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Semisimple conjugacy classes and classes in the Weyl group

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Abstract

We discuss a map θ from the semisimple conjugacy classes of a finite group G^F of Lie type to the F-conjugacy classes of its Weyl group. We obtain two expressions for the number of semisimple classes mapped by θ into a given F-conjugacy class of W. The first involves distinguished coset representatives in the affine Weyl group and the second is the number of elements in the coroot lattice satisfying certain conditions. The Brauer complex plays a key role in the proof. The map θ has recently proved of interest in connection with probabilistic and combinatorial group theory.

Keywords: Semisimple classes; Classes in Weyl group; Affine Weyl group; Coroot lattice; Brauer complex; Card shuffling

1. Introduction

Let G be a simple simply-connected algebraic group over the algebraic closure K of the prime field F_p, and let $F : G \to G$ be a Frobenius endomorphism. Let G^F be the corresponding finite group of Lie type. It was shown by Steinberg [8, 14.8], that the number of conjugacy classes of semisimple elements in G^F is q^ℓ where ℓ is the rank of G and q is the absolute value of all eigenvalues of F on the co-character group of an F-stable maximal torus of G.

Let $s' \in G^F$ be semisimple and let T' be a maximally split maximal torus of the centralizer $C(s')$. Then T' is an F-stable maximal torus of G but is not necessarily maximally split in G. Let T be a maximally split torus of G. Then $T' = {}^gT$ for some

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$g \in G$. Since T, T' are both F-stable we have $g^{-1}F(g) \in \mathcal{N}(T)$. Let $W = \mathcal{N}(T)/T$ be the Weyl group and $w \in W$ be the image of $g^{-1}F(g)$ under the natural homomorphism from $\mathcal{N}(T)$ to W. The element $w \in W$ is not uniquely determined by the semisimple conjugacy class of G^F containing s', but its F-conjugacy class is uniquely determined. Here $w_1, w_2 \in W$ are F-conjugate if $w_2 = x^{-1}w_1F(x)$ for some $x \in W$. (The F-action on W is induced from that on T.) Thus we have a map θ from semisimple conjugacy classes of G^F to F-conjugacy classes in W.

This map θ has recently proved to be of interest in probabilistic group theory in a number of special cases. When $G^F = SL_n(q)$ each semisimple conjugacy class in G^F determines a polynomial of degree n in $F_q[t]$, the characteristic polynomial of the elements in the class. The F-conjugacy classes of W are the conjugacy classes in the symmetric group S_n, so correspond to partitions of n. A given polynomial of degree n in $F_q[t]$ will factorize into irreducible polynomials whose degrees form a partition of n. The map θ takes the semisimple conjugacy class associated to the given polynomial into this partition of n. In this way, we obtain a measure on the set of partitions of n given by the number of semisimple conjugacy classes of $SL_n(q)$ mapping to a given partition under θ. J. Fulman [7] has obtained interpretations of this measure on the F-conjugacy classes of W in terms of card shuffling in the cases when G^F is $SL_n(q)$ and $Sp_{2n}(q)$.

The purpose of this paper is to obtain two expressions for the number of semisimple conjugacy classes of G^F mapped by θ into a given F-conjugacy class C of W. Let X, Y be the character and co-character groups of T respectively and let $V = Y \otimes \mathbb{R}$. For $\gamma \in V$ let $\tau(\gamma): V \to V$ be the translation $v \mapsto v + \gamma$. Then $W_a = W\tau(Y)$ is the affine Weyl group acting on V. Let $F: Y \to Y$ be the Frobenius action on Y induced by that on T. Then $F = qF_0$ where $F_0: Y \to Y$ has finite order (cf. [8, 11.14]). Let $W'_a = W\tau(F^{-1}(Y))$. Then W'_a is a group of transformations of V which contains W_a as a subgroup of index q^ℓ. In fact W'_a is isomorphic to W_a, both being isomorphic to the affine Weyl group of G. Each left coset of W_a in W'_a has a unique element of minimal length with respect to the length function on the Coxeter group W'_a. These are called the distinguished coset representatives of W_a in W'_a. We denote by $\pi : W'_a \to W$ the natural homomorphism from the affine Weyl group W'_a to the Weyl group W.

We shall prove the following result.

Theorem 1. *Let C be an F-conjugacy class of W. Then the following three numbers are equal*:

(i) *The number of semisimple conjugacy classes of G^F mapped by θ to C.*
(ii) *The number of distinguished coset representatives d of W_a in W'_a such that $\pi(d) \in C$.*
(iii) $\sum_{w \in C} m_w$, *where m_w is the number of elements $\gamma \in Y$ satisfying the following conditions*:
   (a) $\langle \alpha_j, \gamma \rangle \geqslant 0$ *for $j = 1, \ldots, \ell$ where the α_j are the set of simple roots of G;*
   (b) $\langle F_0^{-1}(\tilde{\alpha}), \gamma \rangle \leqslant q$ *where $\tilde{\alpha}$ is the highest root and $F = qF_0$ on X;*
   (c) $w(\alpha_j)$ *is a positive root for all $j \in J(\gamma)$. Here $\alpha_0 = -\tilde{\alpha}$ and $J(\gamma)$ is the subset of $\{0, 1, \ldots, \ell\}$ defined as follows. For $j \in \{1, \ldots, \ell\}$, $j \in J(\gamma)$ if and only if $\langle \alpha_j, \gamma \rangle = 0$. For $j = 0$, $j \in J(\gamma)$ if and only if $\langle F_0^{-1}(\tilde{\alpha}), \gamma \rangle = q$.*

We note that $F_0(\tilde{\alpha}) = \tilde{\alpha}$ when G^F is a Chevalley group or Steinberg twisted group, but not when it is a Suzuki or Ree group.

