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## Idempotent plethories



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## ABSTRACT

Let  $k$  be a commutative ring with identity. A  $k$ -plethory is a commutative  $k$ -algebra  $P$  together with a comonad structure  $\mathbb{W}_P$ , called the  $P$ -Witt ring functor, on the covariant functor that it represents. We say that a  $k$ -plethory  $P$  is *idempotent* if the comonad  $\mathbb{W}_P$  is idempotent, or equivalently if the map from the trivial  $k$ -plethory  $k[e]$  to  $P$  is a  $k$ -plethory epimorphism. We prove several results on idempotent plethories. We also study the  $k$ -plethories contained in  $K[e]$ , where  $K$  is the total quotient ring of  $k$ , which are necessarily idempotent and contained in  $\text{Int}(k) = \{f \in K[e] : f(k) \subseteq k\}$ . For example, for any ring  $l$  between  $k$  and  $K$  we find necessary and sufficient conditions—all of which hold if  $k$  is a integral domain of Krull type—so that the ring  $\text{Int}_l(k) = \text{Int}(k) \cap l[e]$  has the structure, necessarily unique and idempotent, of a  $k$ -plethory with unit given by the inclusion  $k[e] \rightarrow \text{Int}_l(k)$ . Our results, when applied to the binomial plethory  $\text{Int}(\mathbb{Z})$ , specialize to known results on binomial rings.

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## 1. Introduction

In this paper all rings and algebras, unless otherwise stated, are assumed commutative with identity. We denote the category of sets by **Sets** and the category of abelian groups by **Ab**. For any ring  $k$  we let  $k\text{-Mod}$  and  $k\text{-Alg}$  denote the category of  $k$ -modules and the category of  $k$ -algebras, respectively, and for any  $k$ -module  $M$  we denote the  $n$ -th tensor power of  $M$  over  $k$  by  $M^{\otimes_k n}$ , or  $M^{\otimes n}$  if the ring  $k$  is understood.

Let  $k$  be a ring. A  $k$ -plethory is a  $k$ -algebra  $P$  together with a comonad structure  $\mathbb{W}_P$ , called the  $P$ -Witt ring functor, on the covariant functor  $\text{Hom}_{k\text{-Alg}}(P, -)$  that it represents [5]. A  $k$ -plethory is also known as a  $k$ - $k$ -biring monoid (or monad object), a  $k$ - $k$ -biring triple, and a Tall–Wraith monoid (or monad object) in  $k\text{-Alg}$  [3,40]. Trivially, the polynomial ring  $k[X]$  has the structure of a  $k$ -plethory, denoted  $k[e]$  and called the trivial  $k$ -plethory, which is an initial object in the category of  $k$ -plethories.

Motivated by our previous efforts [23] to use the theory of plethories to generalize our results in [20] on binomial rings, we say that a  $k$ -plethory  $P$  is *idempotent* if the comonad  $\mathbb{W}_P$  is idempotent, in the sense of [2], [4, Definition 4.1.1], [18,33]; that is,  $P$  is idempotent if the natural transformation  $\mathbb{W}_P \rightarrow \mathbb{W}_P \circ \mathbb{W}_P$  is an isomorphism, or, equivalently, if the composition map  $P \odot P \rightarrow P$  is an isomorphism. The idempotent  $k$ -plethories are the plethystic analogue of the  $k$ -epimorphisms, which are the  $k$ -algebras  $A$  such that the map  $k \rightarrow A$  is an epimorphism of rings, or equivalently such that the multiplication map  $A \otimes_k A \rightarrow A$  is an isomorphism [38, Theorem 1]. (The  $\mathbb{Z}$ -epimorphisms were classified in [8] and again in [9], and the classification was later generalized in [19] to Dedekind domains.) Not surprisingly, an analogous equivalence holds for plethories: a  $k$ -plethory  $P$  is idempotent if and only if the map  $k[e] \rightarrow P$  from the trivial  $k$ -plethory to  $P$  is an epimorphism of  $k$ -plethories.

This paper represents a first step towards a classification of the idempotent  $k$ -plethories, or more generally the  $k$ -plethory epimorphisms. This problem is embedded in two larger problems: first, to generalize, when possible, results in commutative algebra and algebraic geometry to plethystic algebra, and, second, to classify all  $k$ -plethories, which recently has been solved for fields  $k$  of characteristic zero [13]—all such plethories are linear—and which could be within reach for  $k = \mathbb{Z}$ . Among our results are several equivalent characterizations of the idempotent plethories, namely, Theorems 2.9, 4.3, 6.4, and 6.7 and Propositions 5.2 and 6.6. In Section 2 we provide an overview of the paper, along with motivation for the theory from the standpoint of binomial rings and integer-valued polynomial rings, and in Section 3 we summarize the relevant definitions and theorems from the theory of plethories as presented in [5] by Borger and Wieland. Sections 4 and 5 focus on general results that have analogues for the  $k$ -epimorphisms, and Section 6 is concerned with questions of existence and uniqueness of idempotent plethory structures. Sections 7 and 8 are devoted to the study of  $k$ -plethories contained in  $K[e]$ , where  $K$  is the total quotient ring of  $k$ , which are all necessarily idempotent and contained in  $\text{Int}(k) = \{f \in K[e] : f(k) \subseteq k\}$ . There we provide, for example, some exotic examples of  $k$ -plethories for any Krull domain  $k$ , including not only  $\text{Int}(k)$  but also the

ring  $\text{Int}^{(\infty)}(k)$  of all polynomials in  $K[X]$  all of whose derivatives are integer-valued. Section 8 deals specifically with the well-studied integer-valued polynomial rings  $\text{Int}(k)$ , for integral domains  $k$ , of [10,21,35–37].

Special cases of Theorems 2.4 and 8.9 and Propositions 7.7, 7.8, 8.2, 8.4, and 8.7, along with Problem 2.2, were announced without proofs by the author in [23].

