Journal of Algebra 420 (2014) 201–216

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics

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A R T I C L E   I N F O

Article history:
Received 29 December 2013
Available online xxxx
Communicated by Nicolás Andruskiewitsch

MSC:
16S35
16S99
37B05

Keywords:
Partial skew group ring
Leavitt path algebra
Partial topological dynamics
Simplicity

A B S T R A C T

Let R_0 be a commutative and associative ring (not necessarily unital), G a group and α a partial action of G on ideals of R_0, all of which have local units. We show that R_0 is maximal commutative in the partial skew group ring $R_0 \rtimes_\alpha G$ if and only if R_0 has the ideal intersection property in $R_0 \rtimes_\alpha G$. From this we derive a criterion for simplicity of $R_0 \rtimes_\alpha G$ in terms of maximal commutativity and G-simplicity of R_0. We also provide two applications of our main results. First, we give a new proof of the simplicity criterion for Leavitt path algebras, as well as a new proof of the Cuntz–Krieger uniqueness theorem. Secondly, we study topological dynamics arising from partial actions on clopen subsets of a compact set.

1. Introduction

Partial skew group rings arose as a generalization of skew group rings and as an algebraic analogue of C*-partial crossed products (see [5]). Much in the same way as

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http://dx.doi.org/10.1016/j.jalgebra.2014.07.027

skew group rings, partial skew group rings provide a way to construct non-commutative rings, and recently Leavitt path algebras have been realized as partial skew group rings (see [10]), indicating that the theory of non-commutative rings may benefit from the theory of partial skew group rings. Still, when compared to the well-established theory of skew group rings, the theory of partial skew group rings is still in its infancy. In fact, to our knowledge, [3] and [4] are the only existing papers regarding the ideal structure of partial skew group rings, and [9] is a recent paper describing simplicity conditions for partial skew group rings of abelian groups.

Our main goal in this paper is to derive necessary and sufficient conditions for simplicity of partial skew group rings. In general, this is still an open problem, even for skew group rings. In [12] and [14], Öinert has attacked this problem for skew group rings $R_0 \rtimes_\alpha G$, where either the group G, or the ring R_0, is abelian. Recently, in [9], a criterion for simplicity of partial skew group rings of abelian groups has been described. In our case, we will extend results of [12] to partial skew group rings $R_0 \rtimes_\alpha G$, where R_0 is assumed to be commutative and associative (not necessarily unital) and α is a partial action on ideals of R_0, all of which have local units. More specifically, we will show that $R_0 \rtimes_\alpha G$ is simple if and only if R_0 is G-simple and maximal commutative in $R_0 \rtimes_\alpha G$. In particular, our results can be applied to Leavitt path algebras, by realizing them as partial skew group rings (see [10]), and to partial skew group rings associated with partial topological dynamics.

Our work is organized in the following way: In Section 2 we present our main results, preceded by a quick overview of the key concepts involved below. In Section 3 we apply the results of Section 2 to derive a new proof of the simplicity criterion for Leavitt path algebras, as well as a new proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras. In Section 4 we show an application of the results of Section 2 to partial topological dynamics, namely to partial actions by clopen subsets of a compact set.

Recall that a partial action of a group G (with identity element denoted by e) on a set Ω, is a pair $\alpha = (\{D_t\}_{t\in \mathrm{G}}, \{\alpha_t\}_{t\in \mathrm{G}})$, where for all $s, t \in \mathrm{G}$, D_t is a subset of Ω and $\alpha_t : D_{t^{-1}} \to D_t$ is a bijection such that $D_e = \Omega$, α_e is the identity map on Ω, $\alpha_t(D_{t^{-1}} \cap D_s) = D_t \cap D_{ts}$ and $\alpha_t(\alpha_s(x)) = \alpha_{ts}(x)$, for all $x \in D_{s^{-1}} \cap D_{s^{-1}t^{-1}}$. In case Ω is a ring (algebra) then, for each $t \in G$, the subset D_t should be an ideal and the map α_t should be a ring (algebra) isomorphism. In the topological setting, each D_t should be an open set and each α_t a homeomorphism, and in the C*-algebra setting each D_t should be a closed ideal and each α_t should be a *-isomorphism.

Associated with a partial action of a group G on a ring A, we have the partial skew group ring, $A \rtimes_\alpha G$, which is the set of all finite formal sums $\sum_{t\in G} a_t\delta_t$, where, for each $t \in G$, $a_t \in D_t$ and δ_t is a symbol. Addition is defined in the usual way and multiplication is determined by $(a_t\delta_t)(b_s\delta_s) = \alpha_t(\alpha_{t^{-1}}(a_t)b_s)\delta_{ts}$. An ideal I of A is said to be *G-invariant* if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I$ holds for all $g \in G$. If A and $\{0\}$ are the only G-invariant ideals of A, then A is said to be *G-simple*.

For $a = \sum_{t\in G} a_t\delta_t \in A \rtimes_\alpha G$, the support of a, which we denote by $\mathrm{supp}(a)$, is the finite set $\{t \in G : a_t \neq 0\}$, and the cardinality of $\mathrm{supp}(a)$ is denoted by $\#\mathrm{supp}(a)$.

For $g \in G$, the projection map onto the g coordinate, $P_g : A \rtimes_\alpha G \to A$, is given by $P_g(\sum_{t\in G} a_t\delta_t) = a_g$ and the augmentation map $\mathcal{T} : A \rtimes_\alpha G \to A$ is defined by $\mathcal{T}(\sum_{t\in G} a_t\delta_t) = \sum_{t\in G} a_t$.

Recall also that the centralizer of a nonempty subset S of a ring R, which we denote by $C_R(S)$, is the set of all elements of R that commute with each element of S. If $C_R(S) = S$ holds, then S is said to be a *maximal commutative subring* of R. Notice that a maximal commutative subring is necessarily commutative. Following [13], a subring S of a ring R is said to have the *ideal intersection property* in R, if $S \cap I \neq \{0\}$ holds for each non-zero ideal I of R.

By abuse of notation, the identity element of an arbitrary group G will be denoted by 0.

2. Maximal commutativity, the ideal intersection property and simplicity

This is the key section of our paper. Recall from [2] that a ring S is said to have *local units*, if there exists a set U of idempotents in S such that, for every finite subset X of S, there exists an $f \in U$ such that $X \subseteq fSf$. From this it follows that $x = fx = xf$ holds for each $x \in X$. If such a set U exists, then it will be referred to as a *set of local units* for S and the idempotent f is then said to be a *local unit* for the subset X.

Throughout this section we will assume that R_0 is a commutative and associative ring and that $\alpha = (\{R_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a partial action of a group G on the ring R_0 such that, for each $t \in G$, the ideal R_t, viewed as a ring, has a set of *local units*. In particular, this implies that R_t is an idempotent ring, for each $t \in G$, and thus, by [5, Corollary 3.2], we are ensured that the partial skew group ring $R_0 \rtimes_\alpha G$ is associative. We begin by showing the relationship between maximal commutativity of R_0 and the ideal intersection property of R_0 in $R_0 \rtimes_\alpha G$.

