



On linear equations in free Lie algebras

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ABSTRACT

We investigate equations of the form $[x_1, u_1] + \cdots + [x_k, u_k] = 0$ over a free Lie algebra L . In the case where the coefficients $u_1, \dots, u_k$ are free generators of L , we generalize a number of earlier results on equations with two variables to equations with an arbitrary number of indeterminates. Our main results refer to the case where the coefficients coincide with the free generators of L . We give a detailed description of the solution space and we obtain an explicit basis for its multilinear fine homogeneous component.

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1. Introduction

Let L be a free Lie algebra over an arbitrary field K . In this paper we study linear equations of the form

$$[x_1, u_1] + [x_2, u_2] + \cdots + [x_k, u_k] = 0, \quad (1.1)$$

where $u_1, u_2, \dots, u_k \in L$ and $x_1, x_2, \dots, x_k$ are indeterminates. This is a continuation of the investigations in [6] which focused on Eq. (1.1) in the case where $k = 2$.

The theory of equations in free Lie algebras is still in its infancy. Motivated by considerable advances in the theory of equations over groups, and, in particular, over free groups, in the last two decades or so, a systematic approach towards a general theory of equations over Lie algebras was developed by È. Daniyarova. An account of her work can be found in the survey [4]. More specific problems about equations over free metabelian Lie algebras were studied in [1] and [2], and the paper [3] deals with systems of equations over free Lie algebras whose solution sets are contained in a finite-dimensional affine subspace. A detailed survey of the results in [1–3] is given in [4]. The aim

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of the above mentioned paper [6] was to begin the study of very concrete equations over free Lie algebras such as Eq. (1.1). This paper dealt exclusively with the case of two indeterminates ($k = 2$). In the present paper we generalize some of the results in [6] to equations with an arbitrary finite number of variables. Our main results refer to the case where the coefficients are precisely the free generators of L . In this case we obtain a detailed description of the solution space in terms of generators. Moreover, we exhibit an explicit basis of the “multilinear” fine homogeneous component of this space.

More specifically, for the greater part of this paper we will assume that the coefficients $u_1, u_2, \dots, u_k$ in (1.1) form an independent set, that is, they form a set of free generators for the Lie subalgebra they generate. In Section 2 we set up notation and recall some relevant facts about free Lie algebras. Then we introduce the notion of the associated linear map of Eq. (1.1) which allows us to identify the solution space of the equation with the kernel of this linear map. In Section 3 we first consider the case where L is of rank k and the coefficients $u_1, u_2, \dots, u_k$ form a free generating set of L . In this case we obtain a formula for the dimension of the homogeneous components of the solution space of (1.1) (Theorem 3.1). Then we consider the case where $u_1, u_2, \dots, u_k$ are free generators of L , but the rank of L may be greater than k . In this case we show that all the solutions of (1.1) are inner solutions, that is, they belong to the subalgebra generated by $u_1, u_2, \dots, u_k$ in L (Theorem 3.2). Finally, we consider the case where $\{u_1, u_2, \dots, u_k\}$ is an arbitrary independent set in L . In this case we show that the space of inner solutions is of finite codimension in the full solution space of (1.1) (Theorem 3.3). For the case where $k = 2$, all results in this section were obtained in [6]. In the present paper we generalize these results from [6] to arbitrary k . For Theorem 3.3 this is easily done by adopting the original proof to the more general situation, but for Theorems 3.1 and 3.2 we offer genuinely new proofs. In particular, in the proof of Theorem 3.2, we have been able to replace the highly technical argument used in [6] by a shorter and more elegant proof based on a powerful lemma of Shirshov’s. In Section 4 we return to the situation where the coefficients $u_1, u_2, \dots, u_k \in L$ in (1.1) are free generators of L . We exhibit the fine homogeneous structure of the solution space, and in the main case where the rank of L is k we give formulae for the dimensions of its fine homogeneous components. In the following Section 5 we reinterpret the solution space in the main case as the kernel of the linear map $L \otimes L_1 \rightarrow L$, where L_1 is the degree one homogeneous component of L (the span of the free generators). This point of view will be exploited in the final two sections. These contain our main results. In Section 6 we exhibit explicit spanning sets of the homogeneous components of the solution space of (1.1), see Theorem 6.1. In its strongest form this requires restrictions on the characteristic of the field K , but it is valid in full generality over fields of characteristic zero. In a Corollary to this theorem we point out how to obtain generating sets (as a K -space) for the homogeneous components of the solution space over an arbitrary field K . In the final Section 7 we exhibit an explicit basis for the multilinear fine homogeneous component of the solution space in arbitrary characteristic.

2. Preliminaries

Throughout this paper we let L denote the free Lie algebra with (finite or countably infinite) free generating set $\mathcal{A} = \{a_1, a_2, \dots\}$, $|\mathcal{A}| \geq 2$, over a field K . By L_n we denote the degree n homogeneous component of L . Thus

$$L = \bigoplus_{n \geq 1} L_n.$$

It is well known that L_n is the span of all left normed Lie monomials $[b_1, b_2, \dots, b_n]$ with $b_i \in \mathcal{A}$. Recall that left normed Lie products are defined inductively by $[b_1] = b_1$ and $[b_1, \dots, b_{n-1}, b_n] = [[b_1, \dots, b_{n-1}], b_n]$. If $\mathcal{A} = \{a_1, \dots, a_r\}$ is a finite set, and hence L the free Lie algebra of finite rank r , then the homogeneous components of L are finite-dimensional, and the dimension d_n of L_n is given by Witt’s formula

