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On higher level Kirillov–Reshetikhin crystals, Demazure crystals, and related uniform models



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We show that a tensor product of nonexceptional type Kirillov–Reshetikhin (KR) crystals is isomorphic to a direct sum of Demazure crystals; we do this in the mixed level case and without the perfectness assumption, thus generalizing a result of Naoi. We use this result to show that, given two tensor products of such KR crystals with the same maximal weight, after removing certain 0-arrows, the two connected components containing the minimal/maximal elements are isomorphic. Based on the latter fact, we reduce a tensor product of higher level perfect KR crystals to one of single-column KR crystals, which allows us to use the uniform models available in the literature in the latter case. We also use our results to give a combinatorial interpretation of the Q -system relations. Our results are conjectured to extend to the exceptional types.

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1. Introduction

Kirillov–Reshetikhin (KR) *modules* are a family of finite-dimensional representations of an affine quantum group without derivation $U'_q(\mathfrak{g})$ that are characterized by their Drinfel'd polynomials. They have been the subject of intense study, with numerous applications and properties, some of which are still conjectural to various extents. For example, see [5–10,13,27,30,34–37,39,46,49,55] and the references therein. One of the most important conjectural properties [14,15] is that KR modules admit *crystal bases* in the sense of Kashiwara [20–22]. These crystals are called *Kirillov–Reshetikhin* (KR) *crystals* and are denoted $B^{r,s}$, where r is an index of the classical Dynkin diagram of \mathfrak{g} and $s \in \mathbb{Z}_{>0}$. KR crystals have been shown to exist in all nonexceptional types by Okado and Schilling [46], in types $G_2^{(1)}$ and $D_4^{(3)}$ by Naoi [40], for certain r in exceptional types [3,18,41], and for $r = 1$ in all types by Kashiwara [24].

KR crystals and their tensor products are known to be connected with *Demazure crystals* of affine highest weight representations. A precise description is known for a tensor product of KR crystals in nonexceptional types such that they are all *perfect* of the same level [11,26,55]; namely, this tensor product is isomorphic, up to certain 0-arrows, to a specified Demazure crystal. When the (nonexceptional type) KR crystals in the tensor product are perfect of mixed levels, Naoi showed that one obtains a direct sum of Demazure crystals [39]. In addition, this relationship was given for $B^{r,1}$ in all types [25] and can be extended to tensor products by the techniques of [39] (see also [38] for an alternative proof). Further connections of KR crystals, viewed as classical crystals, were described in [7,8].

One important unsolved problem involving KR crystals $B^{r,s}$ (and their tensor products) is constructing a uniform model in all types. For $B^{r,1}$ and their tensor products, such a model, based on *projected level-zero Lakshmibai–Seshadri* (LS) *paths*, was given by Naito and Sagaki [42,43]. In untwisted types, an explicit description of these piecewise-linear paths was given as *quantum Lakshmibai–Seshadri* (LS) *paths* in [35]; the alternative *quantum alcove model* was given in the same paper (see also [31,32]), while the quantum LS paths for type $A_{2n}^{(2)}$ were developed in [44]. A partially uniform model for $B^{r,1}$ using *Nakajima monomials* was given by Hernandez and Nakajima [16]. A uniform model for the classical crystal structure of $B^{r,s}$ was given in terms of *rigged configurations* in [28,47,53,54]. However, the affine crystal structure on rigged configurations has currently only been explicitly constructed for type $A_n^{(1)}$ [56], $D_n^{(1)}$ [48], and $A_{2n-1}^{(2)}$, $B_n^{(1)}$ [54]. Also in the nonexceptional case, a type-specific construction, based on virtual crystals and tableaux, is found in [9].

One goal of this paper is to construct the perfect KR crystals $B^{r,s}$ of nonexceptional type, as well as their tensor products of a fixed level, up to certain 0-arrows called *non-level ℓ Demazure arrows*, respectively *non-level ℓ dual Demazure arrows*. This construction is done by identifying them with specific subcrystals of certain tensor products of single-column KR crystals $B^{r,1}$. Then, for the latter, we can use the uniform models mentioned above, *i.e.*, quantum LS paths and the quantum alcove model. We focus on

the latter, as it is purely combinatorial and easier to use. Furthermore, we are currently working on a very explicit combinatorial description of the mentioned subcrystals.

The paper also achieves several other goals as follows. First, we derive as our main tool a generalization of Naoi's result mentioned above [39] to the nonperfect case. Secondly, our reduction theorem used to construct $B^{r,s}$ in terms of $B^{r,1}$ is proved in much larger generality, as an identification between two tensor products of KR crystals of mixed levels (again, of nonexceptional type, and possibly nonperfect). Thirdly, another special case of this relationship is shown to realize combinatorially a part of the Q -system relations, which are satisfied by the classical characters of KR crystals [13]; we are led to a conjecture about a combinatorial realization of the entire Q -system relations.

We conjecture that our results extend to the exceptional types. In particular, this would immediately lead to a uniform model for all (level ℓ dual Demazure portions of) tensor products of perfect KR crystals with a fixed level. Other problems are stated as well.

Let us describe our results in more detail. For a tensor product of KR crystals B , there exists a unique (classical) weight λ , called the maximal weight, and a unique element of weight $w_0(\lambda)$, called the minimal element; here w_0 is the longest element of the corresponding finite Weyl group. We say that a 0-arrow is a non-level ℓ Demazure arrow if it is one of the first ℓ 0-arrows in its 0-string. Given two tensor products of KR crystals B and B' with the same maximal weight λ and of level bounded by ℓ , our main theorem states that, after removing all non-level ℓ Demazure arrows, the connected components containing the corresponding minimal elements are isomorphic. A contragredient dual version of this result also holds.

The main tool in proving our construction is showing for a tensor product of KR crystals B that $B \otimes u_{\ell\Lambda_0}$ (this tensor product is equivalent to removing all the non-level ℓ Demazure arrows) is isomorphic to a direct sum of Demazure crystals; this is the mentioned generalization of Naoi's result. As a consequence, we show that all tensor products of KR crystals are isomorphic to some direct sum of Demazure crystals. We note that our results do not imply that $B^{r,s}$, when perfect of level ℓ , is isomorphic to a single Demazure crystal; however, this does follow as a consequence when the two crystals have the same classical characters, like in the cases discussed in [7].

This paper is organized as follows. In Section 2, we provide the necessary background. In Section 3, we prove our main results. In Section 4 we describe the reduction to single-column KR crystals and explain the way in which the quantum alcove model applies to higher level KR crystals. In Section 5, we refer to the Q -system relations and the mentioned conjecture involving them.

