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Infinite branching in the first syzygy

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ABSTRACT

The first syzygy $\Omega_1(\mathbf{Z})$ of a group G consists of the isomorphism classes of modules which are stably equivalent to the augmentation ideal $\mathcal{I} = \mathrm{Ker}(\epsilon : \mathbf{Z}[G] \to \mathbf{Z})$. When G is finitely generated $\Omega_1(\mathbf{Z})$ admits the structure of an infinite tree whose roots do not extend infinitely downward. We show that the minimal level $\Omega_1^{min}(\mathbf{Z})$ is infinite for certain groups of the form $G = C_\infty^N \times \Phi$ where Φ is finite.

Let G be a group for which the trivial module $\mathbf{Z}$ admits a truncated resolution

$$0 \to J \to E_{k-1} \to \cdots \to E_0 \to \mathbf{Z} \to 0$$

where each E_r is a finitely generated stably free module over $\mathbf{Z}[G]$. The kth-*syzygy* $\Omega_k(\mathbf{Z})$ is the class of $\mathbf{Z}[G]$-modules stably equivalent to J; it has the structure of a tree whose roots do not extend infinitely downward. Beyond that general fact however, very little is known about the detailed structure of $\Omega_k(\mathbf{Z})$ even for quite familiar groups. In this paper we exhibit cases where the first syzygy $\Omega_1(\mathbf{Z})$ has infinitely many roots; that is, where the minimal level $\Omega_1^{min}(\mathbf{Z})$ is infinite. If C_∞ denotes the infinite cyclic group and $Q(8m)$ is the quaternion group of order $8m$ then, for any $N \geqslant 1$ and $m \geqslant 1$, we show:

Theorem I. *$\Omega_1^{min}(\mathbf{Z})$ is infinite when $G \cong C_\infty^N \times Q(8m)$.*

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We aim to parametrize $\Omega_1^{min}(\mathbf{Z})$ by the more familiar class SF_1 of stably free modules of rank 1; up to sign, each stably free $\mathbf{Z}[G]$-module S of rank 1 gives a unique surjective $\mathbf{Z}[G]$-homomorphism $\epsilon_S : S \to \mathbf{Z}$ and the correspondence $S \mapsto \kappa(S) = \mathrm{Ker}(\epsilon_S)$ determines a mapping $\kappa : SF_1 \to \Omega_1(\mathbf{Z})$. Observe that $\kappa(\mathbf{Z}[G])$ is simply $\mathcal{I}$, the kernel of the augmentation homomorphism $\epsilon : \mathbf{Z}[G] \to \mathbf{Z}$. To show infinite branching in $\Omega_1(\mathbf{Z})$ at the level of $\mathcal{I}$ it is enough, since each $\kappa(S)$ is at same height as $\kappa(\mathbf{Z}[G]) = \mathcal{I}$, to show that $\mathrm{Im}(\kappa)$ is infinite. In this connection we shall prove:

Theorem II. *Let G be a finitely generated infinite group for which the induced mapping $\epsilon_* : \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z}[G]) \to \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathbf{Z}, \mathbf{Z})$ is injective; then $\kappa : SF_1 \to \Omega_1(\mathbf{Z})$ is injective.*

The hypotheses of Theorem II are satisfied by $G = C_\infty^N \times \Phi$ for any finite group Φ. Determining whether $\mathrm{Im}(\kappa) \subset \Omega_1^{min}(\mathbf{Z})$ leads to the question:

$(*)$ Is $\mathcal{I}$ a minimal element of $\Omega_1(\mathbf{Z})$?

Perhaps surprisingly, the answer in general is 'No'; $\mathcal{I}$ fails to be minimal when G is a free product of the form $G = \Gamma * C_\infty$. There are, however, criteria which guarantee minimality:

Theorem III. *$\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$ if* either $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) = 0$ or G^{ab} *is finite.*

In conjunction with the main result of [9], this establishes the cases $N \geqslant 2$ of Theorem I. The case $N = 1$ is more difficult however, as then neither condition holds; to complete the proof of Theorem I we use a much more delicate argument to show:

Theorem IV. *$\mathcal{I}$ is minimal when $G = F_m \times \Phi$ where F_m is the free group of rank m and Φ is a nontrivial finite group.*

1. The tree structure on a stable module

For a ring Λ, the stability relation '$\sim$' on Λ-modules is defined by

$$M_1 \sim M_2 \iff M_1 \oplus \Lambda^{n_1} \cong M_2 \oplus \Lambda^{n_2}$$

for some $n_1, n_2 \geqslant 0$. When M is a Λ-module $[M]$ will denote the corresponding *stable module*, that is, the set of isomorphism classes of modules N such that $N \sim M$; then $[M]$ has a natural structure of a directed graph in which the vertices are the isomorphism classes of modules N for which $N \sim M$ and where edges take the form $N \to N \oplus \Lambda$. The ring Λ has the *surjective rank property* when, given integers $n, N \geqslant 1$ and a surjective Λ-homomorphism $\varphi : \Lambda^N \to \Lambda^n$ then $n \leqslant N$. It is a comparatively mild restriction as the following shows (cf. [3]).

Proposition 1.1. *Let Λ be a ring for which there exists a (nontrivial) ring homomorphism $\psi : \Lambda \to \mathbf{F}$ where $\mathbf{F}$ is a field. Then Λ has the surjective rank property.*

In particular, this is true for any group ring $\Lambda = A[G]$ where A is commutative. At one point we shall also need to appeal to a slightly stronger property. A ring Λ is said to be *weakly finite* (see [3]) when any surjective Λ-homomorphism $\Lambda^m \to \Lambda^m$ is necessarily an isomorphism. By a theorem of Montgomery and Kaplansky [11], for *any* group G the integral group ring $\mathbf{Z}[G]$ is weakly finite.

Assuming that Λ has the surjective rank property, it is straightforward to show that if M is finitely generated then $M \oplus \Lambda^a \cong M$ only when $a = 0$. It follows that the stable module $[M]$ has the structure of a tree. Moreover there is a 'gap function' $g : [M] \times [M] \to \mathbf{Z}$ defined by means of

$$g(N_1, N_2) = p - q \iff N_1 \oplus \Lambda^p \cong N_2 \oplus \Lambda^q$$

which satisfies the following properties:

$$g(N, N \oplus \Lambda^b) = b, \tag{1.2}$$

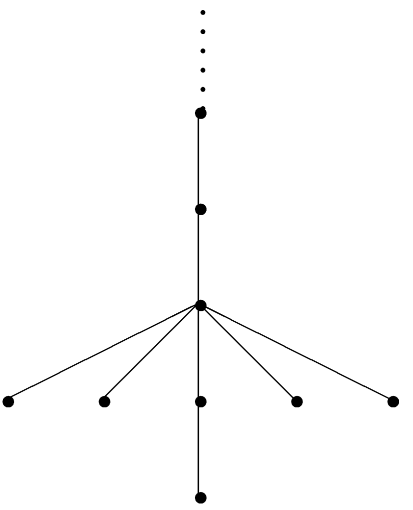
$$g(N_2, N_1) = -g(N_1, N_2), \tag{1.3}$$

$$g(N_1, N_3) = g(N_1, N_2) + g(N_2, N_3). \tag{1.4}$$

For a *nonzero* finitely generated Λ-module M we define $\rho_\Lambda(M)$ to be the least positive integer a for which there exists a surjective Λ-homomorphism $\varphi : \Lambda^a \to M$.

