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The Chern class map on abelian surfaces

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ABSTRACT

We examine the Chern class map $c_1 : \mathrm{NS}(S)/p\,\mathrm{NS}(S) \to \mathrm{H}^1(S, \Omega_S^1)$ for an abelian surface S in characteristic $p \geq 3$, and give a basis of the kernel c_1 for the superspecial abelian surface.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and S be a nonsingular complete algebraic surface over k. We denote by $\mathrm{H}^2_{dR}(S)$ the second de Rham cohomology group of S, and by $\mathrm{NS}(S)$ the Néron–Severi group of S. $\mathrm{NS}(S)$ is a finitely generated abelian group, and the rank $\rho(S)$ of $\mathrm{NS}(S)$ is called the Picard number. We have the Chern class map $\mathrm{NS}(S)/p\,\mathrm{NS}(S) \to \mathrm{H}^2_{dR}(S)$ and this map is injective if the Hodge-to-de Rham spectral sequence of S degenerates at E_1-term (cf. Ogus [9]). We also have the Chern class map $c_1 : \mathrm{NS}(S)/p\,\mathrm{NS}(S) \to \mathrm{H}^1(S, \Omega_S^1)$. This map is not necessarily injec-

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tive, even if the Hodge-to-de Rham spectral sequence of S degenerates at E_1-term (cf. Ogus [9]).

In this paper, we examine this map c_1 in the case of abelian surfaces. For abelian surfaces, the Chern class map c_1 is injective if and only if the abelian surface is not superspecial (for the definition, see Section 2). This fact was implicitly proved in Ogus [9] by using the notion of K3 crystal. We give here a down-to-earth proof of this fact and determine a basis of the kernel of the Chern class map c_1 for the superspecial abelian surface. To calculate a basis of $\operatorname{Ker} c_1$, in Section 2 we examine the structure of the Néron–Severi group of the superspecial abelian surface. Using theory of quaternion algebra, problems on divisors on superspecial abelian surfaces are translated into problems in matrix algebras over quaternion algebras. As an example, we give an explicit description of our theory in the case of characteristic 3. Finally, we examine the Chern class map for Kummer surfaces and show results similar to those for abelian surfaces.

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2. The Néron–Severi group

Let k be an algebraically closed field of characteristic $p > 0$. An abelian surface is said to be supersingular if it is isogenous to a product of two supersingular elliptic curves. An abelian surface is said to be superspecial if it is isomorphic to a product of two supersingular elliptic curves. By definition, if an abelian surface is superspecial, then the abelian surface is supersingular. But the converse does not necessarily hold (cf. Oort [11]). Note that a superspecial abelian surface is unique up to isomorphism (cf. Shioda [12]). In this section, we examine the structure of the Néron–Severi group of the superspecial abelian surface.

Let E be a supersingular elliptic curve defined over k, and we consider the superspecial abelian surface $A = E_1 \times E_2$ with $E_1 = E_2 = E$. We denote by O_E the zero point of E. We take a divisor $X = E_1 \times \{O_{E_2}\} + \{O_{E_1}\} \times E_2$, which gives a principal polarization on A. We also denote $E_1 \times \{O_{E_2}\}$ (resp. $\{O_{E_1}\} \times E_2$) by E_1 (resp. by E_2) for the sake of simplicity. We set $\mathcal{O} = \operatorname{End}(E)$ and $B = \operatorname{End}^0(E) = \operatorname{End}(E) \otimes \mathbf{Q}$. Then B is a quaternion division algebra over the rational number field $\mathbf{Q}$ with discriminant p, and $\mathcal{O}$ is a maximal order of B (cf. Mumford [8], Section 22 and Deuring [3], Section 2). For an element $a \in B$, we denote by $\bar{a}$ the image under the canonical involution.

We have a natural identification of $\operatorname{End}(A)$ with the ring $\mathrm{M}_2(\mathcal{O})$ of two-by-two matrices with coefficients in $\mathcal{O}$:

$$\operatorname{End}(A) = \mathrm{M}_2(\mathcal{O}).$$

Here, the action of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O})$ is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : A = E \times E \longrightarrow A = E \times E$$
$$(x, y) \mapsto \big(\alpha(x) + \beta(y), \gamma(x) + \delta(y)\big).$$

From here on, by a divisor L we often mean the divisor class represented by L in $\mathrm{NS}(A)$ if confusion is unlikely to occur. For a divisor L, we have a homomorphism

$$\begin{aligned} \varphi_L : A &\longrightarrow \mathrm{Pic}^0(A) \\ x &\mapsto T_x^* L - L, \end{aligned}$$

where T_x is the translation by $x \in A$ (cf. Mumford [8]). We set

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \;\middle|\; \alpha, \delta \in \mathbf{Z},\ \gamma, \beta \in \mathcal{O},\ \gamma = \bar{\beta} \right\}.$$

The main part of the following theorem may be known to specialists (cf. Mumford [8], and Ibukiyama, Katsura and Oort [6]), but since we cannot find a convenient reference, we give here a proof for it.

Theorem 2.1. *The homomorphism*

$$\begin{aligned} j : \mathrm{NS}(A) &\longrightarrow H \\ L &\mapsto \varphi_X^{-1} \circ \varphi_L \end{aligned}$$

is bijective. By this correspondence, we have

$$j(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad j(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $L_1, L_2 \in \mathrm{NS}(A)$ such that

$$j(L_1) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \qquad j(L_2) = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

the intersection number $L_1 \cdot L_2$ is given by

$$L_1 \cdot L_2 = \alpha_2\delta_1 + \alpha_1\delta_2 - \gamma_1\beta_2 - \gamma_2\beta_1.$$

In particular, for $L \in \mathrm{NS}(A)$ such that $j(L) = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ we have

$$L^2 = 2 \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$L \cdot E_1 = \alpha, \qquad L \cdot E_2 = \delta.$$

We have also $j(nD) = nj(D)$ for an integer n.

The first and the final statements of this theorem are given in Mumford [8]. In particular, the final statement follows easily from the definition of φ_L. To prove the others, we need some lemmas.

Lemma 2.2. *The restriction homomorphism*

$$\begin{aligned} \mathrm{Res} : \mathrm{Pic}^0(A) &\longrightarrow \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) \\ L &\mapsto (L|_{E_1}, L|_{E_2}) \end{aligned}$$

is an isomorphism, and the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi_X} & \mathrm{Pic}^0(A) & \ni & L \\ \| & & \downarrow \mathrm{Res} & & \downarrow \\ E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) & \ni & (L|_{E_1}, L|_{E_2}) \end{array}$$

Proof. The first statement is well-known (cf. Mumford [8]). For $x = (x_1, x_2) \in A$, we have

$$\begin{aligned} \mathrm{Res} \circ \varphi_X(x) &= \mathrm{Res}\big(T_x^* X - X\big) \\ &= \big(T_{x_1}^* O_{E_1} - O_{E_1}, T_{x_2}^* O_{E_2} - O_{E_2}\big) = (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})(x) \qquad \square \end{aligned}$$

We now examine the canonical involution of B. Since we have $B = \mathrm{End}(E) \otimes \mathbf{Q}$, it suffices to define it for the elements of $\mathrm{End}(E)$. Then, for $g \in \mathrm{End}(E)$, the canonical involution is given by

$$\bar{g} = \varphi_{O_E}^{-1} \circ g^* \circ \varphi_{O_E},$$

which is the Rosati involution of $\mathrm{End}(E) \otimes \mathbf{Q}$ (cf. Mumford [8], Section 21, and Tate [13], Section 4). For the elliptic curve E, we have

$$\bar{g} \circ g = \varphi_{O_E}^{-1} \circ g^* \circ \varphi_{O_E} \circ g = (\deg g)\mathrm{id}_E.$$

