(e_i B e_j) = e_j D(B) e_i$  and

$$\bigoplus_{(m,i) \in \mathbb{Z} \times \{1, \dots, n\}} \widehat{B}(e_m, i, e_r, j) = e_j B \oplus D(B e_j),$$

for any  $r \in \mathbb{Z}$  and  $j \in \{1, \dots, n\}$ . We denote by  $\nu_{\widehat{B}}$  the *Nakayama automorphism* of  $\widehat{B}$  defined by

$$\nu_{\widehat{B}}(e_m, i) = e_{m+1, i} \quad \text{for all } (m, i) \in \mathbb{Z} \times \{1, \dots, n\}.$$

An automorphism  $\varphi$  of the  $K$ -category  $\widehat{B}$  is said to be:

- *positive* if, for each pair  $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$ , we have  $\varphi(e_m, i) = e_{p, j}$  for some  $p \geq m$  and some  $j \in \{1, \dots, n\}$ ;
- *rigid* if, for each pair  $(m, i) \in \mathbb{Z} \times \{1, \dots, n\}$ , there exists  $j \in \{1, \dots, n\}$  such that  $\varphi(e_m, i) = e_{m, j}$ ;
- *strictly positive* if it is positive but not rigid.

Then the automorphisms  $\nu_{\widehat{B}}^r$ ,  $r \geq 1$ , are strictly positive automorphisms of  $\widehat{B}$ .

A group  $G$  of automorphisms of  $\widehat{B}$  is said to be *admissible* if  $G$  acts freely on the set of objects of  $\widehat{B}$  and has finitely many orbits. Then, following P. Gabriel [12], we may consider the orbit category  $\widehat{B}/G$  of  $\widehat{B}$  with respect to  $G$  whose objects are the  $G$ -orbits of objects in  $\widehat{B}$ , and the morphism spaces are given by

$$(\widehat{B}/G)(a, b) = \left\{ (f_{y,x}) \in \prod_{(x,y) \in a \times b} \widehat{B}(x, y) \mid gf_{y,x} = f_{gy,gx}, \text{ for all } g \in G, x \in a, y \in b \right\}$$

for all objects  $a, b$  of  $\widehat{B}/G$ . Since  $\widehat{B}/G$  has finitely many objects and the morphism spaces in  $\widehat{B}/G$  are finite-dimensional, we have the associated finite-dimensional selfinjective  $K$ -algebra  $\bigoplus(\widehat{B}/G)$  which is the direct sum of all morphism spaces in  $\widehat{B}/G$ , called the *orbit algebra* of  $\widehat{B}$  with respect to  $G$ . We will identify  $\widehat{B}/G$  with  $\bigoplus(\widehat{B}/G)$ . For example, for each positive integer  $r$ , the infinite cyclic group  $(\nu_{\widehat{B}}^r)$  generated by the  $r$ -th power  $\nu_{\widehat{B}}^r$  of  $\nu_{\widehat{B}}$  is an admissible group of automorphisms of  $\widehat{B}$ , and we have the associated selfinjective orbit algebra

$$T(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & f_1 & b_1 \end{bmatrix} \mid \begin{matrix} b_1, \dots, b_{r-1} \in B, \\ f_1, \dots, f_{r-1} \in D(B) \end{matrix} \right\},$$

called the  $r$ -fold *trivial extension algebra* of  $B$ . In particular,  $T(B)^{(1)} \cong T(B) = B \ltimes D(B)$  is the *trivial extension algebra* of  $B$  by the injective cogenerator  $D(B)$ .

Let  $A$  be a selfinjective algebra. For a subset  $X$  of  $A$ , we may consider the left annihilator  $l_A(X) = \{a \in A \mid aX = 0\}$  of  $X$  in  $A$  and the right annihilator  $r_A(X) = \{a \in A \mid Xa = 0\}$  of  $X$  in  $A$ . Then by a theorem due to T. Nakayama (see [41, Theorem IV.6.10]) the annihilator operation  $l_A$  induces a Galois correspondence from the lattice of right ideals of  $A$  to the lattice of left ideals of  $A$ , and  $r_A$  is the inverse Galois correspondence to  $l_A$ . Let  $I$  be an ideal of  $A$ ,  $B = A/I$ , and  $e$  an idempotent of  $A$  such that  $e + I$  is the identity of  $B$ . We may assume that  $1_A = e_1 + \dots + e_r$  with  $e_1, \dots, e_r$  pairwise orthogonal primitive idempotents of  $A$ ,  $e = e_1 + \dots + e_n$  for some  $n \leq r$ , and  $\{e_i \mid 1 \leq i \leq n\}$  is the set of all idempotents in  $\{e_i \mid 1 \leq i \leq r\}$  which are not in  $I$ . Then such an idempotent  $e$  is uniquely determined by  $I$  up to an inner automorphism of  $A$ , and is called a *residual identity* of  $B = A/I$ . Observe also that  $B \cong eAe/eIe$ .

Let  $A$  be a selfinjective algebra,  $I$  an ideal of  $A$ ,  $B = A/I$ ,  $e$  a residual identity of  $B$  and assume that  $r_A(I) = eI$ . Then we have a canonical isomorphism of algebras  $eAe/eIe \rightarrow A/I$  and  $I$  can be considered as an  $(eAe/eIe)$ – $(eAe/eIe)$ -bimodule. Following [33], we denote by  $A[I]$  the direct sum of  $K$ -vector spaces  $(eAe/eIe) \oplus I$  with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for  $b, c \in eAe/eIe$  and  $x, y \in I$ . Then  $A[I]$  is a  $K$ -algebra with the identity  $(e + eIe, 1_A - e)$ , and, by identifying  $x \in I$  with  $(0, x) \in A[I]$ , we may consider  $I$  to be the ideal of  $A[I]$ . Observe that  $e = (e + eIe, 0)$  is a residual identity of  $A[I]/I = eAe/eIe \cong A/I$ . We also note that  $\text{soc}(A) \subseteq I$  and  $l_{eAe}(I) = eIe = r_{eAe}(I)$ , by [33, Proposition 2.3].

The following theorem is a consequence of results established in [33] and [35].