$$d_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d} \quad (2.1)$$

where μ is the Möbius function [5, Theorem 5.11]. The fine homogeneous components of L are spanned by all left normed Lie monomials of the same multidegree in the free generators. More precisely, a composition q of n in r parts ($q \models n$) is a sequence $q = (q_1, q_2, \dots, q_r)$ of non-negative integers such that $\sum_{j=1}^r q_j = n$. Note that our use of compositions is slightly different from the usual conventions: We do allow zero parts and assume throughout that the number of parts is r . For a fixed composition $q = (q_1, q_2, \dots, q_r)$ of n , let L_q be the subspace of L generated by all Lie products of partial degree q_j with respect to a_j for $1 \leq j \leq r$. Then

$$L_n = \bigoplus_{q \models n} L_q.$$

The subspaces L_q are known as the fine homogeneous components of L . Note that, for $n > 1$, $L_q = 0$ if $q_j = n$ for some j . The dimension d_q of L_q with $q \models n$ is given by the fine Witt formula

$$d_q = \frac{1}{n} \sum_{d | \gcd(q_1, \dots, q_r)} \mu(d) \frac{(n/d)!}{(q_1/d)!(q_2/d)! \dots (q_r/d)!}$$

where $\gcd(q_1, q_2, \dots, q_r)$ denotes the greatest common divisor of $q_1, q_2, \dots, q_r$ [5, Theorem 5.11].

Now consider Eq. (1.1). We will regard the solutions $(s_1, \dots, s_k)$ of this equation as elements of the direct sum

$$L^{\oplus k} = L^1 \oplus L^2 \oplus \dots \oplus L^k$$

where $L^1, \dots, L^k$ are isomorphic copies of L . It is clear that the set of all solutions V is a subspace of $L^{\oplus k}$. Moreover, we introduce the coefficient vector $u = (u_1, \dots, u_k)$ of Eq. (1.1). Then the solution space V may be identified with the kernel of the linear map $\phi_u : L^{\oplus k} \rightarrow L$ defined by

$$(v_1, v_2, \dots, v_k) \phi_u = [v_1, u_1] + [v_2, u_2] + \dots + [v_k, u_k]$$

for all $(v_1, \dots, v_k) \in L^{\oplus k}$. We call ϕ_u the linear map associated with Eq. (1.1). We say that a solution $(s_1, \dots, s_k) \in V$ is an inner solution if each s_i belongs to the subalgebra generated by the coefficients $u_1, \dots, u_k$ in L . It is plain that the inner solutions form a subspace of the full solution space V .

3. The homogeneous structure of the solution space

In this section we consider Eq. (1.1) under certain special conditions on the coefficients. In the first two subsections we deal with the case where the coefficients are actually distinct free generators of L : $u = (a_1, \dots, a_k)$. In that case Eq. (1.1) turns into

$$[x_1, a_1] + [x_2, a_2] + \dots + [x_k, a_k] = 0, \quad (3.1)$$

and the associated map $\phi = \phi_u$ (we suppress the subscript u in this special case) is given by

$$(v_1, v_2, \dots, v_k) \phi = [v_1, a_1] + [v_2, a_2] + \dots + [v_k, a_k]. \quad (3.2)$$

Its restriction to $L_n^1 \oplus \cdots \oplus L_n^k$ maps this direct summand of $L^{\oplus k}$ into the homogeneous component L_{n+1} of L . We denote this restriction by ϕ_{n+1} :

$$\phi_{n+1} : L_n^1 \oplus \cdots \oplus L_n^k \rightarrow L_{n+1}. \quad (3.3)$$

It follows easily that the solution space V decomposes into the direct sum of its subspaces

$$V = \bigoplus_{n \geq 1} V_n, \quad \text{with } V_n = \ker \phi_{n+1}.$$

We call V_n the degree n homogeneous component of the solution space V .

3.1. $k = r$ and $u_1 = a_1, \dots, u_k = a_k$

In this case the linear map (3.3) is surjective for all $n \geq 1$. Indeed, as mentioned in Section 2, the homogeneous component L_{n+1} is spanned by all left normed Lie monomials $[b_1, \dots, b_n, b_{n+1}]$ with $b_i \in \mathcal{A}$. In the case under consideration, any such Lie monomial is in the image of ϕ : If $b_{n+1} = a_i$ for some i with $1 \leq i \leq r$, then for $(v_1, \dots, v_k)$ with $v_i = [b_1, \dots, b_n]$ and $v_j = 0$ for $j \neq i$ we have in view of (3.2) $(v_1, \dots, v_k)\phi = [b_1, \dots, b_n, b_{n+1}]$. The dimension of the domain in (3.3) is kd_n , and the dimension of the codomain is d_{n+1} . Since this linear map is surjective, the dimension of the kernel is $kd_n - d_{n+1}$. We summarize our discussion in the following

Theorem 3.1. *Let L be the free Lie algebra of rank k over a field K with free generators $a_1, \dots, a_k$, and let V_n denote the degree n homogeneous component of the solution space of Eq. (3.1). Then*

$$\dim V_n = kd_n - d_{n+1}$$

for all $n \geq 1$, where d_n and d_{n+1} are as in (2.1) with $r = k$.

3.2. $k < r$ and $u_1 = a_1, \dots, u_k = a_k$

In this case we show that all solutions of Eq. (1.1) are inner solutions. The proof is based on a special case of a powerful lemma due to Shirshov. Recall that a set of elements in L is called independent if it is a free generating set of the Lie subalgebra of L it generates. Of course, any subalgebra of L is itself free by the famous Shirshov–Witt Theorem. Shirshov’s Lemma is in essence the key ingredient of Shirshov’s original proof of this theorem, see [9]. An English translation of this celebrated paper is now available in [10]. An element of L is called homogeneous if it is contained in some homogeneous component. A set $\mathcal{S}$ of homogeneous elements in L is reduced if and only if no element $s \in \mathcal{S}$ belongs to the subalgebra generated by the set $\mathcal{S} \setminus \{s\}$ in L . We mention that in this definition of a reduced set it is not required that all the elements of the set belong to one and the same homogeneous component. The result reads as follows.

