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Parameters for which the Lawrence–Krammer representation is reducible

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ABSTRACT

We show that the representation, introduced by Lawrence and Krammer to show the linearity of the braid group, is generically irreducible. However, for some values of its two parameters when these are specialized to complex numbers, it becomes reducible. We construct a representation of degree $\frac{n(n-1)}{2}$ of the BMW algebra of type A_{n-1}. As a representation of the braid group on n strands, it is equivalent to the Lawrence–Krammer representation where the two parameters of the BMW algebra are related to those appearing in the Lawrence–Krammer representation. We give the values of the parameters for which the representation is reducible and give the proper invariant subspaces in some cases. We use this representation to show that for these special values of the parameters, the BMW algebra of type A_{n-1} is not semisimple.

1. Introduction

1.1. Introduction and main results

In [8], Daan Krammer constructed a faithful linear representation of the braid group. Since this representation was earlier introduced by Ruth Lawrence in [9], it is called the Lawrence–Krammer representation. Stephen Bigelow uses this same representation in [1] to show independently from Krammer that the braid group is linear. A generalization of the linearity result for the braid group to the other Artin groups of finite type is given in [5] and independently in [6]. In this paper, we examine a representation of degree $\frac{n(n-1)}{2}$ of the BMW algebra of type A_{n-1}. As a representation of the braid group on n strands, this representation is equivalent to the Lawrence–Krammer repre-

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sentation (abbreviated L–K representation). By studying this representation we show that the L–K representation is generically irreducible. However, for some values of its two parameters when these are specialized to complex numbers, it becomes reducible. Throughout the paper, we let l, m and r be three nonzero complex numbers, where m and r are related by $m = \frac{1}{r} - r$. We define $\mathcal{H}_{F,r^2}(n)$ as the Iwahori–Hecke algebra of the symmetric group $Sym(n)$ over the field $F = \mathbb{Q}(l, r)$ with generators $g_1, \ldots, g_{n-1}$. They satisfy the braid relations and the quadratic relation $g_i^2 + mg_i = 1$ for all i. Our definition is the same as the definition of [11] after the generators have been rescaled by a factor $\frac{1}{r}$. Our main result is as follows.

Theorem 1 *(Main theorem). Let n be an integer with $n \geqslant 3$ and let m, l and r be three nonzero complex numbers, where m and r are related by $m = \frac{1}{r} - r$. Assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple, and so assume that $r^{2k} \neq 1$ for every integer $k \in \{1, \ldots, n\}$.*

When $n \geqslant 4$, the Lawrence–Krammer representation of the BMW algebra of type A_{n-1} with parameters l and m over the field $\mathbb{Q}(l, r)$ is irreducible, except when $l \in \{r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, when it is reducible.

When $n = 3$, the Lawrence–Krammer representation of the BMW algebra of type A_2 with parameters l and m over the field $\mathbb{Q}(l, r)$ is irreducible, except when $l \in \{-r^3, \frac{1}{r^3}, 1, -1\}$, when it is reducible.

As will appear in the proof of the main theorem, the assumption that $\mathcal{H}_{F,r^2}(n)$ is semisimple is crucial. It is equivalent to the condition that $r^{2k} \neq 1$ for every $k \in \{1, \ldots, n\}$: see Corollary 3.44, p. 48 of [11].

Some cases of reducibility of the Lawrence–Krammer representation have been studied in the past, but no systematic study is done. For instance in [2], Stephen Bigelow studies the case of reducibility $l = r$ by topological methods.

A consequence of our result and of the method that we use is the following.

Theorem 2. *Let n be an integer and let l, m and r be three nonzero complex numbers, where m and r are related by $m = \frac{1}{r} - r$.*

Suppose $n \geqslant 4$. If $r^{2k} = 1$ for some $k \in \{2, \ldots, n\}$ or if l belongs to the set of values $\{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}, -r^{2n-3}, r^{n-3}, -r^{n-3}, \frac{1}{r^3}, -\frac{1}{r}\}$, the BMW algebra of type A_{n-1} with parameters l and m over the field $\mathbb{Q}(l, r)$ is not semisimple.

Suppose $n = 3$. If $r^4 = 1$ or $r^6 = 1$ or if $l \in \{-r^3, \frac{1}{r^3}, 1, -1\}$, the BMW algebra of type A_2 with parameters l and m over the field $\mathbb{Q}(l, r)$ is not semisimple.

In [14], Hans Wenzl states that the BMW algebra of type A_{n-1} is semisimple except possibly if r is a root of unity or $l = r^n$, for some $n \in \mathbb{Z}$. Here, Theorem 2 gives instances in which the algebra is not semisimple. The result of this theorem is also contained in the recent work of Hebing Rui and Mei Si (see [13]). They use the representation theory of cellular algebras.

1.2. Definitions

1.2.1. The BMW algebra

We recall below the defining relations of the BMW algebra $B(A_{n-1})$ (or simply B) of type A_{n-1} with nonzero complex parameters l and m over the field $\mathbb{Q}(l, r)$, where r is a root of the quadratic $X^2 - mX + 1$. This algebra has two sets of $(n-1)$ elements, namely the invertible g_i's that satisfy the braid relations (1) and (2) and generate the algebra and the e_i's that generate an ideal. For nodes i and j with $1 \leqslant i, j \leqslant n-1$, we will write $i \sim j$ if $|i - j| = 1$ and $i \nsim j$ if $|i - j| > 1$. The defining relations of the algebra are as follows

$$g_i g_j = g_j g_i \quad \text{if } i \nsim j, \tag{1}$$

$$g_i g_j g_i = g_j g_i g_j \quad \text{if } i \sim j, \tag{2}$$

$$e_i = \frac{l}{m}\left(g_i^2 + mg_i - 1\right) \quad \text{for all } i, \tag{3}$$

$$g_i e_i = l^{-1} e_i \quad \text{for all } i, \tag{4}$$

$$e_i g_j e_i = l e_i \quad \text{if } i \sim j. \tag{5}$$

We will also use some direct consequences of these defining relations (see [4, Proposition 2.1]):

$$e_i g_i = l^{-1} e_i \quad \text{for all } i, \tag{6}$$

$$g_i^2 = 1 - mg_i + ml^{-1} e_i \quad \text{for all } i, \tag{7}$$

$$g_i^{-1} = g_i + m - m\, e_i \quad \text{for all } i \tag{8}$$

as well as the following "mixed braid relations" (see [4, Proposition 2.3]):

$$g_i g_j e_i = e_j e_i \quad \text{if } i \sim j, \tag{9}$$

$$g_i e_j e_i = g_j e_i + m(e_i - e_j e_i) \quad \text{if } i \sim j. \tag{10}$$

1.2.2. The Lawrence–Krammer space

We now recall some terminology associated with root systems of type A_{n-1}. Let $M = (m_{ij})_{1 \leqslant i \leqslant j \leqslant n-1}$ be the Coxeter matrix of type A_{n-1}.

Let $(\alpha_1, \ldots, \alpha_{n-1})$ be the canonical basis of $\mathbb{R}^{n-1}$ and let's define a bilinear form B_M over $\mathbb{R}^{n-1}$ by

$$B_M(\alpha_i, \alpha_j) = -cos\left(\frac{\pi}{m_{ij}}\right).$$

By the theory in [3], B_M is an inner product that we will simply denote by $(\,|\,)$. Let r_i denote the reflection with respect to the hyperplane $\mathrm{Ker}(\alpha_i|.)$ of $\mathbb{R}^{n-1}$, and so

$$\forall x \in \mathbb{R}^{n-1}, \quad r_i(x) = x - 2(\alpha_i|x)\alpha_i.$$

Finally, let ϕ^+ denote the set of $\frac{n(n-1)}{2}$ positive roots

$$\phi^+ = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_3, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_1, \ldots,$$
$$\alpha_{n-1}, \alpha_{n-1} + \alpha_{n-2}, \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1\}.$$

We define $\mathcal{V}^{(n)}$ as the vector space over the field $F = \mathbb{Q}(l, r)$ with basis the vectors x_β's, indexed by the positive roots $\beta \in \phi^+$. Thus, $\dim_F \mathcal{V}^{(n)} = |\phi^+| = \frac{n(n-1)}{2}$. This space $\mathcal{V}^{(n)}$ is the Lawrence–Krammer space (L–K space).

