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The Kazhdan–Lusztig Basis and the Temperley–Lieb Quotient in Type D

Jozsef Losonczy

Mathematics Department, Long Island University, Brookville, New York 11548
E-mail: losonczy@e-math.ams.org

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Let $\mathscr{H}$ be a Hecke algebra associated with a Coxeter system of type D, and let $\mathscr{TL}$ be the corresponding Temperley–Lieb quotient. The algebra $\mathscr{TL}$ admits a canonical basis, which facilitates the construction of irreducible representations. In this paper, we explain the relationship between the canonical basis of $\mathscr{TL}$ and the Kazhdan–Lusztig basis of $\mathscr{H}$.

Key Words: canonical basis; Coxeter group; Hecke algebra; Kazhdan–Lusztig basis; Temperley–Lieb algebra.

1. INTRODUCTION

Let X be a Coxeter graph and let $W(X)$ be an associated Coxeter group with Coxeter generators $S(X)$ and length function ℓ. Let $\mathscr{H}(X)$ be the corresponding Hecke algebra. This is an associative, unital algebra over the ring $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials. The Hecke algebra $\mathscr{H}(X)$ has generators T_s, one for each $s \in S(X)$, which are subject to the following relations: $T_s^2 = (q-1)T_s + q$, where $q = v^2$; $(T_sT_{s'})^m = (T_{s'}T_s)^m$ if ss' has order $2m$; and $(T_sT_{s'})^m T_s = (T_{s'}T_s)^m T_{s'}$ if ss' has order $2m+1$. When there is no need to specify the underlying Coxeter graph X, we sometimes simplify notation by writing W, S, and $\mathscr{H}$ for the Coxeter group, its distinguished set of generators, and the corresponding Hecke algebra.

The algebra $\mathscr{H}$ has an $\mathscr{A}$-basis $\{T_w : w \in W\}$, where T_w is defined to be the product $T_{s_1}T_{s_2} \cdots T_{s_n}$ for any reduced expression $s_1 s_2 \cdots s_n$ equal to w. (A product $w_1 w_2 \cdots w_n$ of elements from W is called *reduced* if $\ell(w_1 w_2 \cdots w_n) = \sum_i \ell(w_i)$.) The presentation for $\mathscr{H}$ given above ensures that the T_w are well defined.

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Let $\mathscr{I}(X)$ be the two-sided ideal of $\mathscr{H}(X)$ generated by the elements $\sum_{x \in \langle s, s' \rangle} T_x$, where (s, s') runs over all pairs of noncommuting Coxeter generators such that the subgroup $\langle s, s' \rangle$ of $W(X)$ is finite. Define $\mathscr{TL}(X)$ to be the quotient $\mathscr{A}$-algebra $\mathscr{H}(X)/\mathscr{I}(X)$ and let θ denote the canonical map from $\mathscr{H}(X)$ to $\mathscr{TL}(X)$. When X is a Coxeter graph of type A, the quotient $\mathscr{TL}(X)$ is known as the Temperley–Lieb algebra, which emerged from the paper [14] and which has been studied in the context of knot theory by Jones [11].

When X is an arbitrary Coxeter graph, the quotient $\mathscr{TL}(X)$ is sometimes called a generalized Temperley–Lieb algebra. Graham [6] has classified the graphs X for which $\mathscr{TL}(X)$ is finite-dimensional; these graphs fall into seven infinite families, denoted by A, B, D, E, F, H, and I.

A canonical basis for $\mathscr{TL} = \mathscr{TL}(X)$ was introduced in [8]. This basis is defined in a manner similar to that of the Kazhdan–Lusztig basis, relative to a lattice and an involution, and it is uniquely determined by these data together with a pair of conditions (Theorem 2.3 gives a precise statement). When X is simply-laced and $\mathscr{TL}(X)$ is finite-dimensional, the canonical basis can be used to construct the irreducible representations of $\mathscr{TL}$ (see the work of Fan [4]). Various examples suggest the possibility that the canonical basis of $\mathscr{TL}$ can be obtained from a particular subset of the Kazhdan–Lusztig basis by projection to the quotient; in fact, such a relationship is known to exist when the underlying graph is of type A, B, or I [5, 8, 9].

We will show (Theorem 3.4) that the projection relationship described above holds in type D. The presence of a branch node in this context introduces some interesting complications. For example, whereas in type A the kernel of θ is spanned by the Kazhdan–Lusztig basis elements that it contains, this is not true in type D.

Our arguments will rely on some general properties of reduced expressions and on the particular nature of minimum length coset representatives in type D, all of which will be described in Section 3. We remark that our proof of Theorem 3.4 gives the corresponding type A result as a special case.

2. CANONICAL BASES

In this section, the underlying Coxeter graph X is of arbitrary type.

Let $\mathscr{A}^- = \mathbb{Z}[v^{-1}]$. The Hecke algebra $\mathscr{H}$ admits a $\mathbb{Z}$-linear ring automorphism of order 2 that sends v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$; this involution is denoted by $h \mapsto \overline{h}$. Kazhdan and Lusztig [12, Theorem 1.1] have shown that for each $w \in W$, there exists a unique element $C'_w \in \mathscr{H}$ such that $\overline{C'_w} = C'_w$ and

$$C'_w = \sum_{\substack{x \in W \\ x \leq w}} v^{-\ell(x)} \tilde{P}_{x,w} T_x,$$

where $\tilde{P}_{x,w} \in v^{-1}\mathscr{A}^-$ if $x < w$, and $\tilde{P}_{w,w} = 1$. Here, $\leq$ denotes the Bruhat–Chevalley partial ordering on W. The set $\{C'_w : w \in W\}$ is known as the Kazhdan–Lusztig basis of $\mathscr{H}$.

The algebra $\mathscr{TL}$ has a canonical basis, which was introduced in [8]; a few additional definitions are necessary in order to describe it.

DEFINITION 2.1. We say that an element $w \in W(X)$ is *complex* if it can be expressed as a reduced product $w_1 w_P w_2$, where w_P is the longest element of some parabolic subgroup P generated by a pair of noncommuting elements $s, s' \in S(X)$. Let $W_c = W_c(X)$ denote the set of all $w \in W(X)$ that are not complex.

For any $w \in W$, let t_w denote the image of T_w in $\mathscr{TL}$.

THEOREM 2.2 [6, Theorem 6.2]. *The set $\{t_w : w \in W_c\}$ is an $\mathscr{A}$-basis for the generalized Temperley–Lieb algebra $\mathscr{TL}$.*

The basis arising from Theorem 2.2 is sometimes called the *t-basis*. It plays a role in the definition of the canonical basis for $\mathscr{TL}$.