**Theorem 2.1.** *Let  $A$  be a selfinjective algebra,  $I$  ideal of  $A$ ,  $B = A/I$ ,  $e$  a residual identity of  $B$ . Assume that  $r_A(I) = eI$  and the valued quiver  $Q_B$  of  $B$  is acyclic. Then the following statements hold.*

- (i)  $A[I]$  is a selfinjective algebra with the same Nakayama permutation as  $A$ .
- (ii)  $A$  and  $A[I]$  are socle equivalent.
- (iii)  $A[I]$  is isomorphic to an orbit algebra  $\widehat{B}/(\varphi\nu_{\widehat{B}})$ , for some positive automorphism  $\varphi$  of  $\widehat{B}$ .

Moreover, if  $K$  is an algebraically closed field, we may replace in (ii) “socle equivalent” by “isomorphic”.

**Proof.** The statements (i), (ii) and the final part of the theorem follow from [33, Theorems 3.2 and 4.1]. The statement (iii) follows from [35, Theorem 4.1].  $\square$

### 3. Proof of the implication (i) $\Rightarrow$ (iii) of Theorem 1.1

Let  $A$  be a selfinjective algebra over a field  $K$ , and assume that  $\Gamma_A$  admits a hereditary almost regular stable slice  $\Delta$ . Let  $M$  be the direct sum of all indecomposable modules in  $\text{mod } A$  lying on  $\Delta$ ,  $I$  the right annihilator  $r_A(M) = \{a \in A \mid Ma = 0\}$  of  $M$  in  $\text{mod } A$ ,  $B = A/I$ , and  $H = \text{End}_B(M)$ . We note that  $H = \text{End}_A(M)$  and hence is a hereditary algebra. Moreover, the valued quiver  $Q_H$  of  $H$  is the opposite quiver  $\Delta^{\text{op}}$  of  $\Delta$ . This implies that every non-zero non-isomorphism in  $\text{mod } A$  between two modules lying on  $\Delta$  is a finite sum of compositions of irreducible homomorphisms in  $\text{mod } A$  corresponding to valued arrows of  $\Delta^{\text{op}}$ .

**Lemma 3.1.** *Let  $X$  be an indecomposable module from  $\Delta$ . Then the following statements hold.*

- (i) *If  $X$  is the radical of an indecomposable projective  $A$ -module, then  $X$  is an injective  $B$ -module.*
- (ii) *If  $X$  is not the radical of an indecomposable projective  $A$ -module, then  $\tau_B^{-1}X = \tau_A^{-1}X$ .*

**Proof.** (i) Assume  $X = \text{rad } P$  for an indecomposable projective  $A$ -module  $P$ . Since  $X$  lies on  $\Delta$  and  $P$  is not in  $\Delta$ ,  $X$  is the largest right  $B$ -submodule of the injective  $A$ -module  $P$ , and consequently  $X$  is an injective  $B$ -module, because  $B$  is a quotient algebra of  $A$ .

(ii) Assume that  $X$  is not the radical of an indecomposable projective  $A$ -module. Let  $Y = \tau_A^{-1}X$  and  $f : P(Y) \rightarrow Y$  be a projective cover of  $Y$  in  $\text{mod } A$ . Since  $\Delta$  is an almost right regular stable slice of  $\Gamma_A$ , we conclude that  $f$  factors through a module  $M^r$  for

some positive integer  $r$ . But then  $Y$  is a  $B$ -module, and hence  $\tau_B^{-1}X = \tau_A^{-1}X$ . Clearly, then  $X$  is not an injective  $B$ -module.  $\square$

**Lemma 3.2.** *The following statements hold.*

- (i)  $\text{Hom}_B(\tau_B^{-1}M, M) = 0$ .
- (ii)  $\text{id}_B M \leq 1$ .

**Proof.** (i) It follows from the proof of Lemma 3.1 that there is an epimorphism of right  $B$ -modules  $g : M^s \rightarrow \tau_B^{-1}M$  for some positive integer  $s$ . Then we conclude that  $\text{Hom}_B(\tau_B^{-1}M, M) = 0$ , because  $\text{End}_A(M) = \text{End}_B(M)$  is a hereditary algebra whose valued quiver is the opposite quiver  $\Delta^{\text{op}}$  of  $\Delta$ .

(ii) Since  $M$  is a faithful  $B$ -module there is a monomorphism of right  $B$ -modules  $B \rightarrow M^t$  for some positive integer  $t$  (see [41, Lemma II.5.5]), and hence  $\text{Hom}_B(\tau_B^{-1}M, B) = 0$ , by (i). This implies that  $\text{id}_B M \leq 1$  (see [41, Proposition III.5.4]).  $\square$

**Lemma 3.3.** *The following statements hold.*

- (i) For any valued arrow  $U \xrightarrow{(c, c')} V$  in  $\Gamma_B$  with  $U$  in  $\Delta$ ,  $V$  belongs to  $\Delta$  or to  $\tau_B^{-1}\Delta$ .
- (ii) For any valued arrow  $V \xrightarrow{(d, d')} U$  in  $\Gamma_B$  with  $U$  in  $\Delta$ ,  $V$  belongs to  $\Delta$  or to  $\tau_B\Delta$ .

**Proof.** (i) It follows from Lemma 3.1 and the fact that  $\Delta$  is a stable slice in  $\Gamma_A$ .

(ii) Assume that  $V \xrightarrow{(d, d')} U$  is a valued arrow in  $\Gamma_B$  with  $U$  in  $\Delta$ . We may assume that  $V$  is not on  $\Delta$ . We claim that  $V$  is not an injective  $B$ -module. Suppose that  $V$  is injective. Because  $M$  is a faithful  $B$ -module, there is an epimorphism of right  $B$ -modules  $M^p \rightarrow D(B)$  for some positive integer  $p$ , by the dual of [41, Lemma II.5.5], because  $l_B(D(M)) = r_B(M) = 0$ . Then there exist homomorphisms of right  $B$ -modules  $W \xrightarrow{f} V \xrightarrow{g} U$  with  $gf \neq 0$  and  $W$  an indecomposable  $B$ -module lying on  $\Delta$ . Since  $V$  is not in  $\Delta$ , this contradicts the fact that  $\Delta^{\text{op}}$  is the valued quiver of the algebra  $\text{End}_B(M)$ . Therefore,  $V$  is not injective in  $\text{mod } B$ . But then  $\tau_B^{-1}V$  is an indecomposable module and there is a valued arrow  $U \xrightarrow{(d', d)} \tau_B^{-1}V$  in  $\Gamma_B$  (see [41, Lemma III.9.1 and Proposition III.9.6]). Then it follows from (i) that  $\tau_B^{-1}V$  belongs to  $\Delta$  or to  $\tau_B^{-1}\Delta$ . Since  $V$  is not in  $\Delta$ , we conclude that  $\tau_B^{-1}V$  belongs to  $\Delta$ , and hence  $V$  belongs to  $\tau_B\Delta$ .  $\square$

**Lemma 3.4.** *The following statements hold.*

- (i)  $\text{Hom}_B(M, \tau_B M) = 0$ .
- (ii)  $\text{pd}_B M \leq 1$ .