Proposition 1.5. *Let Λ be a ring with the surjective rank property and let M be a finitely generated Λ-module; if $K \in [M]$ is such that $0 \leqslant g(K, M)$ then $g(K, M) \leqslant \rho_\Lambda(M)$.*

The correspondence $K \mapsto g(K, M)$ gives a function $[M] \to \mathbf{Z}$ which is bounded above by $\rho_\Lambda(M)$. Choose $M_0 \in [M]$ to maximize this function. It then follows from (1.3), (1.4) that $0 \leqslant g(M_0, N)$ for all $N \in [M]$. Such a module M_0 is called a *root module* for $[M]$; the function $h : [M] \to \mathbf{N}$; $K \mapsto g(M_0, K)$ is then surjective and measures the height of N above the root level. We may paraphrase the existence of the height function on $[M]$ by saying that $[M]$ is a tree with roots which do not extend infinitely downwards. For example, over the integral group ring of the generalized quaternion group Q_{36} the stable module $[0]$ (that is, the isomorphism classes of finitely generated stably free modules) is represented by the tree below, the deepest root representing the zero module and the remaining four roots representing the nontrivial stably free modules of rank 1 (compare [12] or Chapter 9 of [7]).



2. Syzygies and the corepresentability of cohomology

Given a finitely generated Λ-module M one may construct an exact sequence

$$0 \to J \to \Lambda^a \to M \to 0$$

for some integer $a \geqslant 0$; the kernel J may be regarded as a 'first derivative' of M. A significant consideration in classical invariant theory was to establish the uniqueness of J when a assumes its minimal value. The difficulties inherent in this approach may be avoided, however, by the use of stable modules; given exact sequences $0 \to J \to \Lambda^a \to M \to 0$ and $0 \to J' \to \Lambda^\alpha \to M \to 0$ then by Schanuel's Lemma $J \oplus \Lambda^\alpha \cong J' \oplus \Lambda^a$; thus the stable class $[J]$ of the kernel is uniquely determined by M. We write $\Omega_1(M) = [J]$ and $\Omega_1(M)$ is then called the *first syzygy* of M. More generally, for each $k \geqslant 1$ we may construct a stably free resolution of M truncated at stage $k-1$; that is, an exact sequence

$$0 \to J_k \to E_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\epsilon} M \to 0$$

in which E_r is stably free over Λ for $1 \leqslant r \leqslant k-1$. Although the isomorphism class of J_k is not uniquely determined by M it follows from Swan's extension of Schanuel's Lemma that the *stable isomorphism class* of J_k *is an invariant of* M; the set $\Omega_k(M)$ of isomorphism classes of modules stably equivalent to J_k is called the kth-syzygy of M.

The construction Ω_k has a cohomological interpretation. Recall the notion of the derived module category over Λ; if $f, g : M \to N$ are Λ-homomorphisms where $M, N \in \mathcal{F}(\mathbf{Z}[G])$ we write $f \approx g$ when $f - g$ *factors through a projective module* thus; $f - g = \beta \circ \alpha$ where $\alpha : M \to P$ and $\beta : P \to N$ are $\mathbf{Z}[G]$-homomorphisms and P is projective over Λ; $\approx$ is an equivalence relation compatible with addition and two sided composition. By the *derived module category* $\mathcal{D}\mathrm{er}(\Lambda)$ we mean the category whose objects are modules over Λ, and in which, for any two modules M, N, the set of morphisms $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N)$ is given by

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N) = \mathrm{Hom}_\Lambda(M, N)/\approx.$$

Stably equivalent modules are isomorphic in the derived category so that $\Omega_k(M)$ also denotes an isomorphism class in $\mathcal{D}\mathrm{er}(\Lambda)$. The characterization of $\mathrm{Ext}^k(-,-)$ as the kth derived functor of $\mathrm{Hom}(-,-)$ can be made explicit in this context. Given an extension of Λ-modules $\mathcal{E} = (0 \to J \xrightarrow{i} E \xrightarrow{p} M \to 0)$ in which E is free and a Λ-homomorphism $f : J \to N$ we form the pushout extension $f_*(\mathcal{E}) = (0 \to J \to \varinjlim(f, i) \to M \to 0)$. The natural transformation $\nu : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_1(M), -) \to \mathrm{Ext}^1(M, -)$ is induced by the correspondence $f \mapsto f_*(\mathcal{E})$ and ν is an isomorphism when $\mathrm{Ext}^1(M, \Lambda) = 0$. We may extend this to $k \geqslant 1$ by dimension shifting to obtain the following *corepresentation formula*:

Theorem 2.1. *There is a natural transformation* $\nu : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_k(M), -) \to \mathrm{Ext}^k(M, -)$ *which is an isomorphism when* $\mathrm{Ext}^k(M, \Lambda) = 0$.

The proof is straightforward (cf. [6] or Chapter 4 of [7]). The condition $\mathrm{Ext}^1(M, \Lambda) = 0$ thus guarantees that $\mathrm{Ext}^1(M, N) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_1(M), N)$. It also intervenes in another way; we note the following de-stabilisation result (see [8], Theorem (3.1)):

Proposition 2.2. *Let* $0 \to J \oplus \Lambda^\alpha \xrightarrow{i} \Lambda^\beta \to M \to 0$ *be an exact sequence of* Λ *modules*; *if* $\mathrm{Ext}^1(M, \Lambda) = 0$ *then* $\Lambda^\beta / i_1(\Lambda^\alpha)$ *is projective.*

3. Proof of Theorem II

Given a ring A and a group G the augmentation $\epsilon_{A,G} : A[G] \to A$ is defined by

$$\epsilon_{A,G}\left(\sum_g a_g g\right) = \sum_g a_g.$$

Evidently $\epsilon_{A,G}$ is surjective ring homomorphism; we put $I_A(G) = \mathrm{Ker}(\epsilon_{A,G})$.

Proposition 3.1. *For any $A[G]$-module N on which G acts trivially there is an isomorphism* $\mathrm{Hom}_{A[G]}(I_A(G), N) \cong \mathrm{Ext}^1_{A[G]}(A, N)$.

Proof. The augmentation sequence $0 \to I_A(G) \xrightarrow{i} A[G] \xrightarrow{\epsilon_A} A \to 0$ gives an exact sequence in cohomology

$$\mathrm{Hom}_{A[G]}\big(A[G], N\big) \xrightarrow{i^*} \mathrm{Hom}_{A[G]}\big(I_A(G), N\big) \xrightarrow{\delta} \mathrm{Ext}^1_{A[G]}(A, N) \xrightarrow{\epsilon_A^*} \mathrm{Ext}^1_{A[G]}\big(A[G], N\big).$$

We will show that $i^* : \mathrm{Hom}_{A[G]}(A[G], N) \to \mathrm{Hom}_{A[G]}(I_A(G), N)$ is zero. First suppose that $\alpha \in \mathrm{Hom}_{A[G]}(A[G], N)$; then, for $g \in G$, $i^*(\alpha)(g-1) = \alpha(i(g-1)) = \alpha(g) - \alpha(1)$. As G acts trivially on N then $\alpha(g) = \alpha(1)g = \alpha(1)$ and $i^*(\alpha)(g-1) = 0$ for all $g \in G$. Hence $i^*(\alpha) = 0$ since I_A is generated over A by elements of the form $g-1$ where $g \in G$. However $\mathrm{Ext}^1_{A[G]}(A[G], N) = 0$ so that the exact sequence simplifies to the desired isomorphism $\delta : \mathrm{Hom}_{A[G]}(I_A(G), N) \to \mathrm{Ext}^1_{A[G]}(A, N)$. □

The case of primary interest is when $A = \mathbf{Z}$ and $\Lambda = \mathbf{Z}[G]$; then as in the Introduction, we put $\mathcal{I} = I_{\mathbf{Z}}(G)$. If G has a finite generating set $\{x_r\}_{1 \leqslant r \leqslant m}$ we obtain an exact sequence

$$\Lambda^m \xrightarrow{X} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0 \tag{3.2}$$

where $X = (x_1 - 1, \dots, x_m - 1)$. In particular, $\mathrm{Im}(X) = \mathcal{I}$ so that:

Proposition 3.3. *If G is a finitely generated group then the integral augmentation ideal $\mathcal{I}$ is finitely generated over $\Lambda = \mathbf{Z}[G]$ and defines an element of $\Omega_1(\mathbf{Z})$.*

Suppose S is a stably free Λ module of rank k so that $S \oplus \Lambda^r \cong \Lambda^{k+r}$. On applying $\mathrm{Hom}_\Lambda(-, \mathbf{Z})$ we see that $\mathrm{Hom}_\Lambda(S, \mathbf{Z}) \oplus \mathbf{Z}^r \cong \mathbf{Z}^{k+r}$ so that, by the cancellation property for finitely generated abelian groups, $\mathrm{Hom}_\Lambda(S, \mathbf{Z}) \cong \mathbf{Z}^k$. When $k = 1$ then $\mathrm{Hom}_\Lambda(S, \mathbf{Z}) \cong \mathbf{Z}$ and in this case there is a surjective homomorphism $\kappa_S : S \to \mathbf{Z}$ which is unique up to sign. In particular, $\mathrm{Ker}(\epsilon_S)$ depends only upon S. Applying Schanuel's Lemma to the exact sequences $0 \to \mathcal{I} \to \Lambda \to \mathbf{Z} \to 0$; $0 \to \mathrm{Ker}(\epsilon_S) \to S \to \mathbf{Z} \to 0$ we see that $\mathrm{Ker}(\epsilon_S) \oplus \Lambda \cong \mathcal{I} \oplus S$ so that $\mathrm{Ker}(\epsilon_S) \oplus \Lambda^{r+1} \cong \mathcal{I} \oplus \Lambda^{r+1}$. Thus $\mathrm{Ker}(\epsilon_S) \in \Omega_1(\mathbf{Z})$; moreover, if $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$ then $\mathrm{Ker}(\epsilon_S)$ is also minimal. In summary:

Proposition 3.4. *There is a mapping $\kappa : SF_1 \to \Omega_1(\mathbf{Z})$ determined by the correspondence $S \mapsto \mathrm{Ker}(\epsilon_S)$; furthermore, if $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$ then $\mathrm{Im}(\kappa) \subset \Omega_1^{min}(\mathbf{Z})$.*

For the remainder of this section we assume:

$(*)$ G is finitely generated infinite and $\epsilon_* : \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z})$ is injective.

When G is finitely generated $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong H^1(G, \mathbf{Z}) \cong G^{ab}/\mathrm{Torsion}$ is a finitely generated free abelian group. Thus with the hypotheses $(*)$ we have:

Proposition 3.5. $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)$ *is a finitely generated free abelian group.*

Corollary 3.6. *If S stably free module of rank* 1 *then* $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, S) \cong \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)$.

Proof. If $S \oplus \Lambda^n \cong \Lambda^{n+1}$ then $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, S) \oplus \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)^n \cong \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \oplus \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)^n$. The result follows since $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)$ is a finitely generated free abelian group. □

Theorem 3.7. *Given an extension $\mathcal{S} = (0 \to J \xrightarrow{i} S \xrightarrow{\epsilon_S} \mathbf{Z} \to 0)$ where S is stably free of rank 1 then, under the hypotheses $(*)$, $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \cong \mathbf{Z}$ and $[\mathcal{S}]$ is a generator.*

Proof. First consider the augmentation sequence $(0 \to \mathcal{I} \xrightarrow{i} \mathbf{Z}[G] \xrightarrow{\epsilon} \mathbf{Z} \to 0)$. Since $\epsilon_* : \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z})$ is injective the exact sequence

$$\mathrm{Hom}_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Hom}_\Lambda(\mathbf{Z}, \mathbf{Z}) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathcal{I}) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \xrightarrow{\epsilon_*} \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z})$$

reduces to $\mathrm{Hom}_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Hom}_\Lambda(\mathbf{Z}, \mathbf{Z}) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathcal{I}) \to 0$. However, as G is infinite, $\mathrm{Hom}_\Lambda(\mathbf{Z}, \Lambda) = 0$ so that $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathcal{I}) \cong \mathrm{Hom}_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$. In the general case where $\mathcal{S} = (0 \to J \xrightarrow{i} S \xrightarrow{\epsilon_S} \mathbf{Z} \to 0)$, by Schanuel's Lemma, $J \oplus \Lambda \cong \mathcal{I} \oplus S$. Hence

$$\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \oplus \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \cong \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathcal{I}) \oplus \mathrm{Ext}^1_\Lambda(\mathbf{Z}, S) \cong \mathbf{Z} \oplus \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda).$$

Thus $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \cong \mathbf{Z}$ as $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda)$ is a finitely generated abelian group. Finally suppose that $\mathcal{X} = (0 \to J \to X \to \mathbf{Z} \to 0)$ represents a generator of $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \cong \mathbf{Z}$. We will show that $[\mathcal{S}] = \pm[\mathcal{X}]$. Since S is projective then $\mathrm{Ext}^1_\Lambda(S, J) = 0$ so that from the exact sequence $\mathrm{Hom}_\Lambda(S, J) \xrightarrow{i^*} \mathrm{Hom}_\Lambda(J, J) \xrightarrow{\delta} \mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \to 0$ we see that the mapping

$$\delta : \mathrm{Hom}_\Lambda(J, J) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, J); \quad \delta(\alpha) = \alpha_*(\mathcal{S})$$

is surjective. In particular, we may write $[\mathcal{X}] = [\alpha_*(\mathcal{S})]$ for some $\alpha \in \mathrm{Hom}_\Lambda(J, J)$. However, $[\mathcal{X}]$ generates $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \cong \mathbf{Z}$ so for some $n \in \mathbf{Z}$ we may write $[\mathcal{S}] = n[\mathcal{X}]$. Thus $[\mathcal{X}] = n[\alpha_*(\mathcal{X})]$. Writing $[\alpha_*(\mathcal{X})] = m[\mathcal{X}]$ for some integer m we obtain $[\mathcal{X}] = mn[\mathcal{X}]$. Since $mn \in \mathbf{Z}$ we see that $mn = \pm 1$ and so $n \pm 1$. $\square$

Theorem 3.8. *If G is finitely generated infinite and $\epsilon_* : \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z})$ is injective then $\kappa : SF_1 \to \Omega_1(\mathbf{Z})$ is injective.*

Proof. Let $S, S' \in SF_1$ and suppose that $\kappa(S) = \kappa(S') = J$. We must show that $S \cong S'$. There are exact sequences $\mathcal{S} = (0 \to J \xrightarrow{i} S \xrightarrow{\epsilon} \mathbf{Z} \to 0)$; $\mathcal{S}' = (0 \to J \xrightarrow{i'} S' \xrightarrow{\epsilon'} \mathbf{Z} \to 0)$ and, by (3.7), both $[\mathcal{S}]$, $[\mathcal{S}']$ generate $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, J) \cong \mathbf{Z}$ so that $[\mathcal{S}'] = \pm[\mathcal{S}]$. Replacing ϵ' by $-\epsilon'$ if necessary we may suppose that $[\mathcal{S}'] = [\mathcal{S}]$. Thus there is a congruence

$$\begin{array}{c} \mathcal{S} \\ c\downarrow \\ \mathcal{S}' \end{array} = \left(\begin{array}{ccccc} 0 \to J & \xrightarrow{i} & S & \xrightarrow{\epsilon} & \mathbf{Z} \to 0 \\ \mathrm{Id}\downarrow & & c\downarrow & & \mathrm{Id}\downarrow \\ 0 \to J & \xrightarrow{i'} & S' & \xrightarrow{\epsilon'} & \mathbf{Z} \to 0 \end{array} \right)$$

and $c : S \to S'$ is the required isomorphism. $\square$

Theorem (3.8) is precisely Theorem II of the Introduction.