The $\mathbb{Z}$-linear ring involution $h \mapsto \overline{h}$ of $\mathscr{H}$ induces an involution of $\mathscr{TL}$ (see [8, Lemma 1.4]). We use the bar notation to represent this involution of $\mathscr{TL}$, which is given by $\overline{\sum_{w \in W_c} a_w t_w} = \sum_{w \in W_c} \overline{a_w} t_{w^{-1}}^{-1}$.

Let $\mathscr{L}$ denote the free $\mathscr{A}^-$-submodule of $\mathscr{TL}$ with basis $\{v^{-\ell(w)} t_w : w \in W_c\}$, and let $\pi: \mathscr{L} \to \mathscr{L}/v^{-1}\mathscr{L}$ be the canonical projection.

THEOREM 2.3 [8, Theorem 2.3]. *There exists a unique basis $\{c_w : w \in W_c\}$ for the lattice $\mathscr{L}$ such that $\overline{c_w} = c_w$ and $\pi(c_w) = \pi(v^{-\ell(w)} t_w)$ for all $w \in W_c$.*

DEFINITION 2.4. The basis $\{c_w : w \in W_c\}$ arising from Theorem 2.3 will be called the canonical basis of $\mathscr{TL}$.

We remark that the canonical basis of $\mathscr{TL}$ is an IC basis, as defined by Du in [1, Sect. 1.1]. In [8], the canonical basis of $\mathscr{TL}(X)$ was described for all graphs X of type A, D, or E, and the relationship between the canonical basis and the corresponding Kazhdan–Lusztig basis was discussed for graphs of type A.

Let $\mathscr{C} = \mathscr{C}(X)$ denote the set of all $C'_w \in \mathscr{H}(X)$ indexed by $w \in W_c(X)$. One sees from [8, Lemma 1.5] that the set $\theta(\mathscr{C})$ is a basis for $\mathscr{TL}(X)$. Note that the elements of $\theta(\mathscr{C})$ are fixed by the involution of $\mathscr{TL}(X)$ from above. It is natural to consider the question of whether $\theta(\mathscr{C})$ equals the canonical basis of $\mathcal{TL}(X)$. When these bases do coincide, we say that the graph X possesses the *projection property*. It is known from [5, Theorem 3.8.2] together with [8, Theorem 3.6] that graphs of type A possess the projection property; it was shown in [9] that the property also holds for graphs of type B or I.

Consider the situation where the ideal $\mathcal{J}(X)$ is spanned by the Kazhdan–Lusztig basis elements that it contains. This is equivalent to the condition that $C'_w \in \mathcal{J}(X)$ for all $w \notin W_c(X)$. The graph X must then possess the projection property [8, Proposition 1.2.3]. While this is a useful fact in the type A setting, it is not helpful for studying the case where X is of type D, as the following example demonstrates.

EXAMPLE 2.5. Take the underlying Coxeter graph X to be of type D_4. Denote the Coxeter generators by σ_1, σ_2, σ_3, σ_4, where σ_3 corresponds to the branch node. We claim that $\mathcal{J} = \mathcal{J}(X)$ is not spanned by the Kazhdan–Lusztig basis elements that it contains. Assume the contrary. Then $C'_w \in \mathcal{J}$, where $w = \sigma_2\sigma_3\sigma_4\sigma_3\sigma_1\sigma_2\sigma_3 \notin W_c$. Hence, $C'_{\sigma_1}C'_w \in \mathcal{J}$. But when the product $C'_{\sigma_1}C'_w$ is expressed as an $\mathcal{A}$-linear combination of Kazhdan–Lusztig basis elements (using the identity $C'_{\sigma_1}C'_w = C'_{\sigma_1 w} + \sum \mu(z,w)C'_z$, where the sum is over all z such that $\sigma_1 z < z < w$ and $\mu(z,w)$ is the coefficient of $v^{\ell(w)-\ell(z)-1}$ in the Kazhdan–Lusztig polynomial $P_{z,w} = v^{\ell(w)-\ell(z)}\tilde{P}_{z,w}$), the element C'_x, where $x = \sigma_1\sigma_2\sigma_4\sigma_3$, appears with integer coefficient 1. This means that some nonzero $\mathcal{A}$-linear combination of elements from $\mathcal{C}$ belongs to $\mathcal{J}$, contradicting the fact that $\theta(\mathcal{C})$ is an $\mathcal{A}$-basis for $\mathcal{TL}$. Note that this example applies also to the situation where X is of type D_r, for $r > 4$.

Because of the phenomenon described in Example 2.5, we need a different condition for establishing that $\theta(\mathcal{C})$ equals the canonical basis. The following proposition provides us with such a condition. It will be used later in Section 3.

PROPOSITION 2.6 [9, Proposition 1.2.2]. *If $v^{-\ell(w)}t_w \in \mathcal{L}$ for all $w \in W(X)$, then $\theta(\mathcal{C})$ coincides with the canonical basis of $\mathcal{TL}(X)$.*

It is necessary to introduce one more basis for the algebra $\mathcal{TL}$.

DEFINITION 2.7. Define, for each $s \in S$, $b_s = v^{-1}t_s + v^{-1}$. More generally, for each $w \in W_c$, it makes sense to define $b_w = b_{s_1}b_{s_2}\cdots b_{s_n}$, where $s_1 s_2 \cdots s_n$ is any reduced expression for w. It is known (and follows from [6, Theorem 6.2]) that the set $\{b_w : w \in W_c\}$ is an $\mathcal{A}$-basis for $\mathcal{TL}$. We call it the *monomial basis*.

When X is a graph of type A, D, or E, the canonical basis for $\mathcal{TL}(X)$ equals the monomial basis [8, Theorem 3.6]. We mention in passing that if

X is non-simply-laced, then the canonical basis of $\mathscr{TL}(X)$ does not equal the monomial basis [8, Remark 3.7 (1)].

3. TYPE D

In this section, we restrict our attention to Coxeter graphs of type D_r. Our goal is to prove that the canonical basis of $\mathscr{TL}(D_r)$ equals the image under θ of the set of all Kazhdan–Lusztig basis elements $C'_w \in \mathscr{H}(D_r)$ indexed by $w \in W_c(D_r)$.

It is known (see [2; 7, Sect. 1]) that the algebra $\mathscr{TL}(D_r)$ is generated by the monomial basis elements b_s, with s ranging over $S(D_r)$, subject to the following relations: $b_s^2 = q_c b_s$, where $q_c = v + v^{-1}$; $b_s b_{s'} = b_{s'} b_s$ if ss' has order 2; and $b_s b_{s'} b_s = b_s$ if ss' has order 3.