Lemma 2.3. *Under the Rosati involution the element* $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M_2(B)$ *maps to*

$$g' = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}$$

Proof. We denote by $\hat{g}$ the dual morphism of g. As the action on divisors, we have $\hat{g} = g^*$. The Rosati involution is given by $g' = \varphi_X^{-1} \circ \hat{g} \circ \varphi_X$. We calculate the right-hand side term explicitly. We have a commutative diagram

$$
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) \\
\downarrow \varphi_X & & || \\
\mathrm{Pic}^0(E_1 \times E_2) & \xrightarrow{\mathrm{Res}} & \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) \\
\uparrow \hat{g} & & \uparrow \mathrm{Res} \circ \hat{g} \circ \mathrm{Res}^{-1} \\
\mathrm{Pic}^0(E_1 \times E_2) & \xrightarrow{\mathrm{Res}} & \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2) \\
\uparrow \varphi_X & & || \\
E_1 \times E_2 & \xrightarrow{\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}} & \mathrm{Pic}^0(E_1) \times \mathrm{Pic}^0(E_2).
\end{array}
$$

Using this diagram, for the point $(x_1, x_2) \in E_1 \times E_2$ we have

$$
\begin{aligned}
g' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \varphi_X^{-1} \circ \hat{g} \circ \varphi_X \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \mathrm{Res} \circ \hat{g} \circ \mathrm{Res}^{-1} \circ (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \mathrm{Res} \circ \hat{g} \circ \mathrm{Res}^{-1} \begin{pmatrix} \varphi_{O_{E_1}}(x_1) \\ \varphi_{O_{E_2}}(x_2) \end{pmatrix} \\
&= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \mathrm{Res} \circ \hat{g}\big(p_1^* \varphi_{\mathcal{O}_1}(x_1) + p_2^* \varphi_{O_{E_2}}(x_2)\big) \\
&= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \mathrm{Res}\big((p_1 \circ g)^* \varphi_{O_{E_1}}(x_1) + (p_2 \circ g)^* \varphi_{O_{E_2}}(x_2)\big)
\end{aligned}
$$

We denote by m_i the addition of E_i $(i = 1, 2)$. Then, we have

$$
p_1 \circ g = m_1 \circ (\alpha \times \beta), \qquad p_2 \circ g = m_2 \circ (\gamma \times \delta).
$$

We denote by q_i $(i = 1, 2)$ the i-th projection $E_1 \times E_1 \to E_1$. Then by Mumford [8], for $L \in \mathrm{Pic}^0(E_1)$ we have

$$
m_1^* L \sim q_1^* L + q_2^* L \quad \text{(linearly equivalent)}.
$$

Therefore we have

$$
\begin{aligned}
(p_1 \circ g)^* \varphi_{O_{E_1}}(x_1) &= \big(m_1 \circ (\alpha \times \beta)\big)^* \varphi_{O_{E_1}}(x_1) \\
&= (\alpha \times \beta)^* m_1^* \varphi_{O_{E_1}}(x_1) \\
&= (\alpha \times \beta)^* \big(q_1^* \varphi_{O_{E_1}}(x_1) + q_2^* \varphi_{O_{E_1}}(x_1)\big) \\
&= \big\{q_1 \circ (\alpha \times \beta)\big\}^* \varphi_{O_{E_1}}(x_1) + \big\{q_2 \circ (\alpha \times \beta)\big\}^* \varphi_{O_{E_1}}(x_1).
\end{aligned}
$$

Since we have commutative diagrams

$$
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\alpha \times \beta} & E_1 \times E_1 \\
\downarrow p_1 & & \downarrow q_1 \\
E_1 & \xrightarrow{\alpha} & E_1,
\end{array}
\qquad
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{\alpha \times \beta} & E_1 \times E_1 \\
\downarrow p_2 & & \downarrow q_2 \\
E_1 & \xrightarrow{\beta} & E_1,
\end{array}
$$

we have

$$(p_1 \circ g)^* \varphi_{O_{E_1}(x_1)} = p_1^* \alpha^* \varphi_{O_{E_1}(x_1)} + p_2^* \beta^* \varphi_{O_{E_1}(x_1)}.$$

In a similar way, we have

$$(p_2 \circ g)^* \varphi_{O_{E_2}(x_2)} = p_1^* \gamma^* \varphi_{O_{E_2}(x_2)} + p_2^* \delta^* \varphi_{O_{E_2}(x_2)}.$$

Therefore, we have

$$\begin{aligned} g' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \operatorname{Res}\bigl(p_1^* \alpha^* \varphi_{O_{E_1}}(x_1) + p_2^* \beta^* \varphi_{O_{E_1}}(x_1) \\ &\quad + p_1^* \gamma^* \varphi_{O_{E_2}}(x_2) + p_2^* \delta^* \varphi_{O_{E_2}}(x_2)\bigr) \\ &= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \begin{pmatrix} \alpha^* \varphi_{O_{E_1}}(x_1) + \gamma^* \varphi_{O_{E_2}}(x_2) \\ \beta^* \varphi_{O_{E_1}}(x_1) + \delta^* \varphi_{O_{E_2}}(x_2) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_{O_{E_1}}^{-1} \alpha^* \varphi_{O_{E_1}}(x_1) + \varphi_{O_{E_1}}^{-1} \gamma^* \varphi_{O_{E_2}}(x_2) \\ \varphi_{O_{E_2}}^{-1} \beta^* \varphi_{O_{E_1}}(x_1) + \varphi_{O_{E_2}}^{-1} \delta^* \varphi_{O_{E_2}}(x_2) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_{O_{E_1}}^{-1} \alpha^* \varphi_{O_{E_1}} & \varphi_{O_{E_1}}^{-1} \gamma^* \varphi_{O_{E_2}} \\ \varphi_{O_{E_2}}^{-1} \beta^* \varphi_{O_{E_1}} & \varphi_{O_{E_2}}^{-1} \delta^* \varphi_{O_{E_2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

Since $E_1 = E_2 = E$ and $\varphi_{\mathcal{O}_1} = \varphi_{\mathcal{O}_2}$, we conclude

$$g' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}' = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} \qquad \square$$

Lemma 2.4. *For a divisor $L \in \operatorname{Pic}(E_1 \times E_2)$ with $j(L) = g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, we have*

$$\alpha = L \cdot E_1, \qquad \delta = L \cdot E_2.$$

Proof. Since α is an integer, we have

$$g \begin{pmatrix} x \\ O_{E_2} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ O_{E_2} \end{pmatrix} = \begin{pmatrix} \alpha x \\ \gamma(x) \end{pmatrix}. \tag{1}$$

Now, we examine αx.

$$\begin{aligned} g \begin{pmatrix} x \\ O_{E_2} \end{pmatrix} &= \varphi_X^{-1} \circ \varphi_L \begin{pmatrix} x \\ O_{E_2} \end{pmatrix} = \varphi_X^{-1} \{ T^*_{(x, \mathcal{O}_2)} L - L \} \\ &= (\varphi_{\mathcal{O}_1} \times \varphi_{\mathcal{O}_2})^{-1} \circ \operatorname{Res}\{ T^*_{(x, \mathcal{O}_2)} L - L \} \end{aligned}$$

We restrict the divisor L to E_1 and denote it by e. Then, the divisor is expressed as

$$e \sim \sum_{i=1}^{\lambda} n_i P_i$$

with integers n_i and points P_i on E_1 $(i = 1, 2, \cdots, \lambda)$. We have

$$L \cdot E_1 = \deg e = \sum_{i=1}^{\lambda} n_i.$$

We set $n = \sum_{i=1}^{\lambda} n_i$. Then, we obtain the following form:

$$g\begin{pmatrix} x \\ O_{E_2} \end{pmatrix} = (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \begin{pmatrix} T_x^* e - e \\ * \end{pmatrix}.$$