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2. Background

Let \mathfrak{g} be an affine Kac–Moody Lie algebra with index set I , Cartan matrix $(A_{ij})_{i,j \in I}$, simple roots $(\alpha_i)_{i \in I}$, fundamental weights $(\Lambda_i)_{i \in I}$, weight lattice P , simple coroots $(\alpha_i^\vee)_{i \in I}$, and canonical pairing $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbb{Z}$ given by $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$. We write $i \sim j$ if $A_{ij} \neq 0$ and $i \neq j$. Let $U_q(\mathfrak{g})$ denote the corresponding (Drinfel’d–Jimbo) quantum group. Define $c_i^\vee := \max(a_i^\vee/a_i, 1)$, where a_i and a_i^\vee are the Kac and dual Kac labels, respectively [19, Table Aff1-3]. Let P^+ and P^- denote the positive and negative weight lattices, respectively. We denote by P_ℓ^+ the dominant weights of level ℓ . Let Q be the root lattice, with Q^+ and Q^- being the positive and negative root lattices, respectively. Let W be the Weyl group corresponding to \mathfrak{g} . The (strong) Bruhat order on W has covers $w < ws_\alpha$ with $\ell(ws_\alpha) = \ell(w) + 1$, where $\ell(\cdot)$ denotes the length function.

The extended affine Weyl group is $\widetilde{W} := W \rtimes \Pi \cong W_0 \rtimes P_0$, where Π is the set of length 0 elements (in \widetilde{W}) and corresponds to automorphisms of the Dynkin diagram of \mathfrak{g} . Let $t_\mu \in \widetilde{W}$ be the translation by $\mu \in P_0$. See, e.g., [1,4,19] for more information on the extended affine Weyl group.

Let $U'_q(\mathfrak{g}) := U_q([\mathfrak{g}, \mathfrak{g}])$. Note that the corresponding weight lattice is $P' := P/\mathbb{Z}\delta$, where $\delta = \sum_{i \in I} c_i \alpha_i$ is the null root; in particular, the simple roots in P' are linearly dependent. We will sometimes abuse notation and write P instead of P' when there is no danger of confusion.

Let \mathfrak{g}_0 denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$, and $U_q(\mathfrak{g}_0)$ the corresponding quantum group. Let $\text{cl}: P \rightarrow P_0$ denote the natural classical projection onto the weight lattice P_0 of \mathfrak{g}_0 . Let $\omega_i := \text{cl}(\Lambda_i)$ be the classical projection of the fundamental weight Λ_i . Let Q_0 and W_0 be the root lattice and Weyl group of \mathfrak{g}_0 , respectively. As usual, we denote by w_0 the longest element of W_0 .

The *quantum Bruhat graph* [12] is the directed graph on W_0 with edges labeled by positive roots of \mathfrak{g}_0

$$w \xrightarrow{\alpha} ws_\alpha \quad \text{for } w < ws_\alpha \text{ or } \ell(ws_\alpha) = \ell(w) - 2\langle \rho, \alpha^\vee \rangle + 1; \quad (2.1)$$

here ρ denotes, as usual, half the sum of the positive roots of \mathfrak{g}_0 .

2.1. Crystals

An *abstract $U_q(\mathfrak{g})$ -crystal* is a set B endowed with *crystal operators* $e_i, f_i: B \rightarrow B \sqcup \{0\}$, for $i \in I$, and *weight function* $\text{wt}: B \rightarrow P$ that satisfy the following conditions:

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle$, for all $b \in B$ and $i \in I$,

- (2) $f_i b = b'$ if and only if $b = e_i b'$, for $b, b' \in B$ and $i \in I$,
 (3) $\text{wt}(f_i b) = \text{wt}(b) - \alpha_i$ if $f_i b \neq 0$;

where the statistics $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z}_{\geq 0}$ are defined by

$$\varepsilon_i(b) := \max\{k \mid e_i^k b \neq 0\}, \quad \varphi_i(b) := \max\{k \mid f_i^k b \neq 0\}.$$

Remark 2.1. The definition of an abstract crystal given in this paper is sometimes called a *regular* or *seminormal* abstract crystal in the literature.

Let $e_i^{\max} b := e_i^{\varepsilon_i(b)} b$ and $f_i^{\max} b := f_i^{\varphi_i(b)} b$. From the axioms, we identify B with an I -edge colored weighted directed graph, where there is an i -colored edge $b \rightarrow b'$ in the graph if and only if $f_i b = b'$. Thus an entire i -string through an element $b \in B$ is given diagrammatically by

$$e_i^{\max} b \xrightarrow{i} \cdots \xrightarrow{i} e_i^2 b \xrightarrow{i} e_i b \xrightarrow{i} b \xrightarrow{i} f_i b \xrightarrow{i} f_i^2 b \xrightarrow{i} \cdots \xrightarrow{i} f_i^{\max} b.$$

An element $b \in B$ is *highest (resp. lowest) weight* if $e_i b = 0$ (resp. $f_i b = 0$) for all $i \in I$. We say that $b \in B$ is *classically highest (resp. lowest) weight* if $e_i b = 0$ (resp. $f_i b = 0$) for all $i \in I_0$.

We define the *tensor product* of abstract $U_q(\mathfrak{g})$ -crystals B_1 and B_2 as the crystal $B_2 \otimes B_1$ that is the Cartesian product $B_2 \times B_1$ with the following crystal structure:

$$\begin{aligned} e_i(b_2 \otimes b_1) &:= \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \\ f_i(b_2 \otimes b_1) &:= \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \\ \varepsilon_i(b_2 \otimes b_1) &:= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \alpha_i^\vee, \text{wt}(b_1) \rangle), \\ \varphi_i(b_2 \otimes b_1) &:= \max(\varphi_i(b_2), \varphi_i(b_1) + \langle \alpha_i^\vee, \text{wt}(b_2) \rangle), \\ \text{wt}(b_2 \otimes b_1) &:= \text{wt}(b_2) + \text{wt}(b_1). \end{aligned}$$

Remark 2.2. Our tensor product convention follows [2], which is opposite to that of Kashiwara [21].

