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The Calabi–Yau property of smash products

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ABSTRACT

Let H be a semisimple Hopf algebra and R a braided Hopf algebra in the category of Yetter–Drinfeld modules over H. When R is a Calabi–Yau algebra, a necessary and sufficient condition for $R\#H$ to be a Calabi–Yau Hopf algebra is given. Conversely, when H is the group algebra of a finite group and the smash product $R\#H$ is a Calabi–Yau algebra, we give a necessary and sufficient condition for the algebra R to be a Calabi–Yau algebra.

Introduction

In recent years, Calabi–Yau (CY) algebras have attracted lots of attention due to their applications in Algebraic Geometry and in Mathematical Physics. The study of Calabi–Yau Hopf algebras was initiated by K. Brown and J. Zhang in 2008, cf. [5], where they studied rigid dualizing complexes of Noetherian Hopf algebras. S. Chemla showed in [6] that quantum enveloping algebras are Calabi–Yau. In [8] J. He, F. Van Oystaeyen and Y. Zhang showed that the smash product of a universal enveloping algebra of a finite dimensional Lie algebra is Calabi–Yau if and only if the group is a subgroup of the special linear group and the enveloping algebra itself is Calabi–Yau. Thus they were able to classify the Noetherian cocommutative Calabi–Yau Hopf algebras of dimension less than 4 over an algebraically closed field. The smash product construction of Calabi–Yau Hopf algebras applied in [8] provides in fact an effective method to construct new Calabi–Yau (Hopf) algebras based on existing Calabi–Yau (Hopf) algebras. However, the Calabi–Yau property of the smash product $R\#\Bbbk G$ depends strongly on the action of $\Bbbk G$ on R. For example, the pointed Hopf algebra $U(\mathcal{D},\lambda)$ of finite Cartan type constructed in [1] with Γ an infinite group of finite rank is Calabi–Yau if and only if the associated

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graded Hopf algebra $R\#\Bbbk\Gamma$ is Calabi–Yau, where R is the Nichols algebra of $U(\mathcal{D},\lambda)$. But in this case, if $R\#\Bbbk\Gamma$ is Calabi–Yau, then R cannot be Calabi–Yau; and vice versa cf. [22]. This leads to the question: can we find the "right" action of G on R so that the Calabi–Yau property of an algebra R delivers the Calabi–Yau property of $R\#\Bbbk G$?

The question was answered by Wu and Zhu in [20], where they considered the smash product $R\#\Bbbk G$ of a Koszul Calabi–Yau algebra R by a finite group of automorphisms of R. They showed that the smash product $R\#\Bbbk G$ is Calabi–Yau if and only if the homological determinant (Definition 1.13) of the G-action on R is trivial. Later, this result was generalized to the case where R is a p-Koszul Calabi–Yau algebra and $\Bbbk G$ is replaced by an involutory Calabi–Yau Hopf algebra [13]. A crossed product is a generalization of a smash product. P. Le Meur studied in [12] when the crossed product of a graded Calabi–Yau algebra by a finite group is still Calabi–Yau.

Inspired by the work of Wu and Zhu [20] and the fact that the associated graded Hopf algebra of a pointed Hopf algebra is a smash product of a braided Hopf algebra in a Yetter–Drinfeld module category over the coradical, we consider in this paper the Calabi–Yau property of a smash product Hopf algebra $R\#H$, where R is a braided Hopf algebra in the Yetter–Drinfeld module category over H. We use the homological determinant of the Hopf action to describe the homological integral (Definition 1.10) of $R\#H$. We then give a necessary and sufficient condition for $R\#H$ to be a Calabi–Yau algebra in case R is Calabi–Yau and H is semisimple (Theorem 2.8). We then continue to consider the inverse problem. That is, if $R\#H$ is Calabi–Yau, when is R Calabi–Yau? In Section 3, we answer this question in case $H=\Bbbk G$ is the group algebra of a finite group. We then go on to characterize the Calabi–Yau property of R when $R\#\Bbbk G$ is Calabi–Yau (Theorem 3.10). Applying our characterization theorem we obtain the Calabi–Yau property of $U(\mathcal{D},\lambda)$ in case the datum is not generic. The generic case is completely worked out in [22]. We will provide two examples of Calabi–Yau pointed Hopf algebras with a finite abelian group of group-like elements.

The paper is organized as follows. In Section 1, we review the definition of a braided Hopf algebra, the definition of a Calabi–Yau algebra, the concept of a homological integral and the notion of homological determinants.

In Section 2, we study the Calabi–Yau property under a Hopf action. Our main result in this section is Theorem 2.8, which characterizes the Calabi–Yau property of the smash product Hopf algebra $R\#H$, where H is a semisimple Hopf algebra and R is a braided Hopf algebra in the Yetter–Drinfeld module category over H.

In Section 3, we consider the question when the Calabi–Yau property of $R\#H$ implies that R is Calabi–Yau. We answer this question in case $H=\Bbbk G$ is the group algebra of a finite group. We first construct a bimodule resolution of R from a projective resolution of $\Bbbk$ over the algebra $R\#\Bbbk G$. Based on this, we obtain a rigid dualizing complex of R in case R is AS-Gorenstein (Theorem 3.8). Our main result in this section is Theorem 3.10.

Throughout, we work over a fixed field $\Bbbk$. All vector spaces and algebras are assumed to be over $\Bbbk$.

1. Preliminaries

Given an algebra A, let A^{op} denote the opposite algebra of A and A^e denote the enveloping algebra $A\otimes A^{op}$ of A. The unfurnished tensor $\otimes$ means $\otimes_{\Bbbk}$ in this paper. $\operatorname{Mod}A$ denotes the category of left A-modules. We use $\operatorname{Mod}A^{op}$ to denote the category of right A-modules.

For a left A-module M and an algebra automorphism $\phi: A\to A$, write ${}_{\phi}M$ for the left A-module defined by $a\cdot m=\phi(a)m$ for any $a\in A$ and $m\in M$. Similarly, for a right A-module N, we have N_{ϕ}. Observe that $A_{\phi}\cong {}_{\phi^{-1}}A$ as A–A-bimodules. $A_{\phi}\cong A$ as A–A-bimodules if and only if ϕ is an inner automorphism.

For a Hopf algebra, we use Sweedler's notation (sumless) for its comultiplication and its coactions. Let A be a Hopf algebra, and $\xi: A\to\Bbbk$ an algebra homomorphism. We write $[\xi]$ to be the winding homomorphism of ξ defined by

$$[\xi](a)=\xi(a_1)a_2,$$

for any $a\in A$.

1.1. Braided Hopf algebra

Let H be a Hopf algebra. We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld modules over H with morphisms given by H-linear and H-colinear maps. If Γ is a finite group, then ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ will be abbreviated to ${}^\Gamma_\Gamma\mathcal{YD}$.

The tensor product of two Yetter–Drinfeld modules M and N is again a Yetter–Drinfeld module with the module and comodule structures given as follows

$$h(m\otimes n)=h_1m\otimes h_2n \quad \text{and} \quad \delta(m\otimes n)=m_{(-1)}n_{(-1)}\otimes m_{(0)}\otimes n_{(0)},$$

for any $h\in H$, $m\in M$ and $n\in N$. This turns the category of Yetter–Drinfeld modules ${}^H_H\mathcal{YD}$ into a braided tensor category. For more detail about braided tensor categories, one refers to [10].

For any two Yetter–Drinfeld modules M and N, the braiding $c_{M,N}: M\otimes N\to N\otimes M$ is given by

$$c_{M,N}(m\otimes n)=m_{(-1)}\cdot n\otimes m_{(0)},$$

for any $m\in M$ and $n\in N$.

Definition 1.1. Let H be a Hopf algebra.

(i) An *algebra* in ${}^H_H\mathcal{YD}$ is a $\Bbbk$-algebra (R,m,u) such that R is a Yetter–Drinfeld H-module, and both the multiplication $m: R\otimes R\to R$ and the unit $u:\Bbbk\to R$ are morphisms in ${}^H_H\mathcal{YD}$.

(ii) A *coalgebra* in ${}^H_H\mathcal{YD}$ is a $\Bbbk$-coalgebra (C,Δ,ε) such that C is a Yetter–Drinfeld H-module, and both the comultiplication $\Delta: R\to R\otimes R$ and the counit $\varepsilon: R\to\Bbbk$ are morphisms in ${}^H_H\mathcal{YD}$.

Let R and S be two algebras in ${}^H_H\mathcal{YD}$. Then $R\otimes S$ is a Yetter–Drinfeld module in ${}^H_H\mathcal{YD}$, and becomes an algebra in the category ${}^H_H\mathcal{YD}$ with the multiplication $m_{R\underline{\otimes}S}$ defined by

$$m_{R\underline{\otimes}S}:=(m_R\otimes m_S)(\mathrm{id}\otimes c\otimes\mathrm{id}).$$

Denote this algebra by $R\underline{\otimes}S$.

Definition 1.2. Let H be a Hopf algebra. A *braided bialgebra* in ${}^H_H\mathcal{YD}$ is a collection $(R,m,u,\Delta,\varepsilon)$, where

(i) (R,m,u) is an algebra in ${}^H_H\mathcal{YD}$.
(ii) (R,Δ,ε) is a coalgebra in ${}^H_H\mathcal{YD}$.
(iii) $\Delta: R\to R\underline{\otimes}R$ and $\varepsilon: R\to\Bbbk$ are morphisms of algebras in ${}^H_H\mathcal{YD}$.

If, in addition, the identity is convolution invertible in $\mathrm{End}(R)$, then R is called a *braided Hopf algebra* in ${}^H_H\mathcal{YD}$. The inverse of the identity is called the *antipode* of R.

In order to distinguish comultiplications of braided Hopf algebras from those of usual Hopf algebras, we use Sweedler's notation with upper indices for braided Hopf algebras

$$\Delta(r)=r^1\otimes r^2.$$

Let H be a Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. For $h\in H$ and $r\in R$, we write $h(r)$ for h acting on r. It is an element in R. On the other hand, we write hr for

h multiplying with r. It is an element in $R\#H$. The algebra $R\#H$ is a usual Hopf algebra with the following structure [16]:

The multiplication is given by

$$(r\#g)(s\#h) := rg_1(s)\#g_2h$$

with unit $u_R \otimes u_H$. The comultiplication is given by

$$\Delta(r\#h) := r^1\#\left(r^2\right)_{(-1)}h_1 \otimes \left(r^2\right)_{(0)}\#h_2 \tag{1}$$

with counit $\varepsilon_R \otimes \varepsilon_H$. The antipode is as follows:

$$\mathcal{S}_{R\#H}(r\#h) = \big(1\#\mathcal{S}_H(r_{(-1)}h)\big)\big(\mathcal{S}_R(r_{(0)})\#1\big). \tag{2}$$

The algebra $R\#H$ is called the *Radford biproduct* or *bosonization* of R by H. The algebra R is a subalgebra of $R\#H$ and H is a Hopf subalgebra of $R\#H$.