Theorem 1 was conjectured by Fulman in the case when G^F is split and proved by him in several particular cases. A proof due to the author when G^F is split appears in Fulman's paper [7]. We give a proof here in the general (not necessarily split) case in order to encourage further results on probabilistic and combinatorial group theory of the type already obtained for certain particular groups G^F.

Thanks are due to Jason Fulman for stimulating the author's interest in this question.

2. Semisimple classes in reductive groups

We recall some basic facts about semisimple conjugacy classes. Proofs can be found, for example, in [2, Chapter 3].

Let G be a simple simply-connected algebraic group over the algebraic closure K of F_p. Let ℓ be the rank of G. Let T be a maximal torus of G and $W = \mathcal{N}(T)/T$ be the Weyl group. Let K^* be the multiplicative group of K and $X = \mathrm{Hom}(T, K^*)$, $Y = \mathrm{Hom}(K^*, T)$ be the character group and co-character group of T, respectively. Then X, Y are free abelian groups of rank ℓ and we have a map $X \times Y \to \mathbb{Z}$ given by $\chi, \gamma \mapsto \langle \chi, \gamma \rangle$ where

$$\chi\bigl(\gamma(\lambda)\bigr) = \lambda^{\langle \chi, \gamma \rangle} \quad \text{for } \chi \in X,\ \gamma \in Y,\ \lambda \in K^*.$$

Given an element $w \in W$ let $n_w \in \mathcal{N}(T)$ be an element mapping to w under the natural homomorphism. Then we define a W-action on T by

$$t^w = n_w^{-1} t n_w, \quad t \in T,\ w \in W.$$

We also define W-actions on X and Y by

$$(\gamma^w)(\lambda) = \bigl(\gamma(\lambda)\bigr)^w, \quad \gamma \in Y,\ w \in W,\ \lambda \in K^*,$$
$$\bigl(w(\chi)\bigr)t = \chi(t^w), \quad \chi \in X,\ w \in W,\ t \in T.$$

Let $V = Y \otimes \mathbb{R}$ and for each $\gamma \in V$ let $\tau(\gamma) : V \to V$ be the translation $v \mapsto v + \gamma$. These maps for $\gamma \in Y$ generate the translation group $\tau(Y)$. We have an action of W on V obtained by extending its action on Y. Let $W_a = W\tau(Y)$ be the affine Weyl group. W_a acts on V as a group of affine transformations given by

$$(\gamma \otimes r)^{w\tau(\gamma')} = (\gamma^w \otimes r) + \gamma', \quad \gamma, \gamma' \in Y,\ r \in \mathbb{R}, w \in W.$$

Let $\Phi \subset X$ be the root system of G with respect to T, and let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a system of simple roots. Let $\tilde{\alpha}$ be the highest root and $\alpha_0 = -\tilde{\alpha}$. Let A be the subset of V given by

$$A = \bigl\{\gamma \in V;\ \langle \alpha_i, \gamma \rangle > 0 \text{ for } i = 1, \ldots, \ell,\ \langle -\alpha_0, \gamma \rangle < 1\bigr\}.$$

A is called the fundamental alcove in V. Its closure $\overline{A}$ is a fundamental region for the action of W_a on V.

We next consider semisimple conjugacy classes in G. Since each semisimple element lies in a maximal torus and any two maximal tori of G are conjugate, any semisimple conjugacy class of G contains an element s which lies in our maximal torus T. Moreover, two elements of T are conjugate in G if and only if they lie in the same W-orbit on T.

Now the map $Y \otimes K^* \to T$ determined by $\gamma \otimes \lambda \mapsto \gamma(\lambda)$ is an isomorphism. Moreover there is a (non-canonical) isomorphism between K^* and $\mathbb{Q}_{p'}/\mathbb{Z}$ where $\mathbb{Q}_{p'}$ is the set of rational numbers with denominator prime to p. Thus we have an isomorphism $Y \otimes (\mathbb{Q}_{p'}/\mathbb{Z}) \to T$. This determines a homomorphism $Y \otimes \mathbb{Q}_{p'} \to T$ with kernel $Y \otimes \mathbb{Z} = Y$. Thus we have an isomorphism

$$(Y \otimes \mathbb{Q}_{p'})/Y \to T.$$

This gives a bijection between T and the $\tau(Y)$-orbits on $Y \otimes \mathbb{Q}_{p'}$. There is therefore a bijection between the W-orbits on T and the orbits of $W_a = W\tau(Y)$ on $Y \otimes \mathbb{Q}_{p'}$. Since each W_a-orbit on $V = Y \otimes \mathbb{R}$ contains a unique element of $\overline{A}$, each W_a-orbit on $Y \otimes \mathbb{Q}_{p'}$ will contain a unique element of $\overline{A}_{p'}$, the set of elements of $\overline{A}$ whose coordinates all lie in $\mathbb{Q}_{p'}$. This is a bijection between semisimple conjugacy classes of G and elements of $\overline{A}_{p'}$. We shall make use of this bijection to understand the properties of the semisimple classes.