**Proof.** (i) Consider an injective envelope  $h : \tau_B M \rightarrow E(\tau_B M)$  of  $\tau_B M$  in  $\text{mod } B$ . Since  $\tau_B M$  has no injective direct summands, it follows from Lemma 3.3 that  $h$  factors through

a module  $M^q$  for some positive integer  $q$ , and hence there is a monomorphism of right  $B$ -modules  $u : \tau_B M \rightarrow M^q$ . But then  $\text{Hom}_B(M, \tau_B M) = 0$ , because  $\Delta^{\text{op}}$  is the valued quiver of the hereditary algebra  $\text{End}_B(M)$ .

(ii) Since  $M$  is a faithful  $B$ -module there is an epimorphism of right  $B$ -modules  $M^p \rightarrow D(B)$  for a positive integer  $p$ , and consequently  $\text{Hom}_B(D(B), \tau_B M) = 0$ . This implies that  $\text{pd}_B M \leq 1$  (see [41, Proposition III.5.4]).  $\square$

**Proposition 3.5.** *The following statements hold.*

- (i)  $M$  is a tilting  $B$ -module.
- (ii)  $T = D(M)$  is a tilting module in  $\text{mod } H$ .
- (iii) There is a canonical isomorphism of  $K$ -algebras  $B \xrightarrow{\sim} \text{End}_H(T)$ .
- (iv)  $\Delta$  is the section  $\Delta_T$  of the connecting component  $\mathcal{C}_T$  of  $\Gamma_B$  determined by  $T$ .

**Proof.** (i) Let  $f_1, \dots, f_d$  be a basis of the  $K$ -vector space  $\text{Hom}_B(B, M)$ . Then we have a monomorphism  $f : B \rightarrow M^d$  in  $\text{mod } B$ , induced by  $f_1, \dots, f_d$ , and hence a short exact sequence

$$0 \rightarrow B \xrightarrow{f} M^d \xrightarrow{g} N \rightarrow 0$$

in  $\text{mod } B$ , where  $N = \text{Coker } f$  and  $g$  is a canonical epimorphism. Then, applying standard arguments using  $\text{pd}_B M \leq 1$ , we conclude (see the proof of [42, Proposition 3.8]) that  $M \oplus N$  is a tilting  $B$ -module. We prove now that  $N$  belongs to the additive category  $\text{add}(M)$  of  $M$ . Assume to the contrary that there exists an indecomposable direct summand  $W$  of  $N$  which does not belong to  $\text{add}(M)$ , or equivalently  $W$  does not lie on  $\Delta$ . Clearly,  $\text{Hom}_B(M, W) \neq 0$  because  $W$  is a quotient module of  $M^d$ . Applying now Lemma 3.3, we conclude that  $\text{Hom}_B(\tau_B^{-1} M, W) \neq 0$ . Moreover, by Lemma 3.2, we have  $\text{id}_B M \leq 1$ . Then, applying [41, Corollary III.6.4], we infer that  $\text{Ext}_B^1(W, M) \cong D \text{Hom}_B(\tau_B^{-1} M, W) \neq 0$ , which contradicts  $\text{Ext}_B^1(N, M) = 0$ . Therefore,  $M$  is a tilting module in  $\text{mod } B$ . We also note that the rank of  $K_0(B)$  is the number of indecomposable modules lying on  $\Delta$ .

(ii)–(iv) It follows from tilting theory that  $T = D(M)$  is a tilting module in  $\text{mod } H$  and there is a canonical isomorphism of  $K$ -algebras  $B \xrightarrow{\sim} \text{End}_H(T)$  (see [43, Proposition VIII.3.3]). In particular,  $B$  is a tilted algebra of type  $\Delta^{\text{op}}$ . Moreover, we have isomorphisms of right  $B$ -modules

$$\text{Hom}_H(T, D(H)) = \text{Hom}_H(D(M), D(H)) \cong \text{Hom}_{H^{\text{op}}}(H, M) \cong M,$$

because  $M$  is a right  $H^{\text{op}}$ -module. Therefore,  $\Delta$  is the canonical section  $\Delta_T$  of the connecting component  $\mathcal{C}_T$  of  $\Gamma_B$  determined by  $T$  (see [43, Theorem VIII.6.7]).  $\square$

We may choose a set  $e_1, \dots, e_r$  of pairwise orthogonal primitive idempotents of  $A$  such that  $1_A = e_1 + \dots + e_r$  and, for some  $n \leq r$ ,  $\{e_i \mid 1 \leq i \leq n\}$  is the set of all idempotents



in  $\{e_i \mid 1 \leq i \leq r\}$  which are not in  $I$ . Then  $e = e_1 + \cdots + e_n$  is a *residual identity* of  $B = A/I$ .

Our next aim is to prove that  $r_A(I) = eI$ .

We denote by  $J$  the trace ideal of  $M$  in  $A$ , that is, the ideal of  $A$  generated by the images of all homomorphisms from  $M$  to  $A$  in  $\text{mod } A$ , and by  $J'$  the trace ideal of the left  $A$ -module  $D(M)$  in  $A$ . Observe that  $I = l_A(D(M))$ . Then we have the following lemma.

**Lemma 3.6.** *We have  $J \subseteq I$  and  $J' \subseteq I$ .*

**Proof.** First we show that  $J \subseteq I$ . By definition, there exists an epimorphism  $\varphi : M^s \rightarrow J$  in  $\text{mod } A$  for some positive integer  $s$ . Suppose that  $J$  is not contained in  $I$ . Then there exists a homomorphism  $f : A \rightarrow M$  in  $\text{mod } A$  such that  $f(J) \neq 0$ . Then we have the sequence of homomorphisms in  $\text{mod } A$

$$M^s \xrightarrow{\varphi} J \xrightarrow{u} A \xrightarrow{f} M$$

with  $u$  being the inclusion homomorphism. But then  $f(u\varphi) = fu\varphi \neq 0$  and this contradicts the fact that  $\Delta^{\text{op}}$  is the valued quiver of  $H = \text{End}_A(M)$ . Hence  $J \subseteq I$ .