Conversely, let A and H be two Hopf algebras and $\pi : A \to H$, $\iota : H \to A$ Hopf algebra homomorphisms such that $\pi\iota = \mathrm{id}_H$. In this case the algebra of right coinvariants with respect to π

$$R = A^{co\pi} := \big\{a \in A \;\big|\; (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\big\},$$

is a braided Hopf algebra in ${}^H_H\mathcal{YD}$, with the following structure [16]:

(i) The action of H on R is the restriction of the adjoint action (composed with ι).
(ii) The coaction is $(\pi \otimes \mathrm{id})\Delta$.
(iii) R is a subalgebra of A.
(iv) The comultiplication is given by

$$\Delta_R(r) = r_1\iota\mathcal{S}_H\pi(r_2) \otimes r_3.$$

(v) The antipode is given by

$$\mathcal{S}_R(r) = \pi(r_1)\mathcal{S}_A(r_2).$$

Define a linear map $\rho : A \to R$ by

$$\rho(a) = a_1\iota\mathcal{S}_H\pi(a_2),$$

for all $a \in A$.

Theorem 1.3. *(See [16].) The morphisms $\Psi : A \to R\#H$ and $\Phi : R\#H \to A$ defined by*

$$\Psi(a) = \rho(a_1)\#\pi(a_2) \quad \textit{and} \quad \Phi(r\#h) = r\iota(h)$$

are mutually inverse isomorphisms of Hopf algebras.

1.2. Calabi–Yau algebras

We follow Ginzburg's definition of a Calabi–Yau algebra [7].

Definition 1.4. An algebra A is called a *Calabi–Yau algebra of dimension d* if

(i) A is *homologically smooth*, that is, A has a bounded resolution of finitely generated projective A–A-bimodules;
(ii) there are A–A-bimodule isomorphisms

$$\operatorname{Ext}^i_{A^e}(A, A^e) = \begin{cases} 0, & i \neq d; \\ A, & i = d. \end{cases}$$

In the sequel, Calabi–Yau will be abbreviated to CY for short.

CY algebras form a class of algebras possessing a rigid dualizing complex (ungraded version). The non-commutative version of a dualizing complex was first introduced by Yekutieli.

A Noetherian algebra in this paper means a *left and right* Noetherian algebra.

Definition 1.5. (Cf. [21], [18, Defn. 6.1].) Assume that A is a Noetherian algebra. Then an object $\mathscr{R}$ of $D^b(A^e)$ is called a *dualizing complex* if it satisfies the following conditions:

(i) $\mathscr{R}$ is of finite injective dimension over A and A^{op}.
(ii) The cohomology of $\mathscr{R}$ is given by bimodules which are finitely generated on both sides.
(iii) The natural morphisms $A \to \operatorname{RHom}_A(\mathscr{R}, \mathscr{R})$ and $A \to \operatorname{RHom}_{A^{op}}(\mathscr{R}, \mathscr{R})$ are isomorphisms in $D(A^e)$.

Roughly speaking, a dualizing complex is a complex $\mathscr{R} \in D^b(A^e)$ such that the functor

$$\operatorname{RHom}_A(-, \mathscr{R}) : D^b_{fg}(A) \to D^b_{fg}\big(A^{op}\big) \tag{3}$$

is a duality, with adjoint $\operatorname{RHom}_{A^{op}}(-, \mathscr{R})$ (cf. [21, Prop. 3.4 and Prop. 3.5]). Here $D^b_{fg}(A)$ is the full triangulated subcategory of $D(A)$ consisting of bounded complexes with finitely generated cohomology modules.

Dualizing complexes are not unique up to isomorphism. To overcome this weakness, Van den Bergh introduced the concept of a rigid dualizing complex cf. [18, Defn. 8.1].

Definition 1.6. Let A be a Noetherian algebra. A dualizing complex $\mathscr{R}$ over A is called *rigid* if

$$\operatorname{RHom}_{A^e}(A, {}_A\mathscr{R} \otimes \mathscr{R}_A) \cong \mathscr{R}$$

in $D(A^e)$.

The following lemma can be found in [5, Prop. 4.3] and [18, Prop. 8.4]. Note that if a Noetherian algebra has finite left and right injective dimension, then they are equal cf. [23, Lemma A]. We call this common value the injective dimension of A.

Lemma 1.7. *Let A be a Noetherian algebra. Then the following two conditions are equivalent*:

(i) *A has a rigid dualizing complex $\mathscr{R} = A_\psi[s]$, where ψ is an algebra automorphism and $s \in \mathbb{Z}$.*

(ii) *A has finite injective dimension d and there is an algebra automorphism ϕ such that*

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A_\phi, & i = d \end{cases}$$

as A–A-bimodules.

If one of the two conditions holds, then $\phi\psi$ is an inner automorphism and $s = d$.

The following corollary follows directly from Lemma 1.7 and the definition of a CY algebra.

Corollary 1.8. *Let A be a Noetherian algebra which is homologically smooth. Then A is a CY algebra of dimension d if and only if A has a rigid dualizing complex $A[d]$.*

1.3. Homological integral

In [8], the CY property of Hopf algebras was discussed by using the homological integrals of Artin–Schelter Gorenstein (AS-Gorenstein for short) algebras [8, Thm. 2.3]. The concept of a homological integral for an AS-Gorenstein Hopf algebra was introduced by Lu, Wu and Zhang in [14] to study infinite dimensional Noetherian Hopf algebras. It generalizes the concept of an integral of a finite dimensional Hopf algebra. In [5], homological integrals were defined for general AS-Gorenstein algebras.

Definition 1.9. (Cf. [5, Defn. 1.2].)

(i) Let A be a left Noetherian augmented algebra with a fixed augmentation map $\varepsilon : A \to \Bbbk$. A is said to be *left AS-Gorenstein*, if
 (a) $\operatorname{injdim}_A A = d < \infty$,
 (b) $\dim \operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) = \begin{cases} 0, & i \neq d; \\ 1, & i = d, \end{cases}$
 where injdim stands for injective dimension.
 A *Right AS-Gorenstein algebras* can be defined similarly.
(ii) An algebra A is said to be *AS-Gorenstein* if it is both left and right AS-Gorenstein (relative to the same augmentation map ε).
(iii) An AS-Gorenstein algebra A is said to be *regular* if in addition, the global dimension of A is finite.

Definition 1.10. Let A be a left AS-Gorenstein algebra with $\operatorname{injdim}_A A = d$. Then $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. Any non-zero element in $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$ is called a *left homological integral* of A. We write $\int^l_A$ for $\operatorname{Ext}^d_A({}_A\Bbbk, {}_AA)$. Similarly, if A is right AS-Gorenstein with $\operatorname{injdim} A_A = d$, any non-zero element in $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$ is called a *right homological integral* of A. Write $\int^r_A$ for $\operatorname{Ext}^d_A(\Bbbk_A, A_A)$.

$\int^l_A$ and $\int^r_A$ are called *left and right homological integral modules* of A respectively.

The left integral module $\int^l_A$ is a 1-dimensional right A-module. Thus $\int^l_A \cong \Bbbk_\xi$ for some algebra homomorphism $\xi : A \to \Bbbk$.

Proposition 1.11. *Let A be a Noetherian augmented algebra such that A is CY of dimension d. Then A is AS-regular of global dimension d. In addition, $\int^l_A \cong \Bbbk$ as right A-modules.*

Proof. If A is an augmented algebra, then ${}_A\Bbbk$ is a finite dimensional A-module. By [4, Remark 2.8], A has global dimension d.

It follows from [4, Prop. 2.2] that A admits a projective bimodule resolution

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where each P_i is finitely generated as an A–A-bimodule. Tensoring with functor $\otimes_A \Bbbk$, we obtain a projective resolution of ${}_A\Bbbk$:

$$0 \to P_d \otimes_A \Bbbk \to \cdots \to P_1 \otimes_A \Bbbk \to P_0 \otimes_A \Bbbk \to {}_A\Bbbk \to 0.$$

Since each P_i is finitely generated, the isomorphism

$$\Bbbk \otimes_A \operatorname{Hom}_{A^e}\bigl(P_i, A^e\bigr) \cong \operatorname{Hom}_A(P_i \otimes_A \Bbbk, A)$$

holds in $\operatorname{Mod} A^{op}$. Therefore, the complex $\operatorname{Hom}_A(P_\bullet \otimes_A \Bbbk, A)$ is isomorphic to the complex $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_\bullet, A^e)$. The fact that the algebra A is CY of dimension d implies that the following A–A-bimodule complex is exact:

$$0 \to \operatorname{Hom}_{A^e}\bigl(P_0, A^e\bigr) \to \cdots \to \operatorname{Hom}_{A^e}\bigl(P_{d-1}, A^e\bigr) \to \operatorname{Hom}_{A^e}\bigl(P_d, A^e\bigr) \to A \to 0.$$

Thus the complex $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_\bullet, A^e)$ is exact except at $\Bbbk \otimes_A \operatorname{Hom}_{A^e}(P_d, A^e)$, whose homology is $\Bbbk$. It follows that the isomorphisms

$$\operatorname{Ext}^i_A({}_A\Bbbk, {}_A A) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk, & i = d \end{cases}$$

hold in $\operatorname{Mod} A^{op}$. Similarly, we have isomorphisms

$$\operatorname{Ext}^i_A(\Bbbk_A, A_A) \cong \begin{cases} 0, & i \neq d; \\ \Bbbk, & i = d \end{cases}$$

in $\operatorname{Mod} A$. We conclude that A is AS-regular and $\int^l_A \cong \Bbbk$. □

Remark 1.12. From the proof of Proposition 1.11 we can see that if A is a Noetherian augmented algebra such that

(i) A is homologically smooth, and
(ii) there is an integer d and an algebra automorphism ψ, such that

$$\operatorname{Ext}^i_{A^e}\bigl(A, A^e\bigr) \cong \begin{cases} 0, & i \neq d; \\ A_\psi, & i = d \end{cases}$$

as A–A-bimodules,

then A is AS-regular of global dimension d. In this case, $\int^l_A \cong \Bbbk_\xi$. The algebra homomorphism ξ is defined by $\xi(a) = \varepsilon(\psi(a))$ for all $a \in A$, where ε is the augmentation map of A.

1.4. Homological determinants

The homological determinant was defined by Jørgensen and Zhang [9] for graded automorphisms of an AS-Gorenstein algebra and by Kirkman, Kuzmanovich and Zhang [11] for Hopf actions on an AS-Gorenstein algebra. The homological determinant was used to study the AS-Gorenstein property of invariant subrings.

Definition 1.13. (Cf. [11].) Let H be a Hopf algebra, and R an H-module AS-Gorenstein algebra of injective dimension d. There is a natural H-action on $\operatorname{Ext}_R^d(\Bbbk, R)$ induced by the H-action on R. Let $\mathbf{e}$ be a non-zero element in $\operatorname{Ext}_R^d(\Bbbk, R)$. Then there exists an algebra homomorphism $\eta : H \to \Bbbk$ satisfying $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$.

(i) The composite map $\eta\mathcal{S}_H : H \to \Bbbk$ is called the *homological determinant* of the H-action on R, and it is denoted by hdet (or more precisely hdet_R).
(ii) The homological determinant hdet_R is said to be *trivial* if $\mathrm{hdet}_R = \varepsilon_H$, where ε_H is the counit of the Hopf algebra H.