2. The representation

We define the following map on the generators of the BMW algebra

$$\nu^{(n)} : \begin{array}{rcl} B(A_{n-1}) & \longrightarrow & End_F\left(\mathcal{V}^{(n)}\right), \\ g_i & \longmapsto & \nu_i. \end{array}$$

For each node i, the action of ν_i on x_β is given as follows

$$\nu_i(x_\beta)=\begin{cases} rx_\beta & \text{if } (\beta|\alpha_i)=0\ (a),\\ \frac{1}{l}x_\beta & \text{if } (\beta|\alpha_i)=1\ (b),\\ x_{\beta+\alpha_i} & \text{if } (\beta|\alpha_i)=-\frac{1}{2} \text{ and } (c),\\ x_{\beta+\alpha_i}+mr^{ht(\beta)-1}x_{\alpha_i}-mx_\beta & \text{if } (\beta|\alpha_i)=-\frac{1}{2} \text{ and } (c'),\\ x_{\beta-\alpha_i}+\frac{m}{lr^{ht(\beta)-2}}x_{\alpha_i}-mx_\beta & \text{if } (\beta|\alpha_i)=\frac{1}{2} \text{ and } (d),\\ x_{\beta-\alpha_i} & \text{if } (\beta|\alpha_i)=\frac{1}{2} \text{ and } (d'), \end{cases}$$

where (c), (c'), (d), (d') are the following conditions:

$$\begin{aligned} &(c) \quad \beta=\alpha_t+\cdots+\alpha_{i-1} \quad \text{with } t\leqslant i-1,\\ &(c') \quad \beta=\alpha_{i+1}+\cdots+\alpha_s \quad \text{with } s\geqslant i+1,\\ &(d) \quad \beta=\alpha_t+\cdots+\alpha_i \quad \text{with } t\leqslant i-1,\\ &(d') \quad \beta=\alpha_i+\cdots+\alpha_s \quad \text{with } s\geqslant i+1. \end{aligned}$$

We then define $\nu^{(n)}(e_i)=\frac{l}{m}(\nu_i^2+m\nu_i-id_{\mathcal{V}^{(n)}})$. We have

$$\nu^{(n)}(e_i)(x_\beta)=\begin{cases} 0 & \text{if } (\beta|\alpha_i)=0,\\ (1-\frac{l-\frac{1}{l}}{\frac{1}{r}-r})x_{\alpha_i} & \text{if } (\beta|\alpha_i)=1,\\ \frac{1}{r^{ht(\beta)-1}}x_{\alpha_i} & \text{if } (\beta|\alpha_i)=-\frac{1}{2} \text{ and } (c),\\ r^{ht(\beta)-1}x_{\alpha_i} & \text{if } (\beta|\alpha_i)=-\frac{1}{2} \text{ and } (c'),\\ \frac{1}{lr^{ht(\beta)-2}}x_{\alpha_i} & \text{if } (\beta|\alpha_i)=\frac{1}{2} \text{ and } (d),\\ lr^{ht(\beta)-2}x_{\alpha_i} & \text{if } (\beta|\alpha_i)=\frac{1}{2} \text{ and } (d'). \end{cases}$$

We can check that the map $\nu^{(n)}$ defines a representation of $B(A_{n-1})$ in the L–K space $\mathcal{V}^{(n)}$ (see [10, Chapter 7, pp. 66–68]). We notice that $\nu^{(n)}(e_i)(x_\beta)$ is always a multiple of x_{α_i}. This is an important fact to show the reducibility of the representation for some specializations of its parameters.

3. Reducibility of the representation

We show that when the representation $\nu^{(n)}$ is reducible, the action on a proper invariant subspace of the L–K space is an Iwahori–Hecke algebra action. Indeed, we show that the e_i's act trivially. We then recall some facts about the degrees of the irreducible representations of the Iwahori–Hecke algebra, which we assume to be semisimple. This assumption plays a key role in the proof of the main theorem in Section 4, ruling out some values for r. We will investigate whether the Iwahori–Hecke algebra representations of small degrees may occur in the L–K space and if so for which values of l and r. We show that if there exists a one-dimensional invariant subspace inside $\mathcal{V}^{(n)}$, it forces the value $\frac{1}{r^{2n-3}}$ for l, except when $n=3$ when it forces $l\in\{-r^3,\frac{1}{r^3}\}$. Conversely, we show that for these values of l and r, there exists a one-dimensional invariant subspace of $\mathcal{V}^{(n)}$ and the representation is thus reducible. Similarly, we show that if there exists an irreducible $(n-1)$-dimensional invariant subspace inside $\mathcal{V}^{(n)}$, it forces $l=\frac{1}{r^{n-3}}$ or $l=-\frac{1}{r^{n-3}}$ in the case when $n\neq 4$ and $l\in\{-r^3,\frac{1}{r},-\frac{1}{r}\}$ in the case when $n=4$. Conversely, for each of these values of l and r, there exists an irreducible $(n-1)$-dimensional subspace of $\mathcal{V}^{(n)}$, which shows the reducibility of the representation in these cases as well. We end the section by showing that when $l=r$ or $l=-r^3$, the representation is reducible. We exhibit an invariant subspace of the L–K space that we show to be proper for these values of l and r.

3.1. Action on a proper invariant subspace of the L–K space

We show that if the representation is reducible, then the e_i's act trivially on any proper invariant subspace of the L–K space, which means the action is a Hecke algebra action.

Proposition 1. *For any proper invariant subspace $\mathcal{U}$ of $\mathcal{V}^{(n)}$, we have $\nu^{(n)}(e_i)(\mathcal{U}) = 0$ for all i.*

Proof. Let $\mathcal{U}$ be a proper invariant subspace of $\mathcal{V}^{(n)}$ and let u be a nonzero vector of $\mathcal{U}$ such that $\nu^{(n)}(e_i)(u) \neq 0$. Since $\nu^{(n)}(e_i)(u)$ is a multiple of x_{α_i}, we see that x_{α_i} is in $\mathcal{U}$. From there, we have

$$\nu_{i-1}(x_{\alpha_i}) = x_{\alpha_i+\alpha_{i-1}} + m x_{\alpha_{i-1}} \mod F x_{\alpha_i}.$$

Hence $x_{\alpha_i+\alpha_{i-1}} + m x_{\alpha_{i-1}}$ is in $\mathcal{U}$. Another application of ν_{i-1} now yields

$$\nu_{i-1}(x_{\alpha_i+\alpha_{i-1}} + m x_{\alpha_{i-1}}) = x_{\alpha_i} + \frac{m}{l} x_{\alpha_{i-1}},$$

from which we derive that $x_{\alpha_{i-1}}$ is in $\mathcal{U}$. By induction, we see that all the x_{α_t}'s for $t \leqslant i$ are in $\mathcal{U}$. In particular, x_{α_1} is in $\mathcal{U}$. From there, it is easy to see that all the x_β's are in fact in $\mathcal{U}$. Then $\mathcal{U}$ is the whole L–K space $\mathcal{V}^{(n)}$, in contradiction to our assumption that $\mathcal{U}$ is proper. □

Corollary 1. *Let $\mathcal{W}$ be a proper irreducible invariant subspace of $\mathcal{V}^{(n)}$. Then, $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(n)$-module.*

Proof. Let $\mathcal{W}$ be a proper irreducible invariant subspace of $\mathcal{V}^{(n)}$. By Proposition 1 and defining relation (3), we have

$$\left[g_i^2 + m g_i - 1\right].\mathcal{W} = 0 \quad \text{for all } i.$$

Hence $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(n)$-module. □

We now recall some general facts about the irreducible representations of the Iwahori–Hecke algebra of the symmetric group. The following two propositions were established by James for the symmetric group $Sym(n)$. They remain true for the Iwahori–Hecke algebra $\mathcal{H}_{F,r^2}(n)$ since we work in characteristic zero and assume $\mathcal{H}_{F,r^2}(n)$ to be semisimple (see [11]).

Proposition 2. *Let n be an integer with $n \geqslant 7$. Assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple. Then, every irreducible $\mathcal{H}_{F,r^2}(n)$-module is either isomorphic to one of the Specht modules $S^{(n)}$, $S^{(1^n)}$, $S^{(n-1,1)}$, $S^{(2,1^{n-2})}$ or has dimension greater than $(n-1)$.*

Proof. It follows from Theorem 6, point (i) of [7]. □

We note that the statement is also true when $n = 3$ and $n = 5$. When $n = 4$, the statement fails as $S^{(2,2)}$ has dimension 2 and when $n = 6$, the statement also fails since $S^{(3,3)}$ and $S^{(2,2,2)}$ both have dimension 5.