Suppose now that  $J'$  is not contained in  $I$ . Then there exists a homomorphism  $f' : A \rightarrow D(M)$  in  $\text{mod } A^{\text{op}}$  such that  $f'(J') \neq 0$ . Moreover, we have in  $\text{mod } A^{\text{op}}$  an epimorphism  $\varphi' : D(M)^m \rightarrow J'$  for some positive integer  $m$ . Then  $f'u'\varphi' \neq 0$  for the inclusion homomorphism  $u' : J' \rightarrow A$ . Applying the duality functor  $D : \text{mod } A^{\text{op}} \rightarrow \text{mod } A$  we obtain homomorphisms in  $\text{mod } A$

$$D(D(M)) \xrightarrow{D(f')} D(A) \xrightarrow{D(u')} D(J') \xrightarrow{D(\varphi')} D(D(M)^m),$$

where  $D(D(M)) \cong M$ ,  $D(D(M)^m) \cong M^m$ ,  $D(A) \cong A$  in  $\text{mod } A$ , and  $D(\varphi')D(u')D(f') = D(f'u'\varphi') \neq 0$ . This again contradicts the fact that  $\Delta^{\text{op}}$  is the valued quiver of  $H = \text{End}_A(M)$ .  $\square$

**Lemma 3.7.** *We have  $l_A(I) = J$ ,  $r_A(I) = J'$  and  $I = r_A(J) = l_A(J')$ .*

**Proof.** See [33, Lemma 5.10] or [42, Lemma 3.11].  $\square$

**Lemma 3.8.** *We have  $eIe = eJe$ . In particular,  $(eIe)^2 = 0$ .*

**Proof.** Since  $B \xrightarrow{\sim} eAe/eIe$  canonically,  $M$  is a module in  $\text{mod } eAe$  with  $r_{eAe}(M) = eIe$ . We note also that  $eJe$  is the trace ideal of  $M$  in  $eAe$ , generated by the images of all homomorphisms from  $M$  to  $eAe$  in  $\text{mod } eAe$ . It follows from Lemma 3.6 that  $eJe = eJ$  is an ideal of  $eAe$  with  $eJe \subseteq eIe \subseteq \text{rad } eAe$ . Let  $C = eAe/eJe$ . Then  $M$  is a sincere module in  $\text{mod } C$ . We will prove that  $M$  is a faithful module in  $\text{mod } C$ . Observe that

this implies  $eIe/eJe = r_C(M) = 0$ , and consequently  $eIe = eJe$ . Clearly, then  $(eIe)^2 = (eJe)(eIe) = 0$ , because  $JI = 0$ .

We prove the claim in several steps.

(1) Assume that  $\text{rad } e_i A$  lies on  $\Delta$ , for some  $i \in \{1, \dots, n\}$ . Clearly, then  $e_i = e_i e = e e_i$  and  $e_i(\text{rad } A) = e_i(\text{rad } A)e$ . Moreover, we have  $e_i(\text{rad } A) = e_i J$ , because  $e_i A$  does not lie on  $\Delta$ . On the other hand,  $e_i B = e_i A/e_i I$  is an indecomposable projective  $B$ -module. Since  $M$  is a faithful  $B$ -module, there exists a monomorphism  $e_i B \rightarrow M^r$  for some positive integer  $r$ . Further, since  $\Delta^{\text{op}}$  is the valued quiver of  $\text{End}_B(M)$ , we conclude that the composed homomorphism  $e_i \text{rad } A \hookrightarrow e_i A \twoheadrightarrow e_i B \rightarrow M^r$  is zero. Hence  $e_i I = e_i(\text{rad } A)$ , and consequently  $e_i J = e_i I$ . In particular, we conclude that  $e_i(\text{rad } A)$  is an injective module in  $\text{mod } C$ .

(2) Assume that  $\text{rad } e_i A$  lies on  $\Delta$ , for some  $i \in \{n+1, \dots, r\}$ . Then  $e_i A e \subseteq e_i(\text{rad } A)$ , and hence  $e_i A e = e_i(\text{rad } A)$ , because  $e_i(\text{rad } A)$  is a right  $B$ -module. This shows that the canonical epimorphism  $e_i(\text{rad } A) \rightarrow e_i(\text{rad } A)/\text{soc}(e_i A)$  is a minimal left almost split homomorphism in  $\text{mod } eAe$ , and hence  $e_i(\text{rad } A)$  is an injective  $eAe$ -module. Clearly, then  $e_i(\text{rad } A)$  is also an injective  $C$ -module.

(3) Assume that  $X$  is a module on  $\Delta$  which is not the radical of an indecomposable projective module in  $\text{mod } A$ . Then it follows from Lemma 3.1 that there is an almost split sequence in  $\text{mod } B$

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

which is an almost split sequence in  $\text{mod } A$ . Recall that  $B \xrightarrow{\sim} eAe/eIe$  canonically. Applying now the properties of the restriction functor  $\text{res}_e = (-)e : \text{mod } A \rightarrow \text{mod } eAe$  (see [2, Theorem I.6.8]), we conclude that the above sequence is an almost split sequence in  $\text{mod } eAe$ . In particular, we conclude that  $\tau_{eAe}^{-1} X = \tau_B^{-1} X$ , under the identification  $B = eAe/eIe$ . Clearly, then we have also  $\tau_C^{-1} X = \tau_B^{-1} X$ .