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## 2. Motivation and overview

A ring  $A$  is said to be *binomial* if  $A$  is  $\mathbb{Z}$ -torsion-free and  $a(a-1)(a-2)\cdots(a-n+1)/n! \in A \otimes_{\mathbb{Z}} \mathbb{Q}$  lies in  $A$  for all  $a \in A$  and all positive integers  $n$ . By [20, Theorem 9.1], a binomial ring is equivalently a  $\lambda$ -ring  $A$  whose Adams operations are all the identity on  $A$ . For any ring  $A$ , let  $\Lambda(A)$  denote the universal  $\lambda$ -ring over  $A$ . (As an abelian group the ring  $\Lambda(A)$  is the group  $1 + TA[[T]]$ , and, in another guise, the ring  $\Lambda(A)$  is the ring  $W(A)$  of big Witt vectors over  $A$ .) Let  $\text{Bin}(A)$  for any ring  $A$  denote the subring of  $\Lambda(A)$  of all elements that are fixed by all of the Adams operations on  $\Lambda(A)$ . (See any of [7,32,42] for the relevant definitions.) The motivating problem of this paper is to generalize the following theorem.

**Theorem 2.1** ([20, Theorem 9.1], [3, Section 46]). *The association  $A \mapsto \text{Bin}(A)$  defines a functor from the category of rings to the category of binomial rings that is a right adjoint to the inclusion from binomial rings to rings and is represented by the ring  $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ .*

By [5, 2.10–11], [40], the functor  $\Lambda \cong W$  is isomorphic to the  $P$ -Witt functor  $\mathbb{W}_P$  of a  $\mathbb{Z}$ -plethory structure  $P$  on the ring of symmetric functions over  $\mathbb{Z}$  in countably many variables. In fact, the theory of plethories generalizes the theory of  $\lambda$ -rings. It also provides an alternative construction of the functor  $\text{Bin}$  as the  $P$ -Witt functor  $\mathbb{W}_P$  of the *binomial plethory*  $P = \text{Int}(\mathbb{Z})$  [5, 2.14].

This approach to constructing  $\text{Bin}$  points to a generalization of Theorem 2.1 to other plethories, in particular, to plethory structures on various rings of polynomials, including the *integer-valued polynomial rings* of [10,35–37], which have been studied exclusively for integral domains but can be generalized to arbitrary rings as follows. Let  $k$  be a ring with total quotient ring  $K$ . The ring of *integer-valued* (or  *$k$ -valued*) *polynomials on  $k$*  is the subring

$$\text{Int}(k) = \{f \in K[X] : f(k) \subseteq k\}$$

of the polynomial ring  $K[X]$ . More generally, for any set  $\mathbf{X}$  and any subset  $\mathbf{E}$  of  $K^{\mathbf{X}}$ , the ring of *integer-valued polynomials on  $\mathbf{E}$*  is the subring

$$\text{Int}(\mathbf{E}, k) = \{f(\mathbf{X}) \in K[\mathbf{X}] : f(\mathbf{E}) \subseteq k\}$$

of the polynomial ring  $K[\mathbf{X}]$ . One writes  $\text{Int}(k^{\mathbf{X}}) = \text{Int}(k^{\mathbf{X}}, k)$ . One also writes  $\text{Int}(k^n) = \text{Int}(k^{\mathbf{X}})$  if  $\mathbf{X}$  is a set of cardinality  $n$ .

By [20, Proposition 6.4], for any set  $\mathbf{X}$  the ring  $\text{Int}(\mathbb{Z}^{\mathbf{X}}) \cong \bigotimes_{X \in \mathbf{X}} \text{Int}(\mathbb{Z})$  is the free binomial ring generated by  $\mathbf{X}$ , and therefore a  $\mathbb{Z}$ -torsion-free ring  $A$  is binomial if and only if for every  $a \in A$  there exists a ring homomorphism  $\text{Int}(\mathbb{Z}) \rightarrow A$  sending  $X$  to  $a$ . (In Section 8 we will show that a ring  $A$  is binomial if and only if for every  $a \in A$  there exists a unique ring homomorphism  $\text{Int}(\mathbb{Z}) \rightarrow A$  sending  $X$  to  $a$ .) To generalize Theorem 2.1 to rings other than  $k = \mathbb{Z}$  we need an appropriate  $k$ -algebra analogue of the binomial rings, which should form a full subcategory  $\mathbf{C}$  of the category of  $k$ -algebras. From this perspective the problem of generalizing Theorem 2.1 translates more precisely to the following.

**Problem 2.2** ([23]). Characterize all pairs  $k, \mathbf{C}$ , where  $k$  is a ring and  $\mathbf{C}$  is a full subcategory of  $k\text{-Alg}$ , such that  $\text{Int}(k)$  represents a right adjoint to the inclusion from  $\mathbf{C}$  to  $k\text{-Alg}$ .

To motivate the following slight modification of the problem, note that  $\text{Int}(\mathbb{Z})$  is a binomial ring and therefore the map  $\text{Bin}(\text{Int}(\mathbb{Z})) \rightarrow \text{Int}(\mathbb{Z})$  is an isomorphism.

**Problem 2.3.** Characterize all pairs  $k, \mathbf{C}$ , where  $k$  is a ring and  $\mathbf{C}$  is a full subcategory of  $k\text{-Alg}$ , such that  $\text{Int}(k)$  represents a right adjoint  $F$  to the inclusion from  $\mathbf{C}$  to  $k\text{-Alg}$  for which the map  $F(\text{Int}(k)) \rightarrow \text{Int}(k)$  is an isomorphism.

Theorem 2.10, stated at the end of this section, provides a solution to Problem 2.3.