For abstract $U_q(\mathfrak{g})$ -crystals B_1, \dots, B_L , the action of the crystal operators on the tensor product $B := B_L \otimes \cdots \otimes B_2 \otimes B_1$ can be computed by the *signature rule*. Let $b := b_L \otimes \cdots \otimes b_2 \otimes b_1 \in B$, and for $i \in I$, we write

$$\underbrace{- \cdots -}_{\varphi_i(b_L)} \underbrace{+ \cdots +}_{\varepsilon_i(b_L)} \cdots \underbrace{- \cdots -}_{\varphi_i(b_1)} \underbrace{+ \cdots +}_{\varepsilon_i(b_1)}.$$

Then by successively deleting consecutive $+-$ -pairs (in that order) in the above sequence, we obtain a sequence

$$\operatorname{sgn}_i(b) := \underbrace{- \cdots -}_{\varphi_i(b)} \underbrace{+ \cdots +}_{\varepsilon_i(b)},$$

called the *reduced signature*. Suppose $1 \leq j_- \leq j_+ \leq L$ are such that b_{j_-} contributes the rightmost $-$ in $\operatorname{sgn}_i(b)$ and b_{j_+} contributes the leftmost $+$ in $\operatorname{sgn}_i(b)$. Then, we have

$$\begin{aligned} e_i b &:= b_L \otimes \cdots \otimes b_{j_++1} \otimes e_i b_{j_+} \otimes b_{j_+-1} \otimes \cdots \otimes b_1, \\ f_i b &:= b_L \otimes \cdots \otimes b_{j_-+1} \otimes f_i b_{j_-} \otimes b_{j_--1} \otimes \cdots \otimes b_1. \end{aligned}$$

Let B_1 and B_2 be two abstract $U_q(\mathfrak{g})$ -crystals. A *crystal morphism* $\psi: B_1 \rightarrow B_2$ is a map $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ with $\psi(0) = 0$, such that the following properties hold for all $b \in B_1$ and $i \in I$:

- (1) if $\psi(b) \in B_2$, then $\operatorname{wt}(\psi(b)) = \operatorname{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$;
- (2) we have $\psi(e_i b) = e_i \psi(b)$ if $\psi(e_i b) \neq 0$ and $e_i \psi(b) \neq 0$;
- (3) we have $\psi(f_i b) = f_i \psi(b)$ if $\psi(f_i b) \neq 0$ and $f_i \psi(b) \neq 0$.

An *embedding* (resp. *isomorphism*) is a crystal morphism such that the induced map $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ is an embedding (resp. bijection). A crystal morphism is *strict* if it commutes with all crystal operators. Note that for a strict crystal embedding $\psi: B \rightarrow B'$ and connected components $C \subseteq B$ and $C' \subseteq B'$ such that $\psi(c) \in C'$ for any $c \in C$, the restriction $\hat{\psi}: C \rightarrow C'$ is an isomorphism, i.e., connected components go to connected components under ψ .

A *similarity map* [23] is an embedding of crystals $\sigma = \sigma_m: B \rightarrow \hat{B}$, with $m \in \mathbb{Z}_{>0}$, which satisfies

$$e_i \mapsto e_i^m, \quad f_i \mapsto f_i^m, \quad \varepsilon_i(\sigma(b)) = m\varepsilon_i(b), \quad \varphi_i(\sigma(b)) = m\varphi_i(b), \quad \operatorname{wt}(\sigma(b)) = m \operatorname{wt}(b). \quad (2.2)$$

An abstract crystal B is a *$U_q(\mathfrak{g})$ -crystal* if B is the crystal basis of some $U_q(\mathfrak{g})$ -module. Kashiwara [21] has shown that the irreducible highest (resp. lowest) weight module $V(\lambda)$, for $\lambda \in P^+$ (resp. $\lambda \in P^-$), admits a crystal basis, denoted $B(\lambda)$; this has a unique highest (resp. lowest) weight element u_λ such that $\operatorname{wt}(u_\lambda) = \lambda$. The elements $u_{w\lambda} := wu_\lambda$, for $w \in W$, are called *extremal*; here we used the W -action on the crystal, which was defined by Kashiwara [22] as follows:

$$s_i b := \begin{cases} f_i^{\langle \alpha_i^\vee, \operatorname{wt}(b) \rangle} b & \text{if } \langle \alpha_i^\vee, \operatorname{wt}(b) \rangle > 0, \\ e_i^{-\langle \alpha_i^\vee, \operatorname{wt}(b) \rangle} b & \text{if } \langle \alpha_i^\vee, \operatorname{wt}(b) \rangle \leq 0. \end{cases}$$

2.2. Demazure crystals

Let $\lambda \in P^+$. A Demazure module is a $U_q^+(\mathfrak{g})$ -module generated by an extremal weight vector of weight $w\lambda \in V(\lambda)$. Kashiwara showed that the Demazure module has a crystal basis that is compatible with the crystal basis $B(\lambda)$ of the corresponding highest weight module $V(\lambda)$.

Hence, we can construct the crystal of a Demazure module as a subcrystal of $B(\lambda)$. Fix a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$. A *Demazure crystal* of the highest weight crystal $B(\lambda)$ is the full subcrystal given by

$$B_w(\lambda) := \{b \in B(\lambda) \mid e_{i_1}^{\max} e_{i_2}^{\max} \cdots e_{i_\ell}^{\max} b = u_\lambda\}. \quad (2.3)$$

Theorem 2.3 (Combinatorial excellent filtration [17,33]). *For all $\lambda, \mu \in P^+$, the crystal $B_w(\mu) \otimes u_\lambda$ is a direct sum of Demazure crystals.*

We also require the following fact, which follows from the definition of the Bruhat order on the Weyl group W .

Proposition 2.4. *Let $\lambda \in P^+$. We have $v \leq w$ if and only if $B_v(\lambda) \subseteq B_w(\lambda)$.*

2.3. Kirillov–Reshetikhin crystals

Let $B^{r,s}$ denote the *Kirillov–Reshetikhin (KR) crystal*, where $r \in I_0$ and $s \in \mathbb{Z}_{>0}$. We refer to Section 1 for a review of the cases when the existence of KR crystals was proved, as well as of the related combinatorial models.

KR crystals have a number of conjectural properties. A KR crystal $B^{r,s}$ is conjectured [27] to be *perfect*¹ of level s/c_r if and only if $s/c_r \in \mathbb{Z}$. This has been shown for all nonexceptional types in [10], and in some special cases for other types [29,57]. KR crystals are known to be well-behaved under similarity maps in nonexceptional types, as stated below.

Theorem 2.5 ([45]). *Let \mathfrak{g} be of nonexceptional affine type. There exists a (unique) similarity map $\sigma_m: B^{r,s} \rightarrow B^{r,ms}$.*

There exists a unique classical component $B(s\omega_r) \subseteq B^{r,s}$, and for any other classical component $B(\lambda) \subseteq B^{r,s}$, we have $s\omega_r - \lambda \in Q_0^+$. We remark that $B^{r,s} \cong B(s\omega_r)$ as $U_q(\mathfrak{g}_0)$ -crystals whenever r is the image of 0 for some Dynkin diagram automorphism. Let $u_{\max}(B^{r,s}) := u_{r\omega_s} \in B(s\omega_r) \subseteq B^{r,s}$ denote the *maximal element*. For general $B := \bigotimes_{j=1}^N B^{r_j, s_j}$, the maximal element is defined as $u_{\max}(B) := u_{\max}(B^{r_1, s_1}) \otimes \cdots \otimes u_{\max}(B^{r_N, s_N})$, and note that it is the unique element of classical weight $\sum_{j=1}^N s_j \omega_{r_j}$.