Definition 3.1. For any $w \in W$, we define $c(w)$ to be the set of Coxeter generators $s \in S$ that appear in some (any) reduced expression for w. We call $c(w)$ the *content* of w. We define $\mathscr{R}(w)$ to be the set of all $s \in S$ such that $\ell(ws) < \ell(w)$. We call $\mathscr{R}(w)$ the *right descent* of w.

Let $\sigma_1, \sigma_2, \ldots, \sigma_r$ denote the Coxeter generators of the Coxeter group $W(D_r)$, indexed so that each of the products $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, and $\sigma_i\sigma_{i+1}$ $(i > 2)$ has order 3. Thus, one has $W(D_0) = \{e\}$, $W(D_1) = W(A_1)$, $W(D_2) = W(A_1) \times W(A_1)$, and $W(D_3) = W(A_3)$. Let $W^{(r)}$ denote the set $\{w \in W(D_r) : 1 \le i < r \Rightarrow \ell(\sigma_i w) > \ell(w)\}$. Then $W^{(r)}$ is a system of right coset representatives for the subgroup $W(D_{r-1})$ of $W(D_r)$, and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in W(D_{r-1})$ and $y \in W^{(r)}$; thus, each $y \in W^{(r)}$ is the unique element of minimum length in $W(D_{r-1})y$ (see [10, Sect. 5.12]).

The sets $W^{(r)}$ have a simple description. One has $W^{(1)} = \{e, \sigma_1\}$, $W^{(2)} = \{e, \sigma_2\}$, $W^{(3)} = \{e, \sigma_3, \sigma_3\sigma_2, \sigma_3\sigma_1, \sigma_3\sigma_2\sigma_1, \sigma_3\sigma_2\sigma_1\sigma_3\}$, and for $r > 3$ the elements of $W^{(r)}$ are given by

$$\{e, \sigma_r, \sigma_r\sigma_{r-1}, \ldots, \sigma_r\sigma_{r-1} \cdots \sigma_3\sigma_2, \sigma_r\sigma_{r-1} \cdots \sigma_3\sigma_1, \sigma_r\sigma_{r-1} \cdots \sigma_3\sigma_2\sigma_1,$$
$$\sigma_r\sigma_{r-1} \cdots \sigma_3\sigma_2\sigma_1\sigma_3, \ldots, \sigma_r\sigma_{r-1} \cdots \sigma_3\sigma_2\sigma_1\sigma_3 \cdots \sigma_{r-1}\sigma_r\}.$$

Observe that each $y \in W^{(r)}$ has either a unique reduced expression, if σ_1, σ_2 do not both belong to $c(y)$, or else y has two reduced expressions. The reduced expression that does not contain the subexpression $\sigma_1\sigma_2$ will be called the *normal* reduced expression for the minimum length coset representative.

One can express any $w \in W(D_r)$ uniquely as a reduced product $w = w_1 w_2 \cdots w_r$, where each $w_i \in W^{(i)}$. (To see uniqueness, let $w'_1 w'_2 \cdots w'_r$ be another such product. The products $w_1 w_2 \cdots w_{r-1}$ and $w'_1 w'_2 \cdots w'_{r-1}$ both belong to $W(D_{r-1})$; hence the right coset of $W(D_{r-1})$ in $W(D_r)$ that contains $w = w_1 w_2 \cdots w_r = w'_1 w'_2 \cdots w'_r$ must also contain w_r and w'_r. Since both of these are elements of the system of coset representatives $W^{(r)}$, we have $w_r = w'_r$. Iterating this argument, we find that $w_i = w'_i$ for all $i = 1, 2, \ldots, r$.) By deleting those w_i that equal the identity and replacing each of the remaining w_i with its normal reduced expression, we obtain a normal reduced expression $s_1 s_2 \cdots s_n$ for w. The normal reduced expression has the following property: for each i, either s_i does not appear to the left of the ith position in $s_1 s_2 \cdots s_n$, or s_{i-1}, s_i do not commute, or $s_i = \sigma_1$ and $s_{i-1} = \sigma_2$.

There are some properties of reduced expressions for elements in W that are useful for proving results about multiplication in the generalized Temperley–Lieb algebra. Perhaps the most fundamental is a well-known theorem of Tits, which states that for any $w \in W$, every reduced expression for w can be transformed into any other reduced expression for w by performing a sequence of braid moves (see [15, Théorème 3]). This is valid for an arbitrary Coxeter system. Using this result, one can characterize W_c as the set of $w \in W$ such that every reduced expression for w can be transformed into any other reduced expression for w by performing a sequence of commutation moves [13, Proposition 1.1]. One sometimes calls W_c the set of *fully commutative* elements of W.

There are two additional properties, peculiar to the simply-laced case, which play a role in our work on type D. We list these below. Both have previously appeared in the paper [4, Sect. 2]. Note that the first property can be obtained as a corollary to the theorem of Tits cited above; for a proof of the second property, see [3, Lemma 2].

Property R1. Let $s_1, s_2, \ldots, s_m$ be an arbitrary sequence from S. Then the product $s_1 s_2 \cdots s_m$ is reduced and belongs to W_c if and only if between any two occurrences of a generator s in the sequence, there exist at least two occurrences of generators which do not commute with s.

Property R2. Let $w \in W_c$ and $s \in S$ satisfy $ws \notin W_c$. Then there exists a unique $s' \in S$ such that any reduced expression for w can be parsed as follows: $w = w_1 s w_2 s' w_3$, where ss' has order 3 and s commutes with every member of $c(w_2) \cup c(w_3)$.

Let $w \in W(D_r)$ and let $s_1 s_2 \cdots s_m$ be an arbitrary reduced expression for w. Let $1 \leq i(1) < i(2) < \cdots < i(k) \leq m$. Then the product $b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}}$ equals $q_c^{\mu} b_{w'}$, for some nonnegative integer μ and some $w' \in W_c$; moreover, we have $\ell(w') \leq k$ and $w' \leq w$ and $s_{i(k)} \in \mathscr{R}(w')$. One can establish this by a straightforward induction on k, using Property R2, the subexpression characterization of Bruhat–Chevalley order, and the presentation of $\mathscr{TL}(D_r)$ given at the beginning of the section. The above fact will be invoked in the sequel.

The following lemma is known to hold for any Coxeter graph of type A, D, or E [4, Proposition 5.4.1]. It is possible to give a relatively simple, self-contained proof for the type D case, as we do below.

LEMMA 3.2. *Let $w \in W_c$ and $s \in S$. Then $b_w b_s = q_c^{\mu} b_{w'}$, where $w' \in W_c$ and μ equals 0 or 1. If $\ell(ws) > \ell(w)$, then $\mu = 0$. If $\ell(ws) < \ell(w)$, then $w' = w$ and $\mu = 1$.*

Proof. Observe that the last assertion follows immediately from associativity of multiplication in $\mathcal{TL}$ and the relation $b_s^2 = q_c b_s$.

