Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 320 (2008) 437–441
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The Hermite ring conjecture in dimension one

Ihsen Yengui

Department of Mathematics, Faculty of Sciences of Sfax, 3018 Sfax, Tunisia

Received 29 November 2007
Available online 4 March 2008
Communicated by Michel Broué

Abstract

We prove constructively that for any ring $\mathbf{R}$ of Krull dimension $\leqslant 1$ and $n \geqslant 3$, the group $\mathrm{E}_n(\mathbf{R}[X])$ acts transitively on $\mathrm{Um}_n(\mathbf{R}[X])$. In particular, we obtain that for any ring $\mathbf{R}$ with Krull dimension $\leqslant 1$, all finitely generated stably free modules over $\mathbf{R}[X]$ are free. This settles the long-standing Hermite ring conjecture for rings of Krull dimension $\leqslant 1$.

Keywords: Stably free modules; Unimodular rows; Quillen–Suslin theorem; Hermite rings; Hermite ring conjecture; Constructive mathematics

Introduction

In 1955, J.-P. Serre remarked [18] that it was not known whether there exist finitely generated projective modules over $\mathbf{A} = \mathbf{K}[X_1, \ldots, X_n]$, $\mathbf{K}$ a field, which are not free. This remark turned into the "Serre conjecture," stating that indeed there were no such modules. Proven independently by Quillen [15] and Suslin [18], it became subsequently known as the Quillen–Suslin theorem. The book of Lam [7] is a nice exposition about Serre's conjecture which has been updated recently in [8]. An important related fact worth mentioning is that it has been known well before the settlement of Serre's conjecture (since 1958) that finitely generated projective modules over $\mathbf{A}$ are stably free, i.e., every finitely generated projective $\mathbf{A}$-module is isomorphic to the kernel of an $\mathbf{A}$-epimorphism $T : \mathbf{A}^n \to \mathbf{A}^\ell$. In that situation the matrix T is unimodular, that is the maximal minors of T generate the unit ideal in $\mathbf{A}$.

E-mail address: ihsen.yengui@fss.rnu.tn.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.02.007

Quillen's and Suslin's proofs had a big effect on the subsequent development of the study of projective modules. Nevertheless many old conjectures and open questions still wait for solutions. Our concern here is the following equivalent two conjectures.

Hermite ring conjecture (1976) [7,8]:

Conjecture 1. *If* $\mathbf{R}$ *is a commutative Hermite ring, then* $\mathbf{R}[X]$ *is also Hermite.*

Conjecture 2. *If* $\mathbf{R}$ *is a commutative ring and* $v = (v_0(X), \dots, v_n(X))$ *is a unimodular row over* $\mathbf{R}[X]$ *such that* $v(0) = (1, 0, \dots, 0)$*, then* v *can be completed to a matrix in* $\mathrm{GL}_{n+1}(\mathbf{R}[X])$.

Recall that a ring $\mathbf{A}$ is said to be Hermite if any finitely generated stably free $\mathbf{A}$-module is free [7,8]. Examples of Hermite rings are local rings, rings of Krull dimension $\leqslant 1$ [3,8], polynomial rings over Bezout domains [1,5,9,12], and polynomial rings over zero-dimensional rings [5,8].

In this paper, we give a positive answer to these conjectures in case $\mathbf{R}$ has Krull dimension $\leqslant 1$. The proof we give is short, simple, and constructive. It relies heavily on the very nice paper [17] of Roitman. So, we assume that the reader has a copy of [17] in hands.

Let us fix some notations. For any ring $\mathbf{A}$ and $n \geqslant 1$, $\mathrm{Um}_n(\mathbf{A})$ denotes the set of unimodular rows in $\mathbf{A}$, that is $\mathrm{Um}_n(\mathbf{A}) = \{(x_0, \dots, x_{n-1}) \in \mathbf{A}^n \mid \langle x_0, \dots, x_{n-1} \rangle = \mathbf{A}\}$.