There is a clear connection between Problems 2.2 and 2.3 and the theory of plethories. Let  $k$  be any ring. A  $k$ - $k$ -biring is a  $k$ -algebra  $R$  together with a lift of the functor  $\text{Hom}_{k\text{-Alg}}(R, -)$  from  $k\text{-Alg}$  to **Sets** to a functor  $\mathbb{W}_R$  from  $k\text{-Alg}$  to  $k\text{-Alg}$ . Thus, if  $k, \mathbf{C}$  is a pair satisfying the condition in Problem 2.2, then the functor  $\text{Hom}_{k\text{-Alg}}(\text{Int}(k), -)$  from  $k\text{-Alg}$  to **Sets** lifts to a functor from  $k\text{-Alg}$  to  $\mathbf{C}$ , whence  $\text{Int}(k)$  has the structure of a  $k$ - $k$ -biring. (This *a priori* places restrictions on candidates for  $k$  and  $\mathbf{C}$ .) Moreover, as explained in [5] and in Section 3, a  $k$ -plethory is equivalently a monoid object in the monoidal category, equipped with the *composition product*  $\odot$ , of  $k$ - $k$ -birings; that is, it is a  $k$ - $k$ -biring  $P$  together with a homomorphism  $P \odot P \rightarrow P$  of  $k$ - $k$ -birings, called *composition*, that is associative and possesses a unit  $k[X] \rightarrow P$ . (The functor  $P \odot -$  is a left adjoint to the comonad  $\mathbb{W}_P$  and is therefore a monad on  $k\text{-Alg}$ .) Since  $\text{Int}(k)$  is closed under the operation of composition of polynomials, any  $k$ - $k$ -biring structure on  $\text{Int}(k)$  containing  $k[X]$  as a sub- $k$ - $k$ -biring is unique and extends to a unique  $k$ -plethory structure on  $\text{Int}(k)$ .

It turns out that there are very large classes of rings  $k$  for which  $\text{Int}(k)$  has the structure of a  $k$ -plethory, including, for example, all Krull domains and more generally all domains of Krull type. An integral domain  $D$  is said to be of *Krull type* [29] if  $D$  is a

locally finite intersection of essential valuation overrings, that is, if  $D = \bigcap_{\mathfrak{p} \in \mathcal{P}} D_{\mathfrak{p}}$ , where  $\mathcal{P} \subseteq \text{Spec } D$ , each  $D_{\mathfrak{p}}$  is a valuation domain, and the intersection is *locally finite*, that is, every nonzero element of  $D$  belongs to only finitely many  $\mathfrak{p} \in \mathcal{P}$ . This is the same as the definition of a Krull domain except that the localizations are assumed to be valuation domains rather than DVRs. Just as with Krull domains, the set  $\mathcal{P}$  may be taken to be canonical, namely, as the set  $t\text{-Max}(D) \subseteq \text{Spec } D$  of all  $t$ -maximal ideals of  $D$ , which for a Krull domain are precisely the height one primes. An ideal is *t-maximal* if it is maximal among the proper  $t$ -closed ideals of  $D$ , where  $t$  is the well-studied *t-closure (star) operation*  $t : I \mapsto I^t = \bigcup \{J^v : J \subseteq I \text{ is finitely generated}\}$  on the partially ordered set of ideals  $I$  of  $D$ , where  $v : I \mapsto I^v = (I^{-1})^{-1}$  is the *divisorial closure (star) operation*. In particular, a domain  $D$  is of Krull type if and only if  $D$  is a PVMD (that is,  $D_{\mathfrak{p}}$  is a valuation domain for every  $t$ -maximal ideal  $\mathfrak{p}$  of  $D$ ) [28] and  $D$  is of *finite t-character* (that is, every nonzero element of  $D$  lies in only finitely many  $t$ -maximal ideals of  $D$ , or equivalently the intersection  $D = \bigcap_{\mathfrak{p} \in t\text{-Max}(D)} D_{\mathfrak{p}}$ , which holds generally, is locally finite) [30]. A Krull domain is equivalently a PVMD, or domain of Krull type, that satisfies the ascending chain condition on  $t$ -closed ideals. In fact, more generally any TV PVMD (that is, any PVMD such that  $I^t = I^v$  for all ideals  $I$ ) [30] is a domain of Krull type, and any  $n$ -dimensional discrete valuation domain is a TV PVMD but is a Krull domain if and only if  $n \leq 1$ .

**Theorem 2.4.** *Let  $k$  be a ring. Each of the following conditions implies the next.*

1.  $k$  is a Krull domain.
2.  $k$  is a TV PVMD.
3.  $k$  is a domain of Krull type.
4.  $k$  is a PVMD and  $\text{Int}(k_{\mathfrak{p}}) = \text{Int}(k)_{\mathfrak{p}}$  for every maximal ideal  $\mathfrak{p}$  of  $k$ .
5.  $\text{Int}(k)_{\mathfrak{p}}$  is equal to  $\text{Int}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
6. For every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}(k)^{\otimes n} \rightarrow \text{Int}(k^n)$  is an isomorphism.
7. The canonical  $k$ -algebra homomorphism  $\text{Int}(k)^{\otimes n} \rightarrow \text{Int}(k^n)$  is an isomorphism for  $n = 2$  and an inclusion for  $n = 3$ .
8.  $\text{Int}(k)$  has the structure, necessarily unique, of a  $k$ - $k$ -biring such that the inclusion  $k[X] \rightarrow \text{Int}(k)$  is a homomorphism of  $k$ - $k$ -birings.
9.  $\text{Int}(k)$  has the structure, necessarily unique, of a  $k$ -plethory with unit given by the inclusion  $k[X] \rightarrow \text{Int}(k)$ . Moreover, composition  $\text{Int}(k) \odot \text{Int}(k) \rightarrow \text{Int}(k)$  is an isomorphism and acts by ordinary composition of polynomials on elements of the form  $f \odot g$ .

In particular, if  $D$  is a domain of Krull type, or more generally a PVMD such that  $\text{Int}(D_{\mathfrak{p}}) = \text{Int}(D)_{\mathfrak{p}}$  for all maximal ideals  $\mathfrak{p}$  of  $D$ , then  $\text{Int}(D)$  has a canonical  $D$ -plethory structure. This lends a new dimension to the study (as in [10–12, 17, 21, 22, 24, 25, 27, 41]) of integer-valued polynomial rings over Dedekind domains, almost Dedekind domains, Krull

domains, domains of Krull type, and PVMDs. [Proposition 8.7](#), for instance, provides in the case where  $D$  is a Dedekind domain a plethystic interpretation of Theorems V.2.10 and V.3.1 of [10], which for certain domains  $D$  provide a correspondence between the completion  $\widehat{D}_{\mathfrak{p}}$  and the set of prime ideals of  $\text{Int}(D)$  lying above  $\mathfrak{p}$ , for any maximal ideal  $\mathfrak{p}$  of  $D$ .