So assume $\ell(ws) > \ell(w)$. If $ws \in W_c$, then the definition of the monomial basis gives $b_w b_s = b_{ws}$. We are left with the case $ws \notin W_c$, which we treat by induction on $n = \ell(w) \geq 2$. The basis for induction holds, so let $n > 2$ and let $s \in S$ satisfy $ws \notin W_c$. Let $s_1 s_2 \cdots s_n$ be the normal reduced expression for w. We parse it according to Property R2: $w = w_1 s w_2 s' w_3$, where ss' has order 3 and s commutes with every member of $c(w_2) \cup c(w_3)$. We may assume $w_3 \neq e$; otherwise, $b_w b_s = b_{w_1 w_2} b_s b_{s'} b_s = b_{w_1 w_2} b_s = b_{w_1 s w_2}$, and the inductive step follows. Thus, we have $w_3 = s_j s_{j+1} \cdots s_n$ for some $j \leq n$, and $b_w b_s = b_{u_1} b_{s_j} b_{s_{j+1}} \cdots b_{s_n}$, where $u_1 = w_1 s w_2$. Observe that $u_1 \leq s_1 s_2 \cdots s_{j-1}$.

We have $s_j \notin \mathcal{R}(u_1)$ (otherwise, u_1 has a reduced expression ending in s_j; hence the product $w_1 s w_2 s' s_j \cdots s_n$ is either not reduced, or else it does not belong to W_c, a contradiction). If $u_1 s_j \in W_c$, then we put $u_2 = u_1 s_j$. If $u_1 s_j \notin W_c$, then we apply the inductive hypothesis, obtaining $b_{u_1} b_{s_j} = b_{u'}$ for some $u' \in W_c$; in this case, we put $u_2 = u'$. In either case, the element u_2 has a reduced expression ending in s_j (this is a consequence of the paragraph that immediately follows Properties R1 and R2 above) and $u_2 \leq s_1 s_2 \cdots s_j$. We may now consider $b_w b_s = b_{u_2} b_{s_{j+1}} b_{s_{j+2}} \cdots b_{s_n}$. We claim that $s_{j+1} \notin \mathcal{R}(u_2)$.

Since $s_1 s_2 \cdots s_n$ is the normal reduced expression for w, either (1) s_{j+1} does not occur to the left of the $(j+1)$th position in $s_1 s_2 \cdots s_n$, in which case $s_{j+1} \notin c(u_2)$; or (2) s_j, s_{j+1} do not commute; or (3) $s_{j+1} = \sigma_1$ and $s_j = \sigma_2$. If either (1) or (2) holds, then it is clear that $s_{j+1} \notin \mathcal{R}(u_2)$. Suppose that (3) holds.

Now, any reduced expression $s'_1 s'_2 \cdots s'_m$ belonging to W_c has the property that, if $\sigma_2 \sigma_1$ is a consecutive subexpression, say $s'_i = \sigma_2$ and $s'_{i+1} = \sigma_1$, then neither σ_1 nor σ_2 appears to the left of the ith position in $s'_1 s'_2 \cdots s'_m$. (To see why, consider a minimum length counterexample and use Property R1 repeatedly to derive a contradiction.) Applying this to our normal reduced expression $s_1 s_2 \cdots s_n$, we have $\sigma_1 \notin c(s_1 s_2 \cdots s_j)$; hence $\sigma_1 \notin c(u_2)$. This implies that $s_{j+1} \notin \mathcal{R}(u_2)$. If $u_2 s_{j+1} \in W_c$, then we put $u_3 = u_2 s_{j+1}$. If $u_2 s_{j+1} \notin W_c$, then we apply the inductive hypothesis, obtaining $b_{u_2} b_{s_{j+1}} = b_{u''}$ for some $u'' \in W_c$; in this case, we put $u_3 = u''$.

Observe that in either case, u_3 has a reduced expression ending in s_{j+1} and $u_3 \leq s_1 s_2 \cdots s_{j+1}$. We may now consider $b_w b_s = b_{u_3} b_{s_{j+2}} b_{s_{j+3}} \cdots b_{s_n}$.

Iterating, we eventually find that $b_w b_s = b_{u_{n-j+2}}$, for some $u_{n-j+2} \in W_c$. The inductive step is complete. ▮

Remark 3.3. One can also prove Lemma 3.2 by using the diagram calculus for $\mathcal{TL}(D_r)$ developed by Green in [7].

The following theorem reconciles the Kazhdan–Lusztig basis for $\mathcal{H}(D_r)$ with the canonical basis of $\mathcal{TL}(D_r)$ (cf. [5, Theorem 3.8.2; 9, Theorem 2.2.1]).

THEOREM 3.4. *Let X be a Coxeter graph of type D. Then the canonical basis of $\mathcal{TL}(X)$ equals the image under θ of the set of all Kazhdan–Lusztig basis elements $C'_w \in \mathcal{H}(X)$ indexed by $w \in W_c(X)$.*

Proof. We shall verify that for every $w \in W$, the element $v^{-\ell(w)} t_w$ lies in the lattice $\mathcal{L}$. An application of Proposition 2.6 then gives the theorem.

As a first step, we prove by induction on $n = \ell(w) \geq 0$ that $v^{-\ell(w)} t_w$ equals a linear combination of monomial basis elements b_x ($x \in W_c$ and $x \leq w$) with coefficients in $\mathcal{A}^-$, and if $w \in W_c$ then the coefficient of b_w is 1. (It is known that for any $w \in W_c$, the element $v^{-\ell(w)} t_w - b_w$ is a linear combination of monomial elements b_x with coefficients in $v^{-1}\mathcal{A}^-$ [8, Lemma 3.5]; thus, we are going to prove a weaker statement for the more general case where w is not necessarily in W_c.)

If $n = 0$ then $w = e$, and $v^{-\ell(e)} t_e = b_e$. If $n = 1$ then w is a Coxeter generator, and $v^{-\ell(w)} t_w = b_w - v^{-1} b_e$. Let $n > 1$. Let $r > 1$ be the smallest integer such that $w \in W(D_r)$. Write $w = yz$ (reduced), where $y \in W(D_{r-1})$ and $z \in W^{(r)}$. We apply the inductive hypothesis to $v^{-\ell(y)} t_y$, writing it as a linear combination of b_x ($x \in W_c$ and $x \leq y$) with coefficients in $\mathcal{A}^-$; if $y \in W_c$ then the coefficient of b_y is 1. Now consider, for any $x \in W_c$ satisfying $x \leq y$, the expression $b_x(v^{-\ell(z)} t_z)$. We shall show that this equals a linear combination of $b_{x'}$ ($x' \in W_c$ and $x' \leq w$) with coefficients in $\mathcal{A}^-$, and when $w \in W_c$, the coefficient of b_w is 1 if $x = y$ and is 0 if $x \neq y$. The inductive step will thereby be established.