4. Minimality conditions

We define $\mathcal{M}(1)$ to be the class of finitely generated groups G for which the trivial $\mathbf{Z}[G]$ module $\mathbf{Z}$ satisfies $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) = 0$.

Proposition 4.1. *If $G \in \mathcal{M}(1)$ then $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$.*

Proof. Suppose that $J \in \Omega_1(\mathbf{Z})$ and that $h : J \oplus \Lambda^a \xrightarrow{\simeq} \mathcal{I} \oplus \Lambda^b$ is an isomorphism for some integers $a, b \geqslant 0$. We must show that $a \leqslant b$. From the exact sequence $0 \to \mathcal{I} \to \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0$ we may form successive exact sequences

$$0 \to \mathcal{I} \oplus \Lambda^b \xrightarrow{i} \Lambda^{b+1} \xrightarrow{\epsilon} \mathbf{Z} \to 0; \quad 0 \to J \oplus \Lambda^a \xrightarrow{j} \Lambda^{b+1} \xrightarrow{\epsilon} \mathbf{Z} \to 0; \quad 0 \to J \to S \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

where $j = i \circ h$ and $S = \Lambda^{b+1}/j(\Lambda^a)$. Since $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) = 0$ then S is projective by (2.2) and is evidently nonzero. From the exact sequence $0 \to \Lambda^a \to \Lambda^{b+1} \to S \to 0$ we see that $\Lambda^{b+1} \cong \Lambda^a \oplus S$. Since Λ has the surjective rank property and $S \neq 0$ then $a \leqslant b$. □

Proposition 4.2. *Let A be a ring which is free as a module over $\mathbf{Z}$; if G is finitely generated then $\mathrm{Ext}^1_{A[G]}(A, A) \cong (G^{ab}/\mathrm{Torsion}) \otimes_{\mathbf{Z}} A$.*

Proof. Extension of scalars gives $\mathrm{Hom}_{\mathbf{Z}[G]}(I_{\mathbf{Z}}(G), A) \cong \mathrm{Hom}_{A[G]}(I_A(G), A)$ so from (3.1):

$$\mathrm{Ext}^1_{A[G]}(A, A) \cong \mathrm{Hom}_{A[G]}\big(I_A(G), A\big) \cong \mathrm{Hom}_{\mathbf{Z}[G]}\big(I_{\mathbf{Z}}(G), A\big) \cong \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathbf{Z}, A).$$

As A is free over $\mathbf{Z}$ it follows from the Universal Coefficient Theorem that

$$\mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathbf{Z}, A) \cong H^1(G, A) \cong \mathrm{Hom}_{\mathbf{Z}}\big(H_1(G, \mathbf{Z}), A\big) \cong \big(G^{ab}/\mathrm{Torsion}\big) \otimes_{\mathbf{Z}} A. \qquad \square$$

It now follows from (3.1) and (4.2) that:

Corollary 4.3. *The following conditions on a finitely generated group G are equivalent*:

(i) G^{ab} *is finite*;
(ii) $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z}) = 0$;
(iii) $\mathrm{Hom}_\Lambda(\mathcal{I}, \mathbf{Z}) = 0$.

We define $\mathcal{M}(2)$ to be the class of finitely generated groups G which satisfy (i)–(iii) of (4.3).

Proposition 4.4. *If $G \in \mathcal{M}(2)$ then $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$.*

Proof. Let $J \in \Omega_1(\mathbf{Z})$ and suppose that $J \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$; we will show $a \leqslant b$. Applying $\mathrm{Hom}_\Lambda(-, \mathbf{Z})$ gives $\mathrm{Hom}_\Lambda(J, \mathbf{Z}) \oplus \mathrm{Hom}_\Lambda(\Lambda, \mathbf{Z})^a \cong \mathrm{Hom}_\Lambda(\mathcal{I}, \mathbf{Z}) \oplus \mathrm{Hom}_\Lambda(\Lambda, \mathbf{Z})^b$. However since $G \in \mathcal{M}(2)$ and $\mathrm{Hom}_\Lambda(\Lambda, \mathbf{Z}) \cong \mathbf{Z}$ then $\mathrm{Hom}_\Lambda(J, \mathbf{Z}) \oplus \mathbf{Z}^a \cong \mathbf{Z}^b$. The conclusion $a \leqslant b$ now follows from the cancellation property for free abelian groups. □

Together (4.1) and (4.4) prove Theorem III of the Introduction. Note that the conditions $\mathcal{M}(1)$, $\mathcal{M}(2)$ are independent; if G is a free abelian group of finite rank $N \geqslant 2$ then G satisfies Poincaré Duality in dimension N, and so $\mathrm{Ext}^r(\mathbf{Z}, \Lambda) = 0$ for $r \neq N$ (see [10]). In particular, G satisfies condition $\mathcal{M}(1)$. However, $G^{ab} \cong G$ is infinite and so G fails the condition $\mathcal{M}(2)$. Conversely, take $G = H_1 * H_2$ to be the free product of nontrivial finite groups H_1, H_2; then $G^{ab} \cong H_1^{ab} \oplus H_2^{ab}$ is finite and so G satisfies condition $\mathcal{M}(2)$. If F denotes the kernel of the natural mapping $G \to H_1 \times H_2$ then by the Kurosh subgroup theorem (for example in the form given in [5, p. 118]) F is a free group of rank $(|H_1| - 1)(|H_2| - 1) \geqslant 2$. Put $\Omega = \mathbf{Z}[F]$; F has finite index in G so applying the Eckmann–Shapiro Lemma we conclude that $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \cong \mathrm{Ext}^1_\Omega(\mathbf{Z}, \Omega)$. Since F is a (generalized) duality group of dimension 1 it follows that $\mathrm{Ext}^1_\Omega(\mathbf{Z}, \Omega) \neq 0$; thus $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \neq 0$ and so G fails condition $\mathcal{M}(1)$.

Both conditions $\mathcal{M}(1)$, $\mathcal{M}(2)$ fail when G is a free product of the form $G = \Gamma * C_\infty$; in that case $\mathcal{I}$ also fails to be minimal in $\Omega_1(\mathbf{Z})$ as shown by:

Proposition 4.5. *Let $G = \Gamma * C_\infty$; then $\mathcal{I}$ fails to be minimal in $\Omega_1(\mathbf{Z})$.*

Proof. Write $\mathcal{I}_G$ for the integral augmentation ideal of G; when $G = \Gamma * \Delta$ we see that (see [4, p. 140])

$$\mathcal{I}_G \cong \big(\mathcal{I}_\Gamma \otimes_{\mathbf{Z}[\Gamma]} \mathbf{Z}[G]\big) \oplus \big(\mathcal{I}_\Delta \otimes_{\mathbf{Z}[\Delta]} \mathbf{Z}[G]\big).$$

On taking Δ to be the infinite cyclic group $C_\infty = \langle t | \emptyset\rangle$ the following exact sequence

$$0 \to \mathbf{Z}[C_\infty] \xrightarrow{t-1} \mathbf{Z}[C_\infty] \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

shows that $\mathcal{I}_{C_\infty} \cong \mathbf{Z}[C_\infty]$ and hence $\mathcal{I}_{C_\infty} \otimes_{\mathbf{Z}[C_\infty]} \mathbf{Z}[G] \cong \mathbf{Z}[G]$. On substituting $\Delta = C_\infty$ in the above we see that

$$\mathcal{I}_G \cong \big(\mathcal{I}_\Gamma \otimes_{\mathbf{Z}[\Gamma]} \mathbf{Z}[G]\big) \oplus \mathbf{Z}[G];$$

hence $\mathcal{I}_\Gamma \otimes_{\mathbf{Z}[\Gamma]} \mathbf{Z}[G]$ lies below $\mathcal{I}_G$ in $\Omega_1^G(\mathbf{Z})$. $\square$

Taking $\Gamma = F_{n-1}$ one sees iteratively that $\mathcal{I}_{F_n} \cong \mathbf{Z}[F_n]^n$ so that $\mathcal{I}_{F_n}$ departs progressively from minimality as n increases. Moreover, even when Γ is the trivial group, (4.5) still shows that 0 lies below $\mathcal{I}_{C_\infty}$ in $\Omega_1^{C_\infty}(\mathbf{Z})$.