¹ The property of being perfect is a technical condition related to KR crystals, which is used to construct the Kyoto path model [27]; see also [2].

Similarly, let $u_{\min}(B)$ denote the *minimal element* of B , which is the unique element of classical weight $w_0\left(\sum_{j=1}^N s_j \omega_{r_j}\right)$.

The key property we need is a relationship between tensor products of KR crystals and Demazure crystals of an affine highest weight crystal.

Theorem 2.6 ([11,26,55]). *Let \mathfrak{g} be of nonexceptional type. Let $B := \bigotimes_{j=1}^N B^{r_j, s_j}$ such that there exists $\ell \in \mathbb{Z}$ with $s_j/c_{r_j} = \ell$ for all j . Let $\omega := -(c_{r_1}\omega_{r_1^*} + \cdots + c_{r_N}\omega_{r_N^*})$, where $\omega_{r^*} = -w_0(\omega_r)$. Then, there exists a crystal isomorphism*

$$\psi: B(\ell\Lambda_{\tau(0)}) \rightarrow B \otimes B(\ell\Lambda_0)$$

given by $u_{\ell\Lambda_{\tau(0)}} \mapsto u_g \otimes u_{\ell\Lambda_0}$, where $u_g = u_1 \otimes \cdots \otimes u_N$ is the *ground state element*, the unique element of B such that $u_N = u_{\max}(B^{r_N, s_N})$ and $\varepsilon_i(u_j) = \varphi_i(u_{j+1})$ for all $1 \leq j < N$ and $i \in I$. Moreover, we have

$$B_v(\ell\Lambda_{\tau(0)}) \cong B \otimes u_{\ell\Lambda_0},$$

where $v\tau = t_\omega$ with $v \in W$ and $\tau \in \Pi$.

Theorem 2.6 is conjectured to hold for all affine types [14,15,7] under the assumption that $s_j/c_{r_j} \in \mathbb{Z}$ implies that B^{r_j, s_j} is perfect (cf. the perfectness conjecture). We say that the above tensor product $B := \bigotimes_{j=1}^N B^{r_j, s_j}$ is of *level bounded by ℓ* if ℓ is such that $\lceil s_j/c_{r_j} \rceil \leq \ell$ for all j . Theorem 2.6 was generalized in [39, Prop. 5.16] in the following way: when B (still of nonexceptional type) is a tensor product of level bounded by ℓ , and $s_j/c_{r_j} \in \mathbb{Z}$, then $B \otimes u_{\ell\Lambda_0}$ is isomorphic to a direct sum of Demazure crystals.

We need the following fact from [55, Prop. 8.1] (which was essentially proved in [26]). The claim holds for *all affine types* since [55, Lemma 7.3] holds, by using the general definition of energy [26,14,15]; see [39, Lemma 6.4] as well. We also note that there is *no assumption of perfectness*.

Proposition 2.7 ([55, Prop. 8.1],[26]). *Consider a tensor product of KR crystals B of level bounded by ℓ . Then there exists a sequence $(\Lambda^{(k)} \in P_\ell^+)^N_{k=1}$ such that*

$$B \otimes B(\ell\Lambda_0) \cong \bigoplus_{k=1}^N B\left(\Lambda^{(k)}\right).$$

3. Main results

For the remainder of this paper, we will consider KR crystals of nonexceptional type. It is known that, for any $J \subsetneq I$, under the Levi branching to the canonical subalgebra \mathfrak{g}_J with index set J (i.e., we remove all i -edges for $i \in I \setminus J$), $B^{r,s}$ is a direct sum of highest weight $U_q(\mathfrak{g}_J)$ -crystals.

In this section we prove our main results.

Remark 3.1. There are contragredient dual versions of all our results. All the proofs hold for the contragredient dual by interchanging $e_i \leftrightarrow f_i$.

We start with our main tool, namely the generalization of Naoi's result [39, Prop. 5.16] (which, in turn, is a generalization of Theorem 2.6, as discussed in Section 2.3). Our generalization holds without the perfectness assumption.

Theorem 3.2. *Let B be a tensor product of KR crystals of level bounded by ℓ , having the decomposition in Proposition 2.7. Then there exists a sequence $(\lambda^{(k)} \in P_0^-)_{k=1}^N$ such that*

$$B \otimes u_{\ell\Lambda_0} = \bigoplus_{k=1}^N B_{\lambda^{(k)}} \cong \bigoplus_{k=1}^N B_{w^{(k)}}(\Lambda^{(k)}) ; \quad (3.1)$$

here the following hold:

- there exists a unique element $b_{\min}^{(k)} \otimes u_{\ell\Lambda_0} \in B_{\lambda^{(k)}}$ satisfying $\text{wt}(b_{\min}^{(k)}) = \lambda^{(k)}$ and

$$\text{wt}(b) - \text{wt}(b_{\min}^{(k)}) \in Q_0^+ \setminus \{0\} \quad \text{for all } b \otimes u_{\ell\Lambda_0} \in B_{\lambda^{(k)}} \setminus \{b_{\min}^{(k)} \otimes u_{\ell\Lambda_0}\},$$

- $w^{(k)}(\Lambda^{(k)}) = \lambda^{(k)} + \ell\Lambda_0$, for $w^{(k)} \in W$ of minimal length.

Roughly speaking, our proof follows the proofs of [39, Prop. 5.16] or [11, Thm. 4.7], by reducing the statement to the case when $B = B^{r,s}$. Then we use the similarity map from Theorem 2.5. In order to complete our proof, we use [39, Lemma 4.8] and another elementary fact stated below.

Lemma 3.3 ([39, Lemma 4.8]). *Let $\Lambda \in P^+$ and $w \in \widetilde{W}$, and assume that $\langle \alpha_i^\vee, w\Lambda \rangle \leq 0$ for all $i \in I_0$. Then for any $b \in B_w(\Lambda)$, we have $b = u_{w\Lambda}$ or*

$$\text{cl}(\text{wt}(b)) \in \text{cl}(w\Lambda) + (Q_0^+ \setminus \{0\}).$$

Lemma 3.4. *Let $\sigma_m: B(\Lambda) \rightarrow B(m\Lambda)$ be a similarity map. For any $w \in W$, this induces a similarity map $\sigma_m^D: B_w(\Lambda) \rightarrow B_w(m\Lambda)$. The image of this map consists of those vertices b in $B_w(m\Lambda)$ for which each e_{i_j} in (2.3) is applied a multiple of m number of times.*

Proof. This follows immediately from the definition of a similarity map (2.2) and of a Demazure crystal (2.3). \square

Proof of Theorem 3.2. It is sufficient to restrict to $B = B^{r,s}$, by the reduction argument in the proof of [39, Prop. 5.16] (this is essentially the induction described in [11, Thm. 4.7]). Furthermore, it is sufficient to consider $B^{r,s} \otimes u_{\ell\Lambda_0}$, where $\ell = \lceil s/c_r \rceil$, by a similar argument to the one used in the proof of [39, Prop. 5.16]. When $s/c_r \in \mathbb{Z}$, the claim holds by Theorem 2.6. Therefore, assume $s/c_r \notin \mathbb{Z}$.