Theorem 2.1. *Let R_0 be a commutative associative ring, G a group and $\alpha = (\{R_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ a partial action such that, for each $t \in G$, R_t has a set of local units. Then $R_0\delta_0$ is maximal commutative in $R_0 \rtimes_\alpha G$ if and only if $I \cap R_0\delta_0 \neq \{0\}$ for each non-zero ideal I of $R_0 \rtimes_\alpha G$.*

Proof. First suppose that $R_0\delta_0$ is maximal commutative in $R_0 \rtimes_\alpha G$ and let I be a non-zero ideal of $R_0 \rtimes_\alpha G$. We will show that $I \cap R_0\delta_0 \neq \{0\}$.

Let $x = \sum_{t\in F} x_t\delta_t$ be a non-zero element in I such that $\#\mathrm{supp}(x)$ is minimal among all the non-zero elements of I and assume that $x_t \neq 0$ for each $t \in F \subseteq G$. Pick an $s \in F$, let $e \in R_{s^{-1}}$ be a local unit for $\alpha_{s^{-1}}(x_s)$ and define $y := x \cdot e\delta_{s^{-1}} \in I$. Next we show that $y \in R_0\delta_0$, but first notice that $y \neq 0$ and $\#\mathrm{supp}(y) \leq \#\mathrm{supp}(x)$, since $x_s \neq 0$ and

$$y = x \cdot e\delta_{s^{-1}} = x_s\delta_s \cdot e\delta_{s^{-1}} + \sum_{t\in F\setminus\{s\}} x_t\delta_t \cdot e\delta_{s^{-1}} = x_s\delta_0 + \sum_{t\in F\setminus\{s\}} x_t\delta_t \cdot e\delta_{s^{-1}}.$$

Now, let $a \in R_0$ and $z := a\delta_0 \cdot y - y \cdot a\delta_0 \in I$. Notice that $\#\mathrm{supp}(z) < \#\mathrm{supp}(x)$, since $a\delta_0 \cdot x_s\delta_0 - x_s\delta_0 \cdot a\delta_0 = 0$, and hence, from the minimality of $\#\mathrm{supp}(x)$, we have that $z = 0$. But this implies that $a\delta_0 \cdot y = y \cdot a\delta_0$ for all $a \in R_0$ and so, by the maximal commutativity of $R_0\delta_0$, we obtain that $y \in R_0\delta_0$ and $I \cap R_0\delta_0 \neq \{0\}$ as desired.

Next we show that if $R_0\delta_0$ is not maximal commutative in $R_0 \rtimes_\alpha G$ then there exists a non-zero ideal J of $R_0 \rtimes_\alpha G$ such that $J \cap R_0\delta_0 = \{0\}$.

So, suppose that $R_0\delta_0$ is not maximal commutative. This means that there exists an element $a = \sum_{t\in F} a_t\delta_t \in R_0 \rtimes_\alpha G \setminus R_0\delta_0$ such that $a \cdot b\delta_0 = b\delta_0 \cdot a$ for all $b \in R_0$, which is equivalent to $a_t\delta_t \cdot b\delta_0 = b\delta_0 \cdot a_t\delta_t$ for all $t \in F$ and $b \in R_0$. Evaluating the multiplications in this last equation we obtain that $\alpha_t(\alpha_{t^{-1}}(a_t)b)\delta_t = ba_t\delta_t$, for all $t \in F$ and $b \in R_0$ and hence

$$\alpha_t\big(\alpha_{t^{-1}}(a_t)b\big) = ba_t = a_tb \tag{1}$$

holds for all $t \in F$ and $b \in R_0$.

Now, fix a non-identity $g \in F$ such that $a_g \neq 0$ and let J be the ideal of $R_0 \rtimes_\alpha G$ generated by the element $a_g\delta_0 - a_g\delta_g$.

Notice that each element of J is a finite sum of elements of the form $b_t\delta_t(a_g\delta_0 - a_g\delta_g)c_r\delta_r$, where $b_t\delta_t, c_r\delta_r \in R_0 \rtimes_\alpha G$. Moreover, $J \neq \{0\}$, since if e is a local unit for a_g, then $e\delta_0(a_g\delta_0 - a_g\delta_g)e\delta_0$ is a non-zero element of J.

We will show that J has null intersection with $R_0\delta_0$ by showing that $\mathcal{T}(J) = 0$. In order to do so, notice that, for $b_t\delta_t, c_r\delta_r \in R_0 \rtimes_\alpha G$, we may use Eq. (1) to conclude that

$$\begin{aligned} b_t\delta_t(a_g\delta_0 - a_g\delta_g)c_r\delta_r &= b_t\delta_t \cdot a_g\delta_0 \cdot c_r\delta_r - b_t\delta_t \cdot a_g\delta_g \cdot c_r\delta_r \\ &= b_t\delta_t \cdot a_gc_r\delta_r - b_t\delta_t \cdot \alpha_g\big(\alpha_{g^{-1}}(a_g)c_r\big)\delta_{gr} \\ &= b_t\delta_t \cdot a_gc_r\delta_r - b_t\delta_t \cdot a_gc_r\delta_{gr} = d\delta_{tr} - d\delta_{tgr}, \end{aligned}$$

where $d = \alpha_t(\alpha_{t^{-1}}(b_t)a_gc_r)$, and hence $\mathcal{T}(J) = 0$. Since the restriction of $\mathcal{T}$ to $R_0\delta_0$ is injective we conclude that $J \cap R_0\delta_0 = \{0\}$, as desired. □

The above result generalizes [12, Theorem 3.5].

Remark 2.2. In this paper we are mainly interested in the situation when each ideal R_t, for $t \in G$, has a set of local units. Notice, however, that when it comes to Theorem 2.1 this assumption can be relaxed. In fact, it is enough to assume that R_t is *non-degenerate* for each $t \in G$, in the sense that for each non-zero $a \in R_t$ there is some $b \in R_t$ such that $ab \neq 0$ or $ba \neq 0$. If R_t is non-degenerate for each $t \in G$, then one can easily adapt the proof of Theorem 2.1, replacing the local units by the elements arising from the non-degeneracy of the ideals, to show that maximal commutativity of R_0 in $R_0 \rtimes_\alpha G$ implies that R_0 has the ideal intersection property (alternatively one can realize that the natural G-gradation on $R_0 \rtimes_\alpha G$ is *non-degenerate* in the sense of [13, Definition 2]). Conversely, non-degeneracy of each R_t, for $t \in G$, ensures that $R_0 \rtimes_\alpha G$ is associative, see [5, Corollary 3.2], and this is all that is needed to show that the ideal intersection property of R_0 in $R_0 \rtimes_\alpha G$ implies maximal commutativity of R_0.

We can now prove the simplicity criterion for $R_0 \rtimes_\alpha G$, and thereby generalize [12, Theorem 6.13].