(4) We may decompose  $M = U \oplus V$  in  $\text{mod } B$  with  $V$  being the direct sum of all indecomposable modules on  $\Delta$  which are not radicals of indecomposable projective modules in  $\text{mod } A$ . It follows from (1) and (2) that  $U$  is an injective  $C$ -module. We prove now that  $\text{id}_C V \leq 1$ . We may assume that  $V \neq 0$ . Observe that, by (3), we have  $\tau_{eAe}^{-1} V = \tau_C^{-1} V = \tau_B^{-1} V$ . Consider the exact sequence

$$0 \rightarrow eJe \xrightarrow{u} eAe \xrightarrow{v} C \rightarrow 0$$

in  $\text{mod } C$ , where  $u$  is the inclusion homomorphism and  $v$  is the canonical epimorphism. Applying the functor  $\text{Hom}_{eAe}(\tau_{eAe}^{-1} V, -) : \text{mod } eAe \rightarrow \text{mod } K$  to this sequence, we get the exact sequence in  $\text{mod } K$  of the form

$$\begin{aligned} \text{Hom}_{eAe}(\tau_{eAe}^{-1} V, eJe) &\xrightarrow{\alpha} \text{Hom}_{eAe}(\tau_{eAe}^{-1} V, eAe) \\ &\xrightarrow{\beta} \text{Hom}_{eAe}(\tau_{eAe}^{-1} V, C) \xrightarrow{\gamma} \text{Ext}_{eAe}^1(\tau_{eAe}^{-1} V, eJe), \end{aligned}$$

where  $\alpha = \text{Hom}_{eAe}(\tau_{eAe}^{-1}V, u)$ ,  $\beta = \text{Hom}_{eAe}(\tau_{eAe}^{-1}V, v)$ , and  $\gamma$  is the connecting homomorphism. Since  $\Delta$  is a section of the connecting component of  $\Gamma_B$ , we conclude that there is an epimorphism  $M^s \rightarrow \tau_B^{-1}V$  in  $\text{mod } B$ , for some positive integer  $s$ , and hence in  $\text{mod } eAe$ . Hence  $\alpha$  is an isomorphism, because  $\tau_{eAe}^{-1}V = \tau_B^{-1}V$ . This implies that  $\gamma$  is a monomorphism. Further, every homomorphism from  $M$  to  $eJe$  in  $\text{mod } B$  factors through  $(\tau_B^{-1}V)^t = (\tau_B^{-1}M)^t$  for some positive integer  $t$ , and hence there is an epimorphism  $(\tau_B^{-1}V)^t \rightarrow eJ$ . Then we get  $\text{Hom}_{eAe}(eJe, V) = \text{Hom}_B(eJe, V) = 0$ , because  $\text{Hom}_B(\tau_B^{-1}M, M) = 0$ . Then we obtain  $\text{Ext}_{eAe}^1(\tau_{eAe}^{-1}V, eJe) \cong D\overline{\text{Hom}}_{eAe}(eJe, V) = 0$ . Summing up, we conclude that  $\text{Hom}_C(\tau_C^{-1}V, C) = \text{Hom}_{eAe}(\tau_{eAe}^{-1}V, C) = 0$ , and hence  $\text{id}_C V \leq 1$ .

(5) By (1), (2) and (4), we have  $\text{id}_C M \leq 1$ . Further,  $\text{Ext}_C^1(M, M) \cong D\overline{\text{Hom}}_C(\tau_C^{-1}M, M) = D\overline{\text{Hom}}_B(\tau_B^{-1}M, M) = 0$ . Since the rank of  $K_0(C)$  is the rank of  $K_0(B)$ , which is the number of indecomposable direct summands of  $M$ , we conclude that  $M$  is a cotilting module in  $\text{mod } C$ . Then  $D(M)$  is a tilting module in  $\text{mod } C^{\text{op}}$ . In particular,  $D(M)$  is a faithful module in  $\text{mod } C^{\text{op}}$ . Then we obtain the required fact  $r_C(M) = r_{C^{\text{op}}}(D(M)) = 0$ .  $\square$

**Lemma 3.9.** *Let  $f$  be a primitive idempotent in  $I$  such that  $fJ \neq fAe$ . Then  $L = fAeAf + fJ + fAeAfAe + eAf + eIe$  is an ideal of  $F = (e+f)A(e+f)$ , and  $N = fAe/fLe$  is a module in  $\text{mod } B$  such that  $\text{Hom}_B(N, M) = 0$  and  $\text{Hom}_B(M, N) \neq 0$ .*

**Proof.** It follows from Lemma 3.8 that  $fAeIe \subseteq fJ$ . Then the fact that  $L$  is an ideal of  $F$  is a direct consequence of  $fJ \subseteq fAe$ . Observe also that  $N \neq 0$ . Indeed, if  $fAe = fLe$  then  $fAe = fJ + fAe(\text{rad}(eAe))$ , since  $eAfAe \subseteq \text{rad}(eAe)$ . But then  $fAe = fJ$ , by the Nakayama lemma [41, Lemma I.3.3], which contradicts our assumption. Further,  $B \cong eAe/eIe$  and  $(fAe)(eIe) = fAeJ \subseteq fJ \subseteq fLe$ , and hence  $N$  is a right  $B$ -module. Moreover,  $N$  is also a left module over  $S = fAf/fLf$  and  $F/L$  is isomorphic to the triangular matrix algebra

$$\Lambda = \begin{bmatrix} S & N \\ 0 & B \end{bmatrix}.$$

Let  $X$  be an indecomposable direct summand of  $M$ . Assume first that  $X$  is not the radical of an indecomposable projective module in  $\text{mod } A$ . Then it follows from Lemma 3.1 that we have in  $\text{mod } B$  an almost split sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

which is also an almost split sequence in  $\text{mod } A$ , and consequently in  $\text{mod } F$ . Since  $\Lambda$  is a quotient algebra of  $F$  and  $B$  is a quotient algebra of  $\Lambda$ , we conclude that it is also an almost split sequence in  $\text{mod } \Lambda$ . Applying now [33, Lemma 5.6] (or [43, Theorem VII.10.9]) we conclude that  $\text{Hom}_B(N, X) = 0$ . Assume now that  $X = \text{rad } P$  for an indecomposable projective module  $P$  in  $\text{mod } A$ . Suppose that  $\text{Hom}_B(N, X) \neq 0$ . It

follows from the assumption imposed on  $\Delta$  that every direct predecessor of  $X$  in  $\Gamma_A$  lies on  $\Delta$  and is not the radical of an indecomposable projective module in  $\text{mod } A$ . Moreover, by Proposition 3.5,  $\Delta$  is the canonical section  $\Delta_T$  of the connecting component  $\mathcal{C}_T$  of  $\Gamma_B$ . Then  $\text{Hom}_B(N, X) \neq 0$  forces that  $X$  is an indecomposable direct summand of  $N$ . Recall that  $N = fAe/fLe$  with  $L = fAeAf + fJ + fAeAfAe + eAf + eIe$ . Hence we obtain  $P = fA$  and  $X = \text{rad } P = fAe$ . Then we conclude that  $fLe = 0$ , and hence  $fJ = 0$ . But it is not possible because  $f(\text{rad } A) = \text{rad } fA$  lies on  $\Delta$ , and is equal to  $fJ$ .