By Theorem 2.6, we have

$$B^{r,c_r s} \otimes u_{s\Lambda_{\tau(0)}} \cong B_v(s\Lambda_{\tau(0)}),$$

where $t_{-c_r\omega_{r^*}} = v\tau$. Therefore, we have

$$\begin{aligned} B^{r,c_r s} \otimes u_{c_r\ell\Lambda_{\tau(0)}} &\cong B^{r,c_r s} \otimes u_{s\Lambda_{\tau(0)}} \otimes u_{(c_r\ell-s)\Lambda_{\tau(0)}} \\ &\cong B_v(s\Lambda_{\tau(0)}) \otimes u_{(c_r\ell-s)\Lambda_{\tau(0)}} \cong \bigoplus_{(w,\Lambda)} B_w(\Lambda), \end{aligned} \quad (3.2)$$

where the last isomorphism is by the combinatorial excellent filtration in Theorem 2.3. Now consider the similarity map $\sigma_{c_r} : B^{r,s} \rightarrow B^{r,c_r s}$ from Theorem 2.5. This gives a similarity map

$$\sigma_{c_r}^D : B^{r,s} \otimes u_{\ell\Lambda_{\tau(0)}} \rightarrow B^{r,c_r s} \otimes u_{c_r\ell\Lambda_{\tau(0)}}.$$

Composing the latter map with the crystal isomorphisms in (3.2), we want to identify the image $\text{Im } \sigma_{c_r}^D$ of $B^{r,s} \otimes u_{\ell\Lambda_{\tau(0)}}$ inside the direct sum of Demazure crystals in (3.2).

Now assume that $\text{Im } \sigma_{c_r}^D$ intersects some Demazure crystal $B_w(\Lambda)$. Pick some vertex in the intersection, and a sequence of e_{i_j} to u_Λ as in (2.3). By the definition of a similarity map (2.2), each e_{i_j} is applied a multiple of k number of times, and the upper endpoints of the various strings belong to $\text{Im } \sigma_{c_r}^D \cap B_w(\Lambda)$. In particular, so does u_Λ . We can now see that $\text{Im } \sigma_{c_r}^D \cap B_w(\Lambda)$ is characterized by the condition in Lemma 3.4.

As $\text{wt}(\sigma_{c_r}^D(b)) = c_r \text{wt}(b)$ for all $b \in B^{r,s} \otimes u_{\ell\Lambda_{\tau(0)}}$, by (2.2), we have $\Lambda/c_r \in P^+$. Combining the above facts with Lemma 3.4, we deduce that

$$\text{Im } \sigma_{c_r}^D \cap B_w(\Lambda) \cong B_w(\Lambda/c_r).$$

Thus, we proved the decomposition (3.1).

Note that the multiset of weights $\Lambda^{(k)}$ in (3.1) coincides with the one in Proposition 2.7. Indeed, by the signature rule, all highest weight vertices in $B \otimes B(\ell\Lambda_0)$ are of the form $b \otimes u_{\ell\Lambda_0}$ with $b \in B$, so they are highest weight vertices in $B \otimes u_{\ell\Lambda_0}$; the reverse inclusion is obvious.

Finally, the existence of the vertices $b_{\min}^{(k)}$ with the desired properties follows from Lemma 3.3. Indeed, let us verify the hypothesis of this Lemma. Take an affine Demazure crystal $B_{w^{(k)}}(\Lambda^{(k)})$ in (3.1). It has a decomposition into classically highest weight crystals, because B has such a decomposition, and tensoring with $u_{\ell\Lambda_0}$ does not affect the classical crystal structure. In the mentioned decomposition of $B_{w^{(k)}}(\Lambda^{(k)})$, the unique element of weight $w^{(k)}(\Lambda^{(k)})$ has to be a classically lowest weight element, so $w^{(k)}(\Lambda^{(k)}) - \ell\Lambda_0$ is a finite antidominant weight. \square

Remarks 3.5. (1) We could have proved Theorem 3.2 by applying the similarity map and then by directly appealing to [39, Prop. 5.16]. However, we found it more illuminating to show the role of the similarity map in a very explicit way: in the single factor case.

(2) Theorem 3.2 does not imply that $B \otimes u_{\ell\Lambda_0}$ is a single Demazure crystal. However, it has been shown in [7] that $B = B^{r,s}$ is isomorphic as an $U_q(\mathfrak{g}_0)$ -crystal to a single Demazure crystal for a number of special $r \in I_0$ in exceptional types. If we combine this with Theorem 3.2, we obtain Theorem 2.6 for these cases.

Following [55], we define a *level ℓ Demazure edge* as being an i -edge $b' \rightarrow b$ in the crystal graph such that either $i \in I_0$ or $\varepsilon_0(b) > \ell$. In other words, an edge is not a level ℓ Demazure edge if it is in the length ℓ head of a 0-string. A *level ℓ dual Demazure edge* is defined similarly, using the length ℓ tail of a 0-string. Let $\widetilde{D}_\ell(B)$ and $\widetilde{DD}_\ell(B)$ denote the subcrystals of B obtained by removing all edges that are not level ℓ Demazure edges in B , respectively dual Demazure edges.

The following lemma is well-known to experts. It follows immediately from the tensor product rule, and motivates the terminology of Demazure edge.

Lemma 3.6. *Let B be a tensor product of KR crystals of level bounded by ℓ . The map*

$$\rho_\ell: \widetilde{D}_\ell(B) \rightarrow B \otimes u_{\ell\Lambda_0}$$

given by $\rho_\ell(b) = b \otimes u_{\ell\Lambda_0}$ is a crystal isomorphism (up to a weight shift).