Let $C(s)$ be the centralizer of s in G. Then $C(s)$ is a reductive subgroup of G which is connected since G is assumed simply-connected. In fact, $C(s) = \langle T, X_\alpha, \alpha \in \Phi_1 \rangle$ where $\Phi_1 = \{\alpha \in \Phi;\ \alpha(s) = 1\}$ and X_α is the root subgroup of G corresponding to $\alpha \in \Phi$. A fundamental system of roots for $C(s)$ can be described in terms of the above bijection between semisimple conjugacy classes and elements of $\overline{A}_{p'}$. Let $a \in \overline{A}_{p'}$ be the point corresponding to the semisimple class containing s. Let J be the subset of $\{0, 1, \dots, \ell\}$ given by

$$j \in J \quad \text{if and only if} \quad \begin{cases} \langle \alpha_j, a \rangle = 0 & \text{for } j = 1, \dots, \ell, \\ \langle -\alpha_0, a \rangle = 1 & \text{for } j = 0. \end{cases}$$

We then say that a lies on the J-face of the fundamental alcove A. Let $\Pi_J \subset \Phi$ be defined by $\Pi_J = \{\alpha_j;\ j \in J\}$. Then it is shown in [5] that there exists $w \in W$ such that $w(\Pi_J)$ is a fundamental system in Φ_1. The element w which appears here depends on the choice of s in its W-orbit in T. There exists an s in any given W-orbit such that Π_J is a fundamental system of roots for $C(s)$.

3. The Frobenius action

We now suppose that $F : G \to G$ is a Frobenius map on G (cf. [2, 1.17]). Then there exist F-stable maximal tori in G. Among these we can find a maximally split F-stable maximal torus T. (T is uniquely determined up to conjugacy by an element of G^F.) We define F-actions on X, Y by

$$\big(F(\chi)\big)t = \chi\big(F(t)\big), \quad \chi \in X,\ t \in T,$$
$$\big(F(\gamma)\big)\lambda = F\big(\gamma(\lambda)\big), \quad \gamma \in Y,\ \lambda \in K^*.$$

It follows from these definitions that

$$\big\langle \chi, F(\gamma)\big\rangle = \big\langle F(\chi), \gamma\big\rangle, \quad \chi \in X,\ \gamma \in Y.$$

We also define an action of F on W by

$$F(n_w) \equiv n_{F(w)} \quad \operatorname{mod} T.$$

It follows from this definition that

$$F(t^w) = F(t)^{F(w)}, \quad t \in T,\ w \in W.$$

Now let $G^F = \{g \in G;\ F(g) = g\}$ be the corresponding finite group of Lie type. Let T' be any F-stable maximal torus of G. Then $T' = {}^gT$ for some $g \in G$. Since $F(T') = T'$ we have $F({}^gT) = {}^gT$ and so $g^{-1}F(g) \in \mathcal{N}(T)$. Hence $g^{-1}F(g) = n_w$ for some $w \in W$. We say that T' is obtained from the maximally split torus T by twisting with w. The map $T' \to w$ induces a bijection between the G^F-classes of F-stable maximal tori of G and the F-conjugacy classes of W. (We recall that $w, w' \in W$ are F-conjugate if there exists $x \in W$ such that $w' = x^{-1}wF(x)$.)

We now consider semisimple conjugacy classes in the finite group G^F. Under the given assumptions on G every F-stable semisimple conjugacy class of G contains F-stable elements and any two such elements are conjugate by an element of G^F. Thus there is a bijective correspondence between semisimple conjugacy classes of G^F and F-stable semisimple conjugacy classes of G.

We may define an F-action on $V = Y \otimes \mathbb{R}$ by extending linearly the above F-action on Y. We may also define an action of F on the subset $\overline{A} \subset V$, i.e., the closure of the fundamental above. We shall denote this action by $a \to F.a$, where $F.a$ is the element of $\overline{A}$ in the orbit of $F(a)$ under the affine Weyl group W_a. If $a \in \overline{A}_{p'}$ we have $F.a \in \overline{A}_{p'}$, thus we have an F-action on $\overline{A}_{p'}$. This F-action on $\overline{A}_{p'}$ is compatible with the F-action on semisimple conjugacy classes of G under the bijection between semisimple conjugacy classes of G and points in $\overline{A}_{p'}$. In particular, we obtain a bijection between F-stable semisimple conjugacy classes of G and F-stable points in $\overline{A}_{p'}$. This in turn gives a bijection between semisimple conjugacy classes of G^F and F-stable points in $\overline{A}_{p'}$.