Theorem 2.3. *Let R_0 be a commutative associative ring, G a group and $\alpha = (\{R_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ a partial action of G on R_0 such that, for each $t \in G$, R_t has a set of local units. Then the partial skew group ring $R_0 \rtimes_\alpha G$ is simple if and only if R_0 is G-simple and $R_0\delta_0$ is maximal commutative in $R_0 \rtimes_\alpha G$.*

Proof. Suppose first that $R = R_0 \rtimes_\alpha G$ is simple. By Theorem 2.1, $R_0\delta_0$ is maximal commutative. We show below that R_0 is G-simple.

Let I be a G-invariant non-zero ideal of R_0. Define J as the set of finite sums $\sum_{t\in G} a_t\delta_t$ such that $a_t \in I \cap R_t$ for all $t \in G$, that is, $J = \{\sum_{t\in G} a_t\delta_t \in R : a_t \in I \cap R_t,\ t \in G\}$.

Notice that J is a non-zero ideal of R. Indeed, if $a_r\delta_r \in R$ and $a_t \in I \cap R_t$ then $a_r\delta_r \cdot a_t\delta_t = \alpha_r(\alpha_{r^{-1}}(a_r)a_t)\delta_{rt}$. Since I is G-invariant, $\alpha_r(\alpha_{r^{-1}}(a_r)a_t) \in I$ and by the definition of a partial action $\alpha_r(\alpha_{r^{-1}}(a_r)a_t) \in R_{rt}$ so that $a_r\delta_r \cdot a_t\delta_t \in J$. Similarly, J is a right ideal of R and so, by the simplicity of R we obtain that $J = R$. Now notice that, from the definition of J, $P_0(J) = I$ and from what was done above, $P_0(J) = P_0(R) = R_0$. So $I = R_0$ and R_0 is G-simple.

Suppose now that R_0 is G-simple and that $R_0\delta_0$ is maximal commutative in R. Let I be a non-zero ideal of R. By Theorem 2.1, $I \cap R_0\delta_0 \neq \{0\}$. Let $J = I \cap R_0\delta_0$ and notice that $P_0(J)$ is a non-zero ideal of R_0. Next we show that $P_0(J)$ is G-invariant.

Let $a_t \in P_0(J) \cap R_t$ and pick a local unit e for a_t in R_t. Since $a_t\delta_0 \in J$ we have that $\alpha_{t^{-1}}(e)\delta_{t^{-1}} \cdot a_t\delta_0 \cdot e\delta_t = \alpha_{t^{-1}}(a_t)\delta_0$ is in J and hence $\alpha_{t^{-1}}(a_t) \in P_0(J)$ and $P_0(J)$ is G-invariant.

Now, since R_0 is G-simple we have that $P_0(J) = R_0$ and so $J = R_0\delta_0$. In particular, $R_0\delta_0 \subseteq I$. Take $s \in G$, $a_s \in R_s$ and an arbitrary $a_s\delta_s \in R_0 \rtimes_\alpha G$. Then, letting e be a local unit for a_s in R_s, we have that $a_s\delta_s = e\delta_0 \cdot a_s\delta_s \in I$. This shows that $R_0 \rtimes_\alpha G = I$, as desired. □

Remark 2.4. Notice that the proof of the fact that simplicity of the partial skew group ring $R_0 \rtimes_\alpha G$ implies G-simplicity of R_0 holds for any partial action $\alpha = (\{R_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$.

Inspired by [8, Example 3.4], we provide the following example.

Example 2.5. Let $R_0 = Ke_1 \oplus Ke_2 \oplus Ke_3$, where K is a field and e_1, e_2, e_3 are orthogonal central idempotents of R_0. Let C_4 be the cyclic group of order 4 with generator g and define a partial action of C_4 on R_0 by $\alpha_0 = \mathrm{id}_{R_0}$,

$$\alpha_g : Ke_2 \oplus Ke_3 \to Ke_1 \oplus Ke_2, \qquad \alpha_g(e_2) = e_1 \quad \text{and} \quad \alpha_g(e_3) = e_2;$$
$$\alpha_{g^2} : Ke_1 \oplus Ke_3 \to Ke_1 \oplus Ke_3, \qquad \alpha_{g^2}(e_1) = e_3 \quad \text{and} \quad \alpha_{g^2}(e_3) = e_1;$$
$$\alpha_{g^3} : Ke_1 \oplus Ke_2 \to Ke_2 \oplus Ke_3, \qquad \alpha_{g^3}(e_1) = e_2 \quad \text{and} \quad \alpha_{g^3}(e_2) = e_3.$$

There are exactly six proper (non-zero) ideals of R_0, namely

$$Ke_1, \qquad Ke_2, \qquad Ke_3, \qquad Ke_1 \oplus Ke_2, \qquad Ke_1 \oplus Ke_3 \quad \text{and} \quad Ke_2 \oplus Ke_3,$$

none of which is C_4-invariant. One easily checks this using the definition of α. Thus, R_0 is C_4-simple. Moreover, a short calculation reveals that $R_0\delta_0$ is maximal commutative in the partial skew group ring $R_0 \rtimes_\alpha C_4$. By Theorem 2.3, we conclude that $R_0 \rtimes_\alpha C_4$ is simple.

3. A new proof of the simplicity criterion for Leavitt path algebras

Recently, Leavitt path algebras have been described as partial skew group rings [10]. More precisely, the Leavitt path algebra associated with a graph E has been realized as a partial skew group ring of a commutative algebra by the free group on the edges of E and so we can apply the characterization of simplicity given in Section 2 to Leavitt path algebras. This will lead to a new proof of the simplicity criterion for Leavitt path algebras that rely solely on partial skew group ring theory. Before we proceed, for the convenience of the reader we shall recall some important notation and definitions.

A directed graph $E = (E^0, E^1, r, s)$ consists of a set E^0 of *vertices*, a set E^1 of *edges*, a *range map* $r : E^1 \to E^0$ and a *source map* $s : E^1 \to E^0$ which may be used to read off the direction of an edge. Given a field K and a directed graph E, the so called *Leavitt path algebra* associated with E (see e.g. [1,11]) is denoted by $L_K(E)$. To be more precise, $L_K(E)$ is the universal K-algebra generated by a set $\{v, e, e^* : v \in E^0, e \in E^1\}$ of elements satisfying the following five assertions:

(i) for all $v, w \in E^0$, $v^2 = v$, and $vw = 0$ if $v \neq w$;
(ii) $s(e)e = er(e) = e$ for all $e \in E^1$;
(iii) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$;
(iv) for all $e, f \in E^1$, $e^*e = r(e)$, and $e^*f = 0$ if $e \neq f$;
(v) $v = \sum_{\{e \in E^1 : s(e) = v\}} ee^*$ for each vertex $v \in E^0$ which satisfies $0 < \#\{e \in E^1 : s(e) = v\} < \infty$.

In [10], Gonçalves and Royer showed that each Leavitt path algebra can be realized as a partial skew group ring. We shall review their construction by first defining a partial action at the level of sets.