Remark 3.7. Let $\psi: \bigoplus_{k=1}^N B_{\lambda^{(k)}} \rightarrow B \otimes u_{\ell\Lambda_0}$ be the isomorphism given by Theorem 2.6. By [55, Lemma 7.3] and [39, Lemma 6.4], then there exist constants $(C_k)_{k=1}^N$ such that for all $b \in B_{\lambda^{(k)}}$, if $\text{wt}(b) = \mu + D\delta$ and $\psi(b) = b' \otimes u_{\ell\Lambda_0}$ (note we are considering this as a $U_q(\mathfrak{g})$ -crystal and have to implicitly branch to $U'_q(\mathfrak{g})$, which simply changes the weight), then we have $D = E(b') + C_k$, where $E(b)$ is the energy statistic of [26,14,15]. These constants were explicitly specified when B was a tensor product of perfect crystals in nonexceptional type in [39, Thm. 7.1].

Let $D_\ell(B)$ denote the connected component of $\widetilde{D}_\ell(B)$ that contains $u_{\min}(B)$, and $DD_\ell(B)$ denote the connected component of $\widetilde{DD}_\ell(B)$ that contains $u_{\max}(B)$. We now present our main theorem.

Theorem 3.8. *Let $B := \bigotimes_{j=1}^N B^{r_j, s_j}$ and $B' := \bigotimes_{j=1}^{N'} B^{r'_j, s'_j}$ be of levels bounded by ℓ and*

$$\sum_{j=1}^N s_j \omega_{r_j} = \sum_{j=1}^{N'} s'_j \omega_{r'_j}. \quad (3.3)$$

Then we have

$$D_\ell(B) \cong D_\ell(B'), \quad DD_\ell(B) \cong DD_\ell(B').$$

Proof. Let λ be the weight given by (3.3), and let $\Lambda := w_0(\lambda) + \ell\Lambda_0$. Note that $\text{wt}(u_{\min}(B)) = w_0(\lambda)$. Since we have $\text{wt}(b) \in w_0(\lambda) + (Q_0^+ \setminus \{0\})$ for all $b \in B \setminus \{u_{\min}(B)\}$,

we deduce that $D_\ell(B) \cong B_w(\mu)$ for some $\mu \in P_\ell^+$ and $w \in W$, by Theorem 3.2 and Lemma 3.6; here w and μ are uniquely determined by the condition $w(\mu) = \Lambda$ and the fact that w is of minimum length. Similarly, we have $D_\ell(B') \cong B_w(\mu)$. Hence, we have $D_\ell(B) \cong D_\ell(B')$. The proof of the contragredient dual version is completely similar, cf. Remark 3.1. \square

We conjecture that Theorems 3.2 and 3.8 also hold in the exceptional types. As evidence, we note that Theorem 2.6 is known to hold as $U_q(\mathfrak{g}_0)$ -crystals in certain cases by [7], which hence implies $U_q(\mathfrak{g}_0)$ -crystal versions of Theorems 3.2 and 3.8. Furthermore, Theorem 3.2 is known to hold for $\ell = 1$ in all types [25,8,38].

4. Uniform models

4.1. Reduction to single-column KR crystals

The following corollary of Theorem 3.8 allows us to reduce tensor products of arbitrary KR crystals to tensor products of single-column ones.

Corollary 4.1. *Let $B := \bigotimes_{j=1}^N B^{r_j, s_j}$ and $B' := \bigotimes_{j=1}^N (B^{r_j, 1})^{\otimes s_j}$ be such that there exists $\ell \in \mathbb{Z}$ with $s_j/c_{r_j} = \ell$ for all j . Then we have*

$$\widetilde{D}_\ell(B) \cong D_\ell(B'), \quad \widetilde{DD}_\ell(B) \cong DD_\ell(B').$$

Proof. Since $s_j/c_{r_j} = \ell \in \mathbb{Z}$, all B^{r_j, s_j} are perfect crystals of level ℓ . This property implies that $\widetilde{D}_\ell(B)$ is connected, so $\widetilde{D}_\ell(B) = D_\ell(B)$, and similarly for the contragredient dual case. Now apply Theorem 3.8. \square

Remarks 4.2.

- (1) The isomorphism in Corollary 4.1 realizes the level ℓ Demazure and dual Demazure portions of B in terms of single-column KR crystals.
- (2) We conjecture a similar realization in the exceptional types, assuming the perfectness conjecture. We expect the proof to be completely similar, based on the generalizations of Theorems 3.2 and 3.8 that were conjectured in Section 3. We have verified this conjecture for $B^{r, s}$ for $s = 2, 3, 4$ in types $D_4^{(3)}$ (which is perfect for all s) and $G_2^{(1)}$. (These cases could follow by a similar proof of Theorem 3.2 using a diagram folding of $D_4^{(1)}$ and the corresponding conjectural virtual crystal construction; see, e.g., [47,50,54].)

The following natural question arises.

Problem 4.3. How is the isomorphism in Corollary 4.1 expressed concretely when the corresponding tensor products of KR crystals are realized based on the tableau model [9] and the rigged configuration model [54]?

One particular approach to Problem 4.3 could be through the use of the so-called *Kirillov–Reshetikhin (KR) tableaux* of [48,54]. These arise from the bijection Φ with rigged configurations, which use column splitting to construct classical crystal embeddings as a core part of the bijection Φ . Considering the fact that rigged configurations and the Demazure constructions are combinatorial R -matrix invariant, as well as the relationship with energy and the affine grading from [11,27,55] (see also Remark 3.7), it is likely that Φ , and hence KR tableaux, could be a consequence of Theorem 3.8.

Example 4.4. In type $C_2^{(1)}$, we have two connected components for $\tilde{D}_1(B^{1,1} \otimes B^{1,1})$, and a single connected component for $\tilde{D}_1(B^{1,2})$; see Fig. 1, where the crystal vertices are labeled by the corresponding *Kashiwara–Nakashima tableaux*, see [2]. Note that the leftmost connected component of the former crystal is $D_1(B^{1,1} \otimes B^{1,1})$, and this is isomorphic to $D_1(B^{1,2}) = \tilde{D}_1(B^{1,2})$. The latter is thus realized in terms of single-column KR crystals.

Note that the corresponding element of $\emptyset \in \tilde{D}_1(B^{1,2})$ corresponds to $\boxed{1} \otimes \boxed{1} \in \tilde{D}_1(B^{1,1} \otimes B^{1,1})$, which is the splitting of the corresponding KR tableaux in $B^{1,2}$ (the other elements are also Kashiwara–Nakashima tableaux).

4.2. The quantum alcove model

We now recall the quantum alcove model and the main results related to it. For more details, including examples, we refer to the relevant papers [31,35,32]. The setup is that of a finite root system Φ_0 of rank r and its Weyl group W_0 , but it also includes the associated alcove picture. We denote by θ the highest root in Φ_0 , and let $\alpha_0 := -\theta$. Also, let $[m] := \{1, 2, \dots, m\}$ and $\mathfrak{h}_{\mathbb{R}} := \mathbb{R} \otimes P$.