We call an $n \times n$ matrix elementary if it has 1's on the diagonal and at most one nonzero off-diagonal entry. More precisely, if $a \in \mathbf{A}$ and $i \neq j$, $1 \leqslant i, j \leqslant n$, we define the elementary matrix $E_{i,j}(a)$ to be the $n \times n$ matrix with 1's on the diagonal, with a in the (i, j)-slot, and with 0's elsewhere. In other words, $E_{i,j}(a)$ is the matrix corresponding to the elementary operation $L_i \to L_i + aL_j$. $\mathrm{E}_n(\mathbf{A})$ will denote the subgroup of $\mathrm{SL}_n(\mathbf{A})$ generated by elementary matrices.

For $f, g \in \mathbf{A}[X]$, $\mathrm{Res}(f, g)$ will denote the resultant of f and g.

All the considered rings are unitary and commutative. The undefined terminology is standard as in [8,18], and, for constructive algebra in [11,13].

1. The main result

Lemma 1. *Let* $\mathbf{R}$ *be a ring, and* $f, g \in \mathbf{R}[X]$ *with* f *a monic polynomial. Then*

$$\langle f, g \rangle = \mathbf{R}[X] \quad \Longleftrightarrow \quad \mathrm{Res}(f, g) \in \mathbf{R}^{\times}.$$

Proof. "$\Leftarrow$" This is an immediate consequence of the fact that $\mathrm{Res}(f, g) \in \langle f, g \rangle \cap \mathbf{R}$.

"$\Rightarrow$" Let $u, v \in \mathbf{R}[X]$ such that $uf + vg = 1$. Since f is a monic polynomial, we have

$$\mathrm{Res}(f, vg) = \mathrm{Res}(f, v)\mathrm{Res}(f, g) = \mathrm{Res}(f, vg + uf) = \mathrm{Res}(f, 1) = 1. \qquad \square$$

Now, we give a constructive and elementary proof of a lemma which was used by Roitman [17] in the proof of his Theorem 5. The proof of that lemma given by Lam in [7,8] (Chapter III, Lemma 1.1) is not constructive and relies on the "going-up" property of integral extensions.

Lemma 2. *Let* $\mathbf{R}$ *be a ring, and* I *an ideal in* $\mathbf{R}[X]$ *that contains a monic polynomial. Let* J *be an ideal in* $\mathbf{R}$ *such that* $I + J[X] = \mathbf{R}[X]$*. Then* $(I \cap \mathbf{R}) + J = \mathbf{R}$.

Proof. Let us denote by f a monic polynomial in I. Since $I + J[X] = \mathbf{R}[X]$, there exist $g \in I$ and $h \in J[X]$ such that $g + h = 1$. It follows that $\langle \bar{f}, \bar{g} \rangle = (\mathbf{R}/J)[X]$ where the classes are taken modulo $J[X]$. By virtue of Lemma 1, we obtain that $\mathrm{Res}(\bar{f}, \bar{g}) \in (\mathbf{R}/J)^{\times}$. As f is a monic polynomial, $\mathrm{Res}(\bar{f}, \bar{g}) = \overline{\mathrm{Res}(f, g)}$, and thus $\langle \mathrm{Res}(f, g) \rangle + J = \mathbf{R}$. The desired conclusion follows from the fact that $\mathrm{Res}(f, g) \in I \cap \mathbf{R}$. □

Lemma 3. *Let* $\mathbf{R}$ *be a reduced local ring of dimension* $\leqslant 1$, $n \geqslant 2$, *and let* $v(X) = (v_0(X), \dots, v_n(X)) \in \mathrm{Um}_{n+1}(\mathbf{R}[X])$. *Then there exists* $E \in \mathrm{E}_{n+1}(R[X])$ *such that* $Ev(X) = (v_0(X), v_1(X), c_2, \dots, c_n)$, *where* $c_i \in \mathbf{R}$.