Let $s_1 s_2 \cdots s_m$ be the normal reduced expression for the minimum length coset representative z (where $s_1 = \sigma_r$, $s_2 = \sigma_{r-1}, \ldots$). We have

$$\begin{aligned} b_x\left(v^{-\ell(z)} t_z\right) &= b_x\left(v^{-1} t_{s_1}\right)\left(v^{-1} t_{s_2}\right) \cdots \left(v^{-1} t_{s_m}\right) \\ &= b_x\left(b_{s_1} - v^{-1}\right)\left(b_{s_2} - v^{-1}\right) \cdots \left(b_{s_m} - v^{-1}\right), \end{aligned}$$

and the last expression expands to a sum of terms

$$(-v)^{k-m} b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}}.$$

Note that when $w \in W_c$, the only term which contributes a nonzero coefficient to b_w is the one where $x = y$ and $k = m$; the contributed coefficient is 1.

The remaining part of the inductive step rests on the following claim, the proof of which will be given in the following section.

Claim 3.5. Let $x \in W(D_{r-1})$ and $z \in W^{(r)} \setminus \{e\}$. Let $s_1 s_2 \cdots s_m$ be the normal reduced expression for z. Define $\mathscr{M}$ to be the collection of all k-tuples $I = (i(1), i(2), \ldots, i(k))$ of integers ($k > 0$ varies) that satisfy $1 \leq i(1) < i(2) < \cdots < i(k) \leq m$ together with the following condition:

$$b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}} = q_c^{\mu} b_{x'}, \qquad \textit{where} \quad x' \in W_c \quad \textit{and} \quad \mu > m - k.$$

Suppose $\mathscr{M} \neq \emptyset$. Then the following statements hold:

1. For any $I = (i(1), i(2), \ldots, i(k)) \in \mathscr{M}$, when we write $b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}} = q_c^{\mu} b_{x'}$, we have $\mu = m - k + 1$.

2. There exists $x'' \in W_c$ such that $b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}} = q_c^{\mu(I)} b_{x''}$ for any $I = (i(1), i(2), \ldots, i(k)) \in \mathscr{M}$.

3. The sets $\{I \in \mathscr{M} : |I| \text{ is odd}\}$ and $\{I \in \mathscr{M} : |I| \text{ is even}\}$ have the same cardinality. Here, $|I|$ denotes the number of entries in I.

We explain the relevance of Claim 3.5. The product $b_x(v^{-\ell(z)} t_z)$ is to be expressed as an $\mathscr{A}^-$-linear combination of monomial basis elements. Above, we have expanded $b_x(v^{-\ell(z)} t_z)$ into a sum of terms $(-v)^{k-m} b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}}$, where the sum is taken over all multi-indices $I = (i(1), i(2), \ldots, i(k))$ satisfying $1 \leq i(1) < i(2) < \cdots < i(k) \leq m$. We may ignore those multi-indices I which do not belong to $\mathscr{M}$, since such I must contribute a term with coefficient in $\mathscr{A}^-$. On the other hand, by the various assertions of the claim, if $\mathscr{M}$ is nonempty, then the terms $(-v)^{k-m} b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}}$ arising from $I \in \mathscr{M}$ equal $(-v)^{|I|-m} q_c^{m-|I|+1} b_{x''}$ and sum to $a b_{x''}$, where $a \in \mathscr{A}^-$ (note that the highest degree term of the Laurent polynomial $(-v)^{|I|-m} q_c^{m-|I|+1}$ is $(-1)^{|I|-m} v$, so that the positive degree terms involved in a cancel by the third assertion of Claim 3.5). The inductive step is complete.

We have shown, granting the truth of the claim, that for any $w \in W$, the element $v^{-\ell(w)} t_w$ equals a linear combination of b_x ($x \in W_c$ and $x \leq w$) with coefficients in $\mathscr{A}^-$, and if $w \in W_c$ then the coefficient of b_w equals 1. It follows by a straightforward induction on length that for any $x \in W_c$, the element b_x equals a linear combination of $v^{-\ell(y)} t_y$ ($y \in W_c$ and $y \leq x$) with coefficients in $\mathscr{A}^-$. We conclude that every $v^{-\ell(w)} t_w$ lies in $\mathscr{L}$. ∎

4. COMBINATORICS

This section is devoted to providing a proof of Claim 3.5. Fix an integer $r > 1$. Fix elements $x \in W(D_{r-1})$ and $z \in W^{(r)} \setminus \{e\}$. Let $s_1 s_2 \cdots s_m$ be the normal reduced expression for z. By a *multi-index*, we shall always mean a k-tuple $I = (i(1), i(2), \ldots, i(k))$ of integers ($k > 0$ varies) satisfying $1 \leq i(1) < i(2) < \cdots < i(k) \leq m$. We call $\{i(1), i(2), \ldots, i(k)\}$ the *underlying set* of I; sometimes, we abuse notation and denote the underlying set also by I (and its cardinality by $|I|$).

Given a multi-index $I = (i(1), i(2), \ldots, i(k))$, we shall sometimes denote the product $b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(k)}}$ by b_I. The set $\{1, 2, \ldots, n\}$ will be denoted by $[n]$.

DEFINITION 4.1. Let $I = (i(1), i(2), \ldots, i(k))$ be a multi-index. Let $l \in [k]$. We say that $i(l)$ *contributes* q_c *to* $b_x b_I$ if $b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(l)}} = q_c \cdot (b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(l-1)}})$. Recall that $q_c = v + v^{-1}$.

DEFINITION 4.2. Define $\mathscr{M}'$ to be the set of all multi-indices $I = (i(1), i(2), \ldots, i(k))$ satisfying the following two conditions:

1. There exists $l \in [k]$ such that $s_{i(l)} = \sigma_1$, $s_{i(l-1)} = \sigma_2$, and the entry $i(l)$ contributes q_c to $b_x b_I$.
2. For all $n \in [m]$, if $n \notin I$ then $n + 1 \in I$ and $n + 1$ contributes q_c to $b_x b_I$.

Note that if $I = (i(1), i(2), \ldots, i(k)) \in \mathscr{M}'$, and the integer l is as in condition 1 of Definition 4.2, then $i(l-1) = i(l) - 1$. Observe also that condition 2 guarantees $m \in I$.

DEFINITION 4.3. Let y be an arbitrary element of W_c and let μ be an arbitrary nonnegative integer. We define $\lfloor q_c^{\mu} b_y \rfloor = y$.