Consider the affine hyperplanes $H_{\beta,k} := \{\lambda \in \mathfrak{h}_{\mathbb{R}} \mid \langle \lambda, \beta^\vee \rangle = k\}$. Recall an *alcove* is a connected component of $\mathfrak{h}_{\mathbb{R}} \setminus \left(\bigcup_{\beta \in \Phi_0} \bigcup_{k \in \mathbb{Z}} H_{\beta,k}\right)$, and the *fundamental alcove* is

$$A_o := \{\lambda \in \mathfrak{h}_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha_i^\vee \rangle < 1 \text{ for all } i \in I_0\}.$$

We say that two alcoves are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves A and B , we write $A \xrightarrow{\beta} B$ if the common wall is contained in the affine hyperplane $H_{\beta,k}$, for some $k \in \mathbb{Z}$, and the root $\beta \in \Phi$ points in the direction from A to B .

An *alcove path* is a sequence of alcoves (A_0, A_1, \dots, A_m) such that A_{j-1} and A_j are adjacent, for $j = 1, \dots, m$. We say that an alcove path is *reduced* if it has minimal length among all alcove paths from A_0 to A_m . Let $A_\lambda = A_o + \lambda$ be the translation of the fundamental alcove A_o by the weight λ .

folded path. Like Π , the folded path can be recorded by a sequence of roots, namely $\Gamma(J) = (\gamma_1, \gamma_2, \dots, \gamma_m)$, where

$$\gamma_k := r_{j_1} r_{j_2} \cdots r_{j_p}(\beta_k), \quad (4.1)$$

with j_p the largest folding position less than k . We define $\gamma_\infty := r_{j_1} r_{j_2} \cdots r_{j_s}(\rho)$. Upon folding, the hyperplane separating the alcoves A_{k-1} and A_k in Π is mapped to

$$H_{|\gamma_k|, -l_k^J} = \widehat{r}_{j_1} \widehat{r}_{j_2} \cdots \widehat{r}_{j_p}(H_{\beta_k, -l_k}), \quad (4.2)$$

for some l_k^J , which is defined by this relation.

Given $i \in J$, we say that i is a *positive folding position* if $\gamma_i > 0$, and a *negative folding position* if $\gamma_i < 0$. We denote the positive folding positions by J^+ , and the negative ones by J^- . We call $\text{wt}(J) := -\widehat{r}_{j_1} \widehat{r}_{j_2} \cdots \widehat{r}_{j_s}(-\lambda)$ the *weight* of J .

A subset $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ (possibly empty) is an *admissible subset* if we have the following path in the quantum Bruhat graph on W_0 :

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \cdots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \cdots r_{j_s} =: \phi(J). \quad (4.3)$$

We call $\Gamma(J)$ an *admissible folding* and $\phi(J)$ its *final direction*. We let $\mathcal{A}(\Gamma)$ be the collection of admissible subsets.

Remark 4.5. Positive and negative folding positions correspond to up and down steps (in Bruhat order) in the chain (4.3), respectively.

We now define the crystal operators on $\mathcal{A}(\Gamma)$. Given $J \subseteq [m]$ and $\alpha \in \Phi$, we will use the following notation:

$$I_\alpha = I_\alpha(J) := \{i \in [m] \mid \gamma_i = \pm\alpha\}, \quad \widehat{I}_\alpha = \widehat{I}_\alpha(J) := I_\alpha \cup \{\infty\},$$

and $l_\alpha^\infty := \langle \text{wt}(J), \text{sgn}(\alpha)\alpha^\vee \rangle$. The following graphical representation of the heights l_i^J for $i \in I_\alpha$ and l_α^∞ is useful for defining the crystal operators. Let

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n < i_{n+1} = \infty\} \quad \text{and} \quad \epsilon_i := \begin{cases} 1 & \text{if } i \notin J, \\ -1 & \text{if } i \in J. \end{cases}$$

If $\alpha > 0$, we define the continuous piecewise linear function $g_\alpha: [0, n + \frac{1}{2}] \rightarrow \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \text{sgn}(\gamma_{i_k}) & \text{if } x \in (k-1, k - \frac{1}{2}), k = 1, \dots, n, \\ \epsilon_{i_k} \text{sgn}(\gamma_{i_k}) & \text{if } x \in (k - \frac{1}{2}, k), k = 1, \dots, n, \\ \text{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle) & \text{if } x \in (n, n + \frac{1}{2}). \end{cases} \quad (4.4)$$

If $\alpha < 0$, we define g_α to be the graph obtained by reflecting $g_{-\alpha}$ in the x -axis. For any α we have

$$\operatorname{sgn}(\alpha)l_{i_k}^J = g_\alpha\left(k - \frac{1}{2}\right), \quad k = 1, \dots, n, \quad \text{and} \quad \operatorname{sgn}(\alpha)l_\alpha^\infty := \langle \operatorname{wt}(J), \alpha^\vee \rangle = g_\alpha\left(n + \frac{1}{2}\right). \quad (4.5)$$

Let J be an admissible subset. Let $\delta_{i,j}$ be the Kronecker delta function. Fix p in $\{0, \dots, r\}$, so α_p is a simple root if $p > 0$, or $-\theta$ if $p = 0$. Let M be the maximum of g_{α_p} , which is known to be a nonnegative integer. Let $m := \min\{i \in \widehat{I}_{\alpha_p} \mid \operatorname{sgn}(\alpha_p)l_i^J = M\}$. It turns out that, if $M \geq \delta_{p,0}$, then we have either $m \in J$ or $m = \infty$; furthermore, if $M > \delta_{p,0}$, then m has a predecessor k in \widehat{I}_{α_p} and $k \notin J$. We define

$$f_p(J) := \begin{cases} (J \setminus \{m\}) \cup \{k\} & \text{if } M > \delta_{p,0}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

Now we define e_p . Assuming that $M > \langle \operatorname{wt}(J), \alpha_p^\vee \rangle$, let $k := \max\{i \in I_{\alpha_p} \mid \operatorname{sgn}(\alpha_p)l_i^J = M\}$, and let m be the successor of k in \widehat{I}_{α_p} . Assuming also that $M \geq \delta_{p,0}$, it turns out that we have $k \in J$ and either $m \notin J$ or $m = \infty$. Define

$$e_p(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \operatorname{wt}(J), \alpha_p^\vee \rangle \text{ and } M \geq \delta_{p,0} \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

In the above definitions, we use the convention that $J \setminus \{\infty\} = J \cup \{\infty\} = J$.

We recall one of the main results in [35], cf. also [34,32]. In the setup of untwisted affine root systems, consider the tensor product of KR crystals $B = \bigotimes_{j=1}^N B^{p_j,1}$. Let $\lambda = \omega_{p_1} + \dots + \omega_{p_N}$, and let Γ be any λ -chain.