Lemma 3.1. (See Shirshov [9].) *Any reduced set of homogeneous elements in L is independent.*

Now we are ready for our theorem.

Theorem 3.2. *Let k be a positive integer and let L be the free Lie algebra over a field K of finite or countably infinite rank with more than k free generators $a_1, a_2, \dots, a_k, a_{k+1}, \dots$. Then all solutions of Eq. (3.1) in L are inner solutions.*

Proof. The free Lie algebra L can be written as a vector space direct sum

$$L = L(a_1, a_2, \dots, a_k) \oplus I(\{a_{k+1}, a_{k+2}, \dots\}) = L(\mathcal{B}) \oplus I(\mathcal{C})$$

where $L(\mathcal{B})$ is the free Lie algebra on $\mathcal{B} = \{a_1, a_2, \dots, a_k\}$ and $I(\mathcal{C})$ is the ideal generated by $\mathcal{C} = \{a_{k+1}, a_{k+2}, \dots\}$.

It is sufficient to prove the assertion of the theorem for homogeneous solutions. Let $(s_1, s_2, \dots, s_k)$ be a non-zero homogeneous solution of (3.1) with $\deg s_i = n \geq 1$. Write $s_i = v_i + w_i$ with $v_i \in L(\mathcal{B})$ and $w_i \in I(\mathcal{C})$, $i = 1, \dots, k$. We need to show that $w_i = 0$ for all i . Let $\mathcal{T} = \{t_1, \dots, t_m\}$ be a basis of the K -span of $\{w_1, \dots, w_k\}$, and set $\mathcal{S} = \mathcal{B} \cup \mathcal{T}$. Note that the t_j are homogeneous elements with $\deg t_j = n$. We claim that the set $\mathcal{S}$ is reduced in the sense of Shirshov. Suppose this is not the case. Then one of the elements of $\mathcal{S}$, say, belongs to the subalgebra generated by the remaining elements of $\mathcal{S}$. It is clear that s cannot be in $\mathcal{B}$. Indeed, the elements of $\mathcal{B}$ are of degree 1, and hence the only way of expressing them in terms of the remaining elements of $\mathcal{S}$ would be as a linear combination. But this is impossible since $\mathcal{S}$ is a linearly independent set. Now suppose that $s = t_j \in \mathcal{T}$. Then

$$t_j = \sum_l \alpha_l m_l \quad (3.4)$$

where each m_l is a Lie monomial of degree n in $\mathcal{S} \setminus \{t_j\}$ and $\alpha_l \in K$. None of the m_l can involve factors from both $\mathcal{B}$ and $\mathcal{T} \setminus \{t_j\}$ (since the degree of such monomials would be strictly greater than n). Hence all the Lie monomials in (3.4) are either monomials in $\mathcal{B}$, and hence elements of $L(\mathcal{B})$, or they are actual elements of $\mathcal{T} \setminus \{t_j\}$. Moreover, since L is the direct sum of $L(\mathcal{B})$ and $I(\mathcal{C})$, and $t_j \in I(\mathcal{C})$, the $L(\mathcal{B})$ -part in (3.4) must be zero. But then the right-hand side turns into a linear combination of elements from $\mathcal{T} \setminus \{t_j\}$, and this gives a contradiction since $\mathcal{T}$ is a linearly independent set. Hence $\mathcal{S}$ is a reduced set, and then Shirshov's Lemma gives that $\mathcal{S}$ is independent. Recall that the $s_i = v_i + w_i$ are a solution of (3.1):

$$\sum_{i=1}^k [v_i, a_i] + \sum_{i=1}^k [w_i, a_i] = 0. \quad (3.5)$$

Since L is the direct sum of $L(\mathcal{B})$ and $I(\mathcal{C})$, and since the first sum on the left-hand side of (3.5) is in $L(\mathcal{B})$ and the second sum is in $I(\mathcal{C})$, it follows that either sum is zero, and hence both $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ are solutions of (3.1). In particular,

$$\sum_{i=1}^k [w_i, a_i] = 0. \quad (3.6)$$

Since $\mathcal{T}$ is a basis of the span of $w_1, \dots, w_k$, each w_i can be written as a linear combination

$$w_i = \sum_{j=1}^m \beta_{ij} t_j \quad (3.7)$$

for some $\beta_{ij} \in K$. Substituting this into (3.6) gives

$$\sum_{i=1}^k \sum_{j=1}^m \beta_{ij} [t_j, a_i] = 0.$$

Since $\mathcal{S}$ is an independent set, the Lie products $[t_j, a_i]$ with $1 \leq j \leq m$ and $1 \leq i \leq k$ are linearly independent. Consequently, all β_{ij} are zero. But then (3.7) implies that all w_i are zero as required. $\square$

Since the space of inner solutions in the setting of Theorem 3.2 is the same as the full solution space in the setting of Theorem 3.1, the following is an immediate consequence of the two theorems.

Corollary 3.1. *Under the assumptions of Theorem 3.2,*

$$\dim V_n = kd_n - d_{n+1}$$

where d_n and d_{n+1} are as in (2.1) with $r = k$.