When the integer n is greater than or equal to 9, there exist even better estimates of the dimensions of the irreducible $\mathcal{H}_{F,r^2}(n)$-modules, as follows.

Proposition 3. *Let n be an integer with $n \geqslant 9$. Assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple. Then, every irreducible $\mathcal{H}_{F,r^2}(n)$-module is either isomorphic to one of the Specht modules $S^{(n)}$, $S^{(n-1,1)}$, $S^{(n-2,2)}$, $S^{(n-2,1,1)}$ or to one of their conjugates, or has dimension greater than $\frac{(n-1)(n-2)}{2}$.*

Proof. It follows from Theorem 7 of [7] with $N = 9$. □

We have the corollary on the dimensions.

Corollary 2.

(i) *Let n be an integer with $n \geqslant 3$ and $n \notin \{4, 8\}$. Assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple. Let $\mathcal{D}$ be an irreducible $\mathcal{H}_{F,r^2}(n)$-module. Then, there are two possibilities*:

$$\textit{either} \quad \dim \mathcal{D} \in \left\{1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}\right\},$$
$$\textit{or} \quad \dim \mathcal{D} > \frac{(n-1)(n-2)}{2}.$$

(ii) *Assume that $\mathcal{H}_{F,r^2}(4)$ is semisimple. Let $\mathcal{D}$ be an irreducible $\mathcal{H}_{F,r^2}(4)$-module. Then* $\dim \mathcal{D} \in \{1, 2, 3\}$.

(iii) *Assume that $\mathcal{H}_{F,r^2}(8)$ is semisimple. Let $\mathcal{D}$ be an irreducible $\mathcal{H}_{F,r^2}(8)$-module. Then* $\dim \mathcal{D} \in \{1, 7, 14, 20, 21\}$ *or* $\dim \mathcal{D} > 21$.

Proof. Points (ii) and (iii) can be seen directly by using the Hook formula. Point (i) is for $n \geqslant 9$ a direct consequence of Proposition 3 after noticing that $S^{(n-2,2)}$ has dimension $\frac{n(n-3)}{2}$ and $S^{(n-2,1,1)}$ dimension $\frac{(n-1)(n-2)}{2}$. For smaller n, the statement also holds by direct investigation using the Hook formula. □

Corollary 1 and Corollary 2 imply that any proper irreducible invariant subspace of the L–K space $\mathcal{V}^{(n)}$ has dimension $1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}$ or dimension greater than $\frac{(n-1)(n-2)}{2}$, except in the special cases when $n \in \{4, 8\}$. Next, we investigate the existence of a one-dimensional invariant subspace of $\mathcal{V}^{(n)}$. We define for two nodes i and j with $i < j$

$$w_{ij} = x_{\alpha_i + \cdots + \alpha_{j-1}}.$$

We will sometimes write $w_{i,j}$ instead of w_{ij}.

3.2. The case $l = \frac{1}{r^{2n-3}}$

We will show the existence of a one-dimensional invariant subspace of the L–K space when $l = \frac{1}{r^{2n-3}}$. We prove the following theorem.

Theorem 3. *Let n be an integer with $n \geqslant 3$ and assume $(r^2)^2 \neq 1$.*

Suppose $n \geqslant 4$. There exists a one-dimensional invariant subspace of $\mathcal{V}^{(n)}$ if and only if $l = \frac{1}{r^{2n-3}}$. If so, it is spanned by $\sum_{1 \leqslant s < t \leqslant n} r^{s+t} w_{st}$.

Suppose $n = 3$. There exists a one-dimensional invariant subspace of $\mathcal{V}^{(3)}$ if and only if $l = \frac{1}{r^3}$ or $l = -r^3$.

Moreover, if $r^6 \neq -1$, it is unique and

$$\textit{when} \quad l = \frac{1}{r^3}, \quad \textit{it is spanned by } w_{12} + r w_{13} + r^2 w_{23},$$
$$\textit{when} \quad l = -r^3, \quad \textit{it is spanned by } w_{12} - \frac{1}{r} w_{13} + \frac{1}{r^2} w_{23}.$$

If $r^6 = -1$, there are exactly two one-dimensional invariant subspaces of $\mathcal{V}^{(3)}$ and they are respectively spanned by the vectors above.

Proof. Let $\mathcal{U}$ be a one-dimensional invariant subspace of $\mathcal{V}^{(n)}$ and let u be a spanning vector of $\mathcal{U}$. For each i, let γ_i be the scalar such that $\nu_i(u) = \gamma_i u$. Since by Proposition 1 we have $(\nu_i^2 + m\nu_i - id_{\mathcal{V}^{(n)}})(u) = 0$, it follows that $\gamma_i^2 + m\gamma_i - 1 = 0$. Hence $\gamma_i \in \{r, -\frac{1}{r}\}$. Further, since $(r^2)^2 \neq 1$, the braid relation $\nu_i\nu_j\nu_i = \nu_j\nu_i\nu_j$ when $i \sim j$ forces that γ_i takes the same value as γ_j. Let's denote by γ the common value of the γ_i's. So, for each i, we have $\nu_i(u) = \gamma u$ with $\gamma \in \{r, -\frac{1}{r}\}$. $\square$

A general form for u is

$$u = \sum_{1 \leqslant i < j \leqslant n} \mu_{ij} w_{ij}, \quad \text{where } \mu_{ij} \in F.$$

We look for relations between these coefficients. We will use the following lemma.

Lemma 1. *Let i be some node. Suppose $v = \sum_{1 \leqslant k < f \leqslant n} \mu_{kf} w_{kf}$ is a vector of $\mathcal{V}^{(n)}$ with $\nu_i(v) = \gamma v$, where $\gamma \in \{r, -\frac{1}{r}\}$. Then the following equalities hold for the coefficients of v:*

$$\forall s \geqslant i+2, \quad \mu_{i+1,s} = \gamma \mu_{i,s}, \tag{11}$$

$$\forall t \leqslant i-1, \quad \mu_{t,i+1} = \gamma \mu_{t,i}. \tag{12}$$

When $i = 1$, only (11) holds and when $i = n-1$, only (12) holds.

Proof. To show (11), we look at the coefficient of $w_{i+1,s}$ in $\nu_i(v) = \gamma v$, where $s \geqslant i+2$. We get $\mu_{i,s} - m\mu_{i+1,s} = \gamma \mu_{i+1,s}$. Since $\gamma + m = \frac{1}{\gamma}$, this equality is equivalent to $\mu_{i+1,s} = \gamma \mu_{i,s}$. Similarly, by equating the coefficients of $w_{t,i+1}$ in $\nu_i(v) = \gamma v$, we obtain (12). $\square$

Applying these equalities to the coefficients of u, we see that all the coefficients of u must be nonzero. In particular, when $n \geqslant 4$, the coefficient μ_{34} of u is nonzero. Because an action of g_1 on w_{34} is a multiplication by r and an action of g_1 on the other terms w_{ij} does not create any term in w_{34}, this forces $\gamma = r$. Thus, without loss of generality, we have

$$u = \sum_{1 \leqslant i < j \leqslant n} r^{i+j} w_{ij}.$$

From there, we look at the action of g_1 on u and the resulting coefficient in w_{12}. The action of g_1 on w_{12} is a multiplication by l^{-1} and an action of g_1 on the $w_{2,j}$'s for $3 \leqslant j \leqslant n$ creates new terms in w_{12} with respective coefficients mr^{j-3}. Thus, we get the equation:

$$\frac{r^3}{l} + \frac{m}{r} \sum_{j=3}^{n} (r^2)^j = r^4,$$

from which we derive that $l = \frac{1}{r^{2n-3}}$.

Conversely, if $l = \frac{1}{r^{2n-3}}$, we define u as $\sum_{1 \leqslant i < j \leqslant n} r^{i+j} w_{ij}$ and check that $\nu_i(u) = ru$ for each i. For details, see [10, §8.2].

This ends the proof of the theorem when $n \geqslant 4$. The case $n = 3$ is different in that γ can take either values r or $-\frac{1}{r}$ forcing in one case $l = \frac{1}{r^3}$ and in the other case $l = -r^3$. Details appear in [10, §8.2].