Proof. As stated by Rao in his proof of Proposition 1.4.4 of [16], this is implicit in [17] (Theorem 5). It is worth pointing out, that the hypothesis that for each nonzero-divisor π of $\mathbf{R}$ there exists $E_\pi \in \mathrm{E}_{n+1}(R[X])$ such that $E_\pi v(X) \equiv v(0) \bmod (\pi \mathbf{R}[X])^{n+1}$ is guaranteed by the fact that $\dim(\mathbf{R}/\pi\mathbf{R}) \leqslant 0$. Moreover, there is no need of the Noetherian hypothesis and we can obtain a fully constructive proof of the desired result. To see this, let us reread carefully Roitman's proof of his Theorem 5 in [17] and let us list the intermediary results he used and which we need for our lemma:

– If v_0 is a monic polynomial then there exists $E \in \mathrm{E}_{n+1}(R[X])$ such that $Ev(X) = (1, 0, \dots, 0)$. This is an immediate consequence of a lemma of Suslin [18] (Lemma 2.3). A constructive proof of that lemma is given in [14,19,20].
– Lemma 3 of [17]. Roitman proved it constructively. The proof is free of any Noetherian hypothesis.
– Lemma 1 of [17]. This is also Proposition III.6.1(b) of Lam [8] (page 125). The proofs given by Lam and Roitman are constructive and free of any Noetherian hypothesis.
– In case $\deg(v_0) = 1$ we immediately get that for $i \geqslant 2$, $\deg(v_i) < 1$, and thus v_i is constant. In more details, by Lemma 3 of [17], we can suppose that, for $1 \leqslant i \leqslant n$, $v_i = X^{2k} w_i$ with $\deg(w_i) < \deg(v_0) = 1$, that is, $w_i \in \mathbf{R}$. Now by Lemma 1 of [17] (taking $t = X$), we can suppose that $v_i \in \mathbf{R}$ for $1 \leqslant i \leqslant n$.
– Lemma 2 of [17]. This is the stable range theorem and there is no need of the Noetherian hypothesis. For a constructive proof of the stable range theorem, the reader can refer to [3] (Theorem 2.4).
– Lemma III.1.1 of [7,8]. This is Lemma 2 above.
– Lemma 4.1(b) of [2]. The proof given by Bass is constructive and free of any Noetherian hypothesis. □

Recall that the boundary ideal of an element a of a ring $\mathbf{R}$ is the ideal $\mathcal{I}(a)$ of $\mathbf{R}$ generated by a and all the $y \in \mathbf{R}$ such that ay is nilpotent. Moreover, $\dim R \leqslant d \Leftrightarrow \dim(\mathbf{R}/\mathcal{I}(a)) \leqslant d - 1$ $\forall a \in \mathbf{R}$ (this defines the Krull dimension recursively initializing with "$\dim \mathbf{R} \leqslant -1 \Leftrightarrow \mathbf{R}$ being trivial") [4].

Recall also that for any ring $\mathbf{R}$, the ring $\mathbf{R}\langle X\rangle$ (respectively $\mathbf{R}(X)$) is the localization of $\mathbf{R}[X]$ at monic polynomials (respectively primitive polynomials). We have $\mathbf{R}[X] \subseteq \mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$, and the containment $\mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ becomes an equality if and only if $\dim \mathbf{R} \leqslant 0$ [6] (see [5] for a constructive proof).

Theorem 4. *Let* $\mathbf{R}$ *be a ring of dimension* $\leqslant 1$, $n \geqslant 2$, *and let* $v(X) = (v_0(X), \dots, v_n(X)) \in \mathrm{Um}_{n+1}(\mathbf{R}[X])$. *Then there exists* $E \in \mathrm{E}_{n+1}(R[X])$ *such that* $E\, v(X) = v(0)$.

Proof. By the local–global principle for elementary matrices [8] (see [10] for a constructive proof), we can suppose that $\mathbf{R}$ is local. Moreover by a result of Kumar and Roitman quoted in [17] (Lemma 1.4.2), we can suppose that $\mathbf{R}$ is reduced. By virtue of Lemma 3, there exists $E \in \mathrm{E}_{n+1}(R[X])$ such that $Ev(X) = (v_0(X), v_1(X), c_2, \ldots, c_n)$, where $c_i \in \mathbf{R}$. So we can without loss of generality suppose that $v_0 = a$ is constant.