2. Calabi–Yau property under Hopf actions

Let H be a Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. In this section, we study the CY property of the smash product $R \# H$, when R is a CY algebra and H is a semisimple Hopf algebra.

For a left $R \# H$-module M, the vector space $M \otimes H$ is a left $R \# H$-module defined by

$$(r \# h) \cdot (m \otimes g) := (r \# h_1)m \otimes h_2 g,$$

for all $r \# h \in R \# H$ and $m \otimes g \in M \otimes H$. Denote this $R \# H$-module by $M \# H$.

Let M and N be two $R \# H$-modules. Then there is a natural left H-module structure on $\operatorname{Hom}_R(M, N)$ given by the adjoint action

$$(h \rightharpoonup f)(m) := h_2 f\big(\mathcal{S}_H^{-1}(h_1)m\big),$$

for all $h \in H$, $f \in \operatorname{Hom}_R(M, N)$ and $m \in M$.

Lemma 2.1. *Let M be a left $R \# H$-module. Then $\operatorname{Hom}_R(M, R) \otimes H$ is an H–$R \# H$-bimodule, where the left H-module structure is defined by*

$$h \cdot (f \otimes g) := h_1 \rightharpoonup f \otimes h_2 g$$

and the right $R \# H$-module structure is given by the diagonal action:

$$(f \otimes g) \cdot (r \# h) := f g_1(r) \otimes g_2 h,$$

for all $f \in \operatorname{Hom}_R(M, R)$, $g, h \in H$ and $r \in R$.

Proof. First we show that for all $h \in H$, $f \in \operatorname{Hom}_R(M, R)$ and $r \in R$

$$(h_1 \rightharpoonup f) h_2(r) = h \rightharpoonup (fr). \tag{4}$$

For $m \in M$, we have

$$\begin{aligned}
\big[(h_1 \rightharpoonup f) h_2(r)\big](m) &= (h_1 \rightharpoonup f)(m) h_2(r) \\
&= h_2\big(f\big(\mathcal{S}_H^{-1}(h_1)m\big)\big) h_3(r) \\
&= h_2\big(f\big(\mathcal{S}_H^{-1}(h_1)m\big) r\big) \\
&= h_2\big((fr)\big(\mathcal{S}_H^{-1}(h_1)m\big)\big) \\
&= \big[h \rightharpoonup (fr)\big](m).
\end{aligned}$$

Now we check that for all $f\otimes g\in \mathrm{Hom}_R(M,R)\otimes H$, $h\in H$ and $r\#k\in R\#H$, $(h\cdot(f\otimes g))\cdot(r\#k)=h\cdot((f\otimes g)\cdot(r\#k))$. We have

$$\begin{aligned}\big(h\cdot(f\otimes g)\big)\cdot(r\#k)&=(h_1\rightharpoonup f\otimes h_2g)\cdot(r\#k)\\&=(h_1\rightharpoonup f)(h_2g_1)(r)\otimes h_3g_2k\end{aligned}$$

and

$$\begin{aligned}h\cdot\big((f\otimes g)\cdot(r\#k)\big)&=h\rightharpoonup\big(fg_1(r)\otimes g_2k\big)\\&=h_1\rightharpoonup\big(fg_1(r)\big)\otimes h_2g_2k\\&\overset{(4)}{=}(h_1\rightharpoonup f)(h_2g_1)(r)\otimes h_3g_2k.\qquad\square\end{aligned}$$

Let M be an $R\#H$-module. There is a natural right $R\#H$-module structure on $\mathrm{Hom}_{R\#H}(M\#H, R\#H)$ induced by the right $R\#H$-module structure of $R\#H$. $\mathrm{Hom}_{R\#H}(M\#H,R\#H)$ is also a left H-module defined by

$$(h\cdot f)(m\otimes g):=f(m\otimes gh),\tag{5}$$

for all $h\in H$, $f\in \mathrm{Hom}_{R\#H}(M\#H,R\#H)$ and $m\otimes g\in M\otimes H$. Then $\mathrm{Hom}_{R\#H}(M\#H,R\#H)$ is an H–$R\#H$-bimodule.

Proposition 2.2. *Let P be an $R\#H$-module such that it is finitely generated projective as an R-module. Then*

$$\mathrm{Hom}_R(P,R)\otimes H\cong \mathrm{Hom}_{R\#H}(P\#H,R\#H)$$

as H–$R\#H$-bimodules.

Proof. Let

$$\psi:\mathrm{Hom}_R(P,R)\otimes H\to \mathrm{Hom}_{R\#H}(P\#H,R\#H)$$

be the homomorphism defined by

$$\begin{aligned}\big[\psi(f\otimes h)\big](p\otimes g)&=(g_1\rightharpoonup f)(p)\#g_2h\\&=g_2\big(f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\#g_3h,\end{aligned}$$

for all $f\otimes h\in \mathrm{Hom}_R(P,R)\otimes H$ and $p\otimes g\in P\#H$.

We claim that the image of ψ is contained in $\mathrm{Hom}_{R\#H}(P\#H,R\#H)$. For any $f\otimes h\in \mathrm{Hom}_R(P,R)\otimes H$, $r\#k\in R\#H$ and $p\otimes g\in P\#H$, on one hand, we have

$$\begin{aligned}\big[\psi(f\otimes h)\big]\big((r\#k)(p\otimes g)\big)&=\big[\psi(f\otimes h)\big]\big((r\#k_1)p\otimes k_2g\big)\\&=(k_3g_2)\big(f\big(\mathcal{S}_H^{-1}(k_2g_1)\big((r\#k_1)p\big)\big)\big)\#k_4g_3h\\&=(k_2g_3)\big(f\big(\big(\big(\mathcal{S}_H^{-1}(k_1g_2)\big)(r)\big)\mathcal{S}_H^{-1}(g_1)p\big)\big)\#k_3g_4h.\end{aligned}$$

On the other hand,

$$\begin{aligned}(r\#k)\big[\psi(f\otimes h)\big](p\otimes g) &= (r\#k)\big(g_2\big(f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\#g_3h\big)\\ &= r(k_1g_2)\big(f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\#k_2g_3h\\ &= (k_2g_3)\big(\mathcal{S}_H^{-1}(k_1g_2)(r)f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\#k_3g_4h\\ &= (k_2g_3)\big(f\big(\big(\big(\mathcal{S}_H^{-1}(k_1g_2)\big)(r)\big)\mathcal{S}_H^{-1}(g_1)p\big)\big)\#k_3g_4h,\end{aligned}$$

where the third equation follows from the identity: $r\#h = h_2\cdot\mathcal{S}_H^{-1}(h_1)(r)$. Now we show that ψ is an H–$R\#H$-bimodule homomorphism. We have

$$\begin{aligned}\big[\psi\big((f\otimes h)(r\#k)\big)\big](p\otimes g) &= \big[\psi\big(fh_1(r)\otimes h_2k\big)\big](p\otimes g)\\ &= g_2\big(\big[fh_1(r)\big]\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\otimes g_3h_2k\\ &= g_2\big(f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)(g_3h_1)(r)\otimes g_4h_2k\\ &= \big(g_2\big(f\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\otimes g_3h\big)(r\#k)\\ &= \big[\psi(f\otimes h)(r\#k)\big](p\otimes g)\end{aligned}$$

and

$$\begin{aligned}\big[\psi\big(k(f\otimes h)\big)\big](p\otimes g) &= \big[\psi(k_1\rightharpoonup f\otimes k_2h)\big](p\otimes g)\\ &= g_2\big((k_1\rightharpoonup f)\big(\mathcal{S}_H^{-1}(g_1)p\big)\big)\#g_3k_2h\\ &= (g_2k_2)\big(f\big(\mathcal{S}_H^{-1}(k_1)\mathcal{S}_H^{-1}(g_1)p\big)\big)\#g_3k_3h\\ &= \big((g_1k_1)\rightharpoonup f\big)(p)\otimes g_2k_2h\\ &= \big[\psi(f\otimes h)\big](p\otimes gk)\\ &= \big[k\cdot\psi(f\otimes h)\big](p\otimes g).\end{aligned}$$

So $\mathrm{Hom}_R(P,R)\otimes H\cong\mathrm{Hom}_{R\#H}(P\#H,R\#H)$ as H–$R\#H$-bimodules when P is finitely generated projective as an R-module. $\square$

Proposition 2.3. *Let H be a finite dimensional Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*

$$\mathrm{Ext}^i_{R\#H}(H,R\#H)\cong\mathrm{Ext}^i_R(\Bbbk,R)\otimes H$$

as H–$R\#H$-bimodules for all $i\geqslant 0$.

Proof. Since R is Noetherian and H is finite dimensional, $R\#H$ is also Noetherian. Then ${}_{R\#H}\Bbbk$ admits a projective resolution

$$\cdots\to P_n\to\cdots\to P_1\to P_0\to\Bbbk\to 0$$

such that each P_n is a finitely generated $R\#H$-module. Because H is finite dimensional, each P_n is also finitely generated as an R-module. Tensoring with H, we obtain a projective resolution of H over $R\#H$

$$\cdots \to P_n \# H \to \cdots \to P_1 \# H \to P_0 \# H \to H \to 0.$$

Applying the functor $\mathrm{Hom}_{R\# H}(-, R\# H)$ to this complex, we obtain the following complex

$$\begin{aligned} 0 &\to \mathrm{Hom}_{R\# H}(P_0 \# H, R\# H) \to \mathrm{Hom}_{R\# H}(P_1 \# H, R\# H) \to \cdots \\ &\to \mathrm{Hom}_{R\# H}(P_n \# H, R\# H) \to \cdots. \end{aligned} \tag{6}$$

This is a complex of H–$R\# H$-bimodules, where the left H-module structure is defined as in (5). By Proposition 2.2, one can check that it is isomorphic to the following complex of H–$R\# H$-bimodules,

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(P_0, R) \otimes H \to \mathrm{Hom}_R(P_1, R) \otimes H \to \cdots \\ &\to \mathrm{Hom}_R(P_n, R) \otimes H \to \cdots. \end{aligned} \tag{7}$$

After taking cohomologies of complex (6) and complex (7), we arrive at isomorphisms of H–$R\# H$-bimodules

$$\mathrm{Ext}^i_{R\# H}(H, R\# H) \cong \mathrm{Ext}^i_R(\Bbbk, R) \otimes H$$

for all $i \geqslant 0$. $\square$

The algebra R can be viewed as an augmented right H-module algebra through the right H-action: $r \cdot h := \mathcal{S}_H^{-1}(h) \cdot r$, for all $r \in R$ and $h \in H$. The algebra $H \# R$ can be defined in a similar way. The multiplication is given by

$$(h \# s)(k \# r) := hk_2 \# (s \cdot k_1)r = hk_2 \# \big(\mathcal{S}_H^{-1}(k_1)(s)\big)r,$$

for all $h \# s$ and $k \# r \in H \# R$. The homomorphism $\varphi : R\# H \to H \# R$ defined by

$$\varphi(r \# k) = k_2 \# \mathcal{S}_H^{-1}(k_1)(r)$$

is an algebra isomorphism with its inverse $\psi : H \# R \to R\# H$ defined by

$$\psi(k \# r) = k_1(r) \# k_2.$$

In addition, φ is compatible with the augmentation maps of $R\# H$ and $H \# R$ respectively. Now right $R\# H$-modules can be treated as $H \# R$-modules. Let M and N be two $H \# R$-modules, then $\mathrm{Hom}_R(M, N)$ is a right H-module defined by

$$(f \leftharpoonup h)(m) := f\big(m\mathcal{S}_H(h_1)\big)h_2,$$

for all $h \in H$, $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$.