We denote by $\oplus$ the addition of E_1, and by $\ominus$ the subtraction of E_1. Then, we have

$$T_x^* e \sim \sum_{i=1}^{\lambda} n_i (P_i \ominus x).$$

By Abel's theorem, we see that

$$\begin{aligned} T_x^* e - e &\sim n_1(P_1 \ominus x) \oplus \cdots \oplus n_\lambda (P_\lambda \ominus x) \ominus (n_1 P_1 \oplus \cdots \oplus n_\lambda P_\lambda) - O_{E_1} \\ &\sim (-n)x - O_{E_1} = T_{nx}^* O_{E_1} - O_{E_1} = \varphi_{O_{E_1}}(nx). \end{aligned}$$

Therefore, we have $\varphi_{O_{E_1}}^{-1}((-nx) - O_{E_1}) = nx$, and

$$g\begin{pmatrix} x \\ O_{E_2} \end{pmatrix} = \begin{pmatrix} nx \\ * \end{pmatrix}. \tag{2}$$

Hence, comparing (1) and (2), we have $\alpha = n = L \cdot E_1$. In a similar way, we have $\delta = L \cdot E_2$. $\square$

Lemma 2.5. *We have* $j(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $j(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. For a point $(x_1, x_2) \in E_1 \times E_2$, we have

$$\begin{aligned} \varphi_X^{-1} \circ \varphi_{E_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \varphi_X^{-1}\{T_{(x_1,x_2)}^* E_1 - E_1\} \\ &= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \circ \operatorname{Res}\{T_{(x_1,x_2)}^* E_1 - E_1\} \\ &= (\varphi_{O_{E_1}} \times \varphi_{O_{E_2}})^{-1} \begin{pmatrix} O_{E_1} - O_{E_1} \\ (-x_2) - O_{E_2} \end{pmatrix} = \begin{pmatrix} O_{E_1} \\ x_2 \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$j(E_1) = \varphi_X^{-1} \circ \varphi_{E_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In a similar way, we obtain the second assertion. □

Lemma 2.6. *For $L \in \mathrm{NS}(E_1 \times E_2)$, we set $j(L) = g = \left(\begin{smallmatrix} \alpha & \bar{\gamma} \\ \gamma & \delta \end{smallmatrix}\right)$. Then,*

$$L^2 = 2 \det g.$$

Proof. Since $\alpha, \delta \in \mathbf{Z}$, we have

$$\begin{aligned} \varphi_X^{-1} \circ \varphi_{(L-\alpha E_2-\delta E_1)} &= \varphi_X^{-1} \circ \varphi_L - \alpha\varphi_X^{-1} \circ \varphi_{E_2} - \delta\varphi_X^{-1} \circ \varphi_{E_1} \\ &= \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \delta \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix} \\ &= \begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & 0 \end{pmatrix}. \end{aligned}$$

Since the right-hand side is contained in H, there exists a divisor Z such that

$$\varphi_X^{-1} \circ \varphi_Z = \begin{pmatrix} 0 & \bar{\gamma} \\ \gamma & 0 \end{pmatrix}.$$

If Z is zero, then we have $\gamma = 0$. Therefore, we have $Z = \alpha E_2 + \delta E_1$ and $Z^2 = 2\alpha\delta = \det g$.

Now, we assume $Z \neq 0$. Since φ_X is an isomorphism, by the Riemann–Roch theorem on the abelian surface A, we have

$$\deg(\varphi_X^{-1} \circ \varphi_Z) = \deg \varphi_Z = (Z^2/2)^2.$$

On the other hand,

$$\deg(\varphi_X^{-1} \circ \varphi_Z) = \deg \gamma \cdot \deg \bar{\gamma} = (\deg \gamma)^2 = (\gamma\bar{\gamma})^2$$

By Lemma 2.4, we have

$$Z \cdot E_1 = Z \cdot E_2 = 0.$$

Therefore, we have $Z \cdot (E_1 + E_2) = 0$. Since $(E_1 + E_2)^2 = 2 > 0$, by the Hodge index theorem we see $Z^2 < 0$. Therefore, we have, $Z^2/2 = -\gamma\bar{\gamma}$.

On the other hand, since φ_X is an isomorphism and $\varphi_X^{-1} \circ \varphi_{(L-\alpha E_2-\delta E_1-Z)} = 0$, we have $\varphi_{(L-\alpha E_2-\delta E_1-Z)} = 0$. Therefore, we have

$$0 \equiv L - \alpha E_2 - \delta E_1 - Z,$$

where by $\equiv$ we mean algebraic equivalence. Hence, we have

$$L^2 = 2\alpha\delta + Z^2 = 2(\alpha\delta - \gamma\bar{\gamma}) = 2\det g. \qquad \square$$

For an automorphism g of A, we can regard g as an element of $\mathrm{M}_2(\mathcal{O})$, and then we can consider ${}^t\bar{g}$.

Lemma 2.7. *Let L_1 and L_2 be two divisors with $j(L_1) = g_1$ and $j(L_2) = g_2$. Let g be an automorphism of A. Then, $g^*L_1 \equiv L_2$ if and only if ${}^t\bar{g}g_1g = g_2$.*

Proof. We have

$$\begin{aligned} g^*L_1 \equiv L_2 \quad &\Longleftrightarrow \quad \varphi_{g^*L_1} = \varphi_{L_2} \\ &\Longleftrightarrow \quad \hat{g} \circ \varphi_{L_1} \circ g = \varphi_{L_2} \\ &\Longleftrightarrow \quad \varphi_X^{-1} \circ \hat{g} \circ \varphi_X \circ \left(\varphi_X^{-1} \circ \varphi_{L_1}\right) \circ g = \varphi_X^{-1} \circ \varphi_{L_2} \\ &\Longleftrightarrow \quad g' \circ g_1 \circ g = g_2. \qquad \square \end{aligned}$$

Let $m : E \times E \to E$ be the addition of E, and we set

$$\Delta = \operatorname{Ker} m.$$

We have $\Delta = \{(P, -P) \mid P \in E\}$. Note that this Δ is different from the usual diagonal. For two endomorphisms $a_1, a_2 \in \mathrm{End}(E)$, we set

$$\Delta_{a_1,a_2} = (a_1 \times a_2)^*\Delta.$$

Using this notation, we have $\Delta = \Delta_{1,1}$. We have the following theorem (cf. [7]).

Theorem 2.8.

$$j(\Delta_{a_1,a_2}) = \begin{pmatrix} \bar{a}_1a_1 & \bar{a}_1a_2 \\ \bar{a}_2a_1 & \bar{a}_2a_2 \end{pmatrix}.$$

In particular, we have

$$j(\Delta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof. Let α, β, γ be elements of $\mathcal{O}$ such that

$$\varphi_X^{-1} \circ \varphi_\Delta = \begin{pmatrix} \alpha & \bar{\gamma} \\ \gamma & \delta \end{pmatrix}.$$

Then, since $E_1 \cdot \Delta = E_2 \cdot \Delta = 1$, we have $\alpha = \delta = 1$ by Lemma 2.4. Since we have

$$\varphi_X^{-1} \circ \varphi_\Delta \begin{pmatrix} x \\ -x \end{pmatrix} = \varphi_X^{-1}\{T^*_{(x,-x)}\Delta - \Delta\} = \varphi_X^{-1}(0) = \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix},$$

we have $\gamma(x) = x$ for any $x \in E$. Therefore, we have $\gamma = 1$.