3.3. $\{u_1, \dots, u_k\}$ is an arbitrary independent set

In the previous subsection we have seen that all solutions of Eq. (3.1) are inner. This is no longer true if the coefficient set $\{u_1, \dots, u_k\}$ is an arbitrary independent set. For example, if L is free on a_1 and a_2 , then $x_1 = a_1, x_2 = a_2$ is a solution of the equation $[x_1, [a_2, a_1, a_2]] - [x_2, [a_2, a_1, a_1]] = 0$ that is not inner. However, we have the following concluding result of this section.

Theorem 3.3. *Let L be a free Lie algebra and let $\{u_1, u_2, \dots, u_k\}$ be an independent subset of L . Then the space V_{in} of inner solutions of Eq. (1.1) over the free Lie algebra L has finite codimension in the full solution space V .*

Proof. Let B be the subalgebra generated by $u_1, \dots, u_k$. Then there exists a subalgebra C of finite codimension in L such that B is a free factor of C (see [8, Theorem 8.4]). Thus C is a free subalgebra of L with $C = L(u_1, u_2, \dots, u_k, \dots)$. Let W be a complement of C as a subspace in L . Thus $L = C \oplus W$, where W is of finite dimension. Let $(s_1, \dots, s_k)$ be a solution of (1.1), with $s_i = s_{ic} + s_{iw}$, $1 \leq i \leq k$, $s_{ic} \in C$ and $s_{iw} \in W$. Then

$$\sum_{i=1}^k [s_{ic}, u_i] = - \sum_{i=1}^k [s_{iw}, u_i] \in [W, u_1] + \dots + [W, u_k].$$

Indeed, the map $\psi : C^{\oplus k} \rightarrow C$ given by

$$(c_1, c_2, \dots, c_k) \mapsto [c_1, u_1] + \dots + [c_k, u_k] \quad (c_i \in C)$$

is linear and it maps $(s_{1c}, s_{2c}, \dots, s_{kc})$ into Y , where $Y = \text{im } \psi \cap ([W, u_1] + \dots + [W, u_k])$. Since W has a finite dimension, $\dim[W, u_i]$ is finite for all i , and hence Y is finite-dimensional subspace. Now, since $\dim Y < \infty$, there exists a finite-dimensional subspace $D \subset C^{\oplus k}$ such that $D\psi = Y$, and thus the whole inverse image of Y in $C^{\oplus k}$ under this map is $D + \ker \psi$. Note that $\ker \psi$ is equal to the solution space of (1.1) over C . By Theorem 3.2, this solution space coincides with the solution space of (1.1) over B , so $\ker \psi = V_{in}$. Thus $(s_{1c}, s_{2c}, \dots, s_{kc}) \in D + V_{in}$, and hence $(s_1, \dots, s_k) \in D + V_{in} + W^{\oplus k}$. As a result, the full solution space V of the given equation is contained in $D + W^{\oplus k} + V_{in}$. As D and $W^{\oplus k}$ have finite dimensions, V_{in} is of finite codimension in V . $\square$

4. The fine homogeneous structure of the solution space

In this section we return to the situation considered in Section 3.2, but we assume now that L is of finite rank. Thus L is free of rank r on $\mathcal{A} = \{a_1, \dots, a_r\}$, $2 \leq k \leq r$, and we study Eq. (3.1). In this case the associated linear map decomposes into the direct sum of its restrictions to direct sums of fine homogeneous components as follows: For any composition $q = (q_1, \dots, q_r) \models n+1$ ($n \geq 1$), in r parts, let $(L^{\oplus k})_q$ denote the direct sum

$$(L^{\oplus k})_q = L^1_{(q_1-1, q_2, \dots, q_r)} \oplus L^2_{(q_1, q_2-1, \dots, q_r)} \oplus \cdots \oplus L^k_{(q_1, q_2, \dots, q_k-1, \dots, q_r)},$$

of fine homogeneous components of total degree n with the convention that $L^i_{(q_1, \dots, q_{i-1}, \dots, q_r)} = 0$ if $q_i = 0$, $1 \leq i \leq k$. Then the associated linear map ϕ of Eq. (3.1) decomposes into the direct sum of its restrictions to the subspaces $(L^{\oplus k})_q$,

$$\phi_q : (L^{\oplus k})_q \rightarrow L_q,$$

where q runs over all compositions of $n+1$ in r parts and $n \geq 1$. It follows that the solution space V is the direct sum of the kernels of these restrictions:

$$V = \bigoplus_{n=1}^{\infty} \bigoplus_{q \models n+1} V_q \quad \text{where } V_q = \ker \phi_q.$$

We refer to those kernels as to the fine homogeneous components of multidegree q of the solution space of Eq. (3.1). In particular, the degree n homogeneous component of V is the direct sum of the fine homogeneous components V_q where q runs over all compositions of $n+1$ in r parts. If $k = r$, all ϕ_q are surjective, and then the dimension of V_q can be determined as in the proof of Theorem 3.1 using the fine Witt formula.