PROPOSITION 4.4. *Let $\mathscr{M}$ and $\mathscr{M}'$ be the collections of multi-indices defined in Claim* 3.5 *and Definition* 4.2, *respectively. We have $\mathscr{M} = \mathscr{M}'$. Also, if the multi-index $I = (i(1), i(2), \ldots, i(k))$ belongs to $\mathscr{M}$ and we write $b_x b_I = q_c^{\mu} b_{x'}$, then $\mu = m - k + 1$.*

Proposition 4.4 establishes the first assertion in Claim 3.5.

Proof. Let $I = (i(1), i(2), \ldots, i(k))$ be a multi-index. We claim that if $n \in [k]$ and $s_{i(n)} \neq \sigma_1$, then $i(n)$ can contribute q_c to $b_x b_I$ only if $i(n) - 1 \notin \{0, i(1), i(2), \ldots, i(k)\}$.

For $n = 1$, if $i(1) = 1$ then $s_{i(1)} \notin c(x)$; hence $i(1)$ does not contribute q_c to $b_x b_I$. If $i(1) > 1$ then clearly $i(1) - 1 \notin \{0, i(1), i(2), \ldots, i(k)\}$. For $1 < n \leq k$, the element $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} \rfloor$ has a reduced expression ending in $s_{i(n-1)}$; hence by Lemma 3.2, $i(n)$ can contribute q_c to $b_x b_I$ only if $s_{i(n-1)}$ commutes with $s_{i(n)}$ (possibly $s_{i(n-1)} = s_{i(n)}$). But $s_{i(n-1)}$ and $s_{i(n)}$ can commute only if $s_{i(n)} = \sigma_1$ or $i(n-1) \neq i(n) - 1$. This establishes the claim.

Thus, when we write $b_x b_I = q_c^{\mu} b_{x'}$, we have $\mu \leq m - k + 1$, with equality if and only if both conditions of Definition 4.2 are satisfied. In particular, we have $\mu > m - k$ if and only if $I \in \mathscr{M}'$. ∎

Henceforth, we shall assume $\mathscr{M} \neq \emptyset$. Note that this implies $r > 2$.

PROPOSITION 4.5. *Let x, z be as above, and let $I = (i(1), i(2), \ldots, i(k)) \in \mathscr{M}$. Then for all $n \in [k]$, we have*

$$\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)}} \rfloor = \lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n)}} \rfloor .$$

In particular, since $m \in I$, we have $\lfloor b_x b_I \rfloor = x''$ for all $I \in \mathscr{M}$, where $x'' = \lfloor b_x b_z \rfloor$.

Proposition 4.5 establishes the second assertion in Claim 3.5.

Proof. We proceed by induction on n. Let $n = 1$. The case where $i(1) = 1$ is trivial. If $i(1) \neq 1$, then $i(1) = 2$ and $i(1)$ contributes q_c to $b_x b_I$. Thus, x can be written as a reduced product ending in s_2, say $x = x' s_2$. But then $b_x b_{s_1} b_{s_2} = b_{x' s_2} b_{s_1} b_{s_2} = b_{x'} b_{s_2} b_{s_1} b_{s_2} = b_{x'} b_{s_2} = b_x$; on the other hand, $b_x b_{s_{i(1)}} = b_x b_{s_2} = b_{x' s_2} b_{s_2} = b_{x'} b_{s_2} b_{s_2} = q_c b_{x'} b_{s_2} = q_c b_x$. The basis for induction is established.

Let $n > 1$. We consider two cases.

Case 1. $i(n) = i(n-1) + 1$. By induction, we have $\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n-1)}} \rfloor = \lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} \rfloor$. But then it immediately follows that

$$\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)-1}} b_{s_{i(n)}} \rfloor = \lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} b_{s_{i(n)}} \rfloor .$$

Case 2. $i(n) = i(n-1) + 2$. Then $i(n)$ contributes q_c to $b_x b_I$. Let y be the element $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} \rfloor$. Then y can be written as a reduced product ending in $s_{i(n)}$, say $y = y' s_{i(n)}$. Note that $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} b_{s_{i(n)}} \rfloor$ equals y. We need to show that $\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)}} \rfloor$ equals y, as well.

The inductive hypothesis and our assumption $i(n-1) = i(n) - 2$ together imply that $\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)-2}} \rfloor = y$; let us write $b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)-2}} = q_c^{\mu} b_y$ for some nonnegative integer μ. Recall from the remark following Definition 4.2 that if $s_{i(l)} = \sigma_1$, then $i(l-1) = i(l) - 1$. The assumption $i(n-1) = i(n) - 2$ therefore implies $s_{i(n)} \neq \sigma_1$; hence $s_{i(n)-1}$ and $s_{i(n)}$ do not commute. Therefore,

$$\begin{aligned} b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)-2}} b_{s_{i(n)-1}} b_{s_{i(n)}} &= q_c^{\mu} b_y b_{s_{i(n)-1}} b_{s_{i(n)}} = q_c^{\mu} b_{y'} b_{s_{i(n)}} b_{s_{i(n)-1}} b_{s_{i(n)}} \\ &= q_c^{\mu} b_{y'} b_{s_{i(n)}} \\ &= q_c^{\mu} b_y . \end{aligned}$$

The inductive step is complete. ∎

Let us regard $\mathscr{M}$ as a partially ordered set, with ordinary set-theoretic inclusion as the partial ordering. In the following definition, we construct a multi-index J. It will be shown that J is the unique minimal element of $\mathscr{M}$, and $|J| < m$. Furthermore, if I is any multi-index satisfying $J \subseteq I \subseteq [m]$, then $I \in \mathscr{M}$. From this, we will be able to deduce that $\mathscr{M}$ has as many elements with even cardinality as it has elements with odd cardinality, thereby establishing the third and final assertion of Claim 3.5.

DEFINITION 4.6. Since $\mathscr{M} \neq \emptyset$, there exists $l \in [m]$ such that $s_l = \sigma_1$ and $s_{l-1} = \sigma_2$ (by Proposition 4.4). We define a multi-index J as follows. If x has a reduced expression ending in s_2, then put $j(1) = 2$; otherwise, put $j(1) = 1$. Suppose $j(1), j(2), \ldots, j(n)$ have been defined, and $j(n) < m$. If $j(n)$ equals $l - 2$ or $l - 1$ or $m - 1$, then put $j(n+1) = j(n) + 1$. If instead $j(n) \neq l-2, l-1, m-1$, then consider $\lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(n)}} \rfloor$. If this member of W_c has a reduced expression ending in $s_{j(n)+2}$, then put $j(n+1) = j(n) + 2$; otherwise, put $j(n+1) = j(n) + 1$. This procedure produces a multi-index $(j(1), j(2), \ldots, j(k'))$, which we denote by J.

PROPOSITION 4.7. *We have* $J \in \mathscr{M}$ *and* $|J| < m$.