Our efforts to prove and generalize [Theorem 2.4](#) (see [Theorem 7.11](#) for a generalization) motivated our study of the *idempotent plethories*, which are singled out by the equivalent conditions of the following proposition (proved in [Section 4](#)).

**Proposition 2.5.** *Let  $k$  be a ring and  $P$  a  $k$ -plethory. The following conditions are equivalent.*

1. *The natural transformation  $\mathbb{W}_P \rightarrow \mathbb{W}_P \circ \mathbb{W}_P$  is an isomorphism.*
2. *The natural transformation  $P \odot (P \odot -) \rightarrow (P \odot -)$  is an isomorphism.*
3. *The  $k$ -algebra homomorphism  $P \rightarrow \mathbb{W}_P(P)$  is an isomorphism.*
4. *The  $k$ -algebra homomorphism  $P \odot P \rightarrow P$  is an isomorphism.*

Thus, a  $k$ -plethory  $P$  is idempotent if and only if the comonad  $\mathbb{W}_P$  is idempotent, if and only if the monad  $P \odot -$  is idempotent, both in the sense of [2], [4, [Definition 4.1.1](#)], [18,33]. Indeed, these are restatements of conditions (1) and (2), respectively, of the proposition. Conditions (3) and (4) are equivalent to conditions (1) and (2), respectively, essentially by the fact that  $P$  represents the comonad  $\mathbb{W}_P$ .

Trivially, the trivial  $k$ -plethory  $k[e]$  is idempotent. We also prove in [Section 4](#) that a  $k$ -plethory  $P$  is idempotent if and only if the unit  $k[e] \rightarrow P$  is a  $k$ -plethory epimorphism.

By [Theorem 2.4](#), if  $\text{Int}(k)$  has the structure of a  $k$ -plethory with unit given by the inclusion  $k[X] \rightarrow \text{Int}(k)$ , then the  $k$ -plethory  $\text{Int}(k)$  is idempotent. More generally, we have the following.

**Proposition 2.6.** *Let  $k$  be a ring with total quotient ring  $K$  and let  $P$  be any  $k$ -plethory contained in  $K[e]$ . Then  $P$  is closed under composition of polynomials in  $K[e]$ , and the  $k$ -plethory composition in  $P$  coincides with composition of polynomials in  $K[e]$ . Moreover,  $P$  is a  $k[e]$ -subalgebra of  $\text{Int}(k)$  and  $P$  is idempotent.*

In particular, if  $\text{Int}(k)$  has a canonical  $k$ -plethory structure, then in fact it is the largest  $k$ -plethory contained in  $K[e]$ . This motivates the following problems.

**Problem 2.7.** Let  $k$  be a ring with total quotient ring  $K$ .

1. Classify the idempotent  $k$ -plethories.
2. Classify the  $k$ -plethories contained in  $K[e]$ .
3. For which  $k$  does  $\text{Int}(k)$  have the structure of a  $k$ -plethory?
4. For which  $k$  is  $\text{Int}(k)$  the largest idempotent  $k$ -plethory?

5. For which  $k$  does there exist a largest idempotent  $k$ -plethory (or equivalently an epimorphic hull of  $k[e]$  in the category of  $k$ -plethories), and how can one construct it?
6. For which  $k$  is every idempotent  $k$ -plethory isomorphic to a  $k[e]$ -subalgebra of  $K[e]$ ?

Regarding [Problem 2.7](#)(4–6) above we make the following conjecture.

**Conjecture 2.8.** *Let  $D$  be a Dedekind domain of characteristic zero with quotient field  $K$ . Then every idempotent  $D$ -plethory is contained in  $K[e]$ , or, equivalently,  $\text{Int}(D)$  is the largest idempotent  $D$ -plethory and is therefore the epimorphic hull of the trivial  $D$ -plethory.*

Sections 7 and 8 reveal further connections between idempotent plethories and integer-valued polynomial rings. The moral is that both theories motivate each other. For example, we show in Section 7 that, for any Krull domain  $D$  with quotient field  $K$  and any domain  $D'$  with  $D \subseteq D' \subseteq K$ , the  $D[X]$ -algebras

$$\text{Int}(D) \cap D'[X] \supseteq \text{Int}^{(\infty)}(D) \cap D'[X] \supseteq \text{Int}^{[\infty]}(D) \cap D'[X]$$

are all  $D$ -plethories, where  $\text{Int}^{(\infty)}(D)$  denotes the ring of all polynomials  $f$  in  $K[X]$  such that  $f$  and all of its derivatives lie in  $\text{Int}(D)$ , and where  $\text{Int}^{[\infty]}(D)$  is the ring of all polynomials  $f$  in  $K[X]$  whose finite differences  $\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}f$  of all orders  $n$ , for all  $h_1, \dots, h_n \in D$ , lie in  $\text{Int}(D)$  [[10, Chapter IX](#)]. It is known, for example, that  $\text{Int}^{(\infty)}(\mathbb{Z}) = \text{Int}^{[\infty]}(\mathbb{Z})$  is free as a  $\mathbb{Z}$ -module with  $\mathbb{Z}$ -basis  $c_0, c_1X, c_2\binom{X}{2}, c_3\binom{X}{3}, \dots$ , where  $c_n = \prod_{p \leq n} p^{\lfloor n/p \rfloor}$  for all  $n$ . In particular,  $\text{Int}^{(\infty)}(\mathbb{Z})$  is not of the form  $\text{Int}(\mathbb{Z}) \cap D'[X]$  for any subring  $D'$  of  $\mathbb{Q}$ . Furthermore, one has

$$\text{Int}(\mathbb{Z}[i]) \supsetneq \text{Int}^{(\infty)}(\mathbb{Z}[i]) \supsetneq \text{Int}^{[\infty]}(\mathbb{Z}[i]) \supsetneq \mathbb{Z}[i][X],$$

so likewise these define  $\mathbb{Z}[i]$ -plethories whose study requires nontrivial results from the theory of integer-valued polynomials.