3.3. The case $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$

In Theorem 4, the case $n = 3$ was special. Likewise, in the following theorem, the case $n = 4$ needs to be formulated separately.

Theorem 4. *Let n be a positive integer with $n \geqslant 3$ and $n \neq 4$. Assume $\mathcal{H}_{F,r^2}(n)$ is semisimple. Then, there exists an irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}^{(n)}$ if and only if $l = \frac{1}{r^{n-3}}$ or $l = -\frac{1}{r^{n-3}}$.*

If so, it is spanned by the $v_i^{(n)}$'s, $1 \leqslant i \leqslant n-1$, where $v_i^{(n)}$ is defined by the formula:

$$v_i^{(n)} = \left(\frac{1}{r} - \frac{1}{l}\right) w_{i,i+1} + \sum_{s=i+2}^{n} r^{s-i-2}\left(w_{i,s} - \frac{1}{r} w_{i+1,s}\right) + \epsilon_l \sum_{t=1}^{i-1} r^{n-i-2+t}\left(w_{t,i} - \frac{1}{r} w_{t,i+1}\right)$$

with

$$\begin{cases} \epsilon_{\frac{1}{r^{n-3}}} = 1, \\ \epsilon_{-\frac{1}{r^{n-3}}} = -1. \end{cases}$$

Suppose $n = 4$ and assume $\mathcal{H}_{F,r^2}(4)$ is semisimple. Then, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$ if and only if $l \in \{\frac{1}{r}, -\frac{1}{r}, -r^3\}$.

If $l \in \{-\frac{1}{r}, \frac{1}{r}\}$, it is spanned by $v_1^{(4)}$, $v_2^{(4)}$, $v_3^{(4)}$.

If $l = -r^3$, it is spanned by the vectors:

$$\begin{cases} u_1 = r w_{23} + w_{13} + \left(\frac{1}{r} + \frac{1}{r^3}\right) w_{34} - w_{24} - \frac{1}{r} w_{14}, \\ u_2 = -r w_{12} - r^2 w_{13} - \frac{1}{r} w_{34} - \frac{1}{r^2} w_{24} + \left(r + \frac{1}{r}\right) w_{14}, \\ u_3 = (r + r^3) w_{12} + \frac{1}{r} w_{23} - w_{13} + w_{24} - r w_{14}. \end{cases}$$

Proof. Suppose that there exists an irreducible $(n-1)$-dimensional invariant subspace $\mathcal{U}$ of $\mathcal{V}^{(n)}$.

Claim 1. *Except in the case when $n = 6$, there exists a basis $(v_1, \ldots, v_{n-1})$ of $\mathcal{U}$ such that one of the following two sets of relations holds*

$$(\triangle)\quad \left| \begin{array}{ll} \nu_t(v_i) = r v_i, & \forall t \notin \{i-1, i, i+1\}, \\ \nu_i(v_i) = -\frac{1}{r} v_i, & \forall 1 \leqslant i \leqslant n-1, \\ \nu_{i+1}(v_i) = r(v_i + v_{i+1}), & \forall 1 \leqslant i \leqslant n-2, \\ \nu_{i-1}(v_i) = r v_i + \frac{1}{r} v_{i-1}, & \forall 2 \leqslant i \leqslant n-1, \end{array} \right.$$

$$(\triangledown)\quad \left| \begin{array}{ll} \nu_t(v_i) = -1/r v_i, & \forall t \notin \{i-1, i, i+1\}, \\ \nu_i(v_i) = r v_i, & \forall 1 \leqslant i \leqslant n-1, \\ \nu_{i+1}(v_i) = -1/r(v_i + v_{i+1}), & \forall 1 \leqslant i \leqslant n-2, \\ \nu_{i-1}(v_i) = -1/r v_i - r v_{i-1}, & \forall 2 \leqslant i \leqslant n-1. \end{array} \right.$$

Proof. By Proposition 2, there are exactly two inequivalent irreducible representations of $\mathcal{H}_{F,r^2}(n)$ of degree $(n-1)$, except in the case $n=6$, when there are exactly four inequivalent irreducible representations of $\mathcal{H}_{F,r^2}(6)$ of degree 5. Consider now the set of relations $(\triangle)$ (resp. $(\triangledown)$). For each i, let M_i (resp. N_i) be the matrix of the endomorphism ν_i in the basis $(v_1,\dots,v_{n-1})$. It is a straightforward verification that the M_i's (resp. N_i's) satisfy the braid relations and the relation $M_i^2+mM_i=I_{n-1}$ (resp. $N_i^2+mN_i=I_{n-1}$) for each i, where I_{n-1} is the identity matrix of size $(n-1)$. Hence the M_i's (resp. the N_i's) yield a matrix representation of $\mathcal{H}_{F,r^2}(n)$ of degree $(n-1)$. To show that these two matrix representations are irreducible, relying on Proposition 2, it suffices to check that there is no one-dimensional invariant subspace of F^{n-1}. This is the case when $r^{2n}\neq 1$. When $n=3$, the two matrix representations are equivalent. When $n\geqslant 4$, they are not: visibly, the matrices of one representation all have the same trace $-\frac{(n-2)}{r}+r$ and the matrices of the other representation all have the same trace $(n-2)r-\frac{1}{r}$. These two values are distinct when $(r^2)^2\neq 1$ and $n\geqslant 4$. We conclude that these two matrix representations are the two inequivalent irreducible representations of $\mathcal{H}_{F,r^2}(n)$ when $n\geqslant 4$ and $n\neq 6$. $\square$

For $n\geqslant 4$, we can show that it is impossible to have the second set of relations, except in the case $n=4$ when it forces $l=-r^3$. For a detailed proof of this fact, see [10, Chapter 8, pp. 81–95]. Let $n\geqslant 3$ and suppose that the v_i's satisfy $(\triangle)$. The relation $\nu_i(v_i)=-\frac{1}{r}v_i$ implies that in v_i there are no terms in w_{ts} for $s\leqslant i-1$ or $t\geqslant i+2$ or $t\leqslant i-1$ and $s\geqslant i+2$. Thus, a general form for v_i must be

$$\begin{aligned} v_i = \mu_{i,i+1}w_{i,i+1} + \sum_{s=i+2}^{n}\mu_{i,s}w_{i,s} + \sum_{s=i+2}^{n}\mu_{i+1,s}w_{i+1,s} \\ + \sum_{t=1}^{i-1}\mu_{t,i}w_{t,i} + \sum_{t=1}^{i-1}\mu_{t,i+1}w_{t,i+1}. \end{aligned} \tag{13}$$

Since $\nu_i(v_i)=-\frac{1}{r}v_i$, both equalities (11) and (12) hold with $\gamma=-\frac{1}{r}$. Further, since $\nu_q(v_i)=rv_i$ for $q\notin\{i-1,i,i+1\}$, applying (11) and (12) with $i=q$ and $\gamma=r$ yields

$$\forall j\geqslant q+2,\quad \mu_{q+1,j}=r,\mu_{q,j}, \tag{14}$$

$$\forall k\leqslant q-1,\quad \mu_{k,q+1}=r\mu_{k,q}. \tag{15}$$

Apply (14) with $q\leqslant i-2$ and $j\in\{i,i+1\}$ to get

$$\forall q\leqslant i-2,\quad \mu_{q+1,i}=r\mu_{q,i}\quad \&\quad \mu_{q+1,i+1}=r\mu_{q,i+1}.$$

Apply (15) with $q\geqslant i+2$ and $k\in\{i,i+1\}$ to get

$$\forall q\geqslant i+2,\quad \mu_{i,q+1}=r\mu_{i,q}\quad \&\quad \mu_{i+1,q+1}=r\mu_{i+1,q}.$$

Expression (13) can now be rewritten as follows.

$$v_i=\zeta^{(i)}w_{i,i+1}+\delta^{(i)}\sum_{s=i+2}^{n}r^{s-i-2}\left(w_{i,s}-\frac{1}{r}w_{i+1,s}\right)+\lambda^{(i)}\sum_{t=1}^{i-1}r^{t-1}\left(w_{t,i}-\frac{1}{r}w_{t,i+1}\right),$$

where $\zeta^{(i)}$, $\delta^{(i)}$ and $\lambda^{(i)}$ are three coefficients to determine. First, we show that all the $\delta^{(i)}$ with $i\in\{1,\dots,n-2\}$ may be set to the value one. Notice that if $v_1,\dots,v_{n-1}$ satisfy $(\triangle)$, then $\delta v_1,\dots,\delta v_{n-1}$ also satisfy $(\triangle)$, where δ is any nonzero scalar. Then, without loss of generality, we set $\delta^{(1)}=1$.