Let $E = (E^0, E^1, r, s)$ be a directed graph. A *path of length* n in E is a sequence $\xi_1\xi_2 \ldots \xi_n$ of edges in E such that $r(\xi_i) = s(\xi_{i+1})$ for $i \in \{1, 2, \ldots, n-1\}$. If ξ is a path of length n, then we write $|\xi| = n$. The set of all finite paths in E is denoted by W. An *infinite path* in E is an infinite sequence $\xi_1\xi_2\dots$ of edges in E such that $r(\xi_i) = s(\xi_{i+1})$ for $i \in \mathbb{N}$. The set of all infinite paths in E is denoted by W^∞. Notice that W (respectively W^∞) is a subset of the set of all finite (respectively infinite) words in the alphabet E^1. As usual, the range and source maps can be extended from E^1 to $W \cup W^\infty \cup E^0$ by defining $s(\xi) := s(\xi_1)$ for $\xi = \xi_1\xi_2\dots \in W^\infty$ or $\xi = \xi_1\dots\xi_n \in W$, $r(\xi) := r(\xi_n)$ for $\xi = \xi_1\dots\xi_n \in W$ and $r(v) = s(v) = v$ for $v \in E^0$. A finite path η is said to be an *initial subpath* of a (possibly infinite) path ξ, if there is a path ξ' such that $r(\eta) = s(\xi')$ and $\xi = \eta\xi'$ hold.

The partial action that we are about to define, takes place on the set

$$X = \bigl\{\xi \in W : r(\xi) \text{ is a sink}\bigr\} \cup \bigl\{v \in E^0 : v \text{ is a sink}\bigr\} \cup W^\infty$$

which is acted upon by $\mathbb{F}$, the free group generated by the set E^1.

Since $\mathbb{F}$ is generated by E^1, the set of edges of E, some elements of $\mathbb{F}$ can be thought of as coming directly from W. More concretely, each element $a \in W$ can be viewed as an element of $\mathbb{F}$, and similarly $a \in W$ defines an element $a^{-1} \in \mathbb{F}$. Hence, given $a, b \in W$ we may view a as an element of $\mathbb{F}$ and b^{-1} as an element of $\mathbb{F}$ and their product ab^{-1} will be an element of $\mathbb{F}$. These are three types of elements of $\mathbb{F}$ that will be of particular interest to our construction. Notice that not all elements of $\mathbb{F}$ arise in this way.

In order to have a partial action of $\mathbb{F}$ on X, for each $c \in \mathbb{F}$ we need to define a set X_c and a map $\theta_c : X_{c^{-1}} \to X_c$, such that they comply with the definition of a partial action.

The first step towards the construction of our partial action, is to define the sets X_c, for $c \in \mathbb{F}$. This is done as follows:

- $X_0 := X$, where 0 is the neutral element of $\mathbb{F}$.
- $X_{b^{-1}} := \{\xi \in X : s(\xi) = r(b)\}$, for all $b \in W$.
- $X_a := \{\xi \in X : \xi_1\xi_2\dots\xi_{|a|} = a\}$, for all $a \in W$.
- $X_{ab^{-1}} := \{\xi \in X : \xi_1\xi_2\dots\xi_{|a|} = a\} = X_a$, for $ab^{-1} \in \mathbb{F}$ with $a, b \in W$, $r(a) = r(b)$ and ab^{-1} in its reduced form.
- $X_c := \emptyset$, for all other $c \in \mathbb{F}$.

The second step towards the construction of our partial action, is to define the maps $\theta_c : X_{c^{-1}} \to X_c$, for $c \in \mathbb{F}$.

Let $\theta_0 : X_0 \to X_0$ be the identity map. For $b \in W$, $\theta_b : X_{b^{-1}} \to X_b$ is defined by $\theta_b(\xi) = b\xi$, for $\xi \in X_{b^{-1}}$. Notice that $\theta_b(\xi)$ is well-defined. Indeed, using that $s(\xi) = r(b)$ we may form the path $b\xi$ which obviously contains b as an initial subpath. Hence, $\theta_b(\xi) \in X_b$. For $b \in W$, we now define $\theta_{b^{-1}} : X_b \to X_{b^{-1}}$ for $\eta \in X_{b^{-1}}$, by $\theta_{b^{-1}}(\eta) = \eta_{|b|+1}\eta_{|b|+2}\dots$ if $r(b)$ is not a sink and $\theta_{b^{-1}}(b) = r(b)$, if $r(b)$ is a sink. It is easy to see that θ_b is bijective with inverse $\theta_{b^{-1}}$.

Finally, for $a, b \in W$ with $r(a) = r(b)$ and ab^{-1} in reduced form, $\theta_{ab^{-1}} : X_{ba^{-1}} \to X_{ab^{-1}}$ is defined by $\theta_{ab^{-1}}(\xi) = a\xi_{(|b|+1)}\xi_{(|b|+2)}\dots$ for $\xi \in X_{ba^{-1}}$. This map is well-defined. To see this, notice that $\xi \in X_{ba^{-1}}$ contains b as an initial subpath. Moreover,

$r(a) = r(b) = s(\xi_{|b|+1})$ and hence we may form the path $a\xi_{(|b|+1)}\xi_{(|b|+2)}\dots$ which obviously contains a as an initial subpath. Hence, $\theta_{ab^{-1}}(\xi) \in X_{ab^{-1}}$. It is not difficult to see that the inverse of this map is given by $\theta_{ba^{-1}} : X_{ab^{-1}} \to X_{ba^{-1}}$ which is defined by $\theta_{ba^{-1}}(\eta) = b\eta_{(|a|+1)}\eta_{(|a|+2)}\dots$ for $\eta \in X_{ab^{-1}}$.

Notice that $\{\{X_c\}_{c\in\mathbb{F}}, \{\theta_c\}_{c\in\mathbb{F}}\}$ is a partial action on the level of sets and so it induces a partial action $\{\{F(X_c)\}_{c\in\mathbb{F}}, \{\alpha_c\}_{c\in\mathbb{F}}\}$, where, for each $c \in \mathbb{F}$, $F(X_c)$ denotes the algebra of all functions from X_c to K, and $\alpha_c : F(X_{c^{-1}}) \to F(X_c)$ is defined by $\alpha_c(f) = f \circ \theta_{c^{-1}}$. The partial skew group ring associated with this partial action is not $L_K(E)$ yet. For this one proceeds in the following way:

For each $c \in \mathbb{F}$, and for each $v \in E^0$, define the characteristic maps $1_c := \chi_{X_c}$ and $1_v := \chi_{X_v}$, where $X_v = \{\xi \in X : s(\xi) = v\}$. Notice that 1_c is the multiplicative identity of $F(X_c)$. Finally, let

$$D_0 = \operatorname{span}\bigl\{\bigl\{1_p : p \in \mathbb{F} \setminus \{0\}\bigr\} \cup \bigl\{1_v : v \in E^0\bigr\}\bigr\}$$

(where span means the K-linear span) and, for each $p \in \mathbb{F} \setminus \{0\}$, let $D_p \subseteq F(X_p)$ be defined as $1_p D_0$, that is,

$$D_p = \operatorname{span}\{1_p 1_q : q \in \mathbb{F}\}.$$

Since $\alpha_p(1_{p^{-1}}1_q) = 1_p 1_{pq}$ (see [10]), consider, for each $p \in \mathbb{F}$, the restriction of α_p to $D_{p^{-1}}$. Notice that $\alpha_p : D_{p^{-1}} \to D_p$ is an isomorphism of K-algebras and, furthermore, $\{\{\alpha_p\}_{p\in\mathbb{F}}, \{D_p\}_{p\in\mathbb{F}}\}$ is a partial action.