Now let $s' \in G^F$ be semisimple. Let T' be a maximally split torus of $C(s')$. Then T' is obtained from a maximally split torus T of G by twisting with $w \in W$. We obtain in this way a map θ from semisimple conjugacy classes of G^F to F-conjugacy classes of W. Given an F-conjugacy class C of W we are interested in the question of how many of the q^ℓ semisimple classes of G^F are mapped by θ to C.

4. Distinguished coset representatives in the affine Weyl group

Let T be a maximally split torus of G, Y the co-character group of T and $V = Y \otimes \mathbb{R}$. Then $F : V \to V$ is a non-singular map defined in Section 3. Let $F^{-1}(Y) = \{\gamma \in V;\ F(\gamma) \in Y\}$ and let $\tau(F^{-1}(Y))$ be the group of translations of V by elements of $F^{-1}(Y)$. Let W'_a be the group of affine transformations of V given by $W'_a = W\tau(F^{-1}(Y))$. Since $F(Y) \subset Y$ we have $Y \subset F^{-1}(Y)$ and $W_a \subset W'_a$. Now the map $W'_a \to W_a$ given by $w\tau(\gamma) \mapsto F(w)\tau(F(\gamma))$ is an isomorphism. This follows from the identities

$$F(ww') = F(w)F(w'), \qquad F\big(\gamma^{w'}\big) = F(\gamma)^{F(w')}.$$

The former identity follows from the definition of the F-action on W and the latter from the relations

$$\begin{aligned}
&\big(F\big(\gamma^{w'}\big)\big)\lambda = F\big(\gamma^{w'}(\lambda)\big) = F\big(\gamma(\lambda)^{w'}\big),\\
&\big(\big(F(\gamma)\big)^{F(w')}\big)\lambda = \big(F(\gamma)\lambda\big)^{F(w')} = \big(F\big(\gamma(\lambda)\big)\big)^{F(w')} = F\big(\gamma(\lambda)^{w'}\big).
\end{aligned}$$

Thus W_a and W'_a are isomorphic. Both act on V and the diagram

$$\begin{tikzcd}
V \arrow[r, "F"] \arrow[d, "w\tau(\gamma)"'] & V \arrow[d, "F(w)\tau(F(\gamma))"] \\
V \arrow[r, "F"] & V
\end{tikzcd}$$

for $\gamma \in F^{-1}(Y)$ is commutative and so the map

$$\begin{aligned}
\big(W'_a, V\big) &\to (W_a, V),\\
\big(w\tau(\gamma), \gamma'\big) &\mapsto \big(F(w)\tau\big(F(\gamma)\big), F(\gamma')\big)
\end{aligned}$$

is an isomorphism of permutation groups. Let

$$\begin{aligned}
A' &= \big\{\gamma \in V;\ F(\gamma) \in A\big\}\\
&= \big\{\gamma \in V;\ \big\langle \alpha_i, F(\gamma)\big\rangle > 0 \text{ for } i = 1, \ldots, \ell,\ \big\langle -\alpha_0, F(\gamma)\big\rangle < 1\big\}\\
&= \big\{\gamma \in V;\ \big\langle F(\alpha_i), \gamma\big\rangle > 0 \text{ for } i = 1, \ldots, \ell,\ \big\langle F(-\alpha_0), \gamma\big\rangle < 1\big\}.
\end{aligned}$$

The closure $\overline{A'}$ is a fundamental region for the action of W'_a on V.

Now $W_a \subset W'_a$ so each affine reflection in W_a lies in W'_a. Thus all reflecting hyperplanes for W_a are reflecting hyperplanes for W'_a. Thus the walls of the fundamental alcove A are reflecting hyperplanes for W'_a. It follows that the closure $\overline{A}$ is the union of certain transforms of $\overline{A'}$ by elements of W'_a.

We recall that the action of F on Y is given by $F = qF_0$ where $q > 1$ and F_0 has finite order (when the group G^F is split q is the number of elements in the base field of G^F but in general the real number q need not be an integer). Now

$$A' = F^{-1}(A) = q^{-1}F_0^{-1}(A).$$

Since $\dim V = \ell$, we have $\operatorname{vol} A' = (\operatorname{vol} A)/q^\ell$.

Thus there are q^ℓ transforms of A' which lie in A. Let D be the subset of W_a' given by

$$D = \{d \in W_a';\ (A')^d \subset A\}.$$

Thus $|D| = q^\ell$. In fact, D is a set of left coset representatives of W_a in W_a'. For let $\omega' \in W_a'$. Let B be the alcove for W_a satisfying $(A')^{\omega'} \subset B$. Then there exists a unique $\omega \in W_a$ with $A^\omega = B$. Thus $(A')^{\omega'} \subset A^\omega$ and so $\omega'\omega^{-1} \in D$. So the coset $\omega' W_a$ contains an element of D, and the above argument shows this element of D is unique.