By definition, we have

$$\Delta_{a_1,a_2} = (a_1 \times a_2)^* \Delta.$$

Therefore, we have

$$j(\Delta_{a_1,a_2}) = {}^t\overline{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} \bar{a}_1 a_1 & \bar{a}_1 a_2 \\ \bar{a}_2 a_1 & \bar{a}_2 a_2 \end{pmatrix} \qquad \square$$

3. Non-superspecial cases

In this section, we examine the injectivity of the Chern class map of abelian surfaces. Let α_p be the local–local group scheme of rank p (cf. Oort [10] for the definition and properties). Then, we have $\mathrm{End}(\alpha_p) \simeq k$, and for an abelian variety X, $\mathrm{Hom}(\alpha_p, X)$ is a right vector space over $\mathrm{End}(\alpha_p) \simeq k$ by composition of morphisms. The a-number of X is defined by

$$a = \dim_k \mathrm{Hom}(\alpha_p, X).$$

We denote by $[p]_X$ multiplication of p:

$$\begin{aligned} [p]_X : X &\longrightarrow X \\ x &\mapsto px. \end{aligned}$$

Then, the reduced part of $\mathrm{Ker}[p]_X$ is of the form:

$$\left(\mathrm{Ker}[p]_x\right)_{red} \simeq (\mathbf{Z}/p\mathbf{Z})^{\oplus r}$$

with an integer r ($0 \le r \le \dim X$). We call r the p-rank of X (cf. Mumford [8]). The following theorem follows essentially from the results in Ogus [9], but we give here a down-to-earth proof. For the definition and properties of the Cartier operator, see Cartier [2].

Theorem 3.1. *Let X be an abelian surface defined over k. Then, the Chern class map*

$$c_1 : \mathrm{NS}(X)/p\,\mathrm{NS}(X) \longrightarrow \mathrm{H}^1\bigl(X, \Omega_X^1\bigr)$$

is injective if and only if X is not superspecial.

Proof. The only-if-part will be proved in Theorem 4.4. We prove here the if-part. We denote by $r(X)$ the p-rank of X, and by $a(X)$ the a-number of X. By Oort [11], X is superspecial if and only if $a(X) = 2$. Therefore, we assume $a(X) \neq 2$. Take an affine open covering $\{U_i\}$ of X, and suppose that there is a divisor $D = \{f_{ij}\}$ which is not zero in $\mathrm{NS}(X)/p\,\mathrm{NS}(X)$, such that $c_1(D) = \{df_{ij}/f_{ij}\} \sim 0$ in $\mathrm{H}^1(X, \Omega_X^1)$. Then, there exists $\omega_i \in \mathrm{H}^0(U_i, \Omega_X^1)$ such that

$$df_{ij}/f_{ij} = \omega_j - \omega_i.$$

(i) The first case: $d\omega_i = 0$.

Applying the Cartier operator C, we obtain

$$df_{ij}/f_{ij} = C(\omega_j) - C(\omega_i).$$

Therefore, we have

$$C(\omega_j) - \omega_j = C(\omega_i) - \omega_i \quad \text{on } U_i \cap U_j$$

and we have a regular 1-form ω' on X which is defined by

$$C(\omega_i) - \omega_i \quad \text{on } U_i.$$

Since $C - \mathrm{id} : H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ is surjective, there exists a regular 1-form $\omega \in H^0(X, \Omega_X^1)$ such that $(C - \mathrm{id})(\omega) = \omega'$. Therefore, we have

$$C(\omega_i - \omega) = \omega_i - \omega.$$

By the property of the Cartier operator, there exists a regular function f_i on U_i such that

$$\omega_i - \omega = df_i/f_i,$$

and we have

$$df_{ij}/f_{ij} = df_j/f_j - df_i/f_i.$$

This means $d(f_{ij}f_i/f_j) = 0$. Therefore, there exists a regular function g_{ij} on $U_i \cap U_j$ such that

$$f_{ij}f_i/f_j = g_{ij}^p \quad \text{on } U_i \cap U_j.$$

Since $D = \{f_{ij}\}$ is a cocycle, we see that $\{g_{ij}\}$ is also a cocycle and that this gives an element of $\mathrm{NS}(X)$. Therefore, we conclude $D \in p\,\mathrm{NS}(X)$, which contradicts $D \neq 0$ in $\mathrm{NS}(X)/p\,\mathrm{NS}(X)$.

(ii) The second case: $d\omega_i \neq 0$.

In this case we have $d\omega_i = d\omega_j$ on $U_i \cap U_j$ and we get a non-zero regular 2-form on X. Since this regular 2-form is d-exact and is a basis of $\mathrm{H}^0(X, \Omega_X^2)$, the Cartier operator acts on $\mathrm{H}^0(X, \Omega_X^2)$ as the zero map. Therefore, X is not ordinary, that is, $r(X) \neq 2$. Therefore, we have either $r(X) = 1$ and $a(X) = 1$, or $r(X) = 0$ and $a(X) = 1$.

Now, we consider the absolute Frobenius $F : \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \mathcal{O}_X)$. Since $a(X) = 1$ in both cases, there exists a non-zero element $\beta = \{g_{ij}\}$ in $\mathrm{H}^1(X, \mathcal{O}_X)$ such that $F(\beta) = 0$. This means that there exists a regular function g_i on U_i such that $g_{ij}^p = g_j - g_i$. Since $dg_i = dg_j$ on $U_i \cap U_j$, we have a non-zero regular 1-form η on X given by dg_i on U_i. Since $\dim \mathrm{H}^0(X, \Omega_X^1) = 2$, in both cases there exists a nonzero regular 1-form η' such that $\{\eta, \eta'\}$ gives a basis of $\mathrm{H}^0(X, \Omega_X^1)$ with $C(\eta') \neq 0$. In fact, we can take η' with $C(\eta') = \eta$ if $r(X) = 0$ and $a(X) = 1$, and we can take η' with $C(\eta') = \eta'$ if $r(X) = 1$ and $a(X) = 1$. Since we have $\mathrm{H}^0(X, \Omega_X^2) = \bigwedge^2 \mathrm{H}^0(X, \Omega_X^1)$, $\eta \wedge \eta'$ gives a basis of $\mathrm{H}^0(X, \Omega_X^2)$. Therefore, there exists a non-zero element $a \in k$ such that

$$d\omega_i = a\eta \wedge \eta' = a\big(d\big(g_i\eta'\big)\big).$$

We set $\theta_i = \omega_i - ag_i\eta'$. Then, θ_i is d-closed and we have

$$\begin{aligned} df_{ij}/f_{ij} &= ag_j\eta' - ag_i\eta' + \theta_j - \theta_i \\ &= ag_{ij}^p\eta' + \theta_j - \theta_i \end{aligned}$$

Applying the Cartier operator, we have

$$df_{ij}/f_{ij} = a^{1/p}g_{ij}C\big(\eta'\big) + C(\theta_j) - C(\theta_i).$$

This means that

$$c_1(D) \sim a^{1/p}\beta \otimes C\big(\eta'\big) \in \mathrm{H}^1(X, \mathcal{O}_X) \otimes \mathrm{H}^0\big(X, \Omega_X^1\big) \cong \mathrm{H}^1\big(X, \Omega_X^1\big)$$

Since $\beta \neq 0$ in $\mathrm{H}^1(X, \mathcal{O}_X)$ and $C(\eta') \neq 0$ in $\mathrm{H}^0(X, \Omega_X^1)$, we see $\beta \otimes C(\eta') \neq 0$ in $\mathrm{H}^1(X, \Omega_X^1)$. A contradiction.

Hence, if $a(X) \neq 2$, we conclude that c_1 is injective. $\square$

4. Superspecial cases

Let k be an algebraically closed field of characteristic $p \geq 3$. For an elliptic curve E over k, we examine the action of endomorphisms of E on $\mathrm{H}^0(E, \Omega_E^1)$ and $\mathrm{H}^1(E, \mathcal{O}_E)$.