Suppose $\delta^{(i)}=1$ for some node i with $1\leqslant i\leqslant n-2$. We will show that $\delta^{(i+1)}=1$. Notice that $\delta^{(i+1)}$ is the coefficient of $w_{i+1,i+3}$ in v_{i+1}. Since an action of g_{i+1} on v_i never creates a term in $w_{i+1,i+3}$, by looking at the coefficient of $w_{i+1,i+3}$ in $\nu_{i+1}(v_i)=rv_i+rv_{i+1}$, we get $0=-r\delta^{(i)}+r\delta^{(i+1)}$. After replacing $\delta^{(i)}$ by 1, this yields $\delta^{(i+1)}=1$. Thus, all the $\delta^{(i)}$ may be set to the value 1. It remains to find the coefficients $\zeta^{(i)}$ and $\lambda^{(i)}$. By looking at the coefficient of $w_{i,i+1}$ in $\nu_{i+1}(v_i)=r(v_i+v_{i+1})$, we get

$$r\zeta^{(i)}+r^i\lambda^{(i+1)}=1,\quad \text{for each } i \text{ with } 1\leqslant i\leqslant n-2. \tag{16}$$

Also, by looking at the coefficient of the same term $w_{i,i+1}$ in the relation $\nu_{i-1}(v_i)=rv_i+\frac{1}{r}v_{i-1}$, we get

$$-m\zeta^{(i)}-r^{i-3}\lambda^{(i)}=r\zeta^{(i)}-\frac{1}{r^2},\quad \text{for each } i \text{ with } 2\leqslant i\leqslant n-1.$$

After multiplication by a factor r^2, we obtain

$$r\zeta^{(i)}+r^{i-1}\lambda^{(i)}=1,\quad \text{for each } i \text{ with } 2\leqslant i\leqslant n-1. \tag{17}$$

By (16) and (17), we get $\lambda^{(i)}=\frac{1}{r^{i-2}}\lambda^{(2)}$, for all $i\geqslant 2$. Further, change indices in (16) to get

$$r\zeta^{(i-1)}+r^{i-1}\lambda^{(i)}=1\quad \text{for each } i \text{ with } 2\leqslant i\leqslant n-1. \tag{18}$$

Now (17) and (18) show that $\zeta^{(i)}=\zeta^{(i-1)}$ for each i with $2\leqslant i\leqslant n-1$. In other words, all the $\zeta^{(i)}$ are equal with a certain scalar ζ. The relation between ζ and $\lambda^{(2)}$ is given by Eq. (18) with $i=2$

$$\lambda^{(2)}=\frac{1}{r}-\zeta. \tag{19}$$

Thus, by determining ζ, we will get a complete expression for all the vectors v_i's. Since we have

$$v_1=\zeta w_{12}+\sum_{s=3}^{n}r^{s-3}\left(w_{1,s}-\frac{1}{r}w_{2,s}\right),$$

by looking at the coefficient of w_{12} in the relation $\nu_1(v_1)=-\frac{1}{r}v_1$, we get the equation:

$$\zeta\left(\frac{1}{l}+\frac{1}{r}\right)=\frac{1}{r^2}-\left(r^2\right)^{n-3}. \tag{20}$$

Further, by looking at the coefficient of $w_{i,i+1}$ in $\nu_i(v_i)=-\frac{1}{r}v_i$, we have

$$\zeta\left(\frac{1}{l}+\frac{1}{r}\right)=\sum_{s=i+2}^{n}r^{s-i-3}mr^{s-i-2}+\lambda^{(i)}\sum_{t=1}^{i-1}r^{t-2}\frac{m}{lr^{i-t-1}}$$

i.e.

$$\zeta\left(\frac{1}{l}+\frac{1}{r}\right)=\frac{1}{r^2}-\left(r^2\right)^{n-i-2}+\frac{\lambda^{(i)}}{l}\left(\frac{1}{r^i}-r^{i-2}\right). \tag*{$(\star)_i$}$$

Now write down $(\star)_2$ and $(\star)_3$:

$$\zeta\left(\frac{1}{l}+\frac{1}{r}\right)=\frac{1}{r^2}-\left(r^2\right)^{n-4}+\frac{\lambda^{(2)}}{l}\left(\frac{1}{r^2}-1\right), \tag*{$(\star)_2$}$$

$$\zeta\left(\frac{1}{l}+\frac{1}{r}\right)=\frac{1}{r^2}-\left(r^2\right)^{n-5}+\frac{\lambda^{(2)}}{lr}\left(\frac{1}{r^3}-r\right), \tag*{$(\star)_3$}$$

where $\lambda^{(3)}$ has been replaced with $\frac{\lambda^{(2)}}{r}$. Subtract these two equalities to get

$$\frac{\lambda^{(2)}}{l}\left(\frac{1}{r^2}-\frac{1}{r^4}\right)=\left(r^2\right)^{n-4}\left(1-\frac{1}{r^2}\right). \tag*{$(\star)_2$–$(\star)_3$}$$

After multiplying this equality by $\frac{1}{r^2}$ and dividing it by $\frac{1}{r^2}-\frac{1}{r^4}$ (recall that $m\neq 0$), we obtain

$$\lambda^{(2)}=l\left(r^2\right)^{n-3}.$$

Hence, by (19), $\zeta=\frac{1}{r}-l(r^2)^{n-3}$. Plugging this value for ζ into (20) now yields

$$l^2=\frac{1}{(r^2)^{n-3}},\quad \text{hence } l\in\left\{\frac{1}{r^{n-3}},-\frac{1}{r^{n-3}}\right\}.$$

If $l=\frac{1}{r^{n-3}}$, we get successively $\lambda^{(2)}=r^{n-3}=\frac{1}{l}$, $\zeta=\frac{1}{r}-\frac{1}{l}$ and $\lambda^{(i)}=r^{n-i-1}$.

If $l=-\frac{1}{r^{n-3}}$, $\lambda^{(2)}$ and ζ are still respectively $\frac{1}{l}$ and $\frac{1}{r}-\frac{1}{l}$ and $\lambda^{(i)}=-r^{n-i-1}$.

We obtain the formula announced in Theorem 4.

Conversely, if $l\in\{\frac{1}{r^{n-3}},-\frac{1}{r^{n-3}}\}$, we can show that the $v_i^{(n)}$'s defined in Theorem 4 satisfy the relations $(\triangle)$ (see [10, §8.3]). In particular, their linear span over F is a proper invariant subspace of $\mathcal{V}^{(n)}$, hence is an $\mathcal{H}_{F,r^2}(n)$-module by Corollary 1. When $n\neq 4$, if the vectors $v_i^{(n)}$'s were linearly dependent, then their linear span would either be one-dimensional or would contain a one-dimensional $\mathcal{H}_{F,r^2}(n)$-submodule, as there is no irreducible $\mathcal{H}_{F,r^2}(n)$-module of dimension between 1 and $(n-1)$ by Corollary 2. In any case, by Theorem 3, that would force $l=\frac{1}{r^{2n-3}}$ when $n\neq 3$ and $l\in\{-r^3,\frac{1}{r^3}\}$ when $n=3$. This is impossible with our assumption that $l\in\{\frac{1}{r^{n-3}},-\frac{1}{r^{n-3}}\}$ and the fact that $r^{2n}\neq 1$. As for $n=4$, the freedom over F of the family of vectors $(v_1^{(4)},v_2^{(4)},v_3^{(4)})$ is a straightforward verification. We are now able to conclude: the vector space $\mathrm{Span}_F(v_1^{(n)},\dots,v_{n-1}^{(n)})$ is $(n-1)$-dimensional, is invariant under the action by the g_i's and is an $\mathcal{H}_{F,r^2}(n)$-module since it is a proper invariant subspace of $\mathcal{V}^{(n)}$. Then, by the relations satisfied by the $v_i^{(n)}$'s, it must be irreducible.