Similar to the left case, we have the following proposition.

Proposition 2.4. *Let H be a finite dimensional Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*

$$\mathrm{Ext}^i_{R\# H}(H_{R\# H}, R\# H_{R\# H}) \cong H \otimes \mathrm{Ext}^i_R(\Bbbk_R, R_R)$$

as $R\# H$–H-bimodules for all $i \geqslant 0$.

Lemma 2.5. *Let H be a Hopf algebra and R an H-module algebra. If the left global dimensions of R and H are d_R and d_H respectively, then the left global dimension of $A = R \# H$ is not greater than $d_R + d_H$.*

Proof. Let M and N be two A-modules. We have

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_H\big(\Bbbk, \mathrm{Hom}_R(M, N)\big),$$

that is, the functor $\mathrm{Hom}_A(M, -)$ factors through as follows

$$\begin{tikzcd} \mathrm{Mod}\, A \arrow[rr, "\mathrm{Hom}_R(M,-)"] \arrow[dr, "\mathrm{Hom}_A(M,-)"'] & & \mathrm{Mod}\, H . \arrow[dl, "\mathrm{Hom}_H(\Bbbk,-)"] \\ & \mathrm{Mod}\, \Bbbk & \end{tikzcd}$$

To apply the Grothendieck spectral sequence (see e.g. [19, Section 5.8]), we need to show that if N is an injective A-module, then $\mathrm{Ext}^q_H(\Bbbk, \mathrm{Hom}_R(M, N)) = 0$ for all $q \geqslant 1$.

Let

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0$$

be a projective resolution of $\Bbbk$ over H. $\mathrm{Ext}^*_H(\Bbbk, \mathrm{Hom}_R(M, N))$ are the cohomologies of the complex $\mathrm{Hom}_H(P_\bullet, \mathrm{Hom}_R(M, N))$. The following isomorphisms hold:

$$\begin{aligned} \mathrm{Hom}_H\big(P_\bullet, \mathrm{Hom}_R(M, N)\big) &\cong \mathrm{Hom}_H\big(\Bbbk, \mathrm{Hom}_{\Bbbk}\big(P_\bullet, \mathrm{Hom}_R(M, N)\big)\big) \\ &\cong \mathrm{Hom}_H\big(\Bbbk, \mathrm{Hom}_R(P_\bullet \otimes M, N)\big) \\ &\cong \mathrm{Hom}_{R \# H}(P_\bullet \otimes M, N). \end{aligned}$$

Let P_i be a projective module in the complex $P_\bullet$. Note that the $R \# H$-module structure on $P_i \otimes M$ is given by

$$(r \# h) \cdot (p \otimes h) = h_2 \otimes r h_1 m,$$

for all $r \# h \in R \# H$ and $p \otimes m \in P_i \otimes M$. The complex $P_\bullet$ is exact except at P_0. Since the functors $\mathrm{Hom}_{R \# H}(-, N)$ and $- \otimes M$ are exact, the complex $\mathrm{Hom}_H(P_\bullet, \mathrm{Hom}_R(M, N))$ is also exact except at $\mathrm{Hom}_H(P_0, \mathrm{Hom}_R(M, N))$. It follows that

$$\mathrm{Ext}^q_H\big(\Bbbk, \mathrm{Hom}_R(M, N)\big) = 0$$

for all $q \geqslant 1$.

Now we have

$$\mathrm{Ext}^q_H\big(\Bbbk, \mathrm{Ext}^p_R(M, N)\big) \quad \Rightarrow \quad \mathrm{Ext}^{p+q}_{R \# H}(M, N).$$

Because the left global dimensions of R and H are d_R and d_H, $\mathrm{Ext}^i_{R \# H}(M, N) = 0$ for all $i \geqslant d_R + d_H$. Therefore, the left global dimension of $R \# H$ is not greater than $d_R + d_H$. $\square$

Let H be an involutory CY Hopf algebra and R a p-Koszul CY algebra and a left H-module algebra. As mentioned in the introduction, Liu, Wu and Zhu used the homological determinant of the

H-action to characterize the CY property of $R\#H$ in [13]. They defined an H-module structure on the Koszul bimodule complex of R and computed the H-module structures on the Hochschild cohomologies. Then they proved that $R\#H$ is CY if and only if the homological determinant is trivial. If H is not involutory or R is not a p-Koszul algebra, is it still true that $R\#H$ is a CY algebra when the homological determinant is trivial?

We answer this question in the case where R is a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$, where H is a semisimple Hopf algebra. We use the homological determinant to discuss the homological integral and the rigid dualizing complex of the algebra $A = R\#H$. We then give a necessary and sufficient condition for A to be a CY algebra. The result we will obtain is slightly different from what was obtained by Liu, Wu and Zhu. We first need the following lemma.

Lemma 2.6. *Let H be a Hopf algebra, and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. Then*

$$\mathcal{S}^2_{R\#H}(r) = \mathcal{S}_H(r_{(-1)})\big(\mathcal{S}^2_R(r_{(0)})\big),$$

for any $r \in R$.

Proof. Set $A = R\#H$. By Eq. (2), for any $r \in R$, we have

$$\mathcal{S}_A(r) = \big(1\#\mathcal{S}_H(r_{(-1)})\big)\big(\mathcal{S}_R(r_{(0)})\#1\big).$$

Therefore,

$$\begin{aligned}
\mathcal{S}^2_A(r) &= \mathcal{S}_A\big(\big(1\#\mathcal{S}_H(r_{(-1)})\big)\big(\mathcal{S}_R(r_{(0)})\#1\big)\big)\\
&= \mathcal{S}_A\big(\mathcal{S}_R(r_{(0)})\#1\big)\mathcal{S}_A\big(1\#\mathcal{S}_H(r_{(-1)})\big)\\
&= \big(1\#\mathcal{S}_H\big(\mathcal{S}_R(r_{(0)})_{(-1)}\big)\big)\big(\mathcal{S}_R\big(\mathcal{S}_R(r_{(0)})_{(0)}\big)\#1\big)\big(1\#\mathcal{S}^2_H(r_{(-1)})\big)\\
&= \big(1\#\mathcal{S}_H(r_{(0)(-1)})\big)\big(\mathcal{S}^2_R(r_{(0)(0)})\#1\big)\big(1\#\mathcal{S}^2_H(r_{(-1)})\big)\\
&= \big(1\#\mathcal{S}_H(r_{(-1)2})\big)\big(\mathcal{S}^2_R(r_{(0)})\#1\big)\big(1\#\mathcal{S}^2_H(r_{(-1)1})\big)\\
&= \mathcal{S}_H(r_{(-1)3})\big(\mathcal{S}^2_R(r_{(0)})\big)\#\mathcal{S}_H(r_{(-1)2})\mathcal{S}^2_H(r_{(-1)1})\\
&= \mathcal{S}_H(r_{(-1)2})\big(\mathcal{S}^2_R(r_{(0)})\big)\#\mathcal{S}_H\big(\varepsilon(r_{(-1)1})\big)\\
&= \mathcal{S}_H(r_{(-1)})\big(\mathcal{S}^2_R(r_{(0)})\big). \qquad \square
\end{aligned}$$

Proposition 2.7. *Let H be a semisimple Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. If R is an AS-regular algebra of global dimension d_R, then $A = R\#H$ is also AS-regular of global dimension d_R.*

In this case, $\int^l_A = \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is defined by

$$\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h),$$

for all $r\#h \in R\#H$, where the algebra map $\xi_R : R \to \Bbbk$ defines the left integral module of R, i.e., $\int^l_R = \Bbbk_{\xi_R}$. The rigid dualizing complex of A is isomorphic to ${}_\psi A[d_R]$, where ψ is the algebra automorphism $[\xi]\mathcal{S}^2_A$. To be precise, ψ is defined by

$$\psi(r\#h) = \xi_R\big(r^1\big)\,\mathrm{hdet}\big(\big(r^2\big)_{(-1)1}h_1\big)\mathcal{S}_H\big(\big(r^2\big)_{(-1)2}\big)\big(\mathcal{S}^2_R\big(\big(r^2\big)_{(0)}\big)\big)\#\mathcal{S}^2_H(h_2),$$

for all $r\#h \in R\#H$.

Proof. Let $P_\bullet \to H \to 0$ be a projective A-module resolution of H with each P_i finitely generated. Since H is semisimple, $\Bbbk$ is projective as an H-module. It follows that $\Bbbk \otimes_H P_\bullet \to \Bbbk \to 0$ is a projective A-module resolution of $\Bbbk$. Now the following isomorphism of complexes holds:

$$\mathrm{Hom}_A(\Bbbk \otimes_H P_\bullet, A) \cong \mathrm{Hom}_H\big(\Bbbk, \mathrm{Hom}_A(P_\bullet, A)\big).$$

The fact that the trivial module $\Bbbk$ is a finitely generated projective H-module implies that

$$\begin{aligned}\mathrm{Ext}_A^i(\Bbbk, A) &\cong \mathrm{Hom}_H\big(\Bbbk, \mathrm{Ext}_A^i(H, A)\big)\\ &\cong \mathrm{Hom}_H(\Bbbk, H) \otimes_H \mathrm{Ext}_A^i(H, A)\end{aligned} \tag{8}$$

for all $i > 0$. Following Proposition 2.3, we have $\int_A^l \cong \int_H^l \otimes_H \int_R^l \otimes H$ and

$$\dim \mathrm{Ext}_A^i(\Bbbk, {}_A A) = \begin{cases} 0, & i \neq d_R;\\ 1, & i = d_R.\end{cases}$$

Let $\mathbf{e}$ be a non-zero element in $\int_R^l$ and $\mathbf{h}$ a non-zero element in $\int_H^l$. Since H is semisimple, H is unimodular. That is, we have $\int_H^l = \Bbbk$. Let $\eta : H \to \Bbbk$ be an algebra homomorphism such that $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$. Then the following equations hold

$$\begin{aligned}(\mathbf{h} \otimes \mathbf{e} \otimes 1) \cdot (r \# h) &= \xi_R(r)\mathbf{h} \otimes \mathbf{e} \otimes h\\ &= \xi_R(r)\mathbf{h} \otimes \varepsilon(h_1)\mathbf{e} \otimes h_2\\ &= \xi_R(r)\mathbf{h} \otimes \eta\big(\mathcal{S}_H(h_1)\big)\eta(h_2)\mathbf{e} \otimes h_3\\ &= \xi_R(r)\eta\big(\mathcal{S}_H(h_1)\big)\mathbf{h} \otimes h_2 \cdot (\mathbf{e} \otimes 1)\\ &= \xi_R(r)\eta\big(\mathcal{S}_H(h_1)\big)\varepsilon(h_2)\mathbf{h} \otimes \mathbf{e} \otimes 1\\ &= \xi_R(r)\,\mathrm{hdet}(h)\mathbf{h} \otimes \mathbf{e} \otimes 1.\end{aligned}$$

This implies that $\int_A^l \cong \Bbbk_\xi$, where ξ is the algebra homomorphism defined in the proposition. Similarly, by Proposition 2.4, we have

$$\dim \mathrm{Ext}_A^i(\Bbbk, A_A) = \begin{cases} 0, & i \neq d_R;\\ 1, & i = d_R.\end{cases}$$

Because H is finite dimensional and R is Noetherian, the algebra A is Noetherian as well. Therefore, the left and right global dimensions of A are equal. Since H is semisimple, the global dimension of H is 0. Now it follows from Lemma 2.5 that the global dimension of A is d_R. In conclusion, we have proved that A is an AS-regular algebra.