Lemma 4.1. *Let E be an elliptic curve and $\alpha \in \mathrm{End}(E)$. Assume α acts on $\mathrm{H}^1(E, \mathcal{O}_E)$ as multiplication by $\beta \in k$ ($\beta \neq 0$). Then, α acts on $\mathrm{H}^0(E, \Omega_E^1)$ as multiplication by $\deg \alpha/\beta$.*

Proof. Using the endomorphism $\alpha : E \to E$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{NS}(E)/p\,\mathrm{NS}(E) & \xrightarrow{\alpha^*} & \mathrm{NS}(E)/p\,\mathrm{NS}(E) \\ \downarrow c_1 & & \downarrow c_1 \\ \mathrm{H}^1(E, \Omega_E^1) & \xrightarrow{\alpha^*} & \mathrm{H}^1(E, \Omega_E^1) \\ \downarrow & & \downarrow \\ \mathrm{H}^1(E, \mathcal{O}_E) \otimes \mathrm{H}^0(E, \Omega_E^1) & \xrightarrow{\alpha^* \otimes \alpha^*} & \mathrm{H}^1(E, \mathcal{O}_E) \otimes \mathrm{H}^0(E, \Omega_E^1). \end{array}$$

Take a point $Q \in E$, and bases $\omega \in \mathrm{H}^0(E, \Omega_E^1)$, $\eta \in \mathrm{H}^1(E, \mathcal{O}_E)$. Then, we have $\alpha^*(Q) = (\deg \alpha)Q$, and $(\alpha^* \otimes \alpha^*)(\omega \otimes \eta) = (\beta\omega) \otimes \alpha^*\eta$. The result follows from the diagram. $\square$

For an integer n, we have an endomorphism $[n]_E : E \longrightarrow E$ given by $P \mapsto nP$ ($P \in E$).

Lemma 4.2. *The induced homomorphism*

$$[n]_E^* : \mathrm{H}^0\big(E, \Omega_E^1\big) \longrightarrow \mathrm{H}^0\big(E, \Omega_E^1\big)$$

is multiplication by n, i.e., $[n]_E^*\omega = n\omega$ for $\omega \in H^0(E, \Omega_E^1)$.

Proof. This follows from the fact that $[n]_*$ is given as multiplication by n on the tangent space at the origin (Mumford [8]). $\square$

Assume $p \neq 2$. Following the theory of Ibukiyama (cf. [5]) to construct a quaternion division algebra over $\mathbf{Q}$ with discriminant p, we take a prime number q such that $-q \equiv 5 \pmod 8$ and $(\frac{-q}{p}) = -1$, and take an integer a such that $a^2 \equiv -p \pmod q$. Here, $(\frac{-q}{p})$ is the Legendre symbol. Then, the quaternion division algebra B over $\mathbf{Q}$ with discriminant p and a maximal order $\mathcal{O}$ of B are given by

$$\begin{aligned} &B = \mathbf{Q} \oplus \mathbf{Q}F \oplus \mathbf{Q}\alpha \oplus \mathbf{Q}F\alpha \quad \text{with} \\ &F^2 = -p, \qquad \alpha^2 = -q, \qquad F\alpha = -\alpha F \\ &\mathcal{O} = \mathbf{Z} + \mathbf{Z}\left(\frac{1+\alpha}{2}\right) + \mathbf{Z}\left(\frac{F(1+\alpha)}{2}\right) + \mathbf{Z}\left(\frac{(a+F)\alpha}{q}\right). \end{aligned}$$

Then, we know that there exists a supersingular elliptic curve E over k with $\mathrm{End}(E) = \mathcal{O}$ and $\mathrm{End}^0(E) = B$ (cf. Deuring [3]).

We need the following well-known lemma.

Lemma 4.3. *For a non-singular complete algebraic curve X, the Chern class map*

$$c_1 : \mathrm{Pic}(X)/p\mathrm{Pic}(X) \hookrightarrow \mathrm{H}^1\big(X, \Omega_X^1\big)$$

is injective.

Proof. Let L be a class of $\mathrm{Pic}(X)/p\mathrm{Pic}(X)$. Then, we can lift this class to $\mathrm{Pic}(X)$. We take an open affine covering $\{U_i\}$ that trivializes the corresponding invertible sheaf, and let the invertible sheaf be given by $\{f_{ij}\}$ with a regular function f_{ij} on $U_i \cap U_j$. Then, we have $c_1(L) = \{df_{ij}/f_{ij}\}$.

Suppose $\{df_{ij}/f_{ij}\} \sim 0$. Then, there exists $\omega_i \in \Omega_X^1(U_i)$ such that

$$\frac{df_{ij}}{f_{ij}} = \omega_j - \omega_i.$$

Since X is one-dimensional, ω_i's are d-closed. By the Cartier operator C, we have

$$\frac{df_{ij}}{f_{ij}} = C(\omega_j) - C(\omega_i).$$

Therefore, we have

$$C(\omega_i) - \omega_i = C(\omega_j) - \omega_j.$$

Hence, $C(\omega_i) - \omega_i$ on U_i gives a global regular 1-form $\omega \in \mathrm{H}^0(X, \Omega_X^1)$. Since $C - \mathrm{id}_X$ is surjective on $\mathrm{H}^0(X, \Omega_X^1)$, there exists $\{\tilde{\omega}\}$ such that $(C - \mathrm{id}_X)(\tilde{\omega}) = \omega$. Replace ω_i by $\omega_i - \tilde{\omega}$, we may assume $C(\omega_i) = \omega_i$. Hence, there exists f_i such that $\omega_i = \frac{df_i}{f_i}$. The result follows from this fact (cf. the proof of Theorem 3.1). $\square$

We now compute the Chern class map explicitly for A, where $A = E_1 \times E_2$ with $E_1 = E_2 = E$, the supersingular elliptic curve. The cup product induces a natural isomorphism

$$\mathrm{H}^1\big(A, \Omega_A^1\big) \cong \mathrm{H}^1(A, \mathcal{O}_A) \otimes \mathrm{H}^0\big(A, \Omega_A^1\big)$$

with

$$\mathrm{H}^1(A, \mathcal{O}_A) \cong \mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \oplus \mathrm{H}^1(E_2, \mathcal{O}_{E_2}),$$
$$\mathrm{H}^0\big(A, \Omega_A^1\big) \cong \mathrm{H}^0\big(E_1, \Omega_{E_1}^1\big) \oplus \mathrm{H}^0\big(E_2, \Omega_{E_2}^1\big).$$

Therefore, we have a decomposition

$$\begin{aligned}\mathrm{H}^1\big(A, \Omega_A^1\big) \cong{} & \big(\mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \otimes \mathrm{H}^0(E_1, \mathcal{O}_{E_1})\big) \oplus \big(\mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \otimes \mathrm{H}^0\big(E_2, \Omega_{E_2}^1\big)\big) \\ & \oplus \big(\mathrm{H}^1(E_2, \mathcal{O}_{E_2}) \otimes \mathrm{H}^0\big(E_1, \Omega_{E_1}^1\big)\big) \oplus \big(\mathrm{H}^1(E_2, \mathcal{O}_{E_2}) \otimes \mathrm{H}^0\big(E_2, \Omega_{E_2}^1\big)\big). \qquad (*)\end{aligned}$$

We have projections

$$pr_i : A \longrightarrow E_i \quad (i = 1, 2).$$

Then, we have injective homomorphisms

$$pr_i^* : \mathrm{H}^1\big(E_i, \Omega^1_{E_i}\big) \hookrightarrow \mathrm{H}^1\big(A, \Omega^1_A\big).$$

Note that

$$\begin{aligned}\mathrm{H}^1\big(E_1, \Omega^1_{E_1}\big) &\cong \mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \otimes \mathrm{H}^0\big(E_1, \Omega^1_{E_1}\big)\\ \mathrm{H}^1\big(E_2, \Omega^1_{E_2}\big) &\cong \mathrm{H}^1(E_2, \mathcal{O}_{E_2}) \otimes \mathrm{H}^0\big(E_2, \Omega^1_{E_2}\big),\end{aligned}$$

and we have the following commutative diagram

$$\begin{array}{ccc}\mathrm{NS}(A)/p\,\mathrm{NS}(A) & \stackrel{c_1}{\hookrightarrow} & \mathrm{H}^1(A, \Omega^1_A)\\ \uparrow & & pr_i^* \uparrow\\ \mathrm{Pic}(E_i)/p\mathrm{Pic}(E_i) & \stackrel{c_1}{\hookrightarrow} & \mathrm{H}^1(E_i, \Omega^1_{E_i})\end{array} \qquad (**)$$

The image of the homomorphism pr_i^* is a one-dimensional subspace $\mathrm{H}^1(E_i, \mathcal{O}_{E_i}) \otimes \mathrm{H}^0(E_i, \Omega^1_{E_i})$ $(i = 1, 2)$ in $\mathrm{H}^1(A, \Omega^1_A)$.