5. A complete resolution for $F_m \times C_n$

As above, let F_m denote the free group of rank m. For any group Φ, we may identify $\mathbf{Z}[F_m \times \Phi] = \mathbf{Z}[F_m] \otimes \mathbf{Z}[\Phi]$ where tensor product is taken over $\mathbf{Z}$. Now suppose that

$$\mathbf{A} = \big(\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\epsilon} \mathbf{Z} \to 0\big)$$

is a complete resolution for $\mathbf{Z}$ over $\mathbf{Z}[\Phi]$. We construct a complete resolution $\mathcal{C}$ for $\mathbf{Z}$ over $\mathbf{Z}[F_m \times \Phi]$ as follows: put $R(m) = \underbrace{R \oplus \cdots \oplus R}_{m}$ where $R = \mathbf{Z}[F_m]$. Put $\mathcal{C}_0 = R \otimes A_0$ and write $\mathcal{C}_n^+ = R(m) \otimes A_{n-1}$, $\mathcal{C}_n^- = R \otimes A_n$ for $n \geqslant 1$. When $n = 1$ we put $\Delta_1 = (X \otimes 1, 1 \otimes \partial_1)$. For any $n \geqslant 2$ and any signs σ, τ we define $\mathbf{Z}[F_m \times \Phi]$-linear maps $(\Delta_n)_{\sigma\tau} : \mathcal{C}_n^\tau \to \mathcal{C}_{n-1}^\sigma$ as follows:

$$(\Delta_n)_{++} = -(1 \otimes \partial_{n-1}); \qquad (\Delta_n)_{+-} = 0;$$
$$(\Delta_n)_{-+} = X \times 1; \qquad (\Delta_n)_{--} = 1 \otimes \partial_n$$

and put

$$\Delta_n = \begin{pmatrix} (\Delta_n)_{++} & (\Delta_n)_{+-} \\ (\Delta_n)_{-+} & (\Delta_n)_{--} \end{pmatrix} = \begin{pmatrix} -(1 \otimes \partial_{n-1}) & 0 \\ X \otimes 1 & 1 \otimes \partial_n \end{pmatrix}.$$

We obtain homomorphisms $\Delta_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ over $\mathbf{Z}[F_m \times \Phi]$ where $\mathcal{C}_n = \mathcal{C}_n^+ \oplus \mathcal{C}_n^-$:

Theorem 5.1. $\mathcal{C} = (\cdots \to \mathcal{C}_{n+1} \xrightarrow{\Delta_{n+1}} \mathcal{C}_n \xrightarrow{\Delta_n} \mathcal{C}_{n-1} \xrightarrow{\Delta_{n-1}} \cdots \xrightarrow{\Delta_2} \mathcal{C}_1 \xrightarrow{\Delta_1} \mathcal{C}_0 \xrightarrow{\epsilon} \mathbf{Z} \to 0)$ *is a complete resolution for* $\mathbf{Z}$ *over* $\mathbf{Z}[F_m \times \Phi]$.

We now specialise to the case where $\Phi = C_n = \langle y | y^n = 1\rangle$, the cyclic group of order n. Take the usual periodic resolution of $\mathbf{Z}$ over $\mathbf{Z}[C_n]$

$$\cdots \xrightarrow{\Sigma} \mathbf{Z}[C_n] \xrightarrow{y-1} \mathbf{Z}[C_n] \xrightarrow{\Sigma} \cdots \xrightarrow{y-1} \mathbf{Z}[C_n] \xrightarrow{\Sigma} \mathbf{Z}[C_n] \xrightarrow{y-1} \mathbf{Z}[C_n] \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

where $\Sigma = \sum_{r=1}^{n} y^r$. The tensor product resolution of (5.1) then assumes the form:

$$\mathcal{C} = \left(\cdots \to \Lambda^2 \overset{\Delta_{2k+1}}{\to} \Lambda^2 \overset{\Delta_{2k}}{\to} \Lambda^2 \overset{\Delta_{2k-1}}{\to} \cdots \overset{\Delta_3}{\to} \Lambda^2 \overset{\Delta_2}{\to} \Lambda^2 \overset{\Delta_1}{\to} \Lambda \overset{\epsilon}{\to} \mathbf{Z} \to 0\right)$$

where $\Delta_1 = (X \otimes 1, 1 \otimes (y-1))$ whilst for $k \geqslant 1$

$$\Delta_{2k} = \begin{pmatrix} -1 \otimes (y-1) & 0 \\ X \otimes 1 & 1 \otimes \Sigma \end{pmatrix}; \qquad \Delta_{2k+1} = \begin{pmatrix} -1 \otimes \Sigma & 0 \\ x \otimes 1 & 1 \otimes (y-1) \end{pmatrix}.$$

Evidently this resolution is periodic in dimensions $\geqslant 2$ so that for all $k \geqslant 1$,

$$\Omega_{2k}(\mathbf{Z}) = \Omega_2(\mathbf{Z}) \quad \text{and} \quad \Omega_{2k+1}(\mathbf{Z}) = \Omega_3(\mathbf{Z}).$$

What is less clear is that $\Omega_3(\mathbf{Z}) = \Omega_1(\mathbf{Z})$ so that, at the level of syzygies, the resolution is completely periodic. To see this, we first make an elementary observation: suppose X, M_1, M_2 are modules over a ring Λ and that $h = \binom{h_1}{h_2} : X \to M_1 \oplus M_2$ is a Λ-homomorphism. Let $\pi : M_1 \oplus M_2 \to M_2$ be the projection; then with this notation:

Proposition 5.2. *The sequence* $0 \to \mathrm{Im}(h_{1|\mathrm{Ker}(h_2)}) \to \mathrm{Im}(h) \overset{\pi}{\to} \mathrm{Im}(h_2) \to 0$ *is exact.*

We now obtain:

Theorem 5.3. $\mathrm{Im}(\Delta_{2k+1}) \cong \mathrm{Im}(\Delta_1) = \mathcal{I}$ *for all* $k \geqslant 1$.

Proof. Observe that $\Delta_{2k+1} = \binom{g}{\Delta_1}$ where $g = (-1 \otimes \Sigma, 0)$. Thus we may apply (5.2) to get an exact sequence $0 \to \mathrm{Im}(g_{|\mathrm{Ker}(\Delta_1)}) \to \mathrm{Im}(\Delta_{2k+1}) \overset{\pi}{\to} \mathrm{Im}(\Delta_1) \to 0$. Observe that $\mathrm{Im}(\Delta_1) = \mathcal{I}$. Moreover, one calculates easily that $g \circ \Delta_2 \equiv 0$; that is, $g_{|\mathrm{Im}(\Delta_2)} = 0$. However, $\mathrm{Im}(\Delta_2) = \mathrm{Ker}(\Delta_1)$ by exactness of $\mathcal{C}$ so that the above exact sequence reduces to an isomorphism $\mathrm{Im}(\Delta_{2k+1}) \cong \mathrm{Im}(\Delta_1) = \mathcal{I}$ as claimed. $\square$

Corollary 5.4. *For each* $k \geqslant 0$, $\mathcal{I} \in \Omega_{2k+1}(\mathbf{Z})$ *over the ring* $\Lambda = \mathbf{Z}[F_m \times C_n]$.

6. Two calculations

Given a ring R and a finite group Φ we consider R as a bimodule over the group ring $\Lambda = R[\Phi]$ where Φ acts trivially.