We next observe that each $d \in D$ is the unique element of minimal length in its coset dW_a. For let $\omega' = d\omega$ with $\omega \in W_a$, $\omega' \in W_a'$. Let ℓ be the length function for the Coxeter group W_a'. Then $\ell(d)$ is the distance between the alcoves A' and $(A')^d$. Let $A'^{\omega'} = B'$ and take a sequence of consecutive alcoves from A' to B' of minimal length. The number of steps in this sequence is $\ell(\omega')$. Now for each alcove in this sequence there is a unique alcove which lies in A and is equivalent to the given one under W_a. Consider the sequence of alcoves in A obtained in this way from the given sequence. This is called the derived sequence. Since the original sequence runs from A' to B', the derived sequence runs from A' to $(A')^d$. Neighboring alcoves in the derived sequence are either consecutive alcoves or are equal. So the number of steps in the derived sequence is at least $\ell(d)$. Thus, $\ell(\omega') \geqslant \ell(d)$ for all $\omega' \in dW_a$. Now suppose that $\ell(\omega') = \ell(d)$. Then there are no repetitions in the derived sequence. But any step in the original sequence which takes an alcove for W_a' into a consecutive alcove for W_a' lying in a different alcove for W_a would give a repetition in the derived sequence. Hence, all terms in the original sequence must lie in the same alcove for W_a, viz A. In particular, $B' \subset A$ and so $\omega' \in D$. Thus, $\omega' = d$. Hence $\ell(\omega') \geqslant \ell(d)$ for all $\omega' \in dW_a$ with equality only if $\omega' = d$.

D is called the set of distinguished left coset representatives of W_a in W_a'. (In the case in which G^F is a split group, this situation was considered by Cellini in [3].)

5. The Brauer complex

We consider the walls of the fundamental alcove A for W_a. Let

$$H_i = \{\gamma \in V;\ \langle \alpha_i, \gamma \rangle = 0\} \quad \text{for } 1 \leqslant i \leqslant \ell,$$
$$H_0 = \{\gamma \in V;\ \langle -\alpha_0, \gamma \rangle = 1\}.$$

The H_i for $i \in \{0, 1, \ldots, \ell\}$ are the walls of A. H_i is called the i-wall of A.

Similarly, we consider the walls of the fundamental alcove A' for W_a'. Let

$$H_i' = \{\gamma \in V;\ \langle \alpha_i, F(\gamma)\rangle = 0\} \quad \text{for } 1 \leqslant i \leqslant \ell,$$
$$H_0' = \{\gamma \in V;\ \langle -\alpha_0, F(\gamma)\rangle = 1\}.$$

The H_i' for $i \in \{0, 1, \ldots, \ell\}$ are the walls of A'. H_i' is called the i-wall of A'. We note that if γ lies on the i-wall of A' then $F(\gamma)$ lies on the i-wall of A.

Consider the set of all open faces of the closed simplex $\overline{A'}$. The set of all transforms of such open faces of $\overline{A'}$ by elements of D form a simplicial complex called the Brauer complex. The simplices of maximal dimension in the Brauer complex are the alcoves $(A')^d$ for $d \in D$. The i-wall of $(A')^d$ is defined to be $(H)_i'^d$. The union of all faces of the Brauer complex is $\overline{A}$. General information about the Brauer complex can be found in [2, 3.8] and in [4].

For each $\gamma \in \overline{A}$ we define $J(\gamma)$ by $J(\gamma) = \{i \in \{0, 1, \ldots, \ell\};\ \gamma \in H_i\}$. We say that γ lies in the $J(\gamma)$-face of A. In general the J-face of A is the intersection of the i walls H_i of A for $i \in J$ where $J \subset \{0, 1, \ldots, \ell\}$.

Similarly, for each $\gamma \in \overline{A'}$ we define $J'(\gamma)$ by

$$J'(\gamma) = \{i \in \{0, 1, \ldots, \ell\};\ \gamma \in H_i'\}.$$

Then γ lies in the $J'(\gamma)$-face of A'.

Lemma 5.1. *Let $\omega' = w'\tau(\gamma') \in W_a'$ where $w' \in W$ and $\gamma' \in F^{-1}(Y)$. Then $\omega' \in D$ if and only if $\gamma' \in \overline{A}$ and $w'(\alpha_j) \in \Phi^+$ for all $j \in J(\gamma')$.*

Proof. If $\omega' \in D$ then $\overline{A'}^{\omega'} \subset \overline{A}$. Since $0^{\omega'} = \gamma'$ we have $\gamma' \in \overline{A}$.

Conversely, suppose that $\gamma' \in \overline{A}$. Then $A'^{w'\tau(\gamma')}$ lies in A if and only if

$$\langle \alpha_j, A'^{w'}\rangle > 0 \quad \text{for all } j \in J(\gamma').$$

For this condition ensures that $A'^{w'\tau(\gamma')}$ lies on the same side of the j-wall of A as A does, for each $j \in J(\gamma')$. The above condition can be written as

$$\langle w'(\alpha_j), A'\rangle > 0 \quad \text{for all } j \in J(\gamma'),$$

i.e., as $w'(\alpha_j) \in \Phi^+$ for all $j \in J(\gamma')$. $\square$

Now let $d \in D$ and $B' = (A')^d$. Then B' is an alcove for W_a' contained in the fundamental alcove A for W_a. In general, a wall of B' will not be a wall of A. However, it may happen that a wall of B' coincides with a wall of A. If this happens, the types of this wall for B' and for A need not be the same. We consider when a given wall of B' is also a wall of A and how the types of the wall with respect to B' and A are related.