Now, we consider the Chern class map

$$\mathrm{NS}(A)/p\,\mathrm{NS}(A) \cong \mathrm{Pic}(A)/p\mathrm{Pic}(A) \xrightarrow{c_1} \mathrm{H}^1\big(A, \Omega^1_A\big).$$

For the divisors E_2 (resp. E_1) on A, we set $\Omega_1 = c_1(E_2)$ (resp. $\Omega_4 = c_1(E_1)$). Then, by the diagram (**) Ω_1 (resp. Ω_4) is a basis of $\mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \otimes \mathrm{H}^0(E_1, \mathcal{O}_{E_1})$ (resp. $\mathrm{H}^1(E_2, \mathcal{O}_{E_2}) \otimes \mathrm{H}^0(E_2, \Omega^1_{E_2})$).

We set

$$\Delta_a = \Delta_{\mathrm{id},a}.$$

Here, id is the identity endomorphism of E. Then we have

$$j(\Delta_a) = \begin{pmatrix} 1 & \bar{a}\\ a & \bar{a}a\end{pmatrix}$$

Since $\{\mathrm{id}, \frac{1+\alpha}{2}, F\frac{1+\alpha}{2}, \frac{(a+F)\alpha}{q}\}$ is a basis of $\mathcal{O} = \mathrm{End}(E)$, we see that

$$E_1, E_2, \Delta = \Delta_{\mathrm{id}}, \Delta_{\frac{1+\alpha}{2}}, \Delta_{F\frac{1+\alpha}{2}}, \Delta_{\frac{(a+F)\alpha}{q}}$$

is a basis of $\mathrm{NS}(A)$. Since $\alpha^2 = -q$, we see that α acts on $\mathrm{H}^0(E, \Omega^1_E)$ as multiplication by $\pm\sqrt{-q}$. We can choose α such that the action α on $\mathrm{H}^0(E, \Omega^1_E)$ is multiplication by $\sqrt{-q}$. F acts on $\mathrm{H}^0(E, \Omega_E^1)$ as the zero-map. Therefore, $\frac{1+\alpha}{2}, F\frac{1+\alpha}{2}$ and $\frac{(a+F)\alpha}{q}$ act on $\mathrm{H}^0(E, \Omega_E^1)$ respectively as multiplication by

$$\frac{1+\sqrt{-q}}{2}, \quad 0, \quad \frac{a\sqrt{-q}}{q}.$$

Since $\mathrm{H}^1(E, \mathcal{O}_E)$ is dual to $\mathrm{H}^0(E, \Omega_E^1)$, by Lemma 4.1 the actions of $\frac{1+\alpha}{2}, F\frac{1+\alpha}{2}$ and $\frac{(a+F)\alpha}{q}$ on $\mathrm{H}^1(E, \mathcal{O}_E)$ are respectively given as multiplication by

$$\frac{1-\sqrt{-q}}{2}, \quad 0, \quad -\frac{a\sqrt{-q}}{q}.$$

Therefore, on the decomposition $(*)$ of the space $\mathrm{H}^1(A, \Omega_A^1)$ the endomorphisms $\mathrm{id} \times \frac{1+\alpha}{2}$, $\mathrm{id} \times F\frac{1+\alpha}{2}$, $\mathrm{id} \times \frac{(a+F)\alpha}{q}$ of A act respectively as multiplication by

$$\left(1, \frac{1+\sqrt{-q}}{2}, \frac{1-\sqrt{-q}}{2}, \frac{1+q}{4}\right), \quad (1,0,0,0), \quad \left(1, \frac{\sqrt{-q}}{q}, -\frac{\sqrt{-q}}{q}, \frac{a^2}{q}\right)$$

on each direct summand.

We consider the automorphism τ of A defined by

$$\begin{aligned} \tau : A = E_1 \times E_2 &\longrightarrow A = E_1 \times E_2 \\ (P_1, P_2) &\mapsto (P_2, P_1). \end{aligned}$$

We denote by Ω_2 a basis of $\mathrm{H}^1(E_1, \mathcal{O}_{E_1}) \otimes \mathrm{H}^0(E_2, \Omega_{E_2}^1)$. We set $\Omega_3 = \tau^* \Omega_2$. Then, Ω_3 is a basis of $\mathrm{H}^0(E_1, \Omega_{E_1}^1) \otimes \mathrm{H}^1(E_2, \mathcal{O}_{E_2})$, and there exist coefficients $\alpha_i \in k$ $(i = 1, 2, 3, 4)$ such that

$$c_1(\Delta) = c_1(\Delta_{\mathrm{id}}) = \alpha_1 \Omega_1 + \alpha_2 \Omega_2 + \alpha_3 \Omega_3 + \alpha_4 \Omega_4$$

We consider inclusions

$$\begin{aligned} \epsilon_1 : E_1 &\longrightarrow E_1 \times E_2 = A \\ P &\mapsto (P, O_{E_2}) \end{aligned} \qquad \begin{aligned} \epsilon_2 : E_2 &\longrightarrow E_1 \times E_2 = A \\ P &\mapsto (O_{E_1}, P) \end{aligned}$$

Then, we have the following diagram induced by ϵ_i.

$$\begin{array}{ccc} \mathrm{Pic}(E_i)/p\mathrm{Pic}(E_i) & \xrightarrow{c_1} & \mathrm{H}^1(E_i, \Omega_{E_i}^1) \\ \uparrow & & \uparrow \\ \mathrm{NS}(A)/p\,\mathrm{NS}(A) & \xrightarrow{c_1} & \mathrm{H}^1(A, \Omega_A^1). \end{array}$$

Using this diagram, by $\Delta \cdot E_1 = 1$ and $\Delta \cdot E_2 = 1$ we see $\alpha_1 = \alpha_4 = 1$. Since $\tau^* \Delta = \Delta$, we also have $\alpha_2 = \alpha_3$, which we denote by α.

We show now $\alpha \neq 0$. We consider the natural inclusion $\phi : \Delta \hookrightarrow A = E_1 \times E_2$ and the diagram

$$\begin{array}{ccc} \mathrm{Pic}(\Delta)/p\mathrm{Pic}(\Delta) & \xrightarrow{c_1} & \mathrm{H}^1(\Delta, \Omega^1_\Delta) \\ \uparrow & & \uparrow \\ \mathrm{NS}(A)/p\,\mathrm{NS}(A) & \xrightarrow{c_1} & \mathrm{H}^1(A, \Omega^1_A). \end{array}$$

Since $\Delta^2 = 0$, we have $\phi^* c_1(\Delta) = 0$. On the other hand, since $\Delta \cdot E_1 = \Delta \cdot E_2 = 1$, we have $\phi^* c_1(E_1) = \phi^* c_1(E_2) \neq 0$. Therefore, we see $\alpha \neq 0$. Replacing Ω_2 by $\alpha\Omega_2$, we may assume $\alpha = 1$.

Summarizing these results, we have

$$\begin{aligned} & c_1(E_1) = \Omega_4, \qquad c_1(E_2) = \Omega_1, \\ & c_1(\Delta) = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4, \\ & c_1(\Delta_{\frac{1+\alpha}{2}}) = \Omega_1 + \frac{1+\sqrt{-q}}{2}\Omega_2 + \frac{1-\sqrt{-q}}{2}\Omega_3 + \frac{1+q}{4}\Omega_4, \\ & c_1(\Delta_{F\frac{1+\alpha}{2}}) = \Omega_1, \\ & c_1(\Delta_{\frac{(a+F)\alpha}{q}}) = \Omega_1 + \frac{a\sqrt{-q}}{q}\Omega_2 - \frac{a\sqrt{-q}}{q}\Omega_3 + \frac{a^2}{q}\Omega_4. \end{aligned}$$

Since $2q$ is prime to p, there exists an integer ℓ such that $\ell \equiv \frac{a}{2q} \pmod{p}$. Keeping these notations, we have the following theorem.