Proposition 6.1. $\mathrm{End}_{\mathcal{D}\mathrm{er}(\Lambda)}(R) \cong R/|\Phi|$.

Proof. Any Λ-homomorphism $\beta : \Lambda \to R$ is a multiple $\beta = b\epsilon$ where $b \in R$ and $\epsilon : \Lambda = R[\Phi] \to R$ is the R-augmentation. Any Λ-homomorphism $\gamma : R \to \Lambda$ is a multiple $\gamma = c\epsilon^*$ where $c \in \Lambda$ and $\epsilon^* : R \to \Lambda$ is the R-dual of ϵ; that is $\epsilon^*(1) = \sum_{\phi \in \Phi} \hat{\phi}$ where $\{\hat{\phi}\}_{\phi \in \Phi}$ is the canonical R-basis of $\Lambda = R[\Phi]$. Observe that $\epsilon^*(1)$ lies in the centre of Λ and that $\epsilon\epsilon^*(1) = |\Phi|$. Suppose that $\alpha = \beta\gamma$ is a factorization of α through Λ^m where

$$\gamma = \begin{pmatrix} c_1\epsilon^* \\ \vdots \\ c_m\epsilon^* \end{pmatrix} : R \to \Lambda^m \quad \text{and} \quad \beta = (b_1\epsilon, \ldots, b_m\epsilon) : \Lambda^m \to R.$$

Then α is completely determined by $\alpha(1) = \sum_i b_i \epsilon\epsilon^*(1)c_i = (\sum_i b_i c_i)|\Phi|$. Conversely, if $\alpha = \lambda|\Phi|$ for some $\lambda \in \Lambda$ then α factors through Λ since $\alpha = \lambda\epsilon \circ \epsilon^*$; thus with the above notation

$$\alpha : R \to R \quad \text{factors through } \Lambda^m \quad \Longleftrightarrow \quad \alpha = \lambda|\Phi| \quad \text{for some } \lambda \in \Lambda.$$

The result now follows as $\alpha \in \mathrm{End}_\Lambda(R)$ factorizes through a projective module if and only if it factorizes through some Λ^m. $\square$

We now specialize to the case where R is the integral group ring $R = \mathbf{Z}[F_m]$ where F_m is free group of rank $m \geqslant 1$ and where $\Phi = C_n$ so that $\Lambda = R[C_n] = \mathbf{Z}[F_m \times C_n]$. We denote by $\mathcal{I}$ the integral augmentation ideal of $\mathbf{Z}[F_m \times C_n]$. From the exact sequence $0 \to \mathcal{I} \to \Lambda \to \mathbf{Z} \to 0$ we get, by dimension shifting, that:

Proposition 6.2. $\mathrm{Ext}^{k+1}_\Lambda(\mathbf{Z}, N) \cong \mathrm{Ext}^k_\Lambda(\mathcal{I}, N)$ *for any Λ-module N.*

Proposition 6.3. $\mathrm{Ext}^{k+1}_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong \mathrm{Ext}^{k+1}_\Lambda(\mathcal{I}, \mathcal{I})$ *for $k \geqslant 1$.*

Proof. Clearly $\mathrm{Ext}^k_R(\mathbf{Z}, R) = 0$ for $k \geqslant 2$ since F_m has cohomological dimension one. Moreover, as F_m is a subgroup of finite index in $G = F_m \times \Phi$ it follows by the Eckmann–Shapiro Lemma that $\mathrm{Ext}^k_\Lambda(\mathbf{Z}, \Lambda) = 0$ for $k \geqslant 2$. Thus by dimension shifting as in (6.2), we see that $\mathrm{Ext}^k_\Lambda(\mathcal{I}, \Lambda) = 0$ for $k \geqslant 1$. Hence the exact sequence

$$\mathrm{Ext}^k(\mathcal{I}, \Lambda) \to \mathrm{Ext}^k(\mathcal{I}, \mathbf{Z}) \to \mathrm{Ext}^{k+1}(\mathcal{I}, \mathcal{I}) \to \mathrm{Ext}^{k+1}(\mathcal{I}, \Lambda)$$

reduces to an isomorphism $\mathrm{Ext}^k_\Lambda(\mathcal{I}, \mathbf{Z}) \cong \mathrm{Ext}^{k+1}_\Lambda(\mathcal{I}, \mathcal{I})$. However, again by dimension shifting, $\mathrm{Ext}^{k+1}_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong \mathrm{Ext}^k_\Lambda(\mathcal{I}, \mathbf{Z})$ so that $\mathrm{Ext}^{k+1}_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong \mathrm{Ext}^{k+1}_\Lambda(\mathcal{I}, \mathcal{I})$ for $k \geqslant 1$. $\square$

Proposition 6.4. $\mathrm{Ext}^3_\Lambda(\mathbf{Z}, \mathcal{I}) \cong \mathbf{Z}/n$.

Proof. The Künneth Theorem applied to $G = F_m \times C_n$ shows that $\mathrm{Ext}^2_\Lambda(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}/n$; thus $\mathrm{Ext}^2_\Lambda(\mathcal{I}, \mathcal{I}) \cong \mathbf{Z}/n$ by (6.3); now apply dimension shifting as in (6.2). $\square$

By (5.4) $\mathcal{I}$ is a representative of $\Omega_3(\mathbf{Z})$ over $\Lambda = \mathbf{Z}[F_m \times C_n]$. As $\mathrm{Ext}^3_\Lambda(\mathbf{Z}, \Lambda) = 0$ the corepresentation formula (2.1) gives an isomorphism $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\mathcal{I}, N) \cong \mathrm{Ext}^3_\Lambda(\mathbf{Z}, N)$ for any Λ-module N; on taking $N = \mathcal{I}$ we obtain:

Corollary 6.5. $\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{I}) \cong \mathbf{Z}/n$.

7. Proof of Theorem IV

Let G be a direct product of groups $G = \Psi \times \Phi$ and make the abbreviations

$$\Lambda = \mathbf{Z}[G]; \qquad R = \mathbf{Z}[\Psi]; \qquad \mathcal{I} = I_{\mathbf{Z}}(G).$$

With the identifications $\Lambda = \mathbf{Z}[\Psi \times \Phi] \cong \mathbf{Z}[\Psi] \otimes_{\mathbf{Z}} \mathbf{Z}[\Phi] \cong R[\Phi]$ we may write $\epsilon = \epsilon_{\mathbf{Z},\Psi \times \Phi} = \epsilon_{\mathbf{Z},\Psi}\epsilon_{R,\Phi}$; we obtain a commutative diagram of Λ-homomorphisms in which the rows and the right hand column are exact:

$$\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
0 \to & I_R(\Phi) & \to & \mathcal{I} & \to & \mathrm{Ker}(\epsilon_{\mathbf{Z},\Psi}) & \to 0 \\
 & \| & & \bigcap & & \bigcap & \\
0 \to & I_R(\Phi) & \to & \Lambda & \overset{\epsilon_{R,\Phi}}{\to} & R & \to 0. \\
 & & & & & \downarrow \epsilon_{\mathbf{Z},\Psi} & \\
 & & & & & \mathbf{Z} & \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array}$$

In particular Λ is an extension of the form:

$$0 \to I_R(\Phi) \to \Lambda \to R \to 0. \tag{7.1}$$

Specializing to the case where $\Psi = F_m = \langle x_1, \ldots, x_m \rangle$ is the free group of rank m we obtain a complete resolution $(0 \to R^m \xrightarrow{X} R \xrightarrow{\epsilon_{\mathbf{Z},F_m}} \mathbf{Z} \to 0)$ for $\mathbf{Z}$ over R where $X = (x_1 - 1, \ldots, x_m - 1)$. Then $\mathrm{Ker}(\epsilon_{\mathbf{Z},F_m}) \cong R^m$ so that

$$\mathcal{I} \text{ is an extension of the form } 0 \to I_R(\Phi) \to \mathcal{I} \to R^m \to 0. \tag{7.2}$$

Now specialize further to the case where Φ is a nontrivial finite group and put $n = |\Phi| > 1$.