In [10] it was shown that the partial skew group ring $D_0 \rtimes_\alpha \mathbb{F}$ is isomorphic to the Leavitt path algebra $L_K(E)$. More precisely, the map $\varphi : L_K(E) \to D_0 \rtimes_\alpha \mathbb{F}$ defined by $\varphi(e) = 1_e\delta_e$, $\varphi(e^*) = 1_{e^{-1}}\delta_{e^{-1}}$ for all $e \in E^1$ and $\varphi(v) = 1_v\delta_0$ for all $v \in E^0$, was shown to be a K-algebra isomorphism.

Recall, see [15], that a subset $H \subseteq E^0$ is said to be hereditary if for any $e \in E^1$ we have that $s(e) \in H$ implies $r(e) \in H$. A hereditary subset $H \subseteq E_0$ is called saturated if whenever $0 < \#s^{-1}(v) < \infty$, then $\{r(e) \in H : e \in E^1 \text{ and } s(e) = v\} \subseteq H$ implies $v \in H$. In [15] it is proved that $L_K(E)$ is simple if and only if the graph E satisfies condition (L), that is, each closed path in the graph E has an exit, and the only hereditary and saturated subsets of E^0 are E^0 and $\emptyset$. From now until the end of this section we will focus on the proof of the above simplicity criterion for $D_0 \rtimes_\alpha \mathbb{F}$ via Theorem 2.3, thus giving a new proof of the simplicity criterion for Leavitt path algebras. On the way, we will obtain some useful results that we will also use, together with Theorem 2.1, in order to give a new proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras.

Proposition 3.1. *The set $D_0\delta_0$ is maximal commutative in $D_0 \rtimes_\alpha \mathbb{F}$ if and only if the graph E satisfies condition (L).*

Proof. Suppose first that E satisfies condition (L). We will show that $D_0\delta_0$ is maximal commutative by contradiction. For this, suppose that there exists an element $a_t \in D_t$, with $t \neq 0$ and $a_t \neq 0$, such that $a_t\delta_t \cdot a_0\delta_0 = a_0\delta_0 \cdot a_t\delta_t$ for each $a_0 \in D_0$, that is, such that

$$\alpha_t\big(\alpha_{t^{-1}}(a_t)a_0\big) = a_t a_0 \tag{2}$$

holds for all $a_0 \in D_0$.

Notice that $a_t \neq 0$ implies that either $t \in W$ or $t = r^{-1}$, with $r \in W$, or $t = ab^{-1}$, where $a, b \in W$. Furthermore, if in Eq. (2) we take $a_0 = 1_{t^{-1}}$ we obtain that $a_t = a_t 1_{t^{-1}}$ and hence the support of a_t is contained in $D_t \cap D_{t^{-1}}$ and so t must be a closed path.

Now, taking appropriate functions for a_0 in Eq. (2) and using induction we obtain that, for all $n \in \mathbb{N}$, $a_t = a_t 1_{(t^n)^{-1}}$ and $a_t 1_{t^n} = a_t$. For example, for $a_0 = 1_{t^{-1}t^{-1}}$ we obtain that $a_t 1_{t^{-1}} = a_t 1_{t^{-1}t^{-1}}$ and so $a_t = a_t 1_{t^{-1}t^{-1}}$. On the other hand, for $a_0 = 1_t 1_{t^{-1}}$ we get that $\alpha_t(\alpha_{t^{-1}}(a_t)1_t 1_{t^{-1}}) = a_t 1_t 1_{t^{-1}}$ and hence $a_t 1_{tt} = a_t 1_{t^{-1}} = a_t$.

Before we derive our contradiction, notice that if $\xi \in X_t$ is such that $a_t(\xi) \neq 0$ then, since $a_t \in D_t$, there exists an $m \in \mathbb{N}$ such that for each $\mu \in X_t$ with $\mu_1 \ldots \mu_m = \xi_1 \ldots \xi_m$ it holds that $a_t(\mu) = a_t(\xi)$. We now separate our argument into three cases.

Case 1. Suppose $t \in W$.

Since $a_t = a_t 1_{t^m}$ we have $t^m = \xi_1 \ldots \xi_m \ldots \xi_{m|t|}$. Let s be an exit for t and $\mu \in X_t$ be such that $\mu_1 \ldots \mu_{m|t|} \ldots \mu_k = t^m t_1 \ldots t_l s$. Then $a_t(\mu) = a_t(\xi) \neq 0$, but $a_t(\mu) = a_t(\mu)1_{t^{m+1}}(\mu) = 0$, a contradiction. So t is not an element of W.

Case 2. Suppose $t = r^{-1}$, with $r \in W$.

This case follows as the previous one, by using the equality $a_t = a_t 1_{(t^m)^{-1}}$ instead of $a_t = a_t 1_{t^m}$.

Case 3. Suppose $t = ab^{-1}$, where $a, b \in W$.

We obtain a contradiction by proceeding as in Case 1 if $|a| \geq |b|$ and as in Case 2 if $|a| < |b|$. The details are left to the reader.

We conclude that there is no $a_t \in D_t$, with $t \neq 0$, such that $a_t\delta_t$ commutes with each element of $D_0\delta_0$ and hence $D_0\delta_0$ is maximal commutative.

Suppose now that E does not satisfy condition (L), that is, there exists a closed path $t = t_1 \ldots t_m$ which has no exit. We will show that $1_t\delta_t$ commutes with all of $D_0\delta_0$ and so $D_0\delta_0$ is not maximal commutative.

Recall that $D_0 = \operatorname{span}\{\{1_p : p \in \mathbb{F} \setminus \{0\}\} \cup \{1_v : v \in E^0\}\}$ and so it is enough to show that $1_t\delta_t$ commutes with $1_v\delta_0$ and with $1_p\delta_0$, for each $v \in E^0$ and $p \in \mathbb{F} \setminus \{0\}$.

Let $v \in E^0$. Then $1_t\delta_t \cdot 1_v\delta_0 = \alpha_t(\alpha_{t^{-1}}(1_t)1_v)\delta_t = \alpha_t(1_{t^{-1}}1_v)\delta_t$ which, by [10, Lemma 2.6(2)], is non-zero only if $r(t) = v$, in which case is equal to $1_t\delta_t$. On the other hand, $1_v\delta_0 \cdot 1_t\delta_t = 1_v 1_t\delta_t$, which is non-zero only if $s(t) = v$, in which case is equal to $1_t\delta_t$. Since t is a closed path it follows that $1_t\delta_t$ commutes with $1_v\delta_0$.