Theorem 4.1. *Let L be a free Lie algebra of rank r , $2 \leq k \leq r$. Then the n th homogeneous component of the solution space of Eq. (3.1) decomposes into the direct sum of fine homogeneous components*

$$V_n = \bigoplus_{q \models n+1} V_q$$

where V_q is the kernel of the restriction of the associated linear map ϕ to the direct summand $(L^{\oplus k})_q$ of $L^{\oplus k}$. Moreover, if $k = r$, then, for $q = (q_1, \dots, q_k) \models n+1$,

$$\dim V_q = d_{(q_1-1, q_2, \dots, q_k)} + d_{(q_1, q_2-1, \dots, q_k)} + \cdots + d_{(q_1, q_2, \dots, q_k-1)} - d_{(q_1, q_2, \dots, q_k)}.$$

Let S_r denote the symmetric group of degree r . Then any permutation $\sigma \in S_r$ gives rise to an automorphism τ_σ of the free Lie algebra L given by $a_i \tau_\sigma = a_{i\sigma}$. We call τ_σ the permutation automorphism corresponding to σ . Permutation automorphisms permute the fine homogeneous components of L . In fact, for any composition $q = (q_1, q_2, \dots, q_r) \models n$ in r parts one has clearly

$$L_q \tau_\sigma = L_{q\sigma} \quad \text{where } q\sigma = (q_{1\sigma^{-1}}, q_{2\sigma^{-1}}, \dots, q_{r\sigma^{-1}}).$$

Now let $k = r$ and suppose $(s_1, s_2, \dots, s_k)$ is a solution of Eq. (3.1), that is

$$[s_1, a_1] + [s_2, a_2] + \cdots + [s_k, a_k] = 0.$$

Applying a permutation automorphism τ_σ to this equation gives

$$\begin{aligned} ([s_1, a_1] + [s_2, a_2] + \cdots + [s_k, a_k]) \tau_\sigma &= [s_1 \tau_\sigma, a_{1\sigma}] + [s_2 \tau_\sigma, a_{2\sigma}] + \cdots + [s_k \tau_\sigma, a_{k\sigma}] \\ &= [s_{1\sigma^{-1}} \tau_\sigma, a_1] + [s_{2\sigma^{-1}} \tau_\sigma, a_2] + \cdots + [s_{k\sigma^{-1}} \tau_\sigma, a_k] \\ &= 0. \end{aligned}$$

Consequently, $(s_{1\sigma^{-1}}\tau_\sigma, s_{2\sigma^{-1}}\tau_\sigma, \dots, s_{k\sigma^{-1}}\tau_\sigma)$ too is a solution of (3.1). Moreover, if $(s_1, s_2, \dots, s_k) \in V_q$ for some $q \models n+1$, then $(s_{1\sigma^{-1}}\tau_\sigma, s_{2\sigma^{-1}}\tau_\sigma, \dots, s_{k\sigma^{-1}}\tau_\sigma) \in V_{q\sigma}$. This yields the following.

Theorem 4.2. *Let L be a free Lie algebra of rank $k \geq 2$, let $\sigma \in S_k$ and τ_σ denote the corresponding permutation automorphism of L . Then:*

- (i) *If $(s_1, s_2, \dots, s_k)$ is a solution of Eq. (3.1) then $(s_{1\sigma^{-1}}\tau_\sigma, s_{2\sigma^{-1}}\tau_\sigma, \dots, s_{k\sigma^{-1}}\tau_\sigma)$ is also a solution of this equation.*
- (ii) *The restriction of the K -linear map $\psi_\sigma : L^{\oplus k} \rightarrow L^{\oplus k}$ given by*

$$\psi_\sigma : (s_1, s_2, \dots, s_k) \mapsto (s_{1\sigma^{-1}}\tau_\sigma, s_{2\sigma^{-1}}\tau_\sigma, \dots, s_{k\sigma^{-1}}\tau_\sigma)$$

to a fine homogeneous component V_q of the solution space of (3.1) is a K -space isomorphism between V_q and $V_{q\sigma}$ where for $q = (q_1, \dots, q_k) \models n+1$, $n \geq 1$, $q\sigma$ is defined as $q\sigma = (q_{1\sigma^{-1}}, q_{2\sigma^{-1}}, \dots, q_{k\sigma^{-1}})$.

5. Reinterpreting the solution space

In this section, and for the rest of this paper, L denotes the free Lie algebra of rank $k > 1$ with free generators $a_1, a_2, \dots, a_k$ over a field K . Recall that the degree 1 homogeneous component L_1 is the K -span of the free generators in L . The direct sum $L^{\oplus k} = L^1 \oplus L^2 \oplus \dots \oplus L^k$ is isomorphic (as K -space) to the tensor product $L \otimes L_1$ via the linear map $\theta : L^{\oplus k} \rightarrow L \otimes L_1$ determined by

$$\theta : (u_1, u_2, \dots, u_k) \mapsto \sum_{i=1}^k u_i \otimes a_i$$

where $u_i \in L^i$. Let φ denote the linear map $L \otimes L_1 \rightarrow L$ given by $u \otimes v \mapsto [u, v]$ where $u \in L$ and $v \in L_1$. Then we have a commutative diagram

$$\begin{array}{ccc} L^{\oplus k} & & \\ \theta \downarrow & \searrow \phi & \\ L \otimes L_1 & \xrightarrow{\varphi} & L \end{array}$$

in which the diagonal map $\phi = \phi_u$ with $u = (a_1, a_2, \dots, a_k)$ is the linear map associated with Eq. (3.1). It follows that the linear isomorphism θ maps the solution space of (3.1), that is the kernel of the associated map ϕ , isomorphically onto the kernel of the map φ . This allows us to identify the solution space of (3.1) with the kernel of the linear map φ . The maps in the diagram respect the grading of L , and hence θ maps the homogeneous component V_{n-1} of the solution space of (3.1) isomorphically onto the kernel of the restriction of φ to the subspace $L_{n-1} \otimes L_1$ of $L \otimes L_1$. This restriction maps $L_{n-1} \otimes L_1$ onto L_n , and we write $\varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n$ for the corresponding map. We summarize our discussion as follows.