Proof. To show $J \in \mathscr{M}$, we use Definition 4.2; observe that we need only verify that J satisfies the last part of the first condition in Definition 4.2. Let $l' \in [k']$ satisfy $s_{j(l')} = \sigma_1$ and $s_{j(l'-1)} = \sigma_2$, and let $l = j(l')$. Note that $j(l'-1) = j(l') - 1 = l - 1$. Let $y = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(l'-1)}} \rfloor$. The argument used in the proof of Proposition 4.5 gives us $y = \lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{j(l'-1)}} \rfloor$, and the latter expression can be rewritten as $\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{l-1}} \rfloor$. We claim that the last expression for y has a reduced expression ending in σ_1. To see why, let $I = (i(1), i(2), \ldots, i(k)) \in \mathscr{M}$, and let n satisfy $s_{i(n)} = \sigma_1$ and $s_{i(n-1)} = \sigma_2$. Recall that $i(n)$ must contribute q_c to $b_x b_I$. Thus, the element $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n-1)}} \rfloor$ has a reduced expression ending in $s_{i(n)}$. By Proposition 4.5 and the equality $i(n-1) = i(n) - 1$, this last element equals $\lfloor b_x b_{s_1} b_{s_2} \cdots b_{s_{i(n)-1}} \rfloor$, and since $i(n) = l$, the claim follows.

So $y = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(l'-1)}} \rfloor$ has a reduced expression ending in $\sigma_1 = s_{j(l')}$. This is equivalent to saying that $j(l')$ contributes q_c to $b_x b_J$. The first part of the proposition is established.

We turn to the inequality $|J| < m$. As above, we let $l \geq 3$ denote the integer such that $s_l = \sigma_1$. There exists an integer $1 < n \leq l - 1$ such that $xs_1 s_2 \cdots s_{n-1}$ is reduced and belongs to W_c, but $xs_1 s_2 \cdots s_n \notin W_c$. (Otherwise, $\lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(l-1)}} \rfloor$ equals the reduced product $xs_1 s_2 \cdots s_{l-2} s_{l-1} = xs_1 s_2 \cdots \sigma_3 \sigma_2 \in W_c$; hence l cannot contribute q_c to $b_x b_J$ by Lemma 3.2, contrary to $J \in \mathscr{M}$.) We claim that $s_n \in \mathscr{R}(xs_1 s_2 \cdots s_{n-2})$.

To establish the claim, apply Property R2 to the situation where $w = xs_1 s_2 \cdots s_{n-1}$ and $s = s_n$; since s_{n-1} and s_n do not commute, the reduced expression $xs_1 s_2 \cdots s_{n-1}$ (replace x with any reduced expression for x)

can be parsed $w_1 s_n w_2 s' w_3$, where $s' = s_{n-1}$, $w_3 = e$, and s_n commutes with every element of $c(w_2)$. Thus, we have $w = w_1 w_2 s_n s_{n-1}$ (reduced); hence $xs_1 s_2 \cdots s_{n-2} = ws_{n-1} = w_1 w_2 s_n$, so that $s_n \in \mathscr{R}(xs_1 s_2 \cdots s_{n-2})$, as claimed.

But then, since $\lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(n-2)}} \rfloor = xs_1 s_2 \cdots s_{n-2}$ (recall that $xs_1 s_2 \cdots s_{n-1}$ is reduced and belongs to W_c), the procedure for constructing J gives $j(n-1) = n$; hence $n - 1 \notin J$. ∎

PROPOSITION 4.8. *We have $J \subseteq I$ for all $I \in \mathscr{M}$.*

Proof. Let $I = (i(1), i(2), \dots, i(k)) \in \mathscr{M}$. We first show that $j(1) \in I$, and then we show that for any $1 \leq n' < k'$, if $j(n') \in I$ then $j(n'+1) \in I$.

If $j(1) = 1$ then by the construction of J, the element x does not have a reduced expression ending in s_2. Hence, $i(1) = 1$ (otherwise, $1 \notin I$; yet 2 cannot contribute q_c to $b_x b_I$). If $j(1) = 2$, then by the construction of J, the element x can be expressed as a reduced product ending in s_2, say $x = x' s_2$. Now assume for contradiction that $2 \notin I$. Then $1, 3 \in I$ ($i(1) = 1$, $i(2) = 3$) and 3 contributes q_c to $b_x b_I$. Since $s_1 = \sigma_r \notin c(x)$, the products $xs_1 = x' s_2 s_1$ are reduced and belong to W_c. Hence, $b_x b_{s_{i(1)}} = b_{x' s_2 s_1}$. Since one of the generators s_1, s_2 does not commute with s_3, the entry $i(2) = 3$ cannot contribute q_c to $b_x b_I$, a contradiction. It follows that $j(1) = 2 \in I$.

Now let $1 \leq n' < k'$, and assume $j(n') \in I$. Let $n \in [k]$ satisfy $i(n) = j(n')$.

If $i(n) + 1 \notin I$, then $i(n) + 2 \in I$ and $i(n+1) = i(n) + 2$ contributes q_c to $b_x b_I$. But then since $\lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(n')}} \rfloor = \lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n)}} \rfloor$ (by Proposition 4.5), the procedure for constructing J ensures that $j(n'+1) = i(n+1) \in I$.

Now suppose $i(n) + 1 \in I$, so that $i(n+1) = i(n) + 1$. If also $i(n) + 2 \in I$, then we must have $j(n'+1) \in I$. Suppose instead $i(n) + 2 \notin I$. If $i(n) + 1 = m$ then $j(n'+1) = m \in I$. If instead $i(n) + 1 < m$, then the assumption $i(n) + 2 \notin I$ and Definition 4.2 together imply $i(n) + 3 \in I$. Thus, $i(n+2) = i(n) + 3$. We claim that $j(n'+1) = j(n') + 1 = i(n+1) \in I$. To see this, assume the contrary, so that $j(n'+1) = j(n') + 2 = i(n) + 2$. Let $l \in [k]$ satisfy $s_{i(l)} = \sigma_1$, $s_{i(l-1)} = \sigma_2$. Recall that $i(l-1) = i(l) - 1$. The assumption $j(n'+1) = j(n') + 2$ forces $j(n') \neq i(l-1) - 1, i(l) - 1$. Hence $i(n) + 1 = j(n') + 1 \neq i(l-1), i(l)$. From this we see that $s_{i(n)+1}$ commutes with neither of the generators $s_{i(n)}, s_{i(n)+2}$. Let $y = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(n')}} \rfloor$. Observe that by the construction of J, the element y has a reduced expression ending in $s_{j(n'+1)} = s_{i(n)+2}$; and since $y = \lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n)}} \rfloor$, we see that y has a reduced expression ending in $s_{i(n)}$. Note that $s_{i(n)} \neq s_{i(n)+2}$. Thus, we can write y as a reduced product $y = y' s_{i(n)} s_{i(n)+2}$.