Theorem 4.4. *The kernel* $\mathrm{Ker}\, c_1$ *is* 2*-dimensional over* $\mathbf{F}_p$*, and a basis of* $\mathrm{Ker}\, c_1$ *is given by divisors*

$$\Delta_{F\frac{1+\alpha}{2}} - E_2, \qquad \Delta_{\frac{2+F\alpha}{q}} - \ell\Delta_{\frac{1+\alpha}{2}} + 2\ell\Delta - (\ell+1)E_2 - (1-q+2a)\ell E_1.$$

Proof. With respect to the basis $\langle \Omega_1, \Omega_2, \Omega_3, \Omega_4 \rangle$, the Chern classes $c_1(E_1)$, $c_1(E_2)$, $c_1(\Delta)$, $c_1(\Delta_{\frac{1+\alpha}{2}})$, $c_1(\Delta_{F\frac{1+\alpha}{2}})$, $c_1(\Delta_{\frac{(a+F)\alpha}{q}})$ are respectively represented as the following vectors:

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{u}_4 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{-q}}{2} \\ \frac{1-\sqrt{-q}}{2} \\ \frac{1+q}{4} \end{pmatrix}, \qquad \mathbf{u}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_6 = \begin{pmatrix} 1 \\ \frac{a\sqrt{-q}}{q} \\ \frac{-a\sqrt{-q}}{q} \\ \frac{a^2}{q} \end{pmatrix}.$$

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent over $\mathbf{F}_p$ and we have

$$\begin{aligned}
&\mathbf{u}_5 = \mathbf{u}_2, \\
&\mathbf{u}_6 = \frac{2a}{q}\mathbf{u}_4 - \frac{a}{q}\mathbf{u}_3 + \left(\frac{a}{2q} + 1\right)\mathbf{u}_2 + \left(\frac{a}{2q} - \frac{a}{2} + \frac{a^2}{q}\right)\mathbf{u}_1,
\end{aligned}$$

we see $\dim_{\mathbf{F}_p} \operatorname{Im} c_1 = 4$. Since $\dim_{\mathbf{F}_p} \mathrm{NS}(A)/p\,\mathrm{NS}(A) = 6$, we have $\dim_{\mathbf{F}_p} \operatorname{Ker} c_1 = 2$. Since $\{E_1, E_2, \Delta, \Delta_{\frac{1+\alpha}{2}}, \Delta_{F\frac{1+\alpha}{2}}, \Delta_{\frac{(a+F)\alpha}{2}}\}$ is a basis of $\mathrm{NS}(A)/p\,\mathrm{NS}(A)$, the latter part follows from our construction. □

Using this theorem, we have the following known corollary (cf. van der Geer and Katsura [4], for instance).

Corollary 4.5. *Let A be a superspecial abelian surface. Then, $\mathrm{H}^1(A, \Omega^1_A)$ is generated by algebraic cycles.*

Proof. This follows from the fact that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are linearly independent also over k. □

5. Example

We give here one concrete example. Assume characteristic $p = 3$. Then, there exists only one supersingular elliptic curve up to isomorphism and it is given by

$$E : y^2 = x^3 - x$$

We consider two automorphisms defined by

$$\begin{aligned}
\sigma &: x \mapsto x + 1, \qquad y \mapsto y, \\
\tau &: x \mapsto -x, \qquad y \mapsto \sqrt{-1}y
\end{aligned}$$

We have a morphism defined by

$$\begin{aligned}
\pi : E &\longrightarrow \mathbf{P}^1 \\
(x, y) &\mapsto x
\end{aligned}$$

By the result of Ibukiyama [5], we have

$$\mathrm{End}(E) = Z + Z\tau + Z\iota \circ \tau + Z\tau \circ \iota \circ \sigma.$$

Here, ι is the involution of E.

Let P be the point on $\mathbf{P}^1$ given by the local equation $x = 0$, and $\tilde{P}$ a point on E such that $\pi(\tilde{P}) = P$. We consider an affine open covering $\{U_0, U_1\}$ of $\mathbf{P}^1$ which is given by

$$U_0 = \{x \in \mathbf{P}^1 \mid x \neq \infty\}, \qquad U_1 = \{x \in \mathbf{P}^1 \mid x \neq 0\}.$$

The divisor P is given by the functions

$$x \quad \text{on } U_0, \qquad 1 \quad \text{on } U_1.$$

Under the notation, we have the following diagram.

$$\begin{array}{ccccc} 2\tilde{P} & \in \mathrm{Pic}(E)/3\mathrm{Pic}(E) & \xhookrightarrow{c_1} & \mathrm{H}^1(E, \Omega^1_E) & \cong & k \\ \uparrow & \uparrow & & \uparrow & & \\ P & \in \mathrm{Pic}(\mathbf{P}^1)/3\mathrm{Pic}(\mathbf{P}^1) & \xhookrightarrow{c_1} & \mathrm{H}^1(\mathbf{P}^1, \Omega^1_{\mathbf{P}^1}) & \cong & k \end{array}$$

In this diagram, we have $c_1(P) = \{\frac{dx}{x}\}$, and $c_1(\tilde{P}) = \{\frac{dx}{2x}\}$.

We set $A = E_1 \times E_2$ with $E_1 = E_2 = E$. We consider the Chern class map

$$c_1 : \mathrm{NS}(A)/3\mathrm{NS}(A) \longrightarrow \mathrm{H}^1(A, \Omega^1_A) \cong \mathrm{H}^1(A, \mathcal{O}_A) \otimes \mathrm{H}^0(A, \Omega^1_A)$$

We also consider the natural inclusion defined by

$$\begin{aligned} \varphi : E &\longrightarrow E_1 \times E_2 = A \\ P &\mapsto (P, O_{E_2}) \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{NS}(A)/3\mathrm{NS}(A) & \xrightarrow{\varphi^*} & \mathrm{NS}(E)/3\mathrm{NS}(E) \\ \downarrow c_1 & & \downarrow c_1 \\ \mathrm{H}^1(A, \Omega^1_A) & \xrightarrow{\varphi^*} & \mathrm{H}^1(E, \Omega^1_E) \end{array}$$

Then, we have

$$\varphi^*\bigl(c_1(\Delta)\bigr) = c_1\bigl(\varphi^*(\Delta)\bigr) = c_1(O_E) = \left\{\frac{dx}{2x}\right\} \neq 0.$$

We determine the action of $\mathrm{End}(E)$ on $\mathrm{H}^0(E, \Omega^1_E)$. A basis of $\mathrm{H}^0(E, \Omega^1_E)$ is given by $\frac{dx}{y}$ and we have

$$(\iota \circ \sigma)^* \frac{dx}{y} = -\frac{dx}{y}, \qquad \tau^* \frac{dx}{y} = -\sqrt{-1}\frac{dx}{y}, \qquad (\tau \circ \iota \circ \sigma)^* \frac{dx}{y} = \sqrt{-1}\frac{dx}{y}.$$