By [5, Prop. 4.5], the rigid dualizing complex of A is isomorphic to ${}_{[\xi]\mathcal{S}_A^2}A[d_R]$. For any $r \# h \in R \# H$, we have

$$\begin{aligned}[\xi]\mathcal{S}_A^2(r \# h) &\overset{(a)}{=} \mathcal{S}_A^2[\xi](r \# h)\\ &\overset{(b)}{=} \xi\big(r^1 \# \big(r^2\big)_{(-1)}h_1\big)\mathcal{S}_A^2\big(\big(r^2\big)_{(0)} \# h_2\big)\\ &= \xi_R\big(r^1\big)\,\mathrm{hdet}\big(\big(r^2\big)_{(-1)}h_1\big)\mathcal{S}_A^2\big(\big(r^2\big)_{(0)}\big) \# \mathcal{S}_H^2(h_2)\end{aligned}$$

$$\begin{aligned}&\overset{(c)}{=}\xi_R\big(r^1\big)\,\mathrm{hdet}\big(\big(r^2\big)_{(-1)}h_1\big)\mathcal{S}_H\big(\big(r^2\big)_{(0)(-1)}\big)\big(\mathcal{S}_R^2\big(\big(r^2\big)_{(0)(0)}\big)\big)\,\#\,\mathcal{S}_H^2(h_2)\\&=\xi_R\big(r^1\big)\,\mathrm{hdet}\big(\big(r^2\big)_{(-1)1}h_1\big)\mathcal{S}_H\big(\big(r^2\big)_{(-1)2}\big)\big(\mathcal{S}_R^2\big(\big(r^2\big)_{(0)}\big)\big)\,\#\,\mathcal{S}_H^2(h_2).\end{aligned}$$

Eqs. (a), (b) and (c) follow from [5, Lemma 2.5], Eq. (1) and Lemma 2.6 respectively. Thus the proof is completed. □

Since ξ is an algebra homomorphism, the following equation holds

$$\xi_R(r)\,\mathrm{hdet}(h)=\xi_R\big(h_1(r)\big)\,\mathrm{hdet}(h_2).$$

We show how $\int_{R\#H}^r$ looks like. Let $\mathbf{e}'$ be a non-zero element in $\mathrm{Ext}_R^d(\Bbbk,R)$. There is an algebra homomorphism $\eta':H\to\Bbbk$ satisfying $\mathbf{e}'\cdot h=\eta'(h)\mathbf{e}'$ for all $h\in H$. Applying a similar argument as in the proof of Proposition 2.7, we have that if $\int_R^r={}_{\xi'_R}\Bbbk$, then $\int_A^r={}_{\xi'}\Bbbk$, where ξ' is defined by $\xi'(r\#h)=\xi'_R(\mathcal{S}_H^{-1}(h_1)(r))\eta'(\mathcal{S}_H(h_2))$ for all $r\#h\in R\#H$.

Now we give the main theorem of this section.

Theorem 2.8. *Let H be a semisimple Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}_H^H\mathcal{YD}$. Suppose that the algebra R is CY of dimension d_R. Then $R\#H$ is CY if and only if the homological determinant of R is trivial and the algebra automorphism ϕ defined by*

$$\phi(r\#h)=\mathcal{S}_H(r_{(-1)})\big(\mathcal{S}_R^2(r_{(0)})\big)\mathcal{S}_H^2(h)$$

for all $r\#h\in R\#H$ is an inner automorphism.

Proof. From Proposition 1.11, we have that R is AS-regular with $\int_R^l\cong\Bbbk$. In addition, since H is finite dimensional and semisimple, the algebra H is unimodular. Thus $\int_H^l=\Bbbk$. Set $A=R\#H$. By Proposition 2.7, we obtain that A is AS-regular with $\int_A^l\cong\Bbbk_\xi$, where ξ is the algebra homomorphism defined by $\xi(r\#h)=\varepsilon(r)\,\mathrm{hdet}(h)$ for all $r\#h\in R\#H$. Following from [8, Thm. 2.3], the algebra A is CY if and only if $\xi=\varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism. On one hand, $\xi=\varepsilon_H$ if and only if $\mathrm{hdet}=\varepsilon_H$. On the other hand, by Lemma 2.6, we have $\mathcal{S}_A^2(r\#h)=\mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)}))\mathcal{S}_H^2(h)$, for any $r\#h\in R\#H$. □

In [13] it is proved that if R is p-Koszul CY and H is involutory, then $R\#H$ is CY if and only if the homological determinant is trivial. Thus in Theorem 2.8, if the braided Hopf algebra R is p-Koszul and the Hopf algebra H is involutory, then we have that the homological determinant is trivial implies that the automorphism ϕ is inner. In the following Example 2.9, we see that the automorphism ϕ can be expressed via the homological determinant of the H-action.

Example 2.9. Let

$$\mathcal{D}\big(\varGamma,(g_i)_{1\leqslant i\leqslant\theta},(\chi_i)_{1\leqslant i\leqslant\theta},(a_{ij})_{1\leqslant i,j\leqslant\theta}\big)$$

be a datum of finite Cartan type (see [2] for terminology), where $\varGamma$ is a finite abelian group and (a_{ij}) is of type $A_1\times\cdots\times A_1$. Assume that V is a braided vector space with a basis $\{x_1,\dots,x_\theta\}$ whose braiding is given by

$$c(x_i\otimes x_j)=q_{ij}x_j\otimes x_i,\quad 1\leqslant i,j\leqslant\theta,$$

where $q_{ij}=\chi_j(g_i)$.

Let R be the following algebra:

$$\Bbbk\langle x_1, \ldots, x_\theta \mid x_i x_j = q_{ij} x_j x_i,\ 1 \leqslant i < j \leqslant \theta\rangle.$$

The algebra R is a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Moreover, it is easy to see that R is a Koszul algebra. Assume that $\mathcal{K}$ is the Koszul complex (cf. complex (6) in [17])

$$0 \to R \otimes R_\theta^{!*} \to \cdots \to R \otimes R_j^{!*} \xrightarrow{d_j} R \otimes R_{j-1}^{!*} \to \cdots \to R \otimes R_1^{!*} \to R \to 0.$$

Then we have that $\mathcal{K} \to {}_R\Bbbk \to 0$ is a projective resolution of $\Bbbk$. Each $R_j^{!*}$ is a left $\Bbbk\Gamma$-module with module structure defined by

$$\begin{aligned}
\bigl[g(\beta)\bigr]\bigl(x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*\bigr) &= \beta\bigl(g^{-1}\bigl(x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*\bigr)\bigr) \\
&= \beta\bigl(g^{-1}\bigl(x_{i_1}^*\bigr) \wedge \cdots \wedge g^{-1}\bigl(x_{i_j}^*\bigr)\bigr) \\
&= \left(\prod_{t=1}^{j} \chi_{i_t}(g)\right)\beta\bigl(x_{i_1}^* \wedge \cdots \wedge x_{i_j}^*\bigr),
\end{aligned}$$

where $\beta \in S_j^{!*}$. Thus each $R \otimes R_j^{!*}$ is a left $\Bbbk\Gamma$-module. It is not difficult to see that the differentials in the Koszul complex are also left Γ-module homomorphisms. By [6, Prop. 5.0.7], we have that $\int_R^l \cong R_\theta^{!*}$. Therefore, $\operatorname{hdet}(g) = \prod_{i=1}^{\theta} \chi_i(g^{-1})$ for all $g \in \Gamma$.

Following from [18, Prop. 8.2 and Thm. 9.2], the algebra R has the rigid dualizing complex $R_\varphi[\theta]$, where φ is the algebra automorphism defined by $\varphi(x_i) = q_{1i} \cdots q_{(i-1)i} q_{i(i+1)}^{-1} \cdots q_{i\theta}^{-1} x_i$, for $1 \leqslant i \leqslant \theta$. By Corollary 1.8, the algebra R is a CY algebra if and only if for each $1 \leqslant i \leqslant \theta$, $q_{1i} \cdots q_{(i-1)i} = q_{i(i+1)} \cdots q_{i\theta}$. In this case,

$$\begin{aligned}
\operatorname{hdet}(g_j) &= \prod_{i=1}^{\theta} \chi_i\bigl(g_j^{-1}\bigr) \\
&= \left(\prod_{i=1}^{j-1} \chi_i\bigl(g_j^{-1}\bigr)\right)\chi_j\bigl(g_j^{-1}\bigr)\left(\prod_{k=j+1}^{\theta} \chi_k\bigl(g_j^{-1}\bigr)\right) \\
&= \left(\prod_{i=1}^{j-1} q_{ij}\right)\chi_j\bigl(g_j^{-1}\bigr)\left(\prod_{k=j+1}^{\theta} q_{jk}^{-1}\right) \\
&= \chi_j\bigl(g_j^{-1}\bigr).
\end{aligned}$$

The algebra automorphism ϕ given in Theorem 2.8 is defined by

$$\phi(x_j) = \chi_j\bigl(g_j^{-1}\bigr)x_j = \operatorname{hdet}(g_j)x_j$$

for all $1 \leqslant j \leqslant \theta$ and $\phi(g) = g$ for all $g \in \Gamma$. However, $\chi_j(g_j) \neq 1$ for all $1 \leqslant j \leqslant \theta$. The algebra $R\#\Bbbk\Gamma$ is not a CY algebra.

Example 2.10. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Assume that there is a group homomorphism $\nu : \Gamma \to \mathrm{Aut}_{Lie}(\mathfrak{g})$, where $\mathrm{Aut}_{Lie}(\mathfrak{g})$ is the group of Lie algebra automorphisms of $\mathfrak{g}$. Then it is known that $U(\mathfrak{g}) \# \Bbbk\Gamma$ is a cocommutative Hopf algebra.

It is proved in [8, Cor. 3.6] that the smash product $U(\mathfrak{g}) \# \Bbbk\Gamma$ is CY if and only if $U(\mathfrak{g})$ is CY and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$.