Summing up, we obtain that  $\text{Hom}_B(N, M) = 0$ . Since every indecomposable module in  $\text{mod } B$  is either generated or cogenerated by  $M$ , we conclude that  $\text{Hom}_B(M, N) \neq 0$ .  $\square$

Applying Lemmas 3.6–3.9 as in [33, Proposition 5.9], we obtain the following proposition.

**Proposition 3.10.** *We have  $Ie = J$  and  $eI = J'$ .*

It follows from Lemma 3.7 and Proposition 3.10 that  $r_A(I) = J' = eI$  and  $l_A(I) = J$ . Moreover, since  $B$  is a tilted algebra, the valued quiver  $Q_B$  of  $B$  is acyclic. Then applying Theorem 2.1 we conclude that  $A$  is socle equivalent to the orbit algebra  $\widehat{B}/(\varphi\nu_{\widehat{B}})$  for some positive automorphism  $\varphi$  of  $\widehat{B}$ . Further, if  $K$  is an algebraically closed field, then  $A$  is isomorphic to  $\widehat{B}/(\varphi\nu_{\widehat{B}})$ .

#### 4. Proof of the implication (iii) $\Rightarrow$ (i) of Theorem 1.1

We start with general facts.

Let  $\Lambda$  be a selfinjective algebra. Then for any indecomposable projective module  $P$  in  $\text{mod } \Lambda$  we have a canonical almost split sequence

$$0 \rightarrow \text{rad } P \rightarrow (\text{rad } P / \text{soc } P) \oplus P \rightarrow P / \text{soc } P \rightarrow 0,$$

and hence  $\text{rad } P$  is a unique direct predecessor of  $P$  and  $P / \text{soc } P$  is a unique direct successor of  $P$  in  $\Gamma_\Lambda$ . Hence, the Auslander-Reiten quiver  $\Gamma_{\Lambda / \text{soc}(\Lambda)}$  is obtained from  $\Gamma_\Lambda$  by deleting all projective modules  $P$  and the arrows  $\text{rad } P \rightarrow P$  and  $P \rightarrow P / \text{soc } P$ . We also note that if  $\Delta$  is a stable slice of  $\Gamma_\Lambda$  then  $\Delta$  is a full valued subquiver of  $\Gamma_{\Lambda / \text{soc}(\Lambda)}$ . Hence we have the following fact.

**Proposition 4.1.** *Let  $\Lambda$  and  $A$  be two socle equivalent selfinjective algebras and  $\phi : \text{mod } \Lambda / \text{soc}(\Lambda) \rightarrow \text{mod } A / \text{soc}(A)$  the isomorphism of module categories induced by an algebra isomorphism  $\varphi : \Lambda / \text{soc}(\Lambda) \rightarrow A / \text{soc}(A)$ . Then a full valued subquiver  $\Delta$  of  $\Gamma_\Lambda$  is a hereditary almost right regular slice of  $\Gamma_\Lambda$  if and only if  $\phi(\Delta)$  is a hereditary almost right regular stable slice of  $\Gamma_A$ .*

Therefore, for proving the sufficiency part of Theorem 1.1, we may assume that  $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$  for a tilted algebra  $B$  and a positive automorphism  $\varphi$  of  $\widehat{B}$ . We divide that proof into three cases.

**Proposition 4.2.** *Assume  $A$  is of infinite representation type. Then  $\Gamma_A$  admits a hereditary right regular stable slice  $\Delta$ .*

**Proof.** By [15,16] and the assumption, we conclude that  $B$  is not a tilted algebra of Dynkin type. It follows from general theory (see [1,8,35,36]) that  $\Gamma_A$  admits an acyclic component  $\mathcal{C}$  containing a right stable full translation subquiver  $\mathcal{D}$  which is closed under successor in  $\mathcal{C}$  and generalized standard in the sense of [30] (the restriction of the infinite radical  $\text{rad}_A^\infty$  of  $\text{mod } A$  to  $\mathcal{D}$  is zero). We note that  $\mathcal{D}$  does not contain projective module, because  $\mathcal{D}$  is right stable. Then we may choose in  $\mathcal{D}$  a full valued connected subquiver  $\Delta$  which intersects every  $\tau_A$ -orbit in  $\mathcal{D}$  exactly once. Clearly,  $\Delta$  is a right regular finite stable slice of  $\Gamma_A$ . Moreover, since  $\mathcal{D}$  is generalized standard, we obtain that  $\Delta$  is a hereditary slice of  $\Gamma_A$ .  $\square$

**Proposition 4.3.** *Let  $A$  be a Nakayama algebra. Then  $\Gamma_A$  admits a hereditary almost right regular stable slice  $\Delta$ .*

**Proof.** Let  $P$  be an indecomposable projective module in  $\text{mod } A$ . Then using the structure of almost split sequence over Nakayama algebras (see [41, Theorems I.10.5 and III.8.7]) we conclude that there is a sectional path  $\Delta$  of the form

$$\text{soc } P = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n = \text{rad } P$$

such that the  $\tau_A$ -orbits of these modules exhaust all indecomposable non-projective modules in  $\text{mod } A$ . Moreover, the  $\tau_A$ -orbit of  $\text{rad } P$  consists of the radicals of all indecomposable projective modules in  $\text{mod } A$ . Hence  $\Delta$  is an almost right regular stable slice of  $\Gamma_A$ . Since  $A = \widehat{B}/(\varphi\nu_{\widehat{B}})$  with  $\varphi$  being a positive automorphism of  $\widehat{B}$ , we conclude that  $\text{rk } K_0(A) \geq \text{rk } K_0(\text{T}(B)) = \text{rk } K_0(B)$ , where  $\text{T}(B) = B \ltimes D(B) = \widehat{B}/(\nu_{\widehat{B}})$ . On the other hand, it follows from [15,16] that  $\text{rk } K_0(B)$  is the number of  $\tau_A$ -orbits in  $\Gamma_A^s$ . Therefore, the number of pairwise non-isomorphic indecomposable projective modules in  $\text{mod } A$  is at least  $n$ . This implies that  $\text{rad } P$  has multiplicity-free composition factors. But then the endomorphism algebra of the direct sum of modules on  $\Delta$  is a hereditary algebra and  $\Delta^{\text{op}}$  is its valued quiver. Summing up, we conclude that  $\Delta$  is a hereditary slice of  $\Gamma_A$ .  $\square$