Lemma 5.1. *For all $n \geq 2$, the isomorphism $\theta : L^{\oplus k} \rightarrow L \otimes L_1$ yields an isomorphism*

$$V_{n-1} \cong \ker \varphi_n$$

where $\varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n$ is, for all $u \in L_{n-1}$ and $v \in L_1$, given by $u \otimes v \mapsto [u, v]$.

In the next section we will obtain an explicit splitting for the projection map φ_n (if the field K is of characteristic coprime to n), and this in turn will allow us to exhibit a spanning set for the kernel. But first we need to introduce some more results on free Lie algebras.

Let T denote the tensor algebra on L_1 . Thus

$$T = \bigoplus_{n \geq 0} T_n \quad \text{where } T_n = \underbrace{L_1 \otimes L_1 \otimes \cdots \otimes L_1}_n$$

with the convention that $T_0 = K$. For $u_1, u_2, \dots, u_m \in T$ it will be convenient to write $u_1 u_2 \dots u_m$ rather than $u_1 \otimes u_2 \otimes \cdots \otimes u_m$. In fact, T can be identified with the free associative algebra on $a_1, \dots, a_k$. It is well known that T is the universal envelope of L . The restriction to the homogeneous components of L of the canonical embedding $L \rightarrow T$ gives embeddings $v_n : L_n \rightarrow T_n$ given by

$$[u_1, u_2, \dots, u_n] v_n = [u_1, u_2, \dots, u_n] \in T_n$$

where $u_i \in L_1$ and the Lie bracket on the right-hand side is the usual Lie bracket in the associative algebra T , i.e. $[u, v] = u \otimes v - v \otimes u$ for all $u, v \in T$. On the other hand, there are canonical projections $\rho_n : T_n \rightarrow L_n$ given by

$$(u_1 u_2 \dots u_n) \rho_n = [u_1, u_2, \dots, u_n] \in L_n,$$

where again $u_i \in L_1$. These two maps are related by Wever's formulae (see [5, Chapter 5, Theorem 5.16])

$$v_n \rho_n = n, \tag{5.1}$$

that is the composite of v_n and ρ_n amounts to multiplication by n in L_n , and, for all $u_1, u_2, \dots, u_n \in L_1$ and $1 \leq k < n$,

$$(u_1 u_2 \dots u_k [u_{k+1}, \dots, u_n]) \rho_n = [u_1, u_2, \dots, u_k, [u_{k+1}, \dots, u_n]]. \tag{5.2}$$

6. A spanning set for the solution space

Our first aim in this section is to find an explicit splitting for the projection map $\varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n$, i.e. a map $\psi_n : L_n \rightarrow L_{n-1} \otimes L_1$ such that $\psi_n \varphi_n = 1$. We construct this map as follows. Let $\bar{\psi}_n : L_n \rightarrow L_{n-1} \otimes L_1$ be the composite of

$$v_n : L_n \rightarrow T_n = T_{n-1} \otimes L_1 \quad \text{and} \quad \rho_{n-1} \otimes 1 : T_{n-1} \otimes L_1 \rightarrow L_{n-1} \otimes L_1.$$

Observe that $\rho_n : T_n \rightarrow L_n$ factors through $L_{n-1} \otimes L_1$, i.e. it can be written as the composite of

$$\rho_{n-1} \otimes 1 : T_{n-1} \otimes L_1 \rightarrow L_{n-1} \otimes L_1 \quad \text{and} \quad \varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n.$$

Then we have

$$\bar{\psi}_n \varphi_n = n.$$

Indeed, by using (5.1) we find

$$\bar{\psi}_n \varphi_n = v_n (\rho_{n-1} \otimes 1) \varphi_n = v_n \rho_n = n.$$

Consequently, if the characteristic of K does not divide n , then $\psi_n = (1/n)\bar{\psi}_n$ is a splitting map for φ_n . But then we have that the kernel of φ_n is precisely

$$\ker \varphi_n = (L_{n-1} \otimes L_1)(1 - \varphi_n \psi_n). \quad (6.1)$$

In order to calculate this kernel more explicitly, we need the following result.

Lemma 6.1. *Let A be an arbitrary associative algebra. Then*

$$\begin{aligned} [c_1, c_2, \dots, c_m] &= (-1)^{m-1} c_m c_{m-1} \dots c_2 c_1 + \sum_{i=2}^{m-1} (-1)^{m-i} c_m c_{m-1} \dots c_{i+1} [c_1, c_2, \dots, c_{i-1}] c_i \\ &\quad + [c_1, \dots, c_{m-1}] c_m \end{aligned}$$

for all $c_1, c_2, \dots, c_m \in A$, $m \geq 1$.

Proof. An easy induction on m . $\square$

Now let $u_1, u_2, \dots, u_n \in L_1$. Then, by using (5.1), Lemma 6.1 with $m = n - 1$ and (5.2) we get