Let d be the dimension of $\mathfrak{g}$. By [8, Lemma 3.1], we have $\int_{U(\mathfrak{g})}^{l} \cong \bigwedge^d \mathfrak{g}^*$ as left Γ-modules, where the left Γ-action on $\mathfrak{g}^*$ is defined by $(g \cdot \alpha)(x) = \alpha(g^{-1}x)$ for all $g \in \Gamma$, $\alpha \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, and Γ acts on $\bigwedge^d \mathfrak{g}^*$ diagonally. Let $\{x_1, \dots, x_d\}$ be a basis of $\mathfrak{g}$. Then

$$g\big(x_1^* \wedge \cdots \wedge x_d^*\big) = \det\big(\nu\big(g^{-1}\big)\big)\big(x_1^* \wedge \cdots \wedge x_d^*\big),$$

for all $g \in \Gamma$. So $\mathrm{hdet}(g) = \det(\nu(g))$. That is, if $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$, then the homological determinant is trivial. The algebra $U(\mathfrak{g})$ is a braided Hopf algebra in the category ${}_{\Gamma}^{\Gamma}\mathcal{YD}$ with trivial coaction. So the automorphism ϕ defined in Theorem 2.8 is the identity. Therefore, if $U(\mathfrak{g})$ is a CY algebra and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$, by Theorem 2.8, the algebra $U(\mathfrak{g}) \# \Bbbk\Gamma$ is a CY algebra. This coincides with the result mentioned before.

3. Rigid dualizing complexes of braided Hopf algebras over finite group algebras

In this section, we further assume the characteristic of the base field $\Bbbk$ is 0. Let Γ be a finite group and R a braided Hopf algebra in the category ${}_{\Gamma}^{\Gamma}\mathcal{YD}$ of Yetter–Drinfeld modules over $\Bbbk\Gamma$ such that $R \# \Gamma$ is a CY algebra. In this section, we answer the question when the algebra R is a CY algebra.

Let A be a Hopf algebra. By [15, Appendix, Lemma 11], A can be viewed as a subalgebra of A^e via the algebra homomorphism $\rho : A \to A^e$ defined by

$$\rho(a) = \sum a_1 \otimes \mathcal{S}(a_2). \tag{9}$$

Then A^e is a right A-module via this embedding. We denote this right A-module by $\mathcal{R}(A^e)$. Actually, $\mathcal{R}(A^e)$ is an A^e–A-bimodule. Similarly, A^e is also an A–A^e-bimodule, where the left A-module is induced from the homomorphism ρ. Denote this bimodule by $\mathcal{L}(A^e)$.

From now on, let Γ be a finite group and R a braided Hopf algebra in the category ${}_{\Gamma}^{\Gamma}\mathcal{YD}$ with Γ-coaction δ. The biproduct $A = R \# \Bbbk\Gamma$ is a usual Hopf algebra [16]. Let $\mathscr{D}$ be the subalgebra of A^e generated by the elements of the form $(r \# g) \otimes (s \# g^{-1})$ with $r, s \in R$ and $g \in \Gamma$.

Note that R is a Γ-graded module, i.e., $R = \bigoplus_{g\in\Gamma} R_g$, where $R_g = \{r \in R \mid \delta(r) = g \otimes r\}$. Therefore, for any $r \in R$, it can be written as $r = \sum_{g\in\Gamma} r_g$ with $r_g \in R_g$. Then $\delta(r) = \sum_{g\in\Gamma} g \otimes r_g$.

Lemma 3.1. *The subalgebra $\mathscr{D}$ is a left (resp. right) A-submodule of $\mathcal{L}(A^e)$ (resp. $\mathcal{R}(A^e)$).*

Proof. For any $r \# h \in A$, where $h \in \Gamma$, by Eqs. (1) and (2), we have

$$\Delta(r \# h) = \sum_{g\in\Gamma} r^1 \# gh \otimes \big(r^2\big)_g \# h$$

and

$$\mathcal{S}_A(r \# h) = \sum_{g\in\Gamma} h^{-1}g^{-1}\mathcal{S}_R(r_g).$$

Any element in $\mathscr{D}$ can be written as a linear combination of elements of the form $s \# k \otimes t \# k^{-1} \in \mathscr{D}$ with $s, t \in R$ and $k \in \Gamma$.

$$\begin{aligned}(r\#h)\cdot\big(s\#k\otimes t\#k^{-1}\big) &= \sum_{g\in\Gamma}\big(r^1\#gh\big)(s\#k)\otimes\big(t\#k^{-1}\big)\mathcal{S}_A\big(\big(r^2\big)_g\#h\big)\\ &= \sum_{g\in\Gamma}\big(r^1\#gh\big)(s\#k)\otimes\big(t\#k^{-1}\big)h^{-1}g^{-1}\mathcal{S}_R\big(\big(r^2\big)_g\big)\\ &= \sum_{g\in\Gamma}\big(r^1(gh)(s)\#ghk\big)\otimes\big(t\big(k^{-1}h^{-1}g^{-1}\big)\big(\mathcal{S}_R\big(\big(r^2\big)_g\big)\big)\#k^{-1}h^{-1}g^{-1}\big)\\ &\in\mathscr{D}.\end{aligned}$$

This shows that $\mathscr{D}$ is a left A-submodule of $\mathcal{L}(A^e)$. Similarly, $\mathscr{D}$ is also a right A-submodule of $\mathcal{R}(A^e)$. $\square$

The following lemma is known, we include it for completeness.

Lemma 3.2.

(a) *Both $\mathcal{L}(A^e)$ and $\mathcal{R}(A^e)$ are free as A-modules.*
(b) *$\mathcal{R}(A^e)\otimes_A \Bbbk\cong A$ as left A^e-modules and this isomorphism restricts to a left R^e-isomorphism $\mathscr{D}\otimes_A\Bbbk\cong R$.*
(c) *If $\xi: A\to\Bbbk$ is an algebra homomorphism, then there is an isomorphism $\Bbbk_\xi\otimes_A\mathcal{L}(A^e)\cong A_{[\xi]\mathcal{S}_A^2}$ of right A^e-modules and the isomorphism restricts to a right R^e-isomorphism $\Bbbk_\xi\otimes_A\mathscr{D}\cong R_{([\xi]\mathcal{S}_A^2)|_R}$.*

Proof. (a) was proved in [5, Lemma 2.2]. The module $L(A^e)$ defined in the same paper is isomorphic to $\mathcal{R}(A^e)$ as right A-modules.

(b) This follows from [15, Appendix].

(c) It was proved in [5, Lemma 4.5] that $\Bbbk_\xi\otimes_A\mathcal{L}(A^e)\cong A_{[\xi]\mathcal{S}_A^2}$ as right A^e-modules. Here we give another proof. We construct the isomorphism explicitly. Define a homomorphism $\Phi:\Bbbk_\xi\otimes_A\mathcal{L}(A^e)\to A_{[\xi]\mathcal{S}_A^2}$ by $\Phi(1\otimes a\otimes b)=\xi(a_1)b\mathcal{S}_A^2(a_2)$ and a homomorphism $\Psi: A_{[\xi]\mathcal{S}_A^2}\to\Bbbk_\xi\otimes_A\mathcal{L}(A^e)$ by $\Psi(a)=1\otimes 1\otimes a$. Note that $[\xi]\mathcal{S}^2=\mathcal{S}^2[\xi]$ holds by Lemma 2.5 in [5]. For any $x,a,b\in A$, we have

$$\begin{aligned}\Phi\big(1\otimes x_1a\otimes b\mathcal{S}(x_2)\big) &= \xi(x_1)\xi(a_1)b\mathcal{S}(x_3)\mathcal{S}^2(x_2)\mathcal{S}^2(a_2)\\ &= \xi(x_1)\xi(a_1)b\mathcal{S}\big(\varepsilon(x_2)\big)\mathcal{S}^2(a_2)\\ &= \xi(x)\xi(a_1)b\mathcal{S}^2(a_2)\\ &= \xi(x)\Phi(1\otimes a\otimes b).\end{aligned}$$

This shows that Φ is well defined. Similar calculations show that Φ and Ψ are right A^e-module homomorphisms and they are inverse to each other.

It is straightforward to check that the isomorphism $\Bbbk_\xi\otimes_A\mathcal{L}(A^e)\cong A_{[\xi]\mathcal{S}_A^2}$ restricts to the isomorphism $\Bbbk_\xi\otimes_A\mathscr{D}\cong R_{([\xi]\mathcal{S}_A^2)|_R}$. $\square$

Lemma 3.3. $\mathrm{Hom}_{R^e}(\mathscr{D},R^e)\cong\mathscr{D}$ *as A–R^e-bimodules.*

Proof. The algebra $\mathscr{D}$ is an A–R^e-bimodule. Note that the A-module structure is induced from the homomorphism ρ defined in (9). On the other hand, the A–R^e-bimodule structure on $\mathrm{Hom}_{R^e}(\mathscr{D},R^e)$ is induced from the right A-module structure on $\mathscr{D}$ and the right R^e-module structure on R^e. We have $r\#g=(1\#g)(g^{-1}(r)\#1)$ for any $r\#g\in R\#\Bbbk\Gamma$. Therefore, an element in $\mathscr{D}$ can be expressed of the form $\sum_{g\in\Gamma}(1\#g^{-1})(r^g\#1)\otimes s^g\#g$ with $r^g,s^g\in R$. For simplicity, we write gr for the element $(1\#g)(r\#1)$ with $r\in R$ and $g\in\Gamma$. Let $\Psi:\mathscr{D}\to\mathrm{Hom}_{R^e}(\mathscr{D},R^e)$ be a homomorphism defined by

$$\Big[\Psi\Big(\sum_{g\in\Gamma} g^{-1}r^g\otimes\big(s^g\# g\big)\Big)\Big]\big(h\otimes h^{-1}\big)=r^h\otimes s^h,$$

for $\sum_{g\in\Gamma} g^{-1}r^g\otimes s^g\# g\in\mathscr{D}, h\in\Gamma$. Next define a homomorphism $\Phi:\mathrm{Hom}_{R^e}(\mathscr{D},R^e)\to\mathscr{D}$ as follows:

$$\Phi(f)=\sum_{g\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\otimes g^{-1}\big)$$

for $f\in\mathrm{Hom}_{R^e}(\mathscr{D},R^e)$. It is clear that Φ is a right R^e-homomorphism. On the other hand, we have

$$\begin{aligned}\Phi\big((r\# h)f\big)&=\sum_{g\in\Gamma}\big(g^{-1}\otimes g\big)\big((r\# h)f\big)\big(g\otimes g^{-1}\big)\\&=\sum_{g\in\Gamma}\sum_{k\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\big(r^1\# k\big)h\otimes\mathcal{S}_A\big(\big(r^2\big)_k\# h\big)g^{-1}\big)\\&=\sum_{g\in\Gamma}\sum_{k\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\big(r^1\# k\big)h\otimes h^{-1}k^{-1}\mathcal{S}_R\big(\big(r^2\big)_k\big)g^{-1}\big)\\&=\sum_{g\in\Gamma}\sum_{k\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\big(r^1\big)\# gkh\otimes h^{-1}k^{-1}g^{-1}g\big(\mathcal{S}_R\big(\big(r^2\big)_k\big)\big)\big),\end{aligned}$$

and

$$\begin{aligned}(r\# h)\Phi(f)&=\Big(\sum_{k\in\Gamma}r_1\# kh\otimes h^{-1}k^{-1}\mathcal{S}_R(r_{2k})\Big)\sum_{g\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\otimes g^{-1}\big)\\&=\sum_{k\in\Gamma}\sum_{g\in\Gamma}\big(r^1\# khg^{-1}\otimes gh^{-1}k^{-1}\mathcal{S}_R\big(\big(r^2\big)_k\big)\big)f\big(g\otimes g^{-1}\big)\\&=\sum_{k\in\Gamma}\sum_{g\in\Gamma}\big(khg^{-1}\big(gh^{-1}k^{-1}\big)\big(r^1\big)\otimes\big(gh^{-1}k^{-1}\big)\mathcal{S}_R\big(\big(r^2\big)_k\big)gh^{-1}k^{-1}\big)f\big(g\otimes g^{-1}\big)\\&=\sum_{k\in\Gamma}\sum_{g\in\Gamma}\big(khg^{-1}\otimes gh^{-1}k^{-1}\big)f\big(\big(gh^{-1}k^{-1}\big)\big(r^1\big)\# g\otimes g^{-1}\big(gh^{-1}k^{-1}\big)\big(\mathcal{S}_R\big(\big(r^2\big)_k\big)\big)\big)\\&=\sum_{g\in\Gamma}\sum_{k\in\Gamma}\big(g^{-1}\otimes g\big)f\big(g\big(r^1\big)\# gkh\otimes h^{-1}k^{-1}g^{-1}g\big(\mathcal{S}_R\big(\big(r^2\big)_k\big)\big)\big).\end{aligned}$$