To complete the proof of Theorem 4, we show that there does not exist any irreducible 5-dimensional invariant subspace of $\mathcal{V}^{(6)}$ that is isomorphic to one of the Specht modules $S^{(3,3)}$ or $S^{(2,2,2)}$. Indeed, suppose such a subspace exists and name it $\mathcal{W}$. Since we have assumed that $\mathcal{H}_{F,r^2}(6)$ is semisimple, we may use the branching rule as it appears in Corollary 6.2 of [11]. We have

$$S^{(3,3)}\downarrow_{\mathcal{H}_{F,r^2}(5)}\simeq S^{(3,2)},\qquad S^{(2,2,2)}\downarrow_{\mathcal{H}_{F,r^2}(5)}\simeq S^{(2,2,1)}.$$

We will show that the restriction of $\mathcal{W}$ to $\mathcal{H}_{F,r^2}(5)$ cannot be isomorphic to $S^{(3,2)}$ or $S^{(2,2,1)}$, hence a contradiction. A proof of the following fact is in [10, §8.3].

Fact. Suppose $\mathcal{H}_{F,r^2}(5)$ is semisimple. Then, up to equivalence, the two irreducible matrix representations of degree 5 of $\mathcal{H}_{F,r^2}(5)$ are respectively defined by the matrices P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 given by

$$P_1 := \begin{bmatrix} r & & & & \\ & r & & & \\ & & r & & \\ 1 & & -r^2 & -\frac{1}{r} & \\ & 1 & & & -\frac{1}{r} \end{bmatrix}, \qquad P_2 := \begin{bmatrix} -\frac{1}{r} & & & 1 & \\ & -\frac{1}{r} & 1 & & 1 \\ & & r & & \\ & & & r & \\ & & & & r \end{bmatrix},$$

$$P_3 := \begin{bmatrix} r & & & & \\ & r & & & \\ & 1 & -\frac{1}{r} & & \\ 1 & & & -\frac{1}{r} & -r^2 \\ & & & & r \end{bmatrix}, \qquad P_4 := \begin{bmatrix} & 1 & -r & & \\ 1 & r-\frac{1}{r} & 1 & & \\ & & r & & \\ & & -r^2 & & 1 \\ & & r & 1 & r-\frac{1}{r} \end{bmatrix},$$

where the blanks must be filled with zeros; the matrices Q_i's are defined from the matrices P_i's by replacing r with $-\frac{1}{r}$.

In [10, Chapter 8, pp. 125–128], it is shown that it is impossible to have a basis $(w_1, w_2, w_3, w_4, w_5)$ of $\mathcal{W}$ in which the matrices of the left action by the g_i's, $i = 1, \dots, 4$ are the Q_i's. In particular, by considering $\mathcal{W}$ as a subspace of $\mathcal{V}^{(5)}$ instead of a subspace of $\mathcal{V}^{(6)}$, the computations of [10] also show that:

Result 1. The irreducible matrix representation of degree 5 of $\mathcal{H}_{F,r^2}(5)$ defined by the matrices Q_i's is not a constituent of the Lawrence–Krammer representation of degree 10 of the BMW algebra of type A_4.

Suppose now that there exists a basis $(w_1, w_2, w_3, w_4, w_5)$ of $\mathcal{W}$ in which the matrices of the left action by the g_i's, $i = 1, \dots, 4$ are the P_i's. We read from the matrices P_1 and P_3 that $g_1.w_4 = -\frac{1}{r} w_4$ and $g_3.w_4 = -\frac{1}{r} w_4$. Thus, we have

$$w_4 = \mu_{13}^{(4)} w_{13} + \mu_{23}^{(4)} w_{23} + \mu_{24}^{(4)} w_{24} + \mu_{14}^{(4)} w_{14},$$

where the coefficients are related by $\mu_{14}^{(4)} = \mu_{23}^{(4)} = -\frac{1}{r}\mu_{13}^{(4)} = -r\mu_{24}^{(4)}$. In particular, all these coefficients are nonzero. The other spanning vectors of $\mathcal{W}$ are

$$\begin{aligned} w_5 &= g_4.w_4, \\ w_1 &= g_2.w_4 - r w_4, \\ w_2 &= g_4.w_1, \\ w_3 &= g_3.w_2 - r w_2. \end{aligned}$$

A quick glance at these equations shows that the node number 6 never appears in any of the w_i's. But $\mathcal{W}$ is an invariant subspace of $\mathcal{V}^{(6)}$. In particular, it must be invariant under the action by g_5. This is not compatible with the expression for w_5. We conclude that it is impossible to have

$$\mathcal{W}\downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(3,2)} \quad \text{or} \quad \mathcal{W}\downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(2,2,1)}$$

and so $\mathcal{W}$ cannot be isomorphic to $S^{(3,3)}$ or $S^{(2,2,2)}$. Thus, by the first part of the proof, the existence of an irreducible 5-dimensional invariant subspace of $\mathcal{V}^{(6)}$ implies that $l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\}$. This completes the proof of the theorem. □

As in Result 1, by considering $\mathcal{W}$ as a subspace of $\mathcal{V}^{(5)}$ instead of a subspace of $\mathcal{V}^{(6)}$, we can show the following result.

Result 2. Assume $\mathcal{H}_{F,r^2}(5)$ is semisimple. If there exists an irreducible 5-dimensional invariant subspace of $\mathcal{V}^{(5)}$ then $l = r$.

3.4. The cases $l = r$ and $l = -r^3$

In this section, we show that when $l = r$ the representation $\nu^{(n)}$ is reducible for all $n \geqslant 4$ and when $l = -r^3$, the representation is reducible for all $n \geqslant 3$. We introduce an invariant subspace of $\mathcal{V}^{(n)}$ that we show to be nontrivial for these values of l and r.

Proposition 4. *For any two nodes i and j with $1 \leqslant i < j \leqslant n$, define*

$$\begin{cases} c_{ij} = g_{j-1} \dots g_{i+1} e_i g_{i+1}^{-1} \dots g_{j-1}^{-1} & \text{if } j \geqslant i+2, \\ c_{i,i+1} = e_i. \end{cases}$$

Then, $K(n) = \bigcap_{1 \leqslant i < j \leqslant n} \operatorname{Ker} \nu^{(n)}(c_{ij})$ is an invariant subspace of $\mathcal{V}^{(n)}$. Moreover, any proper invariant subspace of $\mathcal{V}^{(n)}$ must be contained in $K(n)$.

Proof. $K(n)$ is not the whole L–K space, as is visible from the expressions for $\nu^{(n)}(e_i)$. We can check that $K(n)$ is a B-module. Verification of this fact is tedious and can be found in [10, §2]. Let $\mathcal{W}$ be a proper invariant subspace of $\mathcal{V}^{(n)}$. By Proposition 1, we have $\nu^{(n)}(c_{i,i+1})(\mathcal{W}) = 0$ for all i with $1 \leqslant i \leqslant n-1$. Then $\nu^{(n)}(c_{i,j})(\mathcal{W}) = 0$ for all i and j with $1 \leqslant i < j \leqslant n$. Hence $\mathcal{W}$ must be contained in $K(n)$. □

To show that $\nu^{(n)}$ is reducible, it will suffice to exhibit a nontrivial element in $K(n)$ when $l = r$ or $l = -r^3$. The following proposition shows that $K(4)$ is irreducible when $l = r$ and $\mathcal{H}_{F,r^2}(4)$ is semisimple.

Proposition 5. *Assume $\mathcal{H}_{F,r^2}(4)$ is semisimple. There exists an irreducible 2-dimensional invariant subspace of $\mathcal{V}^{(4)}$ if and only if $l = r$. If so it is unique and it is $K(4)$. Moreover, it is spanned over F by the two linearly independent vectors*:

$$v_1 = w_{13} - \frac{1}{r} w_{23} + \frac{1}{r^2} w_{24} - \frac{1}{r} w_{14},$$
$$v_2 = w_{12} - \frac{1}{r} w_{13} - \frac{1}{r} w_{24} + \frac{1}{r^2} w_{34}.$$

Proof. See the proofs of Result 2, p. 154 and Corollary 5, p. 159 of [10]. □

The next proposition shows the reducibility of the representation when $l = r$ and $n \geqslant 4$.