Proposition 7.3. *If $\mathcal{L} \in [\mathcal{I}]$ then $\mathcal{L} \neq 0$.*

Proof. Otherwise one would have $\mathcal{I} \oplus \Lambda^r \cong \Lambda^s$ for some $r, s \geqslant 1$. That is, $\mathcal{I}$ is stably free and so G has cohomological dimension 1. This is a contradiction, since $G = F_m \times \Phi$ has infinite cohomological dimension. □

We note the following:

Proposition 7.4. $\mathrm{Hom}_\Lambda(I_R(\Phi), R) = 0$.

Proof. By (3.1) it suffices to show that $\mathrm{Ext}^1_{R[\Phi]}(R, R) = 0$. However, by (4.2), as R is free over $\mathbf{Z}$, $\mathrm{Ext}^1_{R[G]}(R, R) \cong (\Phi^{ab}/\mathrm{Torsion}) \otimes_{\mathbf{Z}} R = 0$ since Φ is finite. □

Now suppose that $\mathcal{L} \in [\mathcal{I}]$, so that $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$ for some $a, b \geqslant 0$. We shall establish a sequence of increasingly better estimates for the relative sizes of $\mathcal{I}$ and $\mathcal{L}$:

Proposition 7.5. $a \leqslant b + m$.

Proof. From the exact sequence $0 \to \mathrm{Hom}_\Lambda(R^m, R) \to \mathrm{Hom}_\Lambda(\mathcal{I}, R) \to \mathrm{Hom}_\Lambda(I_R(\Phi), R)$ and (7.4) we see that $\mathrm{Hom}_\Lambda(\mathcal{I}, R) \cong R^m$. It follows that $\mathrm{Hom}_\Lambda(\mathcal{I} \oplus \Lambda^b, R) \cong R^{b+m}$; since $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$ then $\mathrm{Hom}_\Lambda(\mathcal{L} \oplus \Lambda^a, R) \cong \mathrm{Hom}_\Lambda(\mathcal{L}, R) \oplus R^a \cong R^{b+m}$. Thus $\mathrm{Hom}_\Lambda(\mathcal{L}, R)$ is a projective R-module. By the Bass–Sheshadri Theorem (see [1]) $\mathrm{Hom}_\Lambda(\mathcal{L}, R)$ is free and so $\mathrm{Hom}_\Lambda(\mathcal{L}, R) \cong R^{b+m-a}$ since R has the invariant basis property [2]. Hence $a \leqslant b + m$. □

Next we show:

Proposition 7.6. *$a < b + m$ and $\mathrm{Hom}_\Lambda(\mathcal{L}, R) \cong R^{b+m-a} \neq 0$.*

Proof. Choose an isomorphism $h : \mathcal{L} \oplus \Lambda^a \to \mathcal{I} \oplus \Lambda^b$. Since $\mathrm{Hom}_\Lambda(\mathcal{I} \oplus \Lambda^b, R) \cong R^{b+m}$ there exists a surjective homomorphism $p : \mathcal{I} \oplus \Lambda^b \to R^{b+m}$. We know from (7.5) that $a \leqslant b + m$, so suppose that $a = b + m$. Then $\mathrm{Hom}_\Lambda(\mathcal{L}, R) = 0$ so that the restriction $p \circ h_{|\mathcal{L}} : \mathcal{L} \to R$ is zero. Likewise, we may choose a surjective homomorphism $q : \Lambda^a \to R^a$ in which $\mathrm{Ker}(q) \cong I_R(\Phi)^a$. Abbreviating $I_R(\Phi)$ to I_R then in the following diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{L} \oplus I_R^a & \xrightarrow{j} & \mathcal{L} \oplus \Lambda^a & \xrightarrow{(0,q)} & R^a & \to 0 \\ & & & \downarrow h & & & \\ 0 \to & I_R \oplus I_R^b & \xrightarrow{i} & \mathcal{I} \oplus \Lambda^b & \xrightarrow{p} & R^{b+m} & \to 0, \end{array}$$

$p \circ h$ vanishes on $\mathcal{L} \oplus I_R^a$. Thus there exist unique homomorphisms $h_- : \mathcal{L} \oplus I_R^a \to I_R \oplus I_R^b$ and $h_+ : R^a \to R^{b+m}$ making the following diagram commute:

$$\begin{array}{ccccccc} 0 \to & \mathcal{L} \oplus I_R^a & \xrightarrow{j} & \mathcal{L} \oplus \Lambda^a & \xrightarrow{(0,q)} & R^a & \to 0 \\ & \downarrow h_- & & \downarrow h & & \downarrow h_+ & \\ 0 \to & I_R \oplus I_R^b & \xrightarrow{i} & \mathcal{I} \oplus \Lambda^b & \xrightarrow{p} & R^{b+m} & \to 0. \end{array}$$

As h is bijective and the rows are exact $h_+ : R^a \to R^{b+m}$ is surjective and, by hypothesis, $a = b + m$. Now $R = \mathbf{Z}[F_m]$, being an integral group ring, is weakly finite [11]. Thus h_+ is an isomorphism. It follows from the Five Lemma (extending the rows to the left by zeroes) that $h_- : \mathcal{L} \oplus I_R^a \to I_R^{b+1}$ is also an isomorphism. Now I_R is free of rank $n - 1$ over R where $n = |\Phi| > 1$. As $\mathcal{L} \oplus I_R^a \cong I_R^{b+1}$ it follows that $\mathcal{L}$ is stably free and hence (by the Bass–Sheshadri Theorem of [1]) free over R. In particular

$$\mathrm{rk}_R(\mathcal{L}) = (n-1)(b + m - a) < (n-1)(b + m - a) = 0.$$

This contradicts (7.3). Hence $a < b + m$ and $\mathrm{Hom}_\Lambda(\mathcal{L}, R) \cong R^{b+m-a} \neq 0$. □

Proposition 7.7. *If $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$ then $a \leqslant b + 1$.*

Proof. Since $\mathrm{Hom}_\Lambda(\mathcal{L}, R) \cong R^{b+m-a}$ choose $\pi : \mathcal{L} \to R^{b+m-a}$ to be a surjective Λ-homomorphism and put $\mathcal{L}_0 = \mathrm{Ker}(\pi)$. Let $g : \mathcal{L} \oplus \Lambda^a \to \mathcal{I} \oplus \Lambda^b$ be the inverse of the isomorphism h considered above, and consider the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & I_R \oplus I_R^b & \xrightarrow{i} & \mathcal{I} \oplus \Lambda^b & \xrightarrow{p} & R^{b+m} & \to 0 \\ & & & \downarrow g & & & \\ 0 \to & \mathcal{L}_0 \oplus I_R^a & \xrightarrow{j} & \mathcal{L} \oplus \Lambda^a & \xrightarrow{(\pi,\mathrm{Id})} & R^{b+m-a} \oplus R^a & \to 0. \end{array}$$

Making the obvious identification of $R^{b+m-a} \oplus R^a$ with R^{b+m}, we note that $(\pi, \mathrm{Id}) \circ g$ vanishes on $I_R \oplus I_R^b$ since $\mathrm{Hom}(I_R, R) = 0$ so that, again using the fact that R is weakly finite, g induces an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 \to & I_R \oplus I_R^b & \xrightarrow{i} & \mathcal{I} \oplus \Lambda^b & \xrightarrow{p} & R^{b+m} & \to 0 \\ & \downarrow g_- & & \downarrow g & & \downarrow g_+ & \\ 0 \to & \mathcal{L}_0 \oplus I_R^a & \xrightarrow{j} & \mathcal{L} \oplus \Lambda^a & \xrightarrow{(\pi,q)} & R^{b+m} & \to 0. \end{array}$$

Thus $\mathcal{L}_0 \oplus I_R^a \cong I_R^{b+1}$. Computing R-ranks we obtain

$$\mathrm{rk}(\mathcal{L}_0) + (n-1)a = (n-1)(b+1)$$

so that $\mathrm{rk}(\mathcal{L}_0) = (n-1)(b+1-a)$. Hence $0 \leqslant b+1-a$ and so $a \leqslant b+1$. $\square$

Now consider the special case of Theorem IV when $\Phi \cong C_n$.