Proposition 5.2. *Let $B' = A'^{w'\tau(\gamma')}$ where $w' \in W, \gamma' \in F^{-1}(Y)$ and $w'\tau(\gamma') \in D$. Let $w = F(w'), \gamma = F(\gamma')$ and let $j \in \{1, \ldots, \ell\}$. Then the j-wall of B' is a wall of A if and only if $\langle w^{-1}(\alpha_j), \gamma\rangle = 0$. If this is so the j-wall of B' coincides with the i-wall of A where $Fw^{-1}(\alpha_j)$ is a positive multiple of α_i.*

Proof. Since $B' = A'^{w'\tau(\gamma')}$ we have $F(B') = A^{w\tau(\gamma)}$. The j-wall of A for $j \in \{1, \dots, \ell\}$ is $H_j = \{v \in V;\ \langle \alpha_j, v\rangle = 0\}$ and $\langle \alpha_j, v\rangle > 0$ for $v \in A$. Thus the j-wall of A^w is

$$H_j^w = \left\{v^w \in V;\ \langle \alpha_j, v\rangle = 0\right\} = \left\{v \in V;\ \left\langle w^{-1}(\alpha_j), v\right\rangle = 0\right\}.$$

Also $\langle w^{-1}(\alpha_j), v\rangle > 0$ for $v \in A^w$. Thus the j-wall of $A^{w\tau(\gamma)}$ is

$$\left\{v + \gamma \in V;\ \left\langle w^{-1}(\alpha_j), v\right\rangle = 0\right\} = \left\{v \in V;\ \left\langle w^{-1}(\alpha_j), v\right\rangle = \left\langle w^{-1}(\alpha_j), \gamma\right\rangle\right\}.$$

Also, $\langle w^{-1}(\alpha_j), v\rangle > \langle w^{-1}(\alpha_j), \gamma\rangle$ for $v \in A^{w\tau(\gamma)}$. Now $A^{w\tau(\gamma)} = F(B')$. Thus the j-wall of B' is

$$\left\{F^{-1}(v) \in V;\ \left\langle w^{-1}(\alpha_j), v\right\rangle = \left\langle w^{-1}(\alpha_j), \gamma\right\rangle\right\} = \left\{v \in V;\ \left\langle Fw^{-1}(\alpha_j), v\right\rangle = \left\langle w^{-1}(\alpha_j), \gamma\right\rangle\right\}.$$

Also $\langle Fw^{-1}(\alpha_j), v\rangle > \langle w^{-1}(\alpha_j), \gamma\rangle$ for $v \in B'$. Now the i-wall of A is $\{v \in V;\ \langle \alpha_i, v\rangle = 0\}$ if $i \in \{1, \dots, \ell\}$. Also $\langle \alpha_i, v\rangle > 0$ for $v \in A$. Thus the j-wall of B' coincides with the i-wall of A if and only if $\langle w^{-1}(\alpha_j), \gamma\rangle = 0$ and $Fw^{-1}(\alpha_j)$ is a positive multiple of α_i. □

Now let X_α, $\alpha \in \Phi$, be the root subgroups of G with respect to the maximally split F-stable maximal torus T. Since $F(T) = T$ we have $F(X_\alpha) = X_{\rho(\alpha)}$ for some permutation ρ of Φ. Moreover, we have $\rho(\Phi^+) = \Phi^+$. It was shown by Chevalley that $F(\alpha)$ is a positive multiple of $\rho^{-1}(\alpha)$ for each $\alpha \in \Phi$, (cf. [8, 11.2]).

Corollary 5.3. *Under the hypotheses of the above proposition, the j-wall of B' coincides with the i-wall of A where $w^{-1}(\alpha_j) = \rho(\alpha_i)$.*

Proof. $Fw^{-1}(\alpha_j)$ is a positive multiple of the root $\rho^{-1}w^{-1}(\alpha_j)$. It is also a positive multiple of α_i. Hence $\rho^{-1}w^{-1}(\alpha_j) = \alpha_i$. □

Now the Brauer complex has the following favorable properties. There is a natural bijection between the q^ℓ F-stable points in $\overline{A}_{p'}$ and the q^ℓ simplices of maximal dimension in the Brauer complex. Each such simplex B' has the property that its closure $\overline{B'}$ contains a unique F-stable point, and this point lies in $\overline{A}_{p'}$. Moreover, distinct simplices B' give rise to distinct F-stable points in their closures.

The F-stable point in $\overline{B'}$ may be given as the fixed point of a contraction map as follows. $F(B')$ is an alcove for W_a, thus there exists a unique $\omega \in W_a$ with $F(B') = A^\omega$. Hence $F(\overline{B'}) = \overline{A^\omega}$ and $F^{-1}(\overline{A^\omega}) = \overline{B'}$.

Let $f : \overline{A} \to \overline{A}$ be the map given by

$$f(a) = F^{-1}(a^\omega).$$

Now $F = qF_0$ on V where $q > 1$ and F_0 has finite order. V may be made into a metric space on which elements of W_a and F_0 act as isometries. With respect to such a metric d we have

$$d\big(f(a_1), f(a_2)\big) = \frac{1}{q} d(a_1, a_2).$$

Thus f is a contraction mapping, so has a unique fixed point. This is the F-stable point in $\overline{B'}$.