Since $s_{i(n+1)} = s_{i(n)+1}$ commutes with neither of the generators $s_{i(n)}, s_{i(n)+2}$, the product $ys_{i(n+1)} = y' s_{i(n)} s_{i(n)+2} s_{i(n)+1}$ is reduced and belongs to W_c (by Property R1). Hence, we have $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(n+1)}} \rfloor = y' s_{i(n)} s_{i(n)+2} s_{i(n)+1}$. But then $i(n+2) = i(n) + 3$ cannot contribute q_c to $b_x b_I$, a contradiction. It follows that $j(n'+1) = j(n') + 1 = i(n+1) \in I$. ∎

DEFINITION 4.9. Let $I = (i(1), i(2), \ldots, i(k))$ be a multi-index and let U be a subset of $[m]$. We define $I \circ U$ to be the unique multi-index with underlying set $\{i(1), i(2), \ldots, i(k)\} \cup U$.

PROPOSITION 4.10. *Let U be any subset of $[m]$. Then $J \circ U \in \mathscr{M}$. Combined with the conclusion of Proposition* 4.8, *this gives $\mathscr{M} = \{J \circ U : U \subseteq ([m] \setminus J)\}$. Therefore, the sets $\{I \in \mathscr{M} : |I| \text{ is odd}\}$ and $\{I \in \mathscr{M} : |I| \text{ is even}\}$ have the same cardinality.*

Proposition 4.10 establishes the third assertion in Claim 3.5.

Proof. We may assume $U \subseteq ([m] \setminus J)$. We verify that $J \circ U$ satisfies both conditions in Definition 4.2. Write $J \circ U = (i(1), i(2), \ldots, i(k))$. For each $p' \in [k']$ there exists $p \in [k]$ such that $i(p) = j(p')$. Moreover, we have

$$\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(p)}} \rfloor = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(p')}} \rfloor.$$

This can be proved by induction on p', by mimicking the proof of Proposition 4.5.

We first verify that $J \circ U$ satisfies condition 1 in Definition 4.2. Observe that since $J \subseteq J \circ U$, there exists $l \in [k]$ such that $s_{i(l)} = \sigma_1$ and $s_{i(l-1)} = \sigma_2$. Let $l' \in [k']$ satisfy $j(l') = i(l)$. Note the equalities $j(l'-1) = j(l') - 1 = i(l) - 1 = i(l-1)$. Let $y = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(l'-1)}} \rfloor$. Since $j(l')$ contributes q_c to $b_x b_J$, the element y has a reduced expression ending in $s_{j(l')} = \sigma_1 = s_{i(l)}$. By the first paragraph, $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(l-1)}} \rfloor = y$. Hence, $i(l)$ contributes q_c to $b_x b_{J \circ U}$, so that the first condition in Definition 4.2 holds relative to $J \circ U$.

For the second condition, suppose we are given $n \in [m]$ satisfying $n \notin J \circ U$. Then in particular $n \notin J$; hence $n + 1 \in J \subseteq J \circ U$ and $n + 1$ contributes q_c to $b_x b_J$. We need to show that $n + 1$ contributes q_c to $b_x b_{J \circ U}$. Let $p \in [k]$ and $p' \in [k']$ satisfy $i(p) = n + 1 = j(p')$. Note that if $n > 1$ then $i(p-1) = n - 1 = j(p'-1)$. Let $y' = \lfloor b_x b_{s_{j(1)}} b_{s_{j(2)}} \cdots b_{s_{j(p'-1)}} \rfloor$. Since $n + 1$ contributes q_c to $b_x b_J$, the element y' has a reduced expression ending in s_{n+1}. But $\lfloor b_x b_{s_{i(1)}} b_{s_{i(2)}} \cdots b_{s_{i(p-1)}} \rfloor = y'$ by the first paragraph; hence $i(p) = n + 1$ contributes q_c to $b_x b_{J \circ U}$, as desired.

Finally, we address the last assertion. Recall from Proposition 4.7 that $[m] \setminus J$ is nonempty. Any nonempty finite set has as many subsets of odd cardinality as it has subsets of even cardinality. Since $\mathscr{M}$ consists of the multi-indices $J \circ U$ for $U \subseteq ([m] \setminus J)$, and since $|J \circ U| = |J| + |U|$, we see that $\mathscr{M}$ has as many elements with odd cardinality as it has elements with even cardinality. ∎

We have established all of the assertions made in Claim 3.5.

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REFERENCES

1. J. Du, IC bases and quantum linear groups, *in* "Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods," Proc. Sympos. Pure Math., Vol. 56, Part 2, pp. 135–148, Am. Math. Soc., Providence, RI, 1994.
2. C. K. Fan, "A Hecke Algebra Quotient and Properties of Commutative Elements of a Weyl Group," Ph.D. thesis, M.I.T., Cambridge, MA, 1995.
3. C. K. Fan, A Hecke algebra quotient and some combinatorial applications, *J. Algebraic Combin.* **5** (1996), 175–189.
4. C. K. Fan, Structure of a Hecke algebra quotient, *J. Amer. Math. Soc.* **10** (1997), 139–167.
5. C. K. Fan and R. M. Green, Monomials and Temperley–Lieb algebras, *J. Algebra* **190** (1997), 498–517.
6. J. J. Graham, "Modular Representations of Hecke Algebras and Related Algebras," Ph.D. thesis, University of Sydney, 1995.
7. R. M. Green, Generalized Temperley–Lieb algebras and decorated tangles, *J. Knot Theory Ramifications* **7** (1998), 155–171.
8. R. M. Green and J. Losonczy, Canonical bases for Hecke algebra quotients, *Math. Res. Lett.* **6** (1999), 213–222.
9. R. M. Green and J. Losonczy, A projection property for Kazhdan–Lusztig bases, *Internat. Math. Res. Notices* **1** (2000), 23–34.
10. J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge Studies in Advanced Mathematics, Vol. 29, Cambridge Univ. Press, Cambridge, UK, 1990.
11. V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* (2) **126** (1987), 335–388.
12. D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
13. J. R. Stembridge, On the fully commutative elements of Coxeter groups, *J. Algebraic Combin.* **5** (1996), 353–385.
14. H. N. V. Temperley and E. H. Lieb, Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: Some exact results for the percolation problem, *Proc. Roy. Soc. London Ser. A* **322** (1971), 251–280.
15. J. Tits, "Le problème des mots dans les groupes de Coxeter," Symposia Mathematica, Istituto Nazionale di Alta Matematica, Vol. 1, pp. 175–185, Academic Press, London/New York, 1969.