Now let $r \in \mathbb{F} \setminus \{0\}$. Notice that, in order to check that $1_t\delta_t$ commutes with $1_r\delta_0$ it is enough to verify that $\alpha_t(1_{t^{-1}}1_r) = 1_t 1_r$, which is equivalent to $1_t 1_{tr} = 1_t 1_r$ (since $\alpha_t(1_{t^{-1}}1_r) = 1_t 1_{tr}$). As before, we now divide our proof into cases:

Case 1. $r \in W$.

If $r = t^n t_1 \dots t_k$ for some $n \geq 0$ and $1 \leq k \leq m$ then, since t has no exit, $X_r = X_t = \{tttt \cdots\}$ and hence $1_t 1_{tr} = 1_t = 1_t 1_r$. If $r \in W$ is not of the above form, then $1_t 1_{tr} = 0 = 1_t 1_r$.

Case 2. $r = s^{-1}$ with $s \in W$.

Suppose first that $r(s) = r(t)$. Then $X_{s^{-1}} = X_t$, since t is a closed path with no exit, and hence $1_t 1_{tr} = 1_t 1_{ts^{-1}} = 1_t = 1_t 1_{s^{-1}} = 1_t 1_r$. If $r(s) \neq r(t)$, then $1_{ts^{-1}} = 0 = 1_t 1_{s^{-1}}$.

Case 3. $r = ab^{-1}$ with $a, b \in W$ and $r(a) = r(b)$.

Since $1_{tr} = 1_{tab^{-1}} = 1_{ta}$ and $1_r = 1_{ab^{-1}} = 1_a$ this case reduces to Case 1.

Case 4. All other $r \in \mathbb{F}$.

In this case $1_r = 0$ and hence both sides of the equation $\alpha_t(1_{t^{-1}}1_r) = 1_t 1_r$ are equal to zero.

We have proved that $1_t\delta_t$ is in the centralizer of $D_0\delta_0$ and hence $D_0\delta_0$ is not maximal commutative, as desired. $\square$

Before we proceed to show the connection between $\mathbb{F}$-simplicity of D_0 and the nonexistence of proper hereditary and saturated subsets of E^0, we shall prove two useful lemmas.

Lemma 3.2. *Let $x_0\delta_0$ be a non-zero element of $D_0\delta_0$ and denote by I the principal ideal of $D_0 \rtimes_\alpha \mathbb{F}$ generated by $x_0\delta_0$. Then there exists a vertex $v \in E^0$ such that $1_v\delta_0 \in I$.*

Proof. We can write x_0 as a linear combination of characteristic functions; $x_0 = \sum_{i=1}^n \lambda_i 1_{a_i b_i^{-1}} + \sum_{j=1}^m \beta_j 1_{v_j}$, where $a_i \in W$ and $b_i \in W \cup \{0\}$ (if $a_i = 0$, then $1_{a_i b_i^{-1}} = 1_{b_i^{-1}} = 1_{r(b_i)}$ since $X_{b_i^{-1}} = X_{r(b_i)}$). Choose some $v \in E^0$ such that $1_v x_0 \neq 0$. If v is a sink, then $1_v 1_{a_i b_i^{-1}} = 0$ for each i, and then

$$0 \neq 1_v x_0 \delta_0 = \sum_{j=1}^n \beta_j 1_v 1_{v_j} \delta_0 = \sum_{j \,:\, v_j = v} \beta_j 1_v \delta_0$$

which shows that $1_v\delta_0 \in I$.

Now, suppose that v is not a sink. Let $m = \max\{|a_i| \mid 1 \le i \le n\}$. Recall that we can write $X_v = \dot\bigcup_{c\in I} X_c$ where the index set I consists of all $c \in W$ such that $s(c) = v$ and $|c| = m$ or $s(c) = v$, $|c| < m$ and $r(c)$ is a sink. If $1_c 1_{a_i b_i^{-1}} \neq 0$, then a_i is an initial subpath of c, and then $1_c 1_{a_i b_i^{-1}} = 1_c 1_{a_i} = 1_c$. Moreover, if $1_c 1_{v_j} \neq 0$, then $1_c 1_{v_j} = 1_c$. Using this, we obtain

$$\begin{aligned} 0 \neq 1_c x_0 \delta_0 &= \sum_{i=1}^{n} \lambda_i 1_c 1_{a_i b_i^{-1}} \delta_0 + \sum_{j=1}^{m} \beta_j 1_c 1_{v_j} \delta_0 \\ &= \sum_{i\,:\,1_c 1_{a_i b_i^{-1}} \neq 0} \lambda_i 1_c 1_{a_i b_i^{-1}} \delta_0 + \sum_{j\,:\,1_c 1_{v_j} \neq 0} \beta_j 1_c 1_{v_j} \delta_0 \\ &= \sum_{i\,:\,1_c 1_{a_i b_i^{-1}} \neq 0} \lambda_i 1_c \delta_0 + \sum_{j\,:\,1_c 1_{v_j} \neq 0} \beta_j 1_c \delta_0 \\ &= \left(\sum_{i\,:\,1_c 1_{a_i b_i^{-1}} \neq 0} \lambda_i + \sum_{j\,:\,1_c 1_{v_j} \neq 0} \beta_j \right) 1_c \delta_0, \end{aligned}$$

which shows that $1_c\delta_0 \in I \setminus \{0\}$. Notice that $1_{r(c)}\delta_0 = 1_{c^{-1}}\delta_0 = 1_{c^{-1}}\delta_{c^{-1}} \cdot 1_c\delta_0 \cdot 1_c\delta_c$. Using that I is an ideal, we conclude that $1_{r(c)}\delta_0 \in I$ which proves the lemma. □

Lemma 3.3. *Let I be an $\mathbb{F}$-invariant ideal of D_0. Then, the set $Z = \{v \in E^0 : 1_v \in I\}$ is hereditary and saturated.*

Proof. Let $e \in E^1$ be such that $s(e) \in Z$. Then $1_e = 1_{s(e)} 1_e \in I \cap D_e$ and, by the $\mathbb{F}$-invariance of I, $\alpha_{e^{-1}}(1_e) = 1_{e^{-1}} = 1_{r(e)} \in I$, so that $r(e) \in Z$.

Now, let $v \in E^0$ be such that $0 < \#s^{-1}(v) < \infty$ and $r(e) \in Z$ for each $e \in s^{-1}(v)$. Notice that $1_{r(e)} = 1_{e^{-1}}$ and so, since I is $\mathbb{F}$-invariant, we have that $1_e = \alpha_e(1_{e^{-1}}) \in I$. This implies that $1_v = \sum_{e\in s^{-1}(v)} 1_e \in I$ and hence $v \in Z$ as desired. □

The following proposition gives us a characterization of $\mathbb{F}$-simplicity of D_0.

Proposition 3.4. *The algebra D_0 is $\mathbb{F}$-simple if and only if the only saturated and hereditary subsets of E^0 are E^0 and $\emptyset$.*

Proof. Suppose first that D_0 is $\mathbb{F}$-simple. Let F be a nonempty saturated and hereditary subset of E^0. We need to show that $F = E^0$.