6. Proof of Theorem 1

The results already described enable us to define a bijection between semisimple conjugacy classes of G^F and distinguished coset representatives of D. Each semisimple conjugacy class of G^F determines an F-stable point of $\overline{A_{p'}}$ and such an F-stable point lies in a unique alcove $B' = (A')^d$ for $d \in D$. This determines our bijection. Now let $d = w'\tau(\gamma')$ for $w' \in W$, $\gamma' \in F^{-1}(Y)$. The key to the proof of Theorem 1 lies in the following proposition.

Proposition 6.1. *Let $d = w'\tau(\gamma') \in D$ and let c be the corresponding semisimple conjugacy class of G^F. Then the F-conjugacy class $\theta(c)$ of W is the one containing w'.*

Proof. Let $B' = A'^d$. Then $F(B') = A^\omega$ where $\omega \in W_a$ is given by $\omega = F(d)$. Thus, $\omega = w\tau(\gamma)$ where $w = F(w') \in W$ and $\gamma = F(\gamma') \in Y$.

Let $a \in \overline{B'}$ be the F-stable point in $\overline{B'}$. Then $f(a) = a$ where $f : \overline{A} \to \overline{A}$ is the contraction map given by $f(v) = F^{-1}(v^\omega)$. Hence, $F(a) = a^\omega = a^{w\tau(\gamma)}$.

Consider the homomorphism $\phi : Y \otimes \mathbb{Q}_{p'} \to T$. We have $a \in Y \otimes \mathbb{Q}_{p'}$. Let $\phi(a) = s$. A comparison of the actions of F on T and on Y shows that $\phi(F(a)) = F(s)$. Similarly, a comparison of the actions of W on T and on Y shows that $\phi(a^w) = s^w$ for $w \in W$. Now $F(a) = a^{w\tau(\gamma)}$ with $\gamma \in Y$, and Y is the kernel of ϕ. Hence $F(s) = s^w$.

Suppose a lies in the J-face of $\overline{A}$ where $J \subset \{0, 1, \ldots, \ell\}$. Then $\Pi_J = \{\alpha_j;\ j \in J\}$ is a fundamental system of roots for $C(s)$ with respect to T. Since we know that $w(\Pi_J)$ is such a fundamental system for some $w \in W$ it is sufficient to check that $s^{s_j} = s$ for each $j \in J$ where s_j is the reflection corresponding to α_j. If $j \in \{1, \ldots, \ell\}$ and $j \in J$ then $\langle \alpha_j, a\rangle = 0$, so $a^{s_j} = a$ and $s^{s_j} = s$. If $0 \in J$ then $\langle -\alpha_0, a\rangle = 1$, and so $a^{s_0} = a - \langle \alpha_0, a\rangle \alpha_0^v = a + \alpha_0^v$. Since $\alpha_0^v \in Y$, this implies that $s^{s_0} = s$. Hence Π_J is a fundamental system of roots for $C(s)$ with respect to T.

By the Lang–Steinberg theorem, there exists $g \in G$ with $g^{-1}F(g) = n_w$ where $n_w \in \mathcal{N}(T)$ is an element mapping to $w \in W$. Let $T' = {}^gT$ and $s' = {}^gs$. Then $F(s') = s'$ and T' is an F-stable maximal torus of G containing s'. The conjugacy class of G^F containing s' is the class c corresponding to d under our bijection.

We wish to show that T' is a maximally split torus of $C(s')$. Now a lies in the J-face of $\overline{A}$. The contraction map f preserves face types since this is true of both $w \in W_a$ and F. Since $f(\overline{A}) = \overline{B'}$ and $f(a) = a$, it follows that a lies in the J-face of $\overline{B'}$ also. Thus the

J-face of $\overline{A}$ coincides with the J-face of $\overline{B'}$. Let $j \in J$. Then we know that the j-wall of $\overline{B'}$ coincides with the i-wall of $\overline{A}$ where $w^{-1}(\alpha_j) = \rho(\alpha_i)$. Hence, $i \in J$. It follows that $w^{-1}(\Pi_J) = \rho(\Pi_J)$.

Now we know that Π_J is a fundamental system of roots for $C(s')$ with respect to T'. Let the fundamental root subgroups of $C(s')$ with respect to T' be X'_α for $\alpha \in \Pi_J$. Let $B_1 = \langle T', X'_\alpha \text{ for } \alpha \in \Pi_J\rangle$. Then B_1 is a Borel subgroup of $C(s')$.