If  $\eta : k[X] \rightarrow R$  is a  $k[X]$ -algebra, then we say that a  $k$ -algebra  $A$  is  $\eta$ -reflective, or  $R$ -reflective if the  $k[X]$ -algebra structure on  $R$  is understood, if for every  $a \in A$  there is a unique  $k$ -algebra homomorphism  $R \rightarrow A$  sending  $\eta(X)$  to  $a$ , or equivalently if every  $k$ -algebra homomorphism  $k[X] \rightarrow A$  factors uniquely through  $\eta$ . For example,  $R$  itself is  $\eta$ -reflective if and only if  $\eta$  is a *reflection map* in  $k\text{-Alg}$ , in the sense of [[9, p. 199](#)], for example, and in [Corollary 8.10](#) we show that a  $\mathbb{Z}$ -algebra is  $\text{Int}(\mathbb{Z})$ -reflective if and only if it is a binomial ring. We denote by  $\eta\text{-Refl}$ , or  $R\text{-Refl}$ , the category of  $R$ -reflective  $k$ -algebras, full in  $k\text{-Alg}$ . If  $P = R$  is a  $k$ -plethory, then we say that  $A$  is  $P$ -reflective if  $A$  is  $\eta$ -reflective, where  $\eta$  is the unit  $k[e] \rightarrow P$ , or equivalently if the  $k$ -algebra homomorphism  $\mathbb{W}_P(A) \rightarrow A$  is an isomorphism. Thus  $P$  is idempotent if and only if  $P$  is  $P$ -reflective.

In Section 6 we show that the forgetful functor from the category of idempotent  $k$ -plethories to  $k[X]\text{-Alg}$  is an isomorphism onto its image. Thus an idempotent  $k$ -plethory structure can be thought of as a property of the underlying  $k[X]$ -algebra rather than as a structure in and of itself. Moreover, if  $P$  is idempotent, then the forgetful functor from the category of  $P$ -rings—which are the (Eilenberg–Moore) algebras of the monad  $P \odot -$  and are studied in Section 5—to the category  $k\text{-Alg}$  is an isomorphism onto  $P\text{-Refl}$ , so likewise a  $P$ -ring may be considered a property of the underlying  $k$ -algebra. This fact allows us to define left and right adjoints to the inclusion from  $P\text{-Refl}$  to  $k\text{-Alg}$ . (For  $k = \mathbb{Z}$  and  $P = \text{Int}(\mathbb{Z})$ , the right adjoint to this inclusion is precisely the functor  $\text{Bin}$ , and the left adjoint is the functor  $\text{Bin}^U$  of [20, Theorem 7.1].) Moreover, it allows us to uniquely characterize any idempotent plethory  $P$  via its category  $P\text{-Refl}$ , and vice versa, using the plethory reconstruction theorem of [5, Introduction], as in Theorem 2.9 below.

A category is said to be *complete* (resp., *cocomplete*, *bicomplete*) if it has all limits (resp., all colimits, all limits and colimits). For any  $k$ -plethory  $P$ , the category  $P\text{-Rings}$  of  $P$ -rings is bicomplete, and the forgetful functor from  $P\text{-Rings}$  to  $k\text{-Alg}$  preserves all limits and colimits [5, 1.10]. (Thus, for example, the tensor product over  $k$  of a collection of  $P$ -rings is a  $P$ -ring.) Moreover, the forgetful function from  $P\text{-Rings}$  to  $k\text{-Alg}$  is an isomorphism onto  $P\text{-Refl}$ . It follows that  $P\text{-Refl}$  is also bicomplete with all limits and colimits computed as they are in  $k\text{-Alg}$ .

A subcategory  $\mathcal{C}$  of a category  $\mathcal{D}$  is said to be *reflective* (resp., *coreflective*, *bireflective*) if the inclusion from  $\mathcal{C}$  to  $\mathcal{D}$  has a left adjoint (resp., a right adjoint, both left and right adjoints). For example, the category of binomial rings is bicomplete and bireflective in  $\mathbb{Z}\text{-Alg}$  [20, Sections 5, 7, and 9], and if the category  $\mathcal{C}$  is as in Problem 2.2, then  $\mathcal{C}$  is a coreflective subcategory of  $D\text{-Alg}$ .

**Theorem 2.9.** *A category  $\mathcal{C}$  is a full, bicomplete, and bireflective subcategory of  $k\text{-Alg}$  if and only if  $\mathcal{C} = P\text{-Refl}$  for a (necessarily unique and idempotent)  $k$ -plethory  $P$ .*

If  $\mathcal{C}$  is a subcategory of  $k\text{-Alg}$ , then we denote by  $\overline{\mathcal{C}}$  the *isomorphic closure* of  $\mathcal{C}$  in  $k\text{-Alg}$ , that is, the full subcategory of  $k\text{-Alg}$  whose objects are the objects of  $k\text{-Alg}$  that are isomorphic to some object in  $\mathcal{C}$ . Our results on idempotent plethories, particularly Theorem 6.7, lead to the following solution to Problem 2.3.

**Theorem 2.10.** *Let  $k$  be a ring. The following conditions are equivalent.*

1.  $\text{Int}(k)$  has the structure, necessarily unique and idempotent, of a  $k$ -plethory with unit given by the inclusion  $k[X] \rightarrow \text{Int}(k)$ .
2.  $\text{Int}(k)$  has the structure, necessarily unique, of a  $k$ - $k$ -biring such that the inclusion  $k[X] \rightarrow \text{Int}(k)$  is a homomorphism of  $k$ - $k$ -birings.



3. There exists a full subcategory  $\mathcal{C}$  of  $k\text{-Alg}$  such that  $\text{Int}(k)$  represents a right adjoint  $F_{\mathcal{C}}$  to the inclusion  $I_{\mathcal{C}}$  from  $\mathcal{C}$  to  $k\text{-Alg}$  for which the map  $F_{\mathcal{C}}(\text{Int}(k)) \rightarrow \text{Int}(k)$  is an isomorphism.
4. There exists a full subcategory  $k\text{-alg}$  of  $k\text{-Alg}$  such that  $\text{Int}(k)$  represents a right adjoint  $F_{\mathcal{D}}$  to the inclusion  $I_{\mathcal{D}}$  from  $\mathcal{D}$  to  $k\text{-Alg}$  for which the counit  $I_{\mathcal{D}} \circ F_{\mathcal{D}} \rightarrow \text{id}_{k\text{-Alg}}$  acts by evaluation at  $X \in \text{Int}(k)$ .
5.  $\text{Int}(k)$  represents an endofunctor  $F$  of  $k\text{-Alg}$  such that evaluation at  $X \in \text{Int}(k)$  defines a natural transformation from  $F$  to  $\text{id}_{k\text{-Alg}}$ .
6. There is an idempotent  $k$ -plethory structure on  $\text{Int}(k)$ .
7. The category of  $\text{Int}(k)$ -reflective  $k$ -algebras is a full, bicomplete, and bireflective subcategory of  $k\text{-Alg}$ .
8. The  $k$ -algebra  $\text{Int}(k)^{\otimes n}$  is  $\text{Int}(k)$ -reflective for all positive integers  $n$ .
9. The  $k$ -algebra  $\text{Int}(k)^{\otimes n}$  is  $\text{Int}(k)$ -reflective for  $n = 2, 3$ .