Now, let us consider the ring $\mathbf{T} := \mathbf{R}/\mathcal{I}(a)$. Since $\dim \mathbf{T} \leqslant 0$, we have that $\mathbf{T}\langle X\rangle = \mathbf{T}(X)$ and thus $\mathbf{T}\langle X\rangle$ is a local ring. It follows that one among $v_1, \ldots, v_n$, say v_1, divides a monic polynomial in $\mathbf{T}[X]$. This means that there exist a monic polynomial $u \in \mathbf{R}[X]$, $w, h_1, h_2 \in \mathbf{R}[X]$ with $ah_2 = 0$, such that

$$wv_1 = u + ah_1 + h_2.$$

This means that $1 \in \langle v_1, a, h_2\rangle$ in the ring $\mathbf{R}\langle X\rangle$ and thus $1 \in \langle v_1, a + h_2\rangle$ by Lemma 2.3 of [3]. That is, $\exists w_1, w_2 \in \mathbf{R}[X] \mid v_1w_1 + (a + h_2)w_2 =: \tilde{u}$ is a monic polynomial.

Let $d \in \mathbb{N}$ and denote by $u_0, \ldots, u_n$ polynomials in $\mathbf{R}[X]$ such that $u_0v_0 + \cdots + u_nv_n = 1$. Denoting by

$$\gamma_1 := E_{1,2}(h_2u_1) \cdots E_{1,n+1}(h_2u_n),$$

$$\gamma_2 := E_{3,2}\bigl(X^d w_1\bigr) E_{3,1}\bigl(X^d w_2\bigr),$$

$$\gamma := \gamma_2\gamma_1,$$

we have

$$\gamma_1 v = (a + h_2, v_1, \ldots, v_n),$$

and

$$\gamma v = \bigl(a + h_2, v_1, v_2 + X^d\tilde{u}, v_3, \ldots, v_n\bigr).$$

So, for sufficiently large d, the third entry of γv becomes a monic polynomial. Thus, as stated in the proof of Lemma 3, we have an algorithm transforming γv into $(1, 0, \ldots, 0)$ using elementary operations [20]. $\square$

Corollary 5. *For any ring $\mathbf{R}$ of Krull dimension $\leqslant 1$, all finitely generated stably free modules over $\mathbf{R}[X]$ are free.*

Proof. It is classical that if $\mathbf{R}$ has Krull dimension $\leqslant 1$ then all finitely generated stably free modules over $\mathbf{R}$ are free (see [3] for a constructive proof). So, we have only to prove that all finitely generated stably free modules over $\mathbf{R}[X]$ are extended from $\mathbf{R}$. For this, let $v = (v_0(X), \ldots, v_n(X)) \in \mathbf{R}[X]^{n+1}$ $(n \geqslant 2)$ be a unimodular row. Our task amounts to prove that there exists $\Gamma \in \mathrm{GL}_{n+1}(\mathbf{R}[X])$ such that $\Gamma V = (1, 0, \ldots, 0)$. This follows from Theorem 4. $\square$

Corollary 6. *The Hermite ring conjecture is true for rings of Krull dimension $\leqslant 1$.*

Corollary 5 encourages us to set the following conjecture.

Conjecture 3. *For any ring* $\mathbf{R}$ *of Krull dimension* $\leqslant 1$, *and* $k \in \mathbb{N}$, *all finitely generated stably free modules over* $\mathbf{R}[X_1, \ldots, X_k]$ *are free.*

Also, Corollary 5 raises the $\mathbf{K}_1$-analogue question. I will state it as a conjecture.

Conjecture 4. *Let* $\mathbf{R}$ *be a ring of Krull dimension* $\leqslant 1$ *and* $n \geqslant 3$. *Then every matrix* $M \in \mathrm{SL}_n(\mathbf{R}[X])$ *is congruent to* $M(0)$ *modulo* $\mathrm{E}_n(\mathbf{R}[X])$.