Now $X'_\alpha = {}^g X_\alpha$ where X_α is the corresponding root subgroup of G with respect to T. Thus the following statements are equivalent:

$$\begin{aligned} F(X'_\alpha) = X'_\beta \quad &\Leftrightarrow \quad F({}^g X_\alpha) = {}^g X_\beta \quad \Leftrightarrow \quad {}^{F(g)}F(X_\alpha) = {}^g X_\beta \\ &\Leftrightarrow \quad {}^{n_w}F(X_\alpha) = X_\beta \quad \Leftrightarrow \quad {}^{n_w}X_{\rho(\alpha)} = X_\beta \\ &\Leftrightarrow \quad w\bigl(\rho(\alpha)\bigr) = \beta. \end{aligned}$$

But we know that $w\rho(\Pi_J) = \Pi_J$, thus F must permute the root subgroups X'_α for $\alpha \in \Pi_J$. It follows that $F(B_1) = B_1$. Hence, B_1 is an F-stable Borel subgroup of $C(s')$ containing T'. Thus T' is a maximally split torus of $C(s')$.

Now T' is obtained from the maximally split torus T of G by twisting with $w \in W$. Hence, the F-conjugacy class $\theta(c)$ of W is the one containing w. Finally, we observe that since $w = F(w')$, w and w' lie in the same F-conjugacy class of W. For $w'^{-1}w'F(w') = w$. Thus the proposition is proved. □

This proposition shows that the number of semisimple conjugacy classes of G^F mapped by θ to C is equal to the number of $d \in D$ with $\pi(d) \in C$. To complete the proof of Theorem 1, we show this is given by

$$\sum_{w \in C} m_w.$$

Now there is a bijection between elements of $d \in D$ and alcoves $(A')^d$ which lie in A. Also the element $w'\tau(\gamma')$ for $w' \in W$, $\gamma' \in F^{-1}(Y)$ lies in D if and only if $\gamma' \in \overline{A}$ and $w'(\alpha_j) \in \Phi^+$ for all $j \in J(\gamma')$. Thus the number of elements of D of the form $w'\tau(\gamma')$ for given $w' \in W$ is the number of elements $\gamma' \in Y \otimes \mathbb{R}$ satisfying the conditions:

$$\begin{gathered} F(\gamma') \in Y, \\ \langle \alpha_j, \gamma' \rangle \geqslant 0 \quad \text{for } j = 1, \dots, \ell, \\ \langle -\alpha_0, \gamma' \rangle \leqslant 1, \\ w'(\alpha_j) \in \Phi^+ \text{ for all } j \in J(\gamma'). \end{gathered}$$

Let $\gamma = F(\gamma')$. Since $J(\gamma) = J(\gamma')$ the above conditions can be written:

$$\begin{gathered} \gamma \in Y, \\ \langle \alpha_j, \gamma \rangle \geqslant 0 \quad \text{for } j = 1, \dots, \ell, \end{gathered}$$

$$\langle F_0^{-1}(\tilde{\alpha}), \gamma \rangle \leqslant q \quad \text{where } \tilde{\alpha} = -\alpha_0 \text{ is the highest root,}$$
$$w'(\alpha_j) \in \Phi^+ \quad \text{for all } j \in J(\gamma).$$

The number of γ satisfying these conditions is $m_{w'}$. Thus the number of elements $d \in D$ with $\pi(d) \in C$ is $\sum_{w' \in C} m_{w'}$. □

7. Examples

7.1. If $G^F = SL_n(q)$, then $W = S_n$ and F acts trivially on W. Thus the F-conjugacy classes of W are the conjugacy classes of S_n and so correspond to partitions of n.

For example, let $G^F = SL_2(5)$. Then $W = S_2$ has two F-conjugacy classes (1) and (s_1). Since G has rank 1, there are five semisimple conjugacy classes in G^F. Under θ three of them map to (1) and two to (s_1).

Now take $G^F = SL_3(5)$. Then $W = S_3$ has three F-conjugacy classes (1) (s_1, s_2, w_0) (s_1s_2, s_2s_1) where $w_0 = s_1s_2s_1 = s_2s_1s_2$. G has rank 2 and so G^F has 25 semisimple conjugacy classes. Under θ 5 of them map to (1), 10 to (s_1, s_2, w_0) and 10 to (s_1s_2, s_2s_1).

7.2. If G^F is the unitary group $SU_n(q^2)$ then again we have $W = S_n$ but this time F acts nontrivially on W. We have $F(w) = w_0ww_0^{-1}$ where $w_0 = (1\,n)(2\,n-1)\ldots$ is the element of maximal length in S_n. Hence $w' = x^{-1}wF(x)$ if and only if $w'w_0 = x^{-1}(ww_0)x$. Thus, w, w' are F-conjugate if and only if ww_0 and $w'w_0$ are conjugate. It follows that F-conjugacy classes of W again correspond to partitions of n.

For example, let $G^F = SU_3(5^2)$. Then $W = S_3$ has three F-conjugacy classes $(1, s_1s_2, s_2s_1)$, (s_1, s_2), (w_0). G has rank 2 and $q = 5$, so G^F has 25 semisimple conjugacy classes. Under θ, 15 of them are mapped to $(1, s_1s_2, s_2s_1)$, 6 to (s_1, s_2) and 4 to (w_0).

In conclusion we mention that there is another measure on partitions of n related to card shuffling, the so called q-shuffles, introduced by Bayer and Diaconis in [1] and studied by Diaconis, McGrath, and Pitman in [6]. It may also be interesting to consider to what extent these results on q-shuffles can be generalized to arbitrary finite groups of Lie type.

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