Proposition 7.8. *$\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$ when $G \cong F_m \times C_n$.*

Proof. Suppose that $\mathcal{L} \in [\mathcal{I}]$ and that $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$; then $a \leqslant b+1$ by (7.7). Suppose that $a = b+1$. Then $b+m-a = m-1$, so that, as in (7.7), there exists a surjection $\pi : \mathcal{L} \to R^{m-1}$ with $\mathrm{Ker}(\pi) = \mathcal{L}_0$. As in the proof of (7.7), $\mathrm{rk}_R(\mathcal{L}_0) = (n-1)(b+1-a) = 0$; thus $\mathcal{L}_0 = 0$ so that the surjection $\pi : \mathcal{L} \to R^{m-1}$ is an isomorphism of Λ-modules. Thus

$$\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{L}) \cong M_{m-1}\big(\mathrm{End}_{\mathcal{D}\mathrm{er}}(R)\big).$$

By (6.1) $\mathrm{End}_{\mathcal{D}\mathrm{er}}(R) \cong R/n$ which is an infinite ring. Thus $M_{m-1}(\mathrm{End}_{\mathcal{D}\mathrm{er}}(R))$ is also infinite. However $\mathcal{L} \cong \mathcal{I}$ so that $\mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{L}) \cong \mathbf{Z}/n$ is finite. From this contradiction we conclude that $a \leqslant b$ and that $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$. $\square$

Before proving Theorem IV when Φ is an arbitrary nontrivial finite group we make a general observation. Suppose G is a group and let $i : H \subset G$ be the inclusion of a subgroup H with finite index $k \geqslant 2$. Let $\mathcal{I} = \mathrm{Ker}(\epsilon_G : \mathbf{Z}[G] \to \mathbf{Z})$; $\mathcal{I}_0 = \mathrm{Ker}(\epsilon_H : \mathbf{Z}[H] \to \mathbf{Z})$ be the respective integral augmentation ideals and let $\natural : i^*(\mathcal{I}) \to i^*(\mathcal{I})/\mathcal{I}_0$ be the canonical mapping. If $\{x_0, x_1, \ldots, x_{k-1}\}$ is a complete set of coset representatives for G/H with $x_0 = 1$ then $i^*(\mathcal{I})/\mathcal{I}_0$ is free of rank $k-1$ over $\mathbf{Z}[H]$ on the basis $\natural(x_r - 1)_{1 \leqslant r \leqslant k-1}$. It follows immediately that:

Proposition 7.9. $i^*(\mathcal{I}) \cong \mathcal{I}_0 \oplus \mathbf{Z}[H]^{k-1}$.

Proof of Theorem IV. Let $G = F_m \times \Phi$ where Φ is a nontrivial finite group. Put $\Lambda = \mathbf{Z}[F_m \times \Phi] \cong R[\Phi]$ where $R = \mathbf{Z}[F_m]$ and let $\mathcal{I} = \mathrm{Ker}(\epsilon : \mathbf{Z}[F_m \times \Phi] \to \mathbf{Z})$ be the integral augmentation ideal. We shall prove that $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$; that is, if $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$ then $a \leqslant b$.

By the special case already established we may suppose that Φ is not cyclic. Let $C_n \subset \Phi$ be a nontrivial cyclic subgroup and put $H = F_m \times C_n$ and $k = |G/H| = |\Phi|/n$. Put $\Lambda_0 = R[C_n]$ and let $\mathcal{I}_0 = \mathrm{Ker}(\epsilon : \mathbf{Z}[F_m \times C_n] \to \mathbf{Z})$ be the integral augmentation ideal of $F_m \times C_n$. From the hypothesis $\mathcal{L} \oplus \Lambda^a \cong \mathcal{I} \oplus \Lambda^b$ it follows that

$$i^*(\mathcal{L}) \oplus i^*(\Lambda)^a \cong i^*(\mathcal{I}) \oplus i^*(\Lambda)^b.$$

However, $i^*(\Lambda) \cong \Lambda_0^k$ and by (7.9), $i^*(\mathcal{I}) \cong \mathcal{I}_0 \oplus \Lambda_0^{k-1}$. Thus $i^*(\mathcal{L}) \oplus \Lambda_0^{ka} \cong \mathcal{I}_0 \oplus \Lambda_0^{kb+k-1}$. Now, by (7.8), $ka \leqslant kb + (k-1)$ and so $a \leqslant b$. $\square$

8. Proof of Theorem I

Let $Q(8m) = \langle x, y \,|\, x^{2m} = y^2,\ xyx = y\rangle$ be the generalized quaternion group of order $8m$. Put $G = C_\infty^N \times Q(8m)$ where $N \geqslant 1$, $\Lambda = \mathbf{Z}[G]$ and $\Lambda_0 = \mathbf{Z}[C_\infty^N]$. Then by the Eckmann–Shapiro Lemma, $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \cong \mathrm{Ext}^1_{\Lambda_0}(\mathbf{Z}, \Lambda_0)$. Since C_∞^N is a Poincaré Duality group of dimension N it follows (see [10]) that

$$\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) = \begin{cases} \mathbf{Z}, & N = 1, \\ 0, & N \geqslant 2. \end{cases} \tag{8.1}$$

By direct calculation one may first show:

$$\epsilon_* : \mathrm{Ext}^1_{\mathbf{Q}[C_\infty]}\big(\mathbf{Q}, \mathbf{Q}[C_\infty]\big) \to \mathrm{Ext}^1_{\mathbf{Q}[C_\infty]}(\mathbf{Q}, \mathbf{Q}) \text{ is an isomorphism.} \tag{8.2}$$

We now prove:

Proposition 8.3. $\epsilon_* : \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \to \mathrm{Ext}^1_\Lambda(\mathbf{Z}, \mathbf{Z})$ *is injective.*

Proof. The statement for $N \geqslant 2$ is trivial by (8.1) so that it suffices to consider the case $N = 1$. In this case, again by (8.1), $\mathrm{Ext}^1_\Lambda(\mathbf{Z}, \Lambda) \cong \mathbf{Z}$ so that it suffices to prove that with rational coefficients the corresponding map $\epsilon_* : \mathrm{Ext}^1_{\mathbf{Q}[G]}(\mathbf{Q}, \mathbf{Q}[G]) \to \mathrm{Ext}^1_{\mathbf{Q}[G]}(\mathbf{Q}, \mathbf{Q})$ is nonzero. This follows from the isomorphism already noted in (8.2) by applying the Künneth Theorem with rational coefficients to $G = C_\infty \times \Phi$ above. $\square$

For $N = 1$, $\mathcal{I}$ is minimal in $\Omega_1(\mathbf{Z})$ by Theorem IV whilst for $N \geqslant 2$ minimality of $\mathcal{I}$ follows from (8.1) and Theorem III. In [9], we showed that $\Lambda = \mathbf{Z}[G]$ admits infinitely many isomorphism types of stably free modules of rank 1. As $\mathcal{I}$ is minimal, to complete the proof of Theorem I it suffices to show that $\kappa : SF_1 \to \Omega_1(\mathbf{Z})$ is injective. This now follows from (8.3) and Theorem II.

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