Proposition 6. *Assume $l = r$. Then the vector $v_1 = w_{13} - \frac{1}{r} w_{23} + \frac{1}{r^2} w_{24} - \frac{1}{r} w_{14}$ of Proposition 5 belongs to $K(n)$ for all $n \geqslant 4$. Thus, $\nu^{(n)}$ is reducible for every $n \geqslant 4$.*

Proof. For $n=4$, the result is contained in Proposition 5. When $i \geqslant 5$, we simply have for any $j \geqslant i+2$, $\nu_{i+1}^{-1}\dots\nu_{j-1}^{-1}(v_1) = \frac{1}{r^{j-i-1}}v_1$ and since $\nu^{(n)}(e_i)(v_1)=0$, we see that v_1 is thus annihilated by all the $\nu^{(n)}(c_{ij})$'s with $i \geqslant 5$. Also, as we saw in Proposition 5, the vector v_1 is in $K(4)$, hence it is annihilated by all the $\nu^{(n)}(c_{ij})$'s with $j \leqslant 4$. Thus, it suffices to check that v_1 is annihilated by $\nu^{(n)}(c_{1j})$, $\nu^{(n)}(c_{2j})$, $\nu^{(n)}(c_{3j})$ and $\nu^{(n)}(c_{4j})$ for all $j \geqslant 5$. We will use the following formulas that give the action of the c_{ij}'s on the basis vectors of the L–K space in some relevant cases here.

$$\left|\begin{array}{ll} \nu^{(n)}(c_{ij})(w_{i,j-k}) = \dfrac{1}{lr^{k-1}}w_{ij}, & (R_k)_{1\leqslant k\leqslant j-i-1}, \\ \nu^{(n)}(c_{ij})(w_{i-k,i}) = \dfrac{1}{r^{(k-1)+(j-i-1)}}w_{ij}, & (L_{j-i,k})_{1\leqslant k\leqslant i-1}, \\ \nu^{(n)}(c_{ij})(w_{i-t,j-s}) = \left(\dfrac{1}{r^{t+s-1}} - \dfrac{1}{r^{t+s-3}}\right)\left(\dfrac{1}{l}-\dfrac{1}{r}\right)w_{ij}, & (C_{t,s})_{1\leqslant t\leqslant i-1, 1\leqslant s\leqslant j-i-1}. \end{array}\right.$$

These formulas can be obtained by using the isomorphism between the BMW algebra and the tangle algebra of Morton and Traczyk (see [12]). The use of the tangles allows us to derive algebraic relations by a geometric approach as in Appendix C of [10].

When $l=r$, we note that the action of $c_{i,j}$ on $w_{i-t,j-s}$ is zero. From there, we have for $j \geqslant 5$, where we replaced l by r:

$$\begin{cases} \nu^{(n)}(c_{1,j})(v_1) = \nu^{(n)}(c_{1,j})\left(w_{13} - \dfrac{1}{r}w_{14}\right) = 0 & \text{by } (R_{j-3}) \text{ and } (R_{j-4}), \\ \nu^{(n)}(c_{2,j})(v_1) = \nu^{(n)}(c_{2,j})\left(-\dfrac{1}{r}w_{23} + \dfrac{1}{r^2}w_{24}\right) = 0 & \text{by } (R_{j-3}) \text{ and } (R_{j-4}), \\ \nu^{(n)}(c_{3,j})(v_1) = \nu^{(n)}(c_{3,j})\left(w_{13} - \dfrac{1}{r}w_{23}\right) = 0 & \text{by } (L_{j-3,2}) \text{ and } (L_{j-3,1}), \\ \nu^{(n)}(c_{4,j})(v_1) = \nu^{(n)}(c_{4,j})\left(\dfrac{1}{r^2}w_{24} - \dfrac{1}{r}w_{14}\right) = 0 & \text{by } (L_{j-4,2}) \text{ and } (L_{j-4,3}). \end{cases}$$

So v_1 is in $K(n)$ for all $n \geqslant 4$, as announced. It will be useful to notice that by the game of the coefficients, the equalities to the right of the first two lines of equations still hold when $l=-r^3$. □

When $l=-r^3$, we have a similar result.

Proposition 7. *When $l=-r^3$, the vector u_1 defined as in Theorem 4 by the expression $u_1 = rw_{23}+w_{13}+(\frac{1}{r}+\frac{1}{r^3})w_{34} - w_{24} - \frac{1}{r}w_{14}$ belongs to $K(n)$ for all $n \geqslant 4$. Thus, when $l=-r^3$, the representation $\nu^{(n)}$ is reducible for every $n \geqslant 3$.*

Proof. When $l=-r^3$, $\nu^{(3)}$ is reducible by Theorem 3 and $\nu^{(4)}$ is also reducible by Theorem 4. Suppose now $n \geqslant 5$. To show that u_1 is in $K(n)$, like in the case $l=r$, it will suffice to check that $\nu^{(n)}(c_{ij})(u_1)=0$ for all $i \leqslant 4$ and $j \geqslant 5$. With $l=-r^3$, the coefficients of type $(C_{t,s})$ are no longer zero. But we have: $\nu^{(n)}(c_{2,j})(w_{13} - \frac{1}{r}w_{14}) = 0$ by $(C_{1,j-3})$ and $(C_{1,j-4})$. For $\nu^{(n)}(c_{3,j})(u_1)$, there is no shortcut and a complete evaluation must be performed. We have, where we respected the same order of the terms in Proposition 7 for the coefficients:

$$\begin{aligned} \nu^{(n)}(c_{3,j})(u_1) = \Bigg[& r\frac{1}{r^{j-4}} + \frac{1}{r^{j-3}} + \left(\frac{1}{r}+\frac{1}{r^3}\right)\left(-\frac{1}{r^3r^{j-5}}\right) + \left(\frac{1}{r^{j-4}} - \frac{1}{r^{j-6}}\right)\left(\frac{1}{r^3}+\frac{1}{r}\right) \\ & + \frac{1}{r}\left(\frac{1}{r^{j-3}} - \frac{1}{r^{j-5}}\right)\left(\frac{1}{r^3}+\frac{1}{r}\right)\Bigg] w_{3,j}. \end{aligned}$$

The rules used are, in the same order: $(L_{j-3,1})$, $(L_{j-3,2})$, (R_{j-4}), $(C_{1,j-4})$ and $(C_{2,j-4})$. All the coefficients cancel nicely to give $\nu^{(n)}(c_{3,j})(u_1) = 0$.

Finally, for $\nu^{(n)}(c_{4,j})(u_1)$, only the terms in u_1 whose last node is node number 4 yield a nonzero contribution, the first one contributing with a coefficient $(\frac{1}{r} + \frac{1}{r^3})\frac{1}{r^{j-5}}$, the second one with a coefficient $-\frac{1}{r^{j-4}}$ and the third one with a coefficient $-\frac{1}{r}\frac{1}{r^{j-3}}$ by rules $(L_{j-4,1})$, $(L_{j-4,2})$ and $(L_{j-4,3})$ respectively. The sum of these three coefficients is zero. Thus, we are done with all the cases and conclude that u_1 belongs to $K(n)$ for all $n \geqslant 4$. $\square$

At this stage, we have shown that when l and r take the values of Theorem 1, the representation $\nu^{(n)}$ is reducible. In the next part, we show conversely that if $\nu^{(n)}$ is reducible, then l and r must be related in the way described in Theorem 1.

4. Proof of the main theorem

We prove the main theorem on the representation $\nu^{(n)}$. We then show that $\nu^{(n)}$ is equivalent to the Lawrence–Krammer representation of the BMW algebra. The idea is to prove the main theorem separately for small values of n and to use induction for larger n. Let's assume that the main theorem is true for $n \in \{3, 4, 5, 6\}$. These cases will be dealt with separately. Given an integer n with $n \geqslant 7$, suppose that the main theorem holds for $\nu^{(n-1)}$ and for $\nu^{(n-2)}$. We saw in part 3 that when $l \in \{r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, the representation $\nu^{(n)}$ is reducible. We will show conversely that if $\nu^{(n)}$ is reducible, it forces these values for l and r. Suppose $\nu^{(n)}$ is reducible and let $\mathcal{W}$ be an irreducible invariant subspace of $\mathcal{V}^{(n)}$. By Corollary 1, $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(n)$-module.