**Proposition 4.4.** *Assume that  $A$  is of finite representation type but not a Nakayama algebra. Then  $\Gamma_A$  admits a hereditary right regular stable slice  $\Delta$ .*

**Proof.** We choose a right regular stable slice  $\Delta$  of  $\Gamma_A$  following the proof of [19, Theorem 3.1]. Namely, since  $A$  is not a Nakayama algebra, there exists an indecomposable

projective module  $P$  such that  $P/\text{soc}(P)$  is not the radical of a projective module. Consider the full valued subquiver  $\Delta_P$  of  $\Gamma_A$  given by  $\tau_A^{-1}(P/\text{soc}(P))$  and all indecomposable modules  $X$  such that there is a non-trivial sectional path in  $\Gamma_A^s$  from  $P/\text{soc}(P)$  to  $X$ . It is shown in [19] that  $\Delta_P$  does not contain  $Q/\text{soc}(Q)$  for any indecomposable projective module  $Q$  in  $\text{mod } A$ . Clearly,  $\Delta_P$  is a stable slice of  $\Gamma_A$ . Then  $\Delta = \tau_A(\Delta_P)$  is a right regular stable slice of  $\Gamma_A$ . We claim that  $\Delta$  is a hereditary slice. Let  $g = \varphi\nu_{\widehat{B}}$  and  $G$  be the infinite cyclic group generated by  $g$ . Consider the Galois covering  $F : \widehat{B} \rightarrow \widehat{B}/G = A$  and the push-down functor  $F_\lambda : \text{mod } \widehat{B} \rightarrow \text{mod } A$  associated to it. Since  $B$  is tilted of Dynkin type,  $\widehat{B}$  is a locally representation-finite locally bounded  $K$ -category (by [15, 16]) and hence  $F_\lambda$  is dense, preserves almost split sequences, and  $\Gamma_A$  is the orbit translation quiver  $\Gamma_{\widehat{B}}/G$  of  $\Gamma_{\widehat{B}}$  with respect to the induced action of  $G$  on  $\Gamma_{\widehat{B}}$  (see [12, Theorem 3.6]). Moreover,  $F_\lambda$  is a Galois covering of module categories, that is, for any indecomposable modules  $X$  and  $Y$  in  $\text{mod } \widehat{B}$ ,  $F_\lambda$  induces an isomorphism of  $K$ -vector spaces

$$\bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\widehat{B}}(X, g^r Y) \rightarrow \text{Hom}_A(F_\lambda(X), F_\lambda(Y)).$$

In particular, we conclude that there exists a right regular stable slice  $\Omega$  of  $\Gamma_{\widehat{B}}$  such that  $F_\lambda(\Omega) = \Delta$ . Let  $N$  be the direct sum of all indecomposable  $\widehat{B}$ -modules lying on  $\Omega$ . Then  $M = F_\lambda(N)$  is the direct sum of all indecomposable  $A$ -modules lying on  $\Delta$ . Consider the annihilator algebra  $C = \widehat{B}/\text{ann}_{\widehat{B}}(\Omega)$  of  $\Omega$  in  $\widehat{B}$ . Since  $N$  is a finite-dimensional  $\widehat{B}$ -module, the support of  $N$  is finite, and hence  $C$  is a finite-dimensional  $K$ -algebra. Further, because  $\Gamma_{\widehat{B}}$  is an acyclic quiver and  $\Omega$  is a right regular stable slice of  $\Gamma_{\widehat{B}}$ , we conclude that  $\Omega$  is a faithful section of  $\Gamma_C$  such that  $\text{Hom}_C(N, \tau_C N) = 0$ . Then it follows from the criterion of Liu and Skowroński [43, Theorem VIII.7.7] that  $N$  is a tilting  $C$ -module,  $H = \text{End}_C(N)$  is a hereditary algebra,  $T = D(N)$  is a tilting module in  $\text{mod } H$ ,  $C \cong \text{End}_H(T)$  canonically, and  $\Omega$  is the section of  $\Gamma_C$  determined by  $T$ . In particular,  $C$  is a tilted algebra of Dynkin type  $\Omega^{\text{op}} = \Delta^{\text{op}}$ . We note also that  $\widehat{B} = \widehat{C}$ , and hence  $\nu_{\widehat{B}} = \nu_{\widehat{C}}$  (see [16]). Since  $g = \varphi\nu_{\widehat{B}}$  with  $\varphi$  a positive automorphism of  $\widehat{B}$ , we conclude that, for any non-zero integer  $r$ , the categories  $C$  and  $g^r(C)$  have no common objects, and consequently  $\text{Hom}_{\widehat{B}}(N, g^r N) = 0$ . Then we obtain isomorphisms of  $K$ -vector spaces

$$H = \text{End}_C(N) = \text{End}_{\widehat{B}}(N) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\widehat{B}}(N, g^r N) \cong \text{End}_A(F_\lambda(N)) = \text{End}_A(M).$$

Hence  $\text{End}_A(M)$  is a hereditary algebra and  $\Delta^{\text{op}}$  is its valued quiver. Therefore,  $\Delta$  is a hereditary stable slice of  $\Gamma_A$ .  $\square$

We end this section with an example illustrating the above considerations.