Consider the ideal I generated by $\{1_v\delta_0 : v \in F\}$ in $D_0 \rtimes_\alpha \mathbb{F}$, that is, I is the linear span of all the elements of the form $a_r\delta_r 1_v \delta_0 b_s \delta_s$, with $v \in F$, $a_r \in D_r$, $b_s \in D_s$ and $r, s \in \mathbb{F}$. Let $J = P_0(D_0\delta_0 \cap I)$ and notice that J is a non-zero $\mathbb{F}$-invariant ideal of D_0 (J is $\mathbb{F}$-invariant since if $a_t \in J \cap D_t$, then $a_t\delta_0 \in I$, so $\alpha_{t^{-1}}(a_t)\delta_0 = 1_{t^{-1}}\delta_{t^{-1}} \cdot a_t\delta_0 \cdot 1_t\delta_t \in I$ and hence $\alpha_{t^{-1}}(a_t) \in J$). Now, since D_0 is $\mathbb{F}$-simple we have that $J = D_0$ and, in particular, $1_u \in J$ for each $u \in E^0$. This means that for each $u \in E^0$, $1_u\delta_0 \in I$, and so we can write

$$1_u\delta_0 = \sum_t x_t\delta_t \cdot 1_{v_t}\delta_0 \cdot y_{t^{-1}}\delta_{t^{-1}} = \sum_t \alpha_t\bigl(\alpha_t^{-1}(x_t)1_{v_t}y_{t^{-1}}\bigr)\delta_0,$$

where the above sum is finite and $v_t \in F$ for each t. Multiplying the above equation by $1_u\delta_0$, we obtain

$$1_u\delta_0 = \sum_{t\in T} 1_u\alpha_t\bigl(\alpha_t^{-1}(x_t)1_{v_t}y_{t^{-1}}\bigr)\delta_0,$$

where

$$T := \bigl\{t \in \mathbb{F} : 1_u\alpha_t\bigl(\alpha_t^{-1}(x_t)1_{v_t}y_{t^{-1}}\bigr) \neq 0\bigr\}.$$

In particular, since $1_u\alpha_t(\alpha_t^{-1}(x_t)1_{v_t}y_{t^{-1}}) \neq 0$ for each $t \in T$, we have that $1_u1_t \neq 0$ and $1_{v_t}1_{t^{-1}} \neq 0$ for all $t \in T$.

Our aim is to show that each $u \in E^0$ belongs to F. So, let $u \in E^0$. If $u = r(b)$ for some path b and $s(b) \in F$ then $u \in F$, since F is hereditary. Moreover, if $0 < \#s^{-1}(u) < \infty$ and $r(e) \in F$ for each $e \in s^{-1}(u)$ then $u \in F$, since F is saturated. So, we are left with the cases when there is no path b with $s(b) \in F$ and $r(b) = u$ and either $s^{-1}(u) = \emptyset$, $\#s^{-1}(u) = \infty$, or $0 < \#s^{-1}(u) < \infty$ but $r(e) \notin F$ for some $e \in s^{-1}(u)$. We handle these three cases below.

Case 1. $s^{-1}(u) = \emptyset$, and there is no path b with $s(b) \in F$ and $r(b) = u$.

First notice that since there is no $b \in W$ such that $s(b) \in F$ and $r(b) = u$, for each $b \in W$, it holds that either $1_u1_{b^{-1}} = 0$ or $1_v1_b = 0$ for each $v \in F$. Then, by the statement right after the definition of T, we obtain that there is no $t \in T$ of the form $t = b^{-1}$ (with $b \in W$). Now, for t of the form $t = ab^{-1} \in \mathbb{F}$, with $a \in W$ and $b \in W \cup \{0\}$, we have that $1_u1_t = 0$, since $s(a) \neq u$, and hence $t = ab^{-1} \notin T$. We conclude that $T = \{0\}$, and so $1_u = 1_u x_0 1_{v_0} y_0$ and it follows that $u = v_0 \in F$.

Case 2. $\#s^{-1}(u) = \infty$, and there is no path b with $s(b) \in F$ and $r(b) = u$.

Here, as in Case 1, there is no $t \in T$ of the form $t = b^{-1}$ with $b \in W$. Suppose that $0 \notin T$. Then each $t \in T$ is of the form $t = ab^{-1}$, with $a \in W$ and $b \in W \cup \{0\}$. Since $\#s^{-1}(u) = \infty$, there is an element $\xi \in X$ with $s(\xi) = u$ and $s(\xi) \neq s(a)$ for each $ab^{-1} \in T$. Notice that $1_t(\xi) = 0$ for all $t \in T$ and so

$$1 = 1_u(\xi) = \sum_{t\in T} 1_u\alpha_t\bigl(\alpha_t^{-1}(x_t)1_{v_t}y_{t^{-1}}\bigr)(\xi) = 0,$$

which is a contradiction. So $0 \in T$ and $1_u x_0 1_{v_0} y_0 \neq 0$, which implies that $u = v_0 \in F$.

Case 3. $0 < \#s^{-1}(u) < \infty$, and there is no path b with $s(b) \in F$ and $r(b) = u$, and there is an edge $e \in s^{-1}(u)$ such that $r(e) \notin F$.

Again, as in Case 1, there is no $t \in T$ of the form $t = b^{-1}$ with $b \in W$. Suppose, as in Case 2, that $0 \notin T$. Then, as before, each $t \in T$ is of the form $t = ab^{-1}$, with $a \in W$ and $b \in W \cup \{0\}$.

Now, for each $t \in T$, let $c_t = 1_u \alpha_t(\alpha_t^{-1}(x_t) 1_{v_t} y_{t^{-1}})$. Since, for each $t = ab^{-1} \in T$, it holds that $1_u 1_t \neq 0$ and $1_{v_t} 1_{t^{-1}} \neq 0$, we have that $s(a) = u$ and $s(b) = v_t \in F$. The heredity of F now implies that $r(b) \in F$ and since $r(a) = r(b)$ we have that $r(a) \in F$. So, we obtain that

$$1_u = \sum_{t \in T} c_t = \sum_{ab^{-1} \in T} c_{ab^{-1}},$$

where $u = s(a)$ and $r(a) \in F$ for all $ab^{-1} \in T$.

Let $z = z_1 \ldots z_m$ be a path of maximum length such that $|z| \leq \max\{|a| : ab^{-1} \in T\}$ with $s(z) = u$ and $r(z_i) \notin F$ for each $i \in \{1, \ldots, m\}$. By the hypothesis, such a z exists. Then multiplying the equation $1_u = \sum_{ab^{-1} \in T} c_{ab^{-1}}$ by 1_z we obtain

$$1_z = \sum_{ab^{-1} \in T \ : \ |z| < |a|, a_1 \ldots a_m = z} c_{ab^{-1}}.$$

Since the sum on the right-hand side is finite, we have that $0 < \# s^{-1}(r(z)) < \infty$. By the maximality of $|z|$, there is no edge $e \in s^{-1}(r(z))$ such that $r(e) \notin F$. Then, $r(e) \in F$ for all $e \in s^{-1}(r(z))$ and, since F is saturated, we obtain that $r(z) \in F$, a contradiction (since $r(z) = r(z_m) \notin F$).