Suppose that the above conditions hold. Then  $\overline{\mathcal{C}} = \overline{\mathcal{D}} = \text{Int}(k)\text{-Refl}$  and  $I_{\mathcal{C}} \circ F_{\mathcal{C}} \cong F = \mathbb{W}_{\text{Int}(k)} = I_{\mathcal{D}} \circ F_{\mathcal{D}}$ . In particular,  $\text{Int}(k)\text{-Refl}$  is the largest subcategory  $\mathcal{C}$  of  $k\text{-Alg}$  satisfying (3) or  $\mathcal{D}$  of  $k\text{-Alg}$  satisfying (4). Moreover, there is a unique  $k$ -algebra automorphism  $\iota$  of  $\text{Int}(k)$  such that the correspondence  $- \circ \iota : I_{\mathcal{C}} \circ F_{\mathcal{C}} \rightarrow I_{\mathcal{D}} \circ F_{\mathcal{D}}$  is a natural isomorphism, and one has  $\iota(X) = uX + b$  for some  $u \in k^*$  and  $b \in k$ .

**Theorem 2.4** provides large classes of domains  $k$  for which  $\text{Int}(k)$  is a  $k$ -plethory (that is, for which the equivalent conditions of the above theorem hold). Moreover, in Section 7 we show that  $\text{Int}(k)$  is not a  $k$ -plethory if  $k = \mathbb{Z}[\varepsilon] = \mathbb{Z}[T]/(T^2)$  is the ring of dual numbers over  $\mathbb{Z}$ , yet  $\text{Int}(\overline{k})$  is a  $\overline{k}$ -plethory, where  $\overline{k} = \mathbb{Z} + \varepsilon\mathbb{Q}[\varepsilon]$  (which is a non-Noetherian ring in which every finitely generated or regular ideal is principal) is the integral closure of  $\mathbb{Z}[\varepsilon]$  in its total quotient ring  $\mathbb{Q}[\varepsilon]$ . Evidently certain questions remain unanswered, namely, [Problems 2.7, 8.5, 8.6, 8.12](#), and, most crucially, the following.

### Problem 2.11.

1. Does there exist a ring  $k$  such that  $\text{Int}(k)$  is not a  $k$ -plethory (that is, such that the equivalent conditions of [Theorem 2.10](#) do not hold) and such that  $k$  is also (a) an integral domain? (b) an integrally closed ring? (c) an integrally closed domain?
2. Is every idempotent  $\mathbb{Z}$ -plethory contained in  $\text{Int}(\mathbb{Z})$ ?
3. Classify the idempotent  $\mathbb{Z}$ -plethories.

## 3. Plethories

In this section we recall some basic definitions from the theory of plethories. The reader familiar with [\[5\]](#) may skip to Section 4. We assume familiarity with the language of monads (or triples), comonads (or cotriples), and the Eilenberg–Moore category of algebras over a monad, and coalgebras over a comonad, as found, for example, in [\[1,3,4\]](#).

Let  $\mathbb{T} = (T, \varepsilon, \mu)$  be a monad on a category  $\mathcal{C}$ , so in particular  $T : \mathcal{C} \rightarrow \mathcal{C}$  is a functor and  $\varepsilon : \text{id}_{\mathcal{C}} \rightarrow T$  and  $\mu : T \circ T \rightarrow T$  are natural transformations. (We often blur the distinction between a monad  $\mathbb{T}$  and the functor  $T$ .) We denote by  $\mathcal{C}^{\mathbb{T}}$  the Eilenberg–Moore category of algebras over the monad  $\mathbb{T}$ . One says that the monad  $\mathbb{T}$  is *idempotent* if it satisfies the equivalent conditions of the following proposition.

**Proposition 3.1** ([4, Proposition 4.2.3]). *Let  $\mathbb{T} = (T, \varepsilon, \mu)$  be a monad on a category  $\mathcal{C}$ . The following conditions are equivalent.*

1. *The multiplication  $\mu : T \circ T \rightarrow T$  of the monad  $\mathbb{T}$  is an isomorphism.*
2. *The forgetful functor  $\mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$  is full and faithful.*
3. *For every algebra  $(X, \xi)$  over the monad  $\mathbb{T}$ , the morphism  $\xi : T(X) \rightarrow X$  in  $\mathcal{C}$  is an isomorphism.*

One also says that a comonad is *idempotent* if it satisfies the equivalent conditions of the dual statement of the above proposition for comonads.

A (commutative unital) ring is equivalently an abelian group  $A$  together with a cocommutative comonad structure on the covariant functor  $\text{Hom}_{\mathbf{Ab}}(A, -)$  that it represents, or equivalently a commutative monad structure on its left adjoint,  $A \otimes -$ . Let  $k$  be a ring. A  $k$ -module is equivalently a coalgebra over the comonad  $\text{Hom}_{\mathbf{Ab}}(k, -)$ , or equivalently an algebra over the monad  $k \otimes -$ . A (commutative unital)  $k$ -algebra is equivalently a  $k$ -module  $M$  together with a cocommutative comonad structure on the covariant functor  $\text{Hom}_{k\text{-Mod}}(M, -)$  that it represents, or equivalently a commutative monad structure on its left adjoint,  $M \otimes_k -$ . Carrying these definitions one step further, one defines a  $k$ -plethory to be a  $k$ -algebra  $P$  together with a comonad structure on the covariant functor  $\text{Hom}_{k\text{-Alg}}(P, -)$  that it represents.