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## References

- [1] I. Assem, J. Nehring, A. Skowroński, Domestic trivial extensions of simply connected algebras, *Tsukuba J. Math.* 150 (2014) 415–452.
- [2] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.
- [3] I. Assem, A. Skowroński, Iterated tilted algebras of type  $\tilde{A}_n$ , *Math. Z.* 195 (1987) 269–290.
- [4] M. Błaszkiwicz, A. Skowroński, On selfinjective algebras of finite representation type with maximal almost split sequences, *J. Algebra* 422 (2015) 450–486.
- [5] R. Bocian, A. Skowroński, Weakly symmetric algebras of Euclidean type, *J. Reine Angew. Math.* 580 (2005) 157–199.
- [6] K. Bongartz, Tilted algebras, in: *Representations of Algebras*, in: *Lecture Notes in Math*, vol. 903, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 26–38.
- [7] P. Dowbor, A. Skowroński, Galois coverings of representation-infinite algebras, *Comment. Math. Helv.* 62 (1987) 311–337.
- [8] K. Erdmann, O. Kerner, A. Skowroński, Self-injective algebras of wild tilted type, *J. Pure Appl. Algebra* 149 (2000) 127–176.
- [9] K. Erdmann, A. Skowroński, On Auslander-Reiten components of blocks and self-injective biserial algebras, *Trans. Amer. Math. Soc.* 330 (1992) 165–189.
- [10] R. Farnsteiner, A. Skowroński, Classification of restricted Lie algebras with tame principal block, *J. Reine Angew. Math.* 546 (2002) 1–45.
- [11] R. Farnsteiner, A. Skowroński, The tame infinitesimal groups of odd characteristic, *Adv. Math.* 205 (2006) 229–274.
- [12] P. Gabriel, The universal cover of a representation-finite algebra, in: *Representations of Algebras*, in: *Lecture Notes in Math.*, vol. 903, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 68–105.
- [13] D. Happel, On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* 62 (1987) 339–389.
- [14] D. Happel, C.M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* 274 (1982) 399–443.
- [15] M. Hoshino, Trivial extensions of tilted algebras, *Comm. Algebra* 10 (1982) 1965–1999.
- [16] D. Hughes, J. Waschbüsch, Trivial extensions of tilted algebras, *Proc. Lond. Math. Soc.* 46 (1983) 347–364.
- [17] A. Jaworska, P. Malicki, A. Skowroński, On Auslander-Reiten components of algebras without external short paths, *J. Lond. Math. Soc.* 85 (2012) 245–268.
- [18] A. Jaworska, P. Malicki, A. Skowroński, Tilted algebras and short chains of modules, *Math. Z.* 273 (2013) 19–27.
- [19] A. Jaworska-Pastuszak, A. Skowroński, Selfinjective algebras without short cycles of indecomposable modules, *J. Pure Appl. Algebra* 222 (2018) 3432–3447.
- [20] O. Kerner, Tilting wild algebras, *J. Lond. Math. Soc.* 39 (1989) 29–47.
- [21] O. Kerner, Stable components of wild tilted algebras, *J. Algebra* 142 (1991) 37–57.
- [22] O. Kerner, A. Skowroński, On module categories with nilpotent infinite radical, *Compos. Math.* 77 (1991) 313–333.
- [23] S. Liu, Tilted algebras and generalized standard Auslander-Reiten components, *Arch. Math. (Basel)* 61 (1993) 12–19.
- [24] L.G. Peng, J. Xiao, Invariability of trivial extensions of tilted algebras under stable equivalences, *J. Lond. Math. Soc.* 52 (1995) 61–72.
- [25] Z. Pogorzały, A. Skowroński, Selfinjective biserial standard algebras, *J. Algebra* 138 (1991) 491–504.
- [26] J. Rickard, Derived categories and stable equivalence, *J. Pure Appl. Algebra* 61 (1989) 303–317.
- [27] C.M. Ringel, *Tame Algebras and Integral Quadratic Forms*, *Lecture Notes in Mathematics*, vol. 1099, Springer-Verlag, Berlin, 1984.
- [28] C.M. Ringel, The regular components of the Auslander-Reiten quiver of a tilted algebra, *Chinese Ann. Math.* 9B (1988) 1–18.



- [29] A. Skowroński, Generalized standard Auslander-Reiten components without oriented cycles, *Osaka J. Math.* 30 (1993) 515–527.
- [30] A. Skowroński, Generalized standard Auslander-Reiten components, *J. Math. Soc. Japan* 46 (1994) 517–543.
- [31] A. Skowroński, Selfinjective algebras of polynomial growth, *Math. Ann.* 285 (1989) 177–199.
- [32] A. Skowroński, Selfinjective algebras: finite and tame type, in: *Trends in Representation Theory of Algebras and Related Topics*, in: *Contemporary Math.*, vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 169–238.
- [33] A. Skowroński, K. Yamagata, Socle deformations of self-injective algebras, *Proc. Lond. Math. Soc.* 72 (1996) 545–566.
- [34] A. Skowroński, K. Yamagata, Stable equivalences of selfinjective algebras of tilted type, *Arch. Math. (Basel)* 70 (1998) 341–350.
- [35] A. Skowroński, K. Yamagata, Galois coverings of selfinjective algebras by repetitive algebras, *Trans. Amer. Math. Soc.* 351 (1999) 715–734.
- [36] A. Skowroński, K. Yamagata, On selfinjective Artin algebras having nonperiodic generalized standard Auslander-Reiten components, *Colloq. Math.* 96 (2003) 235–244.
- [37] A. Skowroński, K. Yamagata, On invariability of selfinjective algebras of tilted type under stable equivalences, *Proc. Amer. Math. Soc.* 132 (2004) 659–667.
- [38] A. Skowroński, K. Yamagata, Positive Galois coverings of selfinjective algebras, *Adv. Math.* 194 (2005) 398–436.
- [39] A. Skowroński, K. Yamagata, Stable equivalences of selfinjective Artin algebras of Dynkin type, *Algebr. Represent. Theory* 9 (2006) 33–45.
- [40] A. Skowroński, K. Yamagata, Selfinjective algebras of quasitilted type, in: *Trends in Representation Theory of Algebras and Related Topics*, in: *European Math. Soc. Series of Congress Reports*, European Math., Soc. Publ. House, Zürich, 2008, pp. 639–708.
- [41] A. Skowroński, K. Yamagata, *Frobenius Algebras i. Basic Representation Theory*, EMS Textbooks in Mathematics, European Math., Soc. Publ. House, Zürich, 2011.
- [42] A. Skowroński, K. Yamagata, On selfinjective algebras of tilted type, *Colloq. Math.* 141 (2015) 89–117.
- [43] A. Skowroński, K. Yamagata, *Frobenius Algebras II. Tilted and Hochschild Extension Algebras*, EMS Textbooks in Mathematics, European Math., Soc. Publ. House, Zürich, 2017.