We conclude that $0 \in T$ and, as in Case 2, it follows that $u \in F$ as desired.

Suppose now, that the only saturated and hereditary subsets of E^0 are E^0 and $\emptyset$. Let I be a non-zero $\mathbb{F}$-invariant ideal of D_0. We need to show that $I = D_0$.

Let J be the (non-zero) ideal of $D_0 \rtimes_\alpha \mathbb{F}$ consisting of all finite sums $\sum a_t \delta_t$, with $a_t \in D_t \cap I$ (J is an ideal since I is $\mathbb{F}$-invariant) and let $Z = \{v \in E^0 : 1_v \in I\}$. By Lemma 3.2, there is some $v \in E^0$ such that $1_v \delta_0 \in J$, so that $1_v \in I$ (since $J \cap D_0 \delta_0 = I\delta_0$) and hence Z is nonempty. By Lemma 3.3, Z is hereditary and saturated, and therefore $Z = E^0$. Thus, $1_v \in I$ for each $v \in E^0$ and hence $I = D_0$, as desired. □

Propositions 3.1 and 3.4 above, enable us to translate the language of Leavitt path algebras into the language of partial skew group rings, and vice versa. Using this, we shall now give a new proof of the simplicity criterion for Leavitt path algebras.

Theorem 3.5. *The partial skew group ring $D_0 \rtimes_\alpha \mathbb{F}$ is simple if and only if the graph E satisfies condition (L) and the only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

Proof. By combining the results from Theorem 2.3, Proposition 3.1 and Proposition 3.4, the desired conclusion follows. □

We end this section by providing an alternative proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras (cf. [10] and [15]).

Theorem 3.6 *(Cuntz–Krieger uniqueness theorem). Let E be a graph that satisfies condition (L). If $\phi : D_0 \rtimes_\alpha \mathbb{F} \to B$ is a K-algebra homomorphism such that $\phi(1_v\delta_0) \neq 0$ for each $v \in E^0$, then ϕ is injective.*

Proof. Suppose that E satisfies condition (L) and that $\phi(1_v\delta_0) \neq 0$ for each $v \in E^0$. Let I denote the ideal $\ker(\phi)$. Seeking a contradiction, suppose that $I \neq \{0\}$. Proposition 3.1 and Theorem 2.1 now yield $D_0\delta_0 \cap I \neq \{0\}$. Let $x_0\delta_0 \in D_0\delta_0 \cap I$ be a non-zero element. By Lemma 3.2, there is some $v \in E^0$ such that $1_v\delta_0 \in I = \ker(\phi)$, but this is a contradiction. Hence $\ker(\phi) = \{0\}$. □

4. Partial topological dynamics

In this final section we use the results of Section 2 to characterize partial actions on a compact space by clopen sets whose associated partial skew group ring is simple. More specifically, we will prove the following theorem.

Theorem 4.1. *Let $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ be a partial action of a group G on a compact space X such that for each $t \in G$, X_t is a clopen set. Then the partial skew group ring $\mathcal{C}(X) \rtimes_\alpha G$, where $\mathcal{C}(X)$ denotes the continuous complex-valued functions on X, is simple if, and only if, θ is topologically free and minimal.*

Remark 4.2. Partial actions on the Cantor set by clopen subsets are exactly the ones for which the enveloping space is Hausdorff (see [6]).

Remark 4.3. Since the partial action acts on clopen sets, each D_t is unital. Hence, we can use Theorem 2.3 to prove the above theorem.

Remark 4.4. In light of Remark 2.2 and Remark 2.4, it follows that the first part of Theorem 4.1 holds for any topological partial action on a locally compact space X, that is, if $\mathcal{C}(X) \rtimes_\alpha G$ is simple then θ is topologically free and minimal.

Before we proceed, recall that there is a correspondence between partial actions on a locally compact Hausdorff space X and partial actions on the C*-algebra of continuous complex-valued functions vanishing at infinity, $\mathcal{C}_0(X)$, (see e.g. [4,7]). Namely, if $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ is a partial action on X, then $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$, where $D_t = \mathcal{C}_0(X_t)$ and $\alpha_t(f) := f \circ h_{t^{-1}}$, is a partial action of G on $\mathcal{C}_0(X)$. Simplicity of the associated C*-partial crossed product was studied in [7], and a version of the above theorem for partial actions of abelian groups was given in [9]. Below we will recall the relevant definitions and make the proper adaptations of the ideas in [9] to the case at hand.

Definition 4.5. A topological partial action $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ is *topologically free* if for all $t \neq 0$ the set $F_t = \{x \in X_{t^{-1}} : h_t(x) = x\}$ has empty interior and is *minimal* if there is no proper open invariant subset of X ($U \subseteq X$ is invariant if $h_t(U \cap X_{t^{-1}}) \subseteq U$ holds for all $t \in G$).

Proposition 4.6. *A partial action $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ on a compact space X is minimal if, and only if, $\mathcal{C}(X)$ is G-simple.*

Proof. The proof of this can be found in [7]. □

Proposition 4.7. *Suppose that $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ is a topologically free partial action. Then $\mathcal{C}(X)\delta_0$ is maximal commutative in $\mathcal{C}(X) \rtimes_\alpha G$.*

Proof. Suppose that $\mathcal{C}(X)\delta_0$ is not maximal commutative. Then there exists a non-zero function f_t and $t \in G$, with $t \neq 0$, such that $f_t\delta_t \cdot f\delta_0 = f\delta_0 \cdot f_t\delta_t$ for all $f \in \mathcal{C}(X)$, which is equivalent to $\alpha_t(\alpha_{t^{-1}}(f_t)f)\delta_t = ff_t\delta_t$, for all $f \in \mathcal{C}(X)$, which in turn is equivalent to

$$f_t(x)f\big(h_{t^{-1}}(x)\big) = f(x)f_t(x), \tag{3}$$

for all $f \in \mathcal{C}(X)$ and $x \in X_t$.

Now, since f_t is non-zero, there exists $x \in X_t$ such that $f_t(x) \neq 0$ and the continuity of f_t implies that there exists an open set $U \subseteq X_t$ such that f_t is non-zero in U. Since the partial action is topologically free there exists $y \in U$ such that $h_{t^{-1}}(y) \neq y$. Let $f \in \mathcal{C}(X)$ be such that $f(y) = 1$ and $f(h_{t^{-1}}(y)) = 0$ (such a function exists by Urysohn's lemma). But then Eq. (3) above implies that $f_t(y) = 0$, a contradiction. □

Proposition 4.8. *If $\mathcal{C}(X) \rtimes_\alpha G$ is simple, then $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ is topologically free.*

Proof. The proof of this proposition is analogous to the proof of Proposition 4.7 in [9]. □

Remark 4.9. The three propositions above, combined with Theorem 2.3, provide the proof of Theorem 4.1.

Acknowledgments

The first author was partially supported by CNPq – Brazil (302157/2011-5). The second author was partially supported by The Swedish Research Council (repatriation grant No. 2012-6113).

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