So Φ is an A–R^e-bimodule homomorphism. It is easy to check that Φ and Ψ are inverse to each other. Thus Φ is an isomorphism. □

Lemma 3.4. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. If $A=R\#\Bbbk\Gamma$ is AS-Gorenstein with $\int_A^l\cong\Bbbk_\xi$, where $\xi:A\to\Bbbk$ is an algebra homomorphism, then we have R–R-bimodule isomorphisms*

$$\mathrm{Ext}^i_{R^e}\big(R,R^e\big)\cong\begin{cases}0, & i\neq d;\\ R_{([\xi]\mathcal{S}_A^2)|_R}, & i=d.\end{cases}$$

Proof. We have the following isomorphisms,

$$\begin{aligned}\mathrm{Ext}^i_{R^e}(R, R^e) &\cong \mathrm{Ext}^i_{R^e}(\mathscr{D}\otimes_A \Bbbk, R^e)\\ &\cong \mathrm{Ext}^i_A\big({}_A\Bbbk, \mathrm{Hom}_{R^e}(\mathscr{D}, R^e)\big)\\ &\cong \mathrm{Ext}^i_A({}_A\Bbbk, \mathscr{D})\\ &\cong \mathrm{Ext}^i_A({}_A\Bbbk, A)\otimes_A \mathscr{D}\\ &\cong \begin{cases} 0, & i \neq d;\\ \Bbbk_\xi \otimes_A \mathscr{D} \cong R_{([\xi]S_A^2)|_R}, & i = d.\end{cases}\end{aligned}$$

The first, the third and the last isomorphisms follow from Lemma 3.2, Lemma 3.3 and Lemma 3.2 respectively. The fourth isomorphism follows from the fact that $\mathscr{D}$ is left A-projective. This is because A^e is free as a left A-module by Lemma 3.2 and A^e is a direct sum of finite copies of $\mathscr{D}$. Indeed, $A^e \cong \bigoplus_{h\in\Gamma}\mathscr{D}^h$, where $\mathscr{D}^h$ is the left A-submodule of A^e generated by elements of the form $(r\#gh)\otimes(s\#g^{-1})$ with $r, s \in R$ and $g \in \Gamma$. Moreover, for every $h \in \Gamma$, $\mathscr{D}^h$ is isomorphic to $\mathscr{D}$ as a left A-module. □

Lemma 3.5. *If the global dimension of $A = R\#\Bbbk\Gamma$ is finite and R is Noetherian, then R is homologically smooth.*

Proof. By assumption, the algebra A is Noetherian, and ${}_A\Bbbk$ has a finite projective resolution

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0,$$

such that each P_i, $0 \leqslant i \leqslant d$, is a finitely generated projective A-module. By a similar argument to the one in the proof of Lemma 3.4, we have that $\mathscr{D}$ is projective as a right A-module. Therefore, the functor $\mathscr{D}\otimes_A -$ is exact. Now we obtain an exact sequence:

$$0 \to \mathscr{D}\otimes_A P_d \to \mathscr{D}\otimes_A P_{d-1} \to \cdots \to \mathscr{D}\otimes_A P_1 \to \mathscr{D}\otimes_A P_0 \to \mathscr{D}\otimes_A \Bbbk \to 0. \tag{10}$$

$\mathscr{D}$ is projective as left R^e-module and $\mathscr{D}\otimes_A \Bbbk \cong R$ as left R^e-modules (Lemma 3.2). So the complex (10) is a projective R–R-bimodule resolution of R. Because each P_i is a finitely generated A-module and Γ is a finite group, each $\mathscr{D}\otimes_A P_i$ is a finitely generated left R^e-module. Therefore, we conclude that R is homologically smooth. □

The homological integral of the skew group algebra $R\#\Bbbk\Gamma$ was discussed by He, Van Oystaeyen and Zhang in [8]. Based on their work, we use the homological determinant of the group action to describe the homological integral of $R\#\Bbbk\Gamma$.

Lemma 3.6. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. If R is an AS-Gorenstein algebra with injective dimension d and $\int_R^l \cong \Bbbk_{\xi_R}$, where $\xi_R : R \to \Bbbk$ is an algebra homomorphism, then the algebra $A = R\#\Bbbk\Gamma$ is AS-Gorenstein with injective dimension d as well, and $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is the algebra homomorphism defined by $\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h)$ for any $r\#h \in R\#\Bbbk\Gamma$.*

Proof. By [8, Props. 1.1 and 1.3], we have that $A = R \# \Bbbk\Gamma$ is AS-Gorenstein of injective dimension d, $\int_R^l$ is a 1-dimensional left Γ-module, and as right A-modules:

$$\int_A^l \cong \left(\int_R^l \otimes \Bbbk\Gamma\right)^\Gamma,$$

where the right A-module structure on $\int_R^l \otimes \Bbbk\Gamma$ is defined by

$$(e \otimes g) \cdot (r \# h) = e\big(g(r)\big) \otimes gh,$$

for $g \in \Bbbk\Gamma$, $r \# h \in R \# \Bbbk\Gamma$ and $e \in \int_R^l$, and the left Γ-action on $\int_R^l \otimes \Bbbk\Gamma$ is the diagonal one. Let $\mathbf{e}$ be a basis of $\int_R^l$. It is not difficult to check that the element $\sum_{g \in \Gamma} g(\mathbf{e}) \otimes g$ is a basis of $(\int_R^l \# \Bbbk\Gamma)^\Gamma$. Let $\eta : \Bbbk\Gamma \to \Bbbk$ be an algebra homomorphism such that $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in \Gamma$. For any $r \# h \in R \# \Bbbk\Gamma$, we have

$$\begin{aligned}
\left(\sum_{g \in \Gamma} g(\mathbf{e}) \# g\right)(r \# h) &= \sum_{g \in \Gamma} g(\mathbf{e}) g(r) \# gh \\
&= \sum_{g \in \Gamma} g(\mathbf{e}r) \# gh \\
&= \xi_R(r) \sum_{g \in \Gamma} g(\mathbf{e}) \# gh \\
&= \xi_R(r)\eta\big(h^{-1}\big) \sum_{g \in \Gamma} (gh)(\mathbf{e}) \# gh \\
&= \xi_R(r)\eta\big(h^{-1}\big) \sum_{g \in \Gamma} g(\mathbf{e}) \# g \\
&= \xi_R(r)\,\mathrm{hdet}(h) \sum_{g \in \Gamma} g(\mathbf{e}) \# g \\
&= \xi(r \# h) \sum_{g \in \Gamma} g(\mathbf{e}) \# g.
\end{aligned}$$

It implies that $\int_A^l \cong \Bbbk_\xi$. $\square$

The following proposition shows that the AS-regularity of $R \# \Bbbk\Gamma$ depends strongly on the AS-regularity of R.

Proposition 3.7. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Then R is AS-regular if and only if $A = R \# \Bbbk\Gamma$ is AS-regular.*

Proof. Assume that R is AS-regular. By Lemma 3.6, the algebra A is AS-Gorenstein. To show that A is AS-regular, it suffices to show that the global dimension of A is finite. Since the global dimension of R is finite, there is a finite projective resolution of $\Bbbk$ over R,

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \Bbbk \to 0.$$

Note that A is projective as a right R-module. Tensoring this resolution with $A\otimes_R -$, we obtain an exact sequence

$$0\to A\otimes_R P_d\to A\otimes_R P_{d-1}\to\cdots\to A\otimes_R P_1\to A\otimes_R P_0\to A\otimes_R \Bbbk\to 0.$$

It is clear that each $A\otimes_R P_i$ is projective. This shows that the projective dimension of $A\otimes_R\Bbbk$ is finite. But ${}_A\Bbbk$ is a direct summand of $A\otimes_R\Bbbk$ as an A-module [3, Lemma III.4.8]. So the projective dimension of ${}_A\Bbbk$ is finite. Since A is a Hopf algebra, the global dimension of A is finite.

Conversely, if A is AS-regular, then R is AS-regular by Lemma 3.4, Lemma 3.5 and Remark 1.12. $\square$

We are ready to give the rigid dualizing complex of an AS-Gorenstein braided Hopf algebra.

Theorem 3.8. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Assume that R is an AS-Gorenstein algebra with injective dimension d. If $\int_R^l\cong\Bbbk_{\xi_R}$, for some algebra homomorphism $\xi_R:R\to\Bbbk$, then R has a rigid dualizing complex ${}_{\varphi}R[d]$, where φ is the algebra automorphism defined by*

$$\varphi(r)=\sum_{g\in\Gamma}\xi_R(r^1)\operatorname{hdet}(g)g^{-1}\big(\mathcal{S}_R^2\big(\big(r^2\big)_g\big)\big),$$

for any $r\in R$.

Proof. Let A be $R\#\Bbbk\Gamma$. By Lemma 3.6, A is AS-Gorenstein with $\int_A^l\cong\Bbbk_\xi$, where $\xi:A\to\Bbbk$ is the algebra homomorphism defined by

$$\xi(r\#h)=\xi_R(r)\operatorname{hdet}(h)$$

for any $r\#h\in R\#\Bbbk\Gamma$. By Lemma 3.4, there are R–R-bimodule isomorphisms

$$\operatorname{Ext}^i_{R^e}\big(R,R^e\big)\cong\begin{cases}0, & i\neq d;\\ R_{([\xi]\mathcal{S}_A^2)|_R}, & i=d.\end{cases}$$

For any $r\in R$,

$$\begin{aligned}[\xi]\mathcal{S}_A^2(r)&=\sum_{g\in\Gamma}\xi\big(r^1\#g\big)\mathcal{S}_A^2\big(\big(r^2\big)_g\big)\\&=\sum_{g\in\Gamma}\xi_R\big(r^1\big)\operatorname{hdet}(g)\mathcal{S}_A^2\big(\big(r^2\big)_g\big)\\&=\sum_{g\in\Gamma}\xi_R\big(r^1\big)\operatorname{hdet}(g)g^{-1}\big(\mathcal{S}_R^2\big(\big(r^2\big)_g\big)\big).\end{aligned}$$

Now the theorem follows from Lemma 1.7. $\square$

Remark 3.9. The algebra $A=R\#\Bbbk\Gamma$ has a rigid dualizing complex ${}_{[\xi]\mathcal{S}_A^2}A[d]$ [5, Prop. 4.5]. Observe that the algebra automorphism φ given in Theorem 3.8 is just the restriction of $[\xi]\mathcal{S}_A^2$ on R.