Since $\mathrm{H}^1(E, \mathcal{O}_E)$ is dual to $\mathrm{H}^0(E, \Omega^1_E)$, the actions of $\iota \circ \sigma$, τ and $\tau \circ \iota$ are respectively given as multiplication by

$$-1, \quad \sqrt{-1}, \quad -\sqrt{-1}$$

by Lemma 4.1. Since we have

$$\begin{aligned}\mathrm{H}^1\big(A,\Omega_A^1\big) &\cong \mathrm{H}^1(A,\mathcal{O}_A)\otimes \mathrm{H}^0\big(A,\Omega_A^1\big)\\ &\cong \big(\mathrm{H}^1(E_1,\mathcal{O}_{E_1})\otimes \mathrm{H}^0(E_1,\mathcal{O}_{E_1})\big)\oplus\big(\mathrm{H}^1(E_1,\mathcal{O}_{E_1})\otimes \mathrm{H}^0\big(E_2,\Omega_{E_2}^1\big)\big)\\ &\quad\oplus\big(\mathrm{H}^1(E_2,\mathcal{O}_{E_2})\otimes \mathrm{H}^0\big(E_1,\Omega_{E_1}^1\big)\big)\oplus\big(\mathrm{H}^1(E_2,\mathcal{O}_{E_2})\otimes \mathrm{H}^0\big(E_2,\Omega_{E_2}^1\big)\big),\end{aligned}$$

the actions $\mathrm{id}\times\iota\circ\sigma$, $\mathrm{id}\times\tau$ and $\mathrm{id}\times\tau\circ\iota\circ\sigma$ are respectively given as multiplication on each summand by

$$\begin{gathered}(1,-1,-1,1)\\ (1,\sqrt{-1},-\sqrt{-1},1)\\ (1,-\sqrt{-1},\sqrt{-1},1).\end{gathered}$$

By our general theory,

$$E_1, E_2, \Delta, \Delta_{\iota\circ\sigma}, \Delta_\tau, \Delta_{\tau\circ\iota\circ\sigma}$$

gives a basis of $\mathrm{NS}(A)$ over $\mathbf{Z}$. Therefore, $\Delta+\Delta_{\iota\circ\sigma}+E_1+E_2$ and $\Delta_\tau+\Delta_{\tau\circ\iota\circ\sigma}+E_1+E_2$ are linearly independent divisors in $\mathrm{NS}(A)/3\,\mathrm{NS}(A)$ over $\mathbf{F}_3$. Moreover, considering the actions of the endomorphisms $\mathrm{id}\times\iota\circ\sigma$, $\mathrm{id}\times\tau$ and $\mathrm{id}\times\tau\circ\iota\circ\sigma$ on $\mathrm{H}^1(A,\Omega_A^1)$ and the commutative diagram

$$\begin{array}{ccc}\mathrm{NS}(A) & \xrightarrow{f^*} & \mathrm{NS}(A)\\ \downarrow c_1 & & \downarrow c_1\\ \mathrm{H}^1(A,\Omega_A^1) & \xrightarrow{f^*} & \mathrm{H}^1(A,\Omega_A^1)\end{array}$$

with $f\in\mathrm{End}(A)$, we conclude that the Chern classes of these two divisors are zero. Therefore, we see that

$$\Delta+\Delta_{\iota\circ\sigma}+E_1+E_2, \Delta_\tau+\Delta_{\tau\circ\iota\circ\sigma}+E_1+E_2$$

gives a basis of $\mathrm{Ker}\,c_1$ over $\mathbf{F}_3$.

6. An application to Kummer surfaces

Let A be an abelian surface defined over k, and ι be the involution $x\mapsto\ominus x$. We denote by $\tilde{A}$ the surface made of 16 blowing-ups at 16 two-torsion points on A. Then, ι induces the action $\tilde{\iota}$ on $\tilde{A}$ and $\mathrm{Km}(A)=\tilde{A}/\tilde{\iota}$ is the Kummer surface. We denote by $\pi:\tilde{A}\to\mathrm{Km}(A)$ the projection. A K3 surface X is called supersingular if the Picard number $\rho(X)$ is equal to the second Betti number $b_2(X)=22$. For a supersingular K3 surface, the discriminant of $\mathrm{NS}(X)$ is equal to the form $-p^{2\sigma_0}$ and σ_0 is called an Artin invariant. We know $1 \leq \sigma_0 \leq 10$ (cf. Artin [1]). A supersingular K3 surface with Artin invariant 1 is said to be superspecial. Such a K3 surface is unique up to isomorphism and is isomorphic to the Kummer surface $\mathrm{Km}(A)$ such that A is superspecial (cf. Ogus [9] and Shioda [12]). We also know that a supersingular K3 surface with $\sigma_0 = 2$ is isomorphic to a Kummer surface $\mathrm{Km}(A)$ such that A is supersingular and non-superspecial (cf. Ogus [9]).

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{NS}(\mathrm{Km}(A))/p\,\mathrm{NS}(\mathrm{Km}(A)) & \xrightarrow{c_1} & \mathrm{H}^1(\mathrm{Km}(A), \Omega^1_{\mathrm{Km}(A)}) \\
\downarrow & & \downarrow \\
\mathrm{NS}(\tilde{A})/p\,\mathrm{NS}(\tilde{A}) & \xrightarrow{c_1} & \mathrm{H}^1(\tilde{A}, \Omega^1_{\tilde{A}}).
\end{array}$$

Since we have $2\mathrm{NS}(\tilde{A}) \subset \pi^*\mathrm{NS}(\mathrm{Km}(A)) \subset \mathrm{NS}(\tilde{A})$ by Shioda [12] and $p \neq 2$, we see $\mathrm{NS}(\mathrm{Km}(A))/p\,\mathrm{NS}(\mathrm{Km}(A)) \cong \mathrm{NS}(\tilde{A})/p\,\mathrm{NS}(\tilde{A})$. Since ι acts on $\mathrm{H}^1(A, \mathcal{O}_A)$ and $\mathrm{H}^0(A, \Omega^1_A)$ as multiplication by -1. Since $\mathrm{H}^1(A, \Omega^1_A) \cong \mathrm{H}^1(A, \mathcal{O}_A) \otimes \mathrm{H}^0(A, \Omega^1_A)$, we see that ι acts as identity on $\mathrm{H}^1(A, \Omega^1_A)$. Therefore, $\tilde{\iota}$ acts as identity on $\mathrm{H}^1(\tilde{A}, \Omega^1_{\tilde{A}})$. Hence, we have $\mathrm{H}^1(\mathrm{Km}(A), \Omega^1_{\mathrm{Km}(A)}) \cong \mathrm{H}^1(\tilde{A}, \Omega^1_{\tilde{A}})$. Summarizing these results, by Theorems 3.1 and 4.4 we have the following theorem.

Theorem 6.1. *For a Kummer surface* $\mathrm{Km}(A)$*, let* c_1 *be the Chern class map*

$$c_1 : \mathrm{NS}\big(\mathrm{Km}(A)\big)/p\,\mathrm{NS}\big(\mathrm{Km}(A)\big) \longrightarrow \mathrm{H}^1\big(\mathrm{Km}(A), \Omega^1_{\mathrm{Km}(A)}\big).$$

Then, we have the following.

(i) *If* $\mathrm{Km}(A)$ *is not superspecial, then* c_1 *is injective.*
(ii) *If* $\mathrm{Km}(A)$ *is superspecial, then* $\dim_{\mathbf{F}_p} \mathrm{Ker}\, c_1 = 2$.

Remark 6.2. For a supersingular K3 surface X, it is known that the homomorphism

$$\mathrm{NS}(X)/p\,\mathrm{NS}(X) \otimes_{\mathbf{F}_p} k \longrightarrow \mathrm{H}^1\big(X, \Omega^1_X\big)$$

induced by c_1 is not injective (cf. Ogus [9]). In particular, if $\mathrm{Km}(A)$ is supersingular,

$$\mathrm{NS}\big(\mathrm{Km}(A)\big)/p\,\mathrm{NS}\big(\mathrm{Km}(A)\big) \otimes_{\mathbf{F}_p} k \longrightarrow \mathrm{H}^1\big(\mathrm{Km}(A), \Omega^1_{\mathrm{Km}(A)}\big)$$

is not injective even if $\mathrm{Km}(A)$ is not superspecial.

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