$$\begin{aligned} &([u_1, u_2, \dots, u_{n-1}] \otimes u_n) \varphi_n \psi_n \\ &= \frac{1}{n} [u_1, u_2, \dots, u_n] \bar{\psi}_n \\ &= \frac{1}{n} [u_1, u_2, \dots, u_n] (\rho_{n-1} \otimes 1) \\ &= \frac{1}{n} ([u_1, u_2, \dots, u_{n-1}] u_n - u_n [u_1, u_2, \dots, u_{n-1}]) (\rho_{n-1} \otimes 1) \\ &= \frac{1}{n} \left([u_1, u_2, \dots, u_{n-1}] u_n \right. \\ &\quad \left. - u_n \left((-1)^{n-2} u_{n-1} u_{n-2} \dots u_2 u_1 + \sum_{i=2}^{n-2} (-1)^{n-1-i} u_{n-1} u_{n-2} \dots u_{i+1} [u_1, u_2, \dots, u_{i-1}] u_i \right. \right. \\ &\quad \left. \left. + [u_1, \dots, u_{n-2}] u_{n-1} \right) \right) (\rho_{n-1} \otimes 1) \\ &= \frac{1}{n} \left([u_1, u_2, \dots, u_{n-1}] \rho_{n-1} \otimes u_n + (-1)^{n-1} (u_n u_{n-1} u_{n-2} \dots u_2) \rho_{n-1} \otimes u_1 \right. \\ &\quad \left. + \sum_{i=2}^{n-2} (-1)^{n-i} (u_n u_{n-1} u_{n-2} \dots u_{i+1} [u_1, u_2, \dots, u_{i-1}]) \rho_{n-1} \otimes u_i \right. \\ &\quad \left. - (u_n [u_1, \dots, u_{n-2}]) \rho_{n-1} \otimes u_{n-1} \right) \end{aligned}$$

$$= \frac{1}{n} \left((n-1)[u_1, u_2, \dots, u_{n-1}] \otimes u_n + (-1)^{n-1}[u_n, u_{n-1}, u_{n-2}, \dots, u_2] \otimes u_1 \right. \\ \left. + \sum_{i=2}^{n-1} (-1)^{n-i}[u_n, u_{n-1}, u_{n-2}, \dots, u_{i+1}, [u_1, u_2, \dots, u_{i-1}]] \otimes u_i \right).$$

Hence

$$([u_1, u_2, \dots, u_{n-1}] \otimes u_n)(1 - \varphi_n \psi_n) \\ = \frac{1}{n} \left((-1)^n[u_n, u_{n-1}, u_{n-2}, \dots, u_2] \otimes u_1 \right. \\ \left. + \sum_{i=2}^{n-1} (-1)^{n+1-i}[u_n, u_{n-1}, u_{n-2}, \dots, u_{i+1}, [u_1, u_2, \dots, u_{i-1}]] \otimes u_i \right. \\ \left. + [u_1, u_2, \dots, u_{n-1}] \otimes u_n \right).$$

In view of (6.1) we can now state our first main result.

Theorem 6.1. *Let L be the free Lie algebra of rank $k > 1$ with free generators $a_1, \dots, a_k$ over a field K . If the characteristic of K does not divide n , then the kernel of the linear map $\varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n$ is spanned by the elements*

$$w_n(u_1, u_2, \dots, u_{n-1}, u_n) = (-1)^n[u_n, u_{n-1}, u_{n-2}, \dots, u_2] \otimes u_1 \\ + \sum_{i=2}^{n-1} (-1)^{n+1-i}[u_n, u_{n-1}, u_{n-2}, \dots, u_{i+1}, [u_1, u_2, \dots, u_{i-1}]] \otimes u_i \\ + [u_1, u_2, \dots, u_{n-1}] \otimes u_n$$

with $u_i \in \{a_1, \dots, a_k\}$.

In order to obtain a description of this kernel in the case where K is a field of arbitrary characteristic, let L be the free Lie algebra on $a_1, \dots, a_k$ over the field of rational numbers $\mathbb{Q}$, and consider the free Lie ring $\mathfrak{L}$ on $a_1, \dots, a_k$ as a subring of L . Then the elements $w_n(u_1, u_2, \dots, u_{n-1}, u_n)$ with $u_i \in \{a_1, \dots, a_k\}$ are in $\mathfrak{L}_{n-1} \otimes \mathfrak{L}_1$, and they belong to the kernel of the restriction $\Phi_n : \mathfrak{L}_{n-1} \otimes \mathfrak{L}_1 \rightarrow \mathfrak{L}_n$ of φ_n to $\mathfrak{L}_{n-1} \otimes \mathfrak{L}_1$. Clearly the rank of the $\mathbb{Z}$ -submodule of $\mathfrak{L}_{n-1} \otimes \mathfrak{L}_1$ spanned by those elements is equal to the dimension of the subspace spanned by those elements in $L_{n-1} \otimes L_1$. Since $\mathfrak{L}_n$ is a free $\mathbb{Z}$ -module, it follows that the kernel of Φ_n is the isolator of the $\mathbb{Z}$ -span of the elements $w_n(u_1, u_2, \dots, u_{n-1}, u_n)$ in $\mathfrak{L}_{n-1} \otimes \mathfrak{L}_1$. But once we have the kernel of Φ_n , we get the kernel of φ_n over an arbitrary field K by tensoring with K . Hence the following is an easy consequence of Theorem 6.1.

Corollary 6.1. *The kernel of the canonical map $\Phi_n : \mathfrak{L}_{n-1} \otimes \mathfrak{L}_1 \rightarrow \mathfrak{L}_n$ is the isolator of the $\mathbb{Z}$ -span of the elements $w_n(u_1, u_2, \dots, u_{n-1}, u_n)$ with $u_i \in \{a_1, \dots, a_k\}$ in $\mathfrak{L}_{n-1} \otimes \mathfrak{L}_1$. For the free Lie algebra L over an arbitrary field K , the kernel of the canonical map $\varphi_n : L_{n-1} \otimes L_1 \rightarrow L_n$ is $\ker \Phi_n \otimes K$.*