Suppose first $n = 7$ or $n \geqslant 9$. So $\dim \mathcal{W} \in \{1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}\}$ or $\dim \mathcal{W} > \frac{(n-1)(n-2)}{2}$ (see Corollary 2). If $\dim \mathcal{W} = 1$, Theorem 3 implies that $l = \frac{1}{r^{2n-3}}$. Also, if $\dim \mathcal{W} = n-1$, Theorem 4 implies that $l \in \{\frac{1}{r^{n-1}}, -\frac{1}{r^{n-1}}\}$. Suppose now that $l \notin \{\frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. Then we have $\dim \mathcal{W} \geqslant \frac{n(n-3)}{2}$. We show that this bound is large enough to make the intersection of $\mathcal{W}$ with the L–K spaces $\mathcal{V}^{(n-1)}$ and $\mathcal{V}^{(n-2)}$ nontrivial. Indeed, we have the following result.

Claim 2. *Let $\mathcal{W}$ be a subspace of $\mathcal{V}^{(n)}$.*
If $\dim \mathcal{W} > n-1$, then $\mathcal{W} \cap \mathcal{V}^{(n-1)} \neq \{0\}$.
If $\dim \mathcal{W} > 2n-3$, then $\mathcal{W} \cap \mathcal{V}^{(n-2)} \neq \{0\}$.

Proof. If $\mathcal{W} \cap \mathcal{V}^{(n-1)} = \{0\}$, the L–K space $\mathcal{V}^{(n)}$ contains the direct sum $\mathcal{W} \oplus \mathcal{V}^{(n-1)}$, which yields on the dimensions: $\dim \mathcal{W} + \frac{(n-1)(n-2)}{2} \leqslant \frac{n(n-1)}{2}$. Then $\dim \mathcal{W} \leqslant n-1$. Similarly, if $\mathcal{W} \cap \mathcal{V}^{(n-2)} = \{0\}$, we get $\dim \mathcal{W} \leqslant \frac{n(n-1)}{2} - \frac{(n-2)(n-3)}{2} = 2n-3$. $\square$

Lemma 2. *When $n > 6$, we have $\frac{n(n-3)}{2} > 2n-3$ and $\frac{n(n-3)}{2} > n-1$.*

By the claim and the lemma, the intersections $\mathcal{W} \cap \mathcal{V}^{(n-1)}$ and $\mathcal{W} \cap \mathcal{V}^{(n-2)}$ are both nontrivial. Since $\mathcal{W}$ is not the whole space $\mathcal{V}^{(n)}$, it cannot contain $\mathcal{V}^{(n-1)}$ by the arguments of the proof of Proposition 1. Similarly, it cannot contain $\mathcal{V}^{(n-2)}$. Hence $\mathcal{W} \cap \mathcal{V}^{(n-1)}$ (resp. $\mathcal{W} \cap \mathcal{V}^{(n-2)}$) is a proper invariant subspace of $\mathcal{V}^{(n-1)}$ (resp. $\mathcal{V}^{(n-2)}$). This implies that $\nu^{(n-1)}$ and $\nu^{(n-2)}$ are both reducible. Since we assumed the main theorem to be true for $\nu^{(n-1)}$ and $\nu^{(n-2)}$, we get

$$l \in \left\{r, -r^3, \frac{1}{r^{2n-5}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}\right\} \cap \left\{r, -r^3, \frac{1}{r^{2n-7}}, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}\right\}.$$

Since $r^2 \neq 1$, $r^{2(n-3)} \neq 1$ and $r^{2n} \neq 1$ when $\mathcal{H}_{F,r^2}(n)$ is semisimple, this only leaves the possibility $l \in \{r, -r^3\}$.

It remains to deal with the case $n=8$. Suppose $n=8$. If $l \notin \{\frac{1}{r^{13}}, \frac{1}{r^5}, -\frac{1}{r^5}\}$, then we have $\dim \mathcal{W} \geqslant 14 > 13 = 2 \times 8 - 3$. Hence, the same arguments as before apply and yield again $l \in \{r, -r^3\}$.

Thus, we have shown that if $\nu^{(n)}$ is reducible and if l and r are such that $l \notin \{\frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, then $l \in \{r, -r^3\}$. So we have proven that if $\nu^{(n)}$ is reducible, then $l \in \{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$.

We now show that the main theorem holds for $\nu^{(n)}$ where $n \in \{3, 4, 5, 6\}$. The case $n=3$ follows from Theorem 3 and Theorem 4 and the case $n=4$ from Theorem 3, Proposition 5 and Theorem 4. For the case $n=5$, we refer the reader to [10, §11, pp. 222–226]. As for the case $n=6$, it must be slightly adapted from the general case. Indeed, suppose $\nu^{(6)}$ is reducible and let $\mathcal{W}$ be an irreducible invariant subspace of $\mathcal{V}^{(6)}$ with $\dim(\mathcal{W}) \geqslant 9$. If $\dim(\mathcal{W}) > 9$, then Claim 2 implies that $\mathcal{W} \cap \mathcal{V}^{(4)} \neq \{0\}$. If $\dim(\mathcal{W}) = 9$, Claim 2 does not apply, but we notice that $\dim(\mathcal{W}) + \dim(\mathcal{V}^{(4)}) = \dim(\mathcal{V}^{(6)})$. Thus, if $\mathcal{W} \cap \mathcal{V}^{(4)} = \{0\}$, we then get $\mathcal{W} \oplus \mathcal{V}^{(4)} = \mathcal{V}^{(6)}$. By Proposition 1, we have $\nu^{(6)}(e_5)(\mathcal{W}) = 0$. But e_5 also acts trivially on $\mathcal{V}^{(4)}$, hence acts trivially on $\mathcal{V}^{(6)}$. This is a contradiction. So again we have $\mathcal{W} \cap \mathcal{V}^{(4)} \neq \{0\}$, and the rest of the proof is the same as in the general case.

To end the proof of the main theorem, we show that $\nu^{(n)}$ is equivalent to the Lawrence–Krammer representation of the BMW algebra. Above, we proved that $\nu^{(n)}$ is reducible if and only if $l \in \{r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. In particular, $\nu^{(n)}$ is generically irreducible over $\mathbb{Q}(l, r)$. Further, we notice that $\nu^{(n)}$ factors through the quotient B/I_2 where I_2 is the two-sided ideal generated by all the products $e_i e_j$ with $|i-j| > 1$. Also, if I_1 denotes the two-sided ideal of B generated by e_1, we observe that I_1 is not in the kernel of $\nu^{(n)}$. Then, $\nu^{(n)}$ is an irreducible representation of I_1/I_2 of degree $\frac{n(n-1)}{2}$. To conclude, we use the work of Cohen, Gijsbers and Wales. In [4], they show that there are only two inequivalent irreducible representations of I_1/I_2 of degree $\frac{n(n-1)}{2}$. One of them is the Lawrence–Krammer representation of the BMW algebra. The representation $\nu^{(n)}$ is equivalent to that representation. Our r is the $\frac{1}{r}$ of [4].

5. Non-semisimplicity of the BMW algebra for some specializations of its parameters

Replacing the L–K representation by one in which r is replaced by its algebraic conjugate $-\frac{1}{r}$ gives another representation. We call it the conjugate L–K representation. By the symmetry of the roles played by r and $-\frac{1}{r}$, when $n \geqslant 4$ and $\mathcal{H}_{F,r^2}(n)$ is semisimple, the conjugate L–K representation is reducible if and only if $l \in \{-\frac{1}{r}, \frac{1}{r^3}, -r^{2n-3}, r^{n-3}, -r^{n-3}\}$. In particular, for $n \geqslant 6$, since $\frac{1}{r^3} \notin \{r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, the two representations are not equivalent. This is also true when $n \in \{4, 5\}$. For instance, for the L–K representation, the trace of the matrix of the left action by g_{n-1} is $\frac{(n-2)(n-3)}{2} r + \frac{1}{l} - (n-2)m$ (see for instance [10, Chapter 6, p. 55]). For the conjugate representation it is $\frac{(n-2)(n-3)}{2}(-\frac{1}{r}) + \frac{1}{l} - (n-2)m$.

We note that Proposition 1 remains valid for the conjugate L–K representation. A consequence of this proposition is that when the representation is reducible, it is indecomposable. Then the BMW algebra is not semisimple for the values of l and r for which the L–K representation or its conjugate representation are reducible. Finally, the Iwahori–Hecke algebra $\mathcal{H}_{F,r^2}(n)$ is a quotient of the BMW algebra $B(A_{n-1})$. Thus, if $\mathcal{H}_{F,r^2}(n)$ is not semisimple, $B(A_{n-1})$ is not semisimple either. Theorem 2 is thus proven.

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