Note that  $\text{Hom}_{\mathbf{Ab}}(A, -)$  and  $\text{Hom}_{k\text{-Mod}}(M, -)$ , for  $A \in \mathbf{Ab}$  and  $M \in k\text{-Mod}$ , respectively, are at least endofunctors of  $\mathbf{Ab}$  and  $k\text{-Mod}$ , as the respective hom sets are enriched with natural abelian group and  $k$ -module structures, both linear, in this sense. By contrast, however,  $\text{Hom}_{k\text{-Alg}}(P, -)$  need not carry with it a natural  $k$ -algebra structure for  $P \in k\text{-Alg}$ . In this sense the  $k$ -plethories are a non-linear analogue of the  $k$ -algebras.

Also note that an endofunctor of the categories  $\mathbf{Ab}$ ,  $k\text{-Mod}$ , and  $k\text{-Alg}$  is representable if and only if it has a left adjoint. Thus, for example, we may define a  $k$ -plethory to be a representable comonad on  $k\text{-Alg}$ , or equivalently a comonad on  $k\text{-Alg}$  that possesses a left adjoint, which by adjunction is a monad. Equivalently still, a  $k$ -plethory is a monad–comonad left–right adjoint pair on  $k\text{-Alg}$ . Under these modified definitions, a  $k$ -plethory is determined only up to unique isomorphism.

The categorical definitions of  $k$ -plethories above can be made more concrete, as follows [5]. Let  $k$  and  $k'$  be rings. A  $k$ - $k'$ -biring is a  $k$ -algebra  $R$  together with a lift of the functor  $\text{Hom}_{k\text{-Alg}}(R, -)$  from  $k\text{-Alg}$  to  $\mathbf{Sets}$  to a functor  $\mathbb{W}_R$ , called the  $R$ -Witt ring functor, from  $k\text{-Alg}$  to  $k'\text{-Alg}$ . A  $k$ - $k'$ -biring is equivalently a  $k$ -algebra  $R$  together with a structure on  $R$  of a  $k'$ -algebra object in the opposite category of  $k\text{-Alg}$ . In other words, a  $k$ - $k'$ -biring

is a  $k$ -algebra  $R$  equipped with two *binary co-operations*  $\Delta^+, \Delta^\times : R \rightarrow R \otimes_k R$ , called *coaddition* and *comultiplication*, a *cozero* and *counit*  $\epsilon^+, \epsilon^\times : R \rightarrow k$ , and a *coadditive coinverse*  $\sigma : R \rightarrow R$ , satisfying laws dual to those defining commutative rings, along with a ring homomorphism  $\beta : k' \rightarrow \mathbb{W}_R(k)$ , which is called the *co- $k'$ -linear structure*. See [3,5,40] for further details.

The polynomial ring  $k[X]$ , for example, has a canonical  $k$ - $k$ -biring structure as it represents the identity functor from  $k\text{-Alg}$  to itself. Coaddition acts by  $X \mapsto X \otimes 1 + 1 \otimes X$ , comultiplication by  $X \mapsto X \otimes X$ , and the co- $k$ -linear structure  $k \rightarrow \mathbb{W}_{k[X]}(k) = \text{Hom}_{k[X]\text{-Alg}}(k[X], k)$  by  $a \mapsto (f \mapsto f(a))$ . We note the following.

**Lemma 3.2.** *Let  $k$  be a ring,  $R$  a  $k$ - $k$ -biring with coaddition  $\Delta^+$ , comultiplication  $\Delta^\times$ , and co- $k$ -linear structure  $\beta$ . Let  $\eta : k[X] \rightarrow R$  be a  $k$ -algebra homomorphism, and let  $e = \eta(X)$ . The following conditions are equivalent.*

1.  $\eta$  is a homomorphism of  $k$ - $k$ -birings.
2.  $e$  is ring-like in  $R$ , that is,  $\Delta^+(e) = e \otimes 1 + 1 \otimes e$ ,  $\Delta^\times(e) = e \otimes e$ , and  $\beta(c)(e) = c$  for all  $c \in k$ .
3. The map  $\mathbb{W}_R(A) \rightarrow A$  acting by  $\varphi \mapsto \varphi(e)$  is a  $k$ -algebra homomorphism for every  $k$ -algebra  $A$ .
4. The map  $\mathbb{W}_R(A) \rightarrow A$  acting by  $\varphi \mapsto \varphi(e)$  is a  $k$ -algebra homomorphism for  $A = k$  and  $A = R^{\otimes 2}$ .

Moreover, every natural transformation from  $\mathbb{W}_R$  to  $\text{id}_{k\text{-Alg}}$  acts by  $\varphi \mapsto \varphi(a)$  for a unique ring-like element  $a$  of  $R$ .

If  $\{R_i\}_{i \in I}$  is an indexed family of  $k$ -algebras, then,  $\text{Hom}_{k\text{-Alg}}(\bigotimes_{i \in I} R_i, -) \cong \prod_{i \in I} \text{Hom}_{k\text{-Alg}}(R_i, -)$ , so if the  $R_i$  are  $k$ - $k$ -birings then the tensor product  $\bigotimes_{i \in I} R_i$  over  $k$  (the coproduct in  $k\text{-Alg}$ ) has a natural  $k$ - $k$ -biring structure. Thus, for example, the polynomial ring  $k[\mathbf{X}]$  over  $k$  in any set  $\mathbf{X}$  of formal variables has a canonical  $k$ - $k$ -biring structure.