Theorem 4.6 ([35,32]). *The (abstract) crystal $\mathcal{A}(\Gamma)$ is isomorphic to $\widetilde{DD}_1(B)$ via a specific weight-preserving bijection Ψ .*

Based on the above discussion and notation, we give a modified crystal structure on $\mathcal{A}(\Gamma)$ such that the result is isomorphic to $\widetilde{DD}_\ell(B)$. Let $\mathcal{A}_\ell(\Gamma)$ be the set $\mathcal{A}(\Gamma)$ with crystal operators defined by

$$f_p(J) := \begin{cases} (J \setminus \{m\}) \cup \{k\} & \text{if } M > \ell\delta_{p,0}, \\ 0 & \text{otherwise,} \end{cases} \quad (4.8a)$$

$$e_p(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \operatorname{wt}(J), \alpha_p^\vee \rangle \text{ and } M \geq \ell\delta_{p,0}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.8b)$$

In particular, we have $\mathcal{A}_1(\Gamma) = \mathcal{A}(\Gamma)$.

Proposition 4.7. *The map Ψ from Theorem 4.6 restricts to a crystal isomorphism $\Psi_\ell: \mathcal{A}_\ell(\Gamma) \rightarrow \widetilde{DD}_\ell(B)$.*

Proof. It is clear that Ψ_ℓ is a bijection since, as sets, $\mathcal{A}_\ell(\Gamma) = \mathcal{A}(\Gamma)$ and $\widetilde{DD}_\ell(B) = \widetilde{DD}_1(B)$. Thus, it remains to show Ψ_ℓ commutes with the crystal operators.

Assuming $\ell \geq 2$, we have $\widetilde{DD}_\ell(B) = \widetilde{DD}_{\ell-1}(\widetilde{DD}_1(B))$, i.e., the crystal $\widetilde{DD}_\ell(B)$ is obtained from $\widetilde{DD}_1(B)$ by removing the last $\ell - 1$ edges in a 0-string. Let φ_0 be the crystal φ -function for $\widetilde{DD}_1(B)$, see Section 2.1. By [31, Theorem 3.9], we have

$$\varphi_0(J) = \max(M - 1, 0).$$

In $\mathcal{A}(\Gamma)$ we have to redefine as 0 every $f_0(J) \neq 0$ with $\varphi_0(J) \leq \ell - 1$, but this condition is equivalent to $M \leq \ell$. Similarly, we have to redefine as 0 every $e_0(J) \neq 0$ with $\varphi_0(J) \leq \ell - 2$, but this condition is equivalent to $M < \ell$. The fact that the crystal operators f_p and e_p in $\mathcal{A}_\ell(\Gamma)$ commute with Ψ_ℓ now follows from (4.6) and (4.7), respectively. \square

Remarks 4.8.

- (1) Using the setup of Corollary 4.1, let $\lambda = \sum_{j=1}^N s_j \omega_{r_j}$, and let Γ be any λ -chain. By the mentioned corollary and the above discussion, we can realize $\widetilde{DD}_\ell(B)$ as the connected component of the admissible subset $J = \emptyset$ in $\mathcal{A}_\ell(\Gamma)$.
- (2) In Remark 4.2, we conjectured that Corollary 4.1 extends to the exceptional types. As the quantum alcove model applies to single-column KR crystals of any untwisted affine type, we would obtain a uniform model for all (level ℓ dual Demazure portions of) tensor products of perfect KR crystals with a fixed level (in the mentioned types).

The following natural question arises.

Problem 4.9. How are the non-level ℓ dual Demazure arrows realized in the quantum alcove model?

In the quantum alcove model for tensor products of single column KR crystals, it is expected that the extra 0-arrows will be slightly more involved. In particular, we see that the p -arrows currently given by (4.6) and (4.7) change only one element in the admissible subsets. It was observed that the extra 0-arrows change more than one entry, but there is no precise conjecture currently. On the other hand, for single columns all the 0-arrows are described in the closely related quantum LS path model, which has been bijected to the quantum alcove model [35]. So it would be interesting to see which of the two models would be better suited for describing the non-level ℓ (dual) Demazure arrows.

5. Conjectures for Q -systems

Recall the Q -system relations (we refer the reader to [30] and references therein):

$$\left(Q_m^{(a)}\right)^2 = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b \sim a} \prod_{k=0}^{-A_{ab}-1} Q_{\left\lfloor \frac{mA_{ba}-k}{A_{ab}} \right\rfloor}^{(b)}, \quad (5.1a)$$

$$\left(Q_m^{(a)}\right)^2 = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b \sim a} (Q_m^{(b)})^{-A_{ba}}, \quad (5.1b)$$

where (5.1a) is for the untwisted Q -system and (5.1b) is for the twisted Q -system.

Conjecture 5.1. Fix some $a \in I_0$, and let $c = \min\{c_b \mid A_{ba} \neq 0\}$. Let $\ell \geq \lceil m/c \rceil$, then we have

$$\tilde{D}_\ell((B^{a,m-1})^{\otimes 2}) \cong \tilde{D}_\ell(B^{a,m} \otimes B^{a,m-2}) \oplus \tilde{D}_\ell\left(\bigotimes_{b \sim a} \bigotimes_{k=0}^{-A_{ab}-1} B^{b,L(k)-1}\right), \quad (5.2a)$$

$$\tilde{D}_\ell((B^{a,m-1})^{\otimes 2}) \cong \tilde{D}_\ell(B^{a,m} \otimes B^{a,m-2}) \oplus \tilde{D}_\ell\left(\bigotimes_{b \sim a} (B^{b,m-1})^{\otimes -A_{ba}}\right), \quad (5.2b)$$

where (5.2a) is for the untwisted types, (5.2b) is for the twisted types, and $L(k) = \left\lfloor \frac{mA_{ba}-k}{A_{ab}} \right\rfloor$.

Note that it is sufficient to prove the case when $\ell \geq \lceil m/c \rceil$.

Conjecture 5.1 is a crystal theoretic interpretation of (5.1) (with renormalized indices). We note that if we branch to $U_q(\mathfrak{g}_0)$ -crystals (or if we took $\ell \gg 1$), then Conjecture 5.1 becomes precisely the statement that classical characters of KR crystals satisfy the Q -system [13]. Thus, Conjecture 5.1 is a strengthening of [13].

Theorem 3.2 says that the crystals in Conjecture 5.1 are isomorphic to a disjoint union of Demazure crystals. Theorem 3.8 implies that the components containing the maximal/minimal elements are isomorphic. However, one would need to precisely enumerate *all* connected components and check their maximal and minimal weights in order to show Conjecture 5.1. We have verified Conjecture 5.1 on a number of examples in different types by using SAGEMATH [51].

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