In fact, by virtue of Theorem 4 and the local–global principle for elementary matrices (see [10] for a constructive proof), Conjecture 4 is equivalent to the following conjecture.

Conjecture 5. *Suppose* $\mathbf{R}$ *is a local ring of Krull dimension* $\leqslant 1$, *and*

$$M = \begin{pmatrix} p & q & 0 \\ r & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}_3\big(\mathbf{R}[X]\big).$$

Then $M \in \mathrm{E}_3(\mathbf{R}[X])$.

References

[1] S. Barhoumi, H. Lombardi, I. Yengui, Projective modules over polynomial rings: A constructive approach, preprint, 2006.
[2] H. Bass, Libération des modules projectifs sur certains anneaux de polynômes, in: Sém. Bourbaki 1973/74, exp. 448, in: Lecture Notes in Math., vol. 431, Springer-Verlag, Berlin/New York, 1975, pp. 228–254.
[3] T. Coquand, H. Lombardi, C. Quitté, Generating non-noetherian modules constructively, Manuscripta Math. 115 (2004) 513–520.
[4] T. Coquand, H. Lombardi, M.-F. Roy, An elementary characterization of Krull dimension, in: L. Corsilla, P. Schuster (Eds.), From Sets and Types to Analysis and Topology: Towards Practicable Foundations for Constructive Mathematics, Oxford University Press, 2005.
[5] A. Ellouz, H. Lombardi, I. Yengui, A constructive comparison of the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$ and application to the Lequain–Simis induction theorem, J. Algebra, in press.
[6] J. Huckaba, Commutative Rings with Zero-Divisors, Marcel Dekker, 1988.
[7] T.Y. Lam, Serre's Conjecture, Lecture Notes in Math., vol. 635, Springer-Verlag, Berlin/New York, 1978.
[8] T.Y. Lam, Serre's Problem on Projective Modules, Springer Monogr. Math., 2006.
[9] Y. Lequain, A. Simis, Projective modules over $R[X_1, \ldots, X_n]$, R a Prüfer domain, J. Pure Appl. Algebra 18 (2) (1980) 165–171.
[10] H. Lombardi, C. Quitté, Constructions cachées en algèbre abstraite (2). Le principe local–global, in: M. Fontana, E. Kabbaj, S. Wiegand (Eds.), Commutative Ring Theory and Applications, in: Lect. Notes Pure Appl. Math., vol. 131, Marcel Dekker, 2002, pp. 461–476.
[11] H. Lombardi, C. Quitté, Théorie constructive élémentaire des modules projectifs de type fini, preprint, 2003.
[12] H. Lombardi, C. Quitté, I. Yengui, Hidden constructions in abstract algebra (6). The theorem of Maroscia, Brewer and Costa, J. Pure Appl. Algebra, in press.
[13] R. Mines, F. Richman, W. Ruitenburg, A Course in Constructive Algebra, Universitext, Springer-Verlag, 1988.
[14] A. Mnif, I. Yengui, An algorithm for unimodular completion over Noetherian rings, J. Algebra 316 (2007) 483–498.
[15] D. Quillen, Projective modules over polynomial rings, Invent. Math. 36 (1976) 167–171.
[16] R.A. Rao, The Bass–Quillen conjecture in dimension three but characteristic $\neq 2, 3$ via a question of A. Suslin, Invent. Math. 93 (1988) 609–618.
[17] M. Roitman, On stably extended projective modules over polynomial rings, Proc. Amer. Math. Soc. 97 (1986) 585–589.
[18] A. Suslin, On the structure of the special linear group over polynomial rings, Math. USSR Izv. 11 (1977) 221–238.
[19] I. Yengui, Making the use of maximal ideals constructive, Theoret. Comput. Sci. 392 (2008) 174–178.
[20] I. Yengui, Suslin's lemma for elimination, preprint, 2007.