Now we can characterize the CY property of R in case $R\#\Bbbk\Gamma$ is CY.

Theorem 3.10. *Let Γ be a finite group and R a Noetherian braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Define an algebra automorphism φ of R by*

$$\varphi(r) = \sum_{g \in \Gamma} g^{-1}\big(\mathcal{S}_R^2(r_g)\big),$$

for any $r \in R$. If $R \# \Bbbk\Gamma$ is a CY algebra, then R is CY if and only if the algebra automorphism φ is an inner automorphism.

Proof. Assume that $A = R \# \Bbbk\Gamma$ is a CY algebra of dimension d. By Proposition 1.11, A is AS-regular of global dimension d and $\int_A^l \cong \Bbbk$. It follows from Lemma 3.5 that R is homologically smooth.

Since $\int_A^l \cong \Bbbk$, by Lemma 3.4 there are R–R-bimodule isomorphisms

$$\mathrm{Ext}^i_{R^e}\big(R, R^e\big) \cong \begin{cases} 0, & i \neq d; \\ R_{\mathcal{S}_A^2|_R}, & i = d. \end{cases}$$

Following Remark 1.12, we obtain that R is AS-regular. Suppose $\int_R^l \cong \Bbbk_{\xi_R}$ for some algebra homomorphism $\xi_R : R \to \Bbbk$. Then by Lemma 3.6, $\int_A^l \cong \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is defined by $\xi(r \# h) = \xi_R(r)\,\mathrm{hdet}(h)$ for any $r \# h \in R \# \Bbbk\Gamma$. But $\int_A^l \cong \Bbbk$. Therefore, $\xi_R = \varepsilon_R$ and $\mathrm{hdet} = \varepsilon_H$. It follows from Theorem 3.8 that the rigid dualizing complex of R is isomorphic to ${}_{\varphi}R[d]$, where φ is defined by

$$\begin{aligned}\varphi(r) &= \sum_{g \in \Gamma} \xi_R\big(r^1\big)\,\mathrm{hdet}(g) g^{-1}\big(\mathcal{S}_R^2\big(\big(r^2\big)_g\big)\big) \\ &= \sum_{g \in \Gamma} g^{-1}\big(\mathcal{S}_R^2(r_g)\big)\end{aligned}$$

for any $r \in R$. Now the theorem follows from Corollary 1.8. $\square$

Corollary 3.11. *Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Assume that R is an AS-regular algebra. Then the following two conditions are equivalent*:

(a) *Both R and $R \# \Bbbk\Gamma$ are CY algebras.*
(b) *The following three conditions are satisfied*:
  (i) $\int_R^l \cong \Bbbk$;
  (ii) *the homological determinant of the group action is trivial*;
  (iii) *the algebra automorphism φ defined by*

$$\varphi(r) = \sum_{g \in \Gamma} g^{-1}\big(\mathcal{S}_R^2(r_g)\big)$$

for all $r \in R$ is an inner automorphism.

Proof. $(a) \Rightarrow (b)$ Since R is a CY algebra, by Proposition 1.11 we have $\int_R^l \cong \Bbbk$. Because both R and $R \# \Bbbk\Gamma$ are CY, (ii) and (iii) are satisfied by Theorem 2.8 and Theorem 3.10.

$(b) \Rightarrow (a)$ Since R is AS-regular, $R \# \Bbbk\Gamma$ is AS-regular by Proposition 3.7. Thus R is homologically smooth (Lemma 3.5). By Theorem 3.8, if the three conditions in (b) are satisfied, then the rigid dualizing complex of R is isomorphic to $R[d]$, where d is the injective dimension of R. So R is a CY algebra. That the algebra $R \# \Bbbk\Gamma$ is a CY algebra follows from Theorem 2.8. $\square$

Example 3.12. Keep the notations from Example 2.10. Assume that Γ is a finite group and that $\mathfrak{g}$ is a finite dimensional Γ-module Lie algebra. Suppose there is a group homomorphism $\nu : \Gamma \to \mathrm{Aut}_{Lie}(\mathfrak{g})$. In Example 2.10, we use Theorem 2.8 to obtain that if $U(\mathfrak{g})$ is a CY algebra and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$ then $U(\mathfrak{g})\#\Bbbk\Gamma$ is a CY algebra. Now by Theorem 3.10, if $U(\mathfrak{g})\#\Bbbk\Gamma$ is a CY algebra, then $U(\mathfrak{g})$ is a CY algebra as well. This is because $U(\mathfrak{g})$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with trivial coaction, the algebra automorphism φ in Theorem 3.10 is the identity.

By [5, Prop. 6.3], we have that $\int^{l}_{U(\mathfrak{g})} = \Bbbk_{\xi}$, where $\xi(x) = \mathrm{tr}(\mathrm{ad}(x))$ for all $x \in \mathfrak{g}$. We calculate in Example 2.10 that $\mathrm{hdet}(g) = \det(\nu(g))$ for $g \in \Gamma$. Therefore, both $U(\mathfrak{g})$ and $U(\mathfrak{g})\#\Bbbk\Gamma$ are CY algebras if and only if $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$ and $\mathrm{Im}(\nu) \subseteq SL(\mathfrak{g})$. This coincides with Corollary 3.5 and Lemma 4.1 in [8].

We refer to [2] for the definition of a datum of finite Cartan type and the definition of the algebras $U(\mathcal{D}, \lambda)$. The algebras $U(\mathcal{D}, \lambda)$ were constructed to classify finite dimensional pointed Hopf algebras whose group-like elements form an abelian group.

Let

$$\mathcal{D}\big(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta}\big)$$

be a datum of finite Cartan type for a finite abelian group Γ and λ a family of linking parameters for $\mathcal{D}$. Let $\{\alpha_1, \ldots, \alpha_\theta\}$ be a set of simple roots of the root system corresponding to the Cartan matrix (a_{ij}). Assume that $w_0 = s_{i_1} \cdots s_{i_p}$ is a reduced decomposition of the longest element in the Weyl group as a product of simple reflections. Then the positive roots are as follows

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \qquad \ldots, \qquad \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}).$$

If $\beta_i = \sum_{i=1}^{\theta} m_i\alpha_i$, then we define $\chi_{\beta_i} = \chi_1^{m_1} \cdots \chi_\theta^{m_\theta}$.

The following proposition characterizes the CY property of the algebra $U(\mathcal{D}, \lambda)$.

Proposition 3.13. (a) *The algebra $A = U(\mathcal{D}, \lambda)$ is AS-regular of global dimension p and $\int^{l}_{A} = \Bbbk_{\xi}$, where ξ is the algebra homomorphism defined by $\xi(g) = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$, for all $g \in \Gamma$ and $\xi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$.*

(b) *The algebra A is CY if and only if $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.*

Proof. (a) can be obtained by applying [2, Thm. 3.3] and a similar argument as in the proof of Theorem 2.2 in [22]. (b) follows from [8, Thm. 2.3]. $\square$

Let R be the algebra generated by $x_1, \ldots, x_\theta$ subject to the relations

$$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, \quad 1 \leqslant i, j \leqslant \theta,\ i \neq j.$$

Then $U(\mathcal{D}, 0) = R\#\Bbbk\Gamma$, where $U(\mathcal{D}, 0)$ is the associated graded algebra of $U(\mathcal{D}, \lambda)$ with respect to its coradical filtration.

Proposition 3.14. *The algebra R is CY if and only if $\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = 1$ for each $1 \leqslant k \leqslant \theta$.*

Proof. By Lemma 3.5 and Theorem 3.8, we have that R is homologically smooth, and that it has a rigid dualizing complex ${}_{\varphi}R[p]$, where φ is the restriction of $[\xi]\mathcal{S}_A^2$ on R. That is, φ is defined by $\varphi(x_k) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)(x_k)$, $1 \leqslant k \leqslant \theta$, where each $1 \leqslant j_k \leqslant p$ is the integer such that $\beta_{j_k} = \alpha_k$. Therefore, R is CY if and only if $\prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k) = 1$ for each $1 \leqslant k \leqslant \theta$. $\square$

One may compare these results with Theorem 2.3, Theorem 3.9 and Lemma 4.1 in [22].

Table 1

	y_1	y_2
χ_1	-1	1
χ_2	-1	-1

In case the algebra $U(\mathcal{D},0) = R\#\Bbbk\Gamma$ is CY, the algebra automorphism φ defined in Theorem 3.10 is $\varphi(x_i) = \chi_i(g_i^{-1})(x_i)$, $1 \leqslant i \leqslant \theta$. However, $\chi_i(g_i) \neq 1$ for all $1 \leqslant i \leqslant \theta$. We conclude that when $R\#\Bbbk\Gamma$ is CY, the algebra R is not a CY algebra.

Now we give two examples of CY pointed Hopf algebra with a finite group of group-like elements.

Example 3.15. Let A be $U(\mathcal{D},\lambda)$ with the datum $(\mathcal{D},\lambda)$ given by

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
- The Cartan matrix is of type A_2;
- $g_i = y_i$, $1 \leqslant i \leqslant 2$;
- χ_i, $1 \leqslant i \leqslant 2$, are given in Table 1.
- $\lambda = 0$.

The algebra A is a CY algebra of dimension 3. Let R be the algebra generated by x_1 and x_2 subject to relations

$$x_1^2x_2 - x_2x_1^2 = 0 \quad \text{and} \quad x_2^2x_1 - x_1x_2^2 = 0.$$

Then $A = R\#\Bbbk\Gamma$. The rigid dualizing complex of R is ${}_{\varphi}R[3]$, where $\varphi = -\mathrm{id}$.

Remark 3.16. From the proof of Proposition 5.8 in [22], we can see that if $A = U(\mathcal{D},\lambda)$ is a CY algebra such that $(\mathcal{D},\lambda)$ is a generic datum, then the Cartan matrix in $\mathcal{D}$ cannot be of type A_2.

Example 3.17. Let A be $U(\mathcal{D},\lambda)$ with the datum $(\mathcal{D},\lambda)$ given by

- $\Gamma = \langle y_1, y_2\rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n$;
- The Cartan matrix is of type $A_1 \times A_1$;
- $g_i = y_i$, $i = 1, 2$;
- $\chi_1(y_i) = q$, $\chi_2(y_i) = q^{-1}$, $i = 1, 2$, where $q \in \Bbbk$ is an n-th root of unity;
- $\lambda = 1$.

The algebra A is a CY algebra of dimension 2.

Let R be the algebra $\Bbbk\langle x_1, x_2 \mid x_1x_2 = q^{-1}x_2x_1\rangle$. Then $\operatorname{Gr} A = U(\mathcal{D},0) = R\#\Bbbk\Gamma$, where $\operatorname{Gr} A$ is the associated graded algebra of A with respect to the coradical filtration of A. The rigid dualizing complex of R is ${}_{\varphi}R[3]$, where φ is defined by $\varphi(x_1) = q^{-1}x_1$ and $\varphi(x_2) = qx_2$.

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