For small n the elements $w_n = w_n(u_1, \dots, u_n)$ are as follows:

$$w_2 = u_1 \otimes u_2 + u_2 \otimes u_1,$$

$$\begin{aligned} w_3 &= [u_1, u_2] \otimes u_3 + [u_3, u_1] \otimes u_2 - [u_3, u_2] \otimes u_1 \\ &= [u_1, u_2] \otimes u_3 + [u_3, u_1] \otimes u_2 + [u_2, u_3] \otimes u_1, \end{aligned}$$

$$\begin{aligned} w_4 &= [u_1, u_2, u_3] \otimes u_4 + [u_4, [u_1, u_2]] \otimes u_3 - [u_4, u_3, u_1] \otimes u_2 + [u_4, u_3, u_2] \otimes u_1 \\ &= [u_1, u_2, u_3] \otimes u_4 - [u_1, u_2, u_4] \otimes u_3 - [u_4, u_3, u_1] \otimes u_2 + [u_4, u_3, u_2] \otimes u_1. \end{aligned}$$

These can be used to get bases of $\ker \varphi_n$ for $n = 2, 3$. The elements

$$a_i \otimes a_i \quad (1 \leq i \leq k); \quad a_i \otimes a_j + a_j \otimes a_i \quad (1 \leq i < j \leq k)$$

form a basis of $\ker \varphi_2$, and the elements

$$[a_i, a_j] \otimes a_m + [a_m, a_i] \otimes a_j + [a_j, a_m] \otimes a_i \quad (1 \leq i < j < m \leq k)$$

form a basis of $\ker \varphi_3$. We mention that the basis elements $a_i \otimes a_i \in \ker \varphi_2$ are not in the span of the elements $w_2(u_1, u_2)$ with $u_1, u_2 \in \{a_1, \dots, a_k\}$ if K is field of characteristic 2. This shows that taking the isolator in Corollary 6.1 is essential. This can also be seen in the case where $n = 4$. The value of $w_4(u_1, \dots, u_4)$ if exactly two of the u_i are equal to a_1 and the other two are equal to a_2 is either 0 or $\pm 2([a_1, a_2, a_1] \otimes a_2 - [a_1, a_2, a_2] \otimes a_1)$. Hence, in characteristic 2, $[a_1, a_2, a_1] \otimes a_2 - [a_1, a_2, a_2] \otimes a_1$ is not in the span of the elements $w_4(u_1, \dots, u_4)$, but it is clearly in the kernel of Φ_4 .

7. A basis for the multilinear fine homogeneous component of the solution space

The fine homogeneous components of the solution space in our new interpretation, i.e. as the kernel of $\varphi : L \otimes L_1 \rightarrow L$, are defined in the obvious way. For a composition $q = (q_1, \dots, q_k) \models n + 1$, the fine homogeneous component $(L \otimes L_1)_q$ of $L \otimes L_1$ is the span of all elements $[b_1, \dots, b_n] \otimes b_{n+1}$ with $b_i \in \mathcal{A}$ and exactly q_j of the b_i being equal a_j , and the fine homogeneous component $(\ker \varphi)_q$ of the solution space of (3.1) is the intersection $(\ker \varphi)_q = \ker \varphi \cap (L \otimes L_1)_q$. It is plain that $(\ker \varphi)_q$ is the image of V_q under the isomorphism θ . The multilinear fine homogeneous component for the free Lie algebra L of rank k is the one determined by the composition $(1, 1, \dots, 1) \models k$. The main result of this section is as follows.

Theorem 7.1. *Let L be the free Lie algebra of rank $k > 1$ with free generators $a_1, \dots, a_k$ over a field K . Then the elements*

$$w_k(a_1, a_{2\sigma}, a_{3\sigma}, \dots, a_{(k-1)\sigma}, a_k) \quad \text{with } \sigma \in S_{k-1} \text{ such that } 1\sigma = 1 \quad (7.1)$$

form a K -basis of the multilinear fine homogeneous component $(\ker \varphi)_{(1,1,\dots,1)}$ of the solution space of (3.1).

Proof. By Theorem 4.1 the dimension of $(\ker \varphi)_{(1,1,\dots,1)}$ is $(k-2)!$, which is precisely the number of elements (7.1). We will work with the free Lie ring $\mathcal{L}$ of rank $k > 1$ with free generators $a_1, \dots, a_k$. In view of Corollary 6.1 it is sufficient to show that the elements (7.1) are linearly independent in $\mathcal{L}_{k-1} \otimes \mathcal{L}_1$, and that their span is isolated in this free $\mathbb{Z}$ -module. Let μ denote the projection map $\mathcal{L}_{k-1} \otimes \mathcal{L}_1 \rightarrow \mathcal{L}_{k-1} \otimes \langle a_k \rangle$ that is induced by the projection of the free $\mathbb{Z}$ -module $\mathcal{L}_1$ onto $\langle a_k \rangle$. Then we have

$$w_k(a_1, a_{2\sigma}, a_{3\sigma}, \dots, a_{(k-1)\sigma}, a_k)\mu = [a_1, a_{2\sigma}, a_{3\sigma}, \dots, a_{(k-1)\sigma}] \otimes a_k. \quad (7.2)$$

It is well known that the elements $[a_1, a_{2\sigma}, a_{3\sigma}, \dots, a_{(k-1)\sigma}]$ where σ runs over all permutation of $\{2, 3, \dots, k-1\}$ form a basis of the multilinear fine homogeneous component of the free Lie ring of rank $k-1$ with free generators $a_1, \dots, a_{k-1}$, see [7, 5.6.2]. It follows that the elements (7.2) can be included into a basis of $\mathfrak{L}_{k-1} \otimes \langle a_k \rangle$, and since this is a direct summand of $\mathfrak{L}_{k-1} \otimes \mathfrak{L}_1$, they can also be included into a basis of this tensor product. Hence they are linearly independent and their span is isolated in $\mathfrak{L}_{k-1} \otimes \mathfrak{L}_1$. $\square$

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