By [5, 1.4–5], for any  $k$ - $k'$ -biring  $R$ , the lifted functor  $\mathbb{W}_R$  from  $k\text{-Alg}$  to  $k'\text{-Alg}$  has a left adjoint, denoted  $R \odot -$ , and  $\odot$  distributes over arbitrary coproducts, both from the left and from the right. The  $k$ -algebra  $R \odot A$  for any  $k'$ -algebra  $A$  is the  $k$ -algebra generated by the symbols  $r \odot a$  for all  $r \in R$  and  $a \in A$ , subject to the relations [5, 1.3.1–2], namely,

$$(r + s) \odot a = (r \odot a) + (s \odot a), \quad (rs) \odot a = (r \odot a)(s \odot a), \quad c \odot a = c,$$

$$r \odot (a + b) = \sum_i (r_{i1}^+ \odot a)(r_{i2}^+ \odot b), \quad r \odot (ab) = \sum_i (r_{i1}^\times \odot a)(r_{i2}^\times \odot b)$$

$$r \odot c' = \beta(c')(r)$$

for  $r, s \in R$ ,  $a, b \in A$ ,  $c \in k$ ,  $c' \in k'$ , where coaddition and comultiplication  $\Delta^+, \Delta^\times : R \rightarrow R \otimes_k R$  act by

$$\Delta^+ : r \mapsto \sum_i r_{i1}^+ \otimes r_{i2}^+, \quad \Delta^\times : r \mapsto \sum_i r_{i1}^\times \otimes r_{i2}^\times,$$

respectively, and where  $\beta : k' \rightarrow \mathbb{W}_R(k)$  is the co- $k'$ -linear structure.

A  $k$ -plethory is equivalently a  $k$ - $k$ -biring  $R$  together with a comonad structure on the endofunctor  $\mathbb{W}_R$  of  $k\text{-Alg}$ . By the adjunction  $(R \odot -) \dashv \mathbb{W}_R$ , a  $k$ -plethory is also equivalently a  $k$ - $k$ -biring  $R$  together with a monad structure on the endofunctor  $R \odot -$  of  $k\text{-Alg}$ .

If  $R$  and  $S$  are  $k$ - $k$ -birings, then  $\text{Hom}_{k\text{-Alg}}(R \odot S, -)$  lifts to the endofunctor  $\mathbb{W}_S \circ \mathbb{W}_R$  of  $k\text{-Alg}$ , so  $R \odot S$  is naturally a  $k$ - $k$ -biring. Moreover, the category of  $k$ - $k$ -birings equipped with the operation  $\odot$  is monoidal with unit  $k[X]$ . It follows that a  $k$ -plethory is equivalently a monoid object in that monoidal category, that is, it is a  $k$ - $k$ -biring  $P$  together with a homomorphism  $\circ : P \odot P \rightarrow P$  of  $k$ - $k$ -birings, called *composition*, that is associative and possesses a unit  $k[X] \rightarrow P$ . We write  $r \circ s$  for  $\circ(r \odot s)$ , and we denote the image of  $X$  in  $P$  by  $e$ .

The *trivial*  $k$ -plethory is the  $k$ -plethory  $P = k[e]$  for which  $\mathbb{W}_P$  is the identity functor on  $k\text{-Alg}$ . It is an initial object in the category of  $k$ -plethories.

If  $P$  is a  $k$ -plethory, then the functor  $P \odot -$  is a monad, and the functor  $\mathbb{W}_P$  a comonad, on the category  $k\text{-Alg}$ . A  $P$ -ring is an (Eilenberg–Moore) algebra of the monad  $P \odot -$ , or equivalently a coalgebra of the comonad  $\mathbb{W}_P$ . A  $P$ -ring is equivalently a  $k$ -algebra  $A$  together with a  $k$ -algebra homomorphism  $\circ : P \odot A \rightarrow A$  such that  $(r \circ s) \circ a = r \circ (s \circ a)$  and  $e \circ a = a$  for all  $r, s \in P$  and all  $a \in A$  [5, 1.9]. Such a map  $\circ$  is said to be a *left action of  $P$  on  $A$* . For example,  $P$  itself has a structure of a  $P$ -ring, as do the  $k$ -algebras  $P \odot A$  and  $\mathbb{W}_P(A)$  for any  $k$ -algebra  $A$ , with left actions given by

$$P \odot (P \odot A) \rightarrow P \odot A, \quad r \odot (s \odot a) \mapsto (r \circ s) \odot a$$

and

$$P \odot \mathbb{W}_P(A) \rightarrow \mathbb{W}_P(A), \quad r \odot \varphi \mapsto \varphi(- \circ r),$$

respectively [5, 1.10].

We let  $P\text{-Rings}$  denote the category of  $P$ -rings, with  $P$ -ring morphisms as  $k$ -algebra homomorphisms that are compatible with the action of  $P$ , in the obvious sense. (This is just the Eilenberg–Moore category of the monad  $P \odot -$ .) The functors  $P \odot -$  and  $\mathbb{W}_P$  from  $k\text{-Alg}$  to  $P\text{-Rings}$  are left and right adjoints, respectively, to the forgetful functor from  $P\text{-Rings}$  to  $k\text{-Alg}$  [5, 1.10]. Therefore  $P \odot A$  is the free  $P$ -ring on  $A$  and  $\mathbb{W}_P(A)$  is the cofree  $P$ -ring on  $A$  for any  $k$ -algebra  $A$ . Thus, for example,  $P \cong P \odot k[X]$  is the free  $P$ -ring on one generator, and  $P^{\otimes \mathbf{X}} \cong P \odot k[X]^{\otimes \mathbf{X}} \cong P \odot k[\mathbf{X}]$  is the free  $P$ -ring generated by  $\mathbf{X}$  for any set  $\mathbf{X}$ . In particular, every  $P$ -ring is isomorphic, for some set  $\mathbf{X}$ , to the

quotient of  $P^{\otimes \mathbf{X}}$  by some  $P$ -ideal [5, Section 5] of  $P^{\otimes \mathbf{X}}$ . The  $P$ -ring  $\mathbb{W}_P(A)$  is called the  $P$ -Witt ring of  $A$ . This terminology comes from the fact that, if  $P$  is the  $\mathbb{Z}$ -plethory  $\Lambda$  of [5, Remark 2.11], then  $\mathbb{W}_P$  is isomorphic to the universal  $\lambda$ -ring functor  $\Lambda$ , and a  $P$ -ring is equivalently a  $\lambda$ -ring.

Plethories may be thought of as a non-linear generalization of the cocommutative bialgebras [5]. In particular, the category of cocommutative  $k$ -bialgebras is naturally equivalent to the category of *linear  $k$ -plethories* [5, 2.2–2.6], which we now define.

For the remainder of this section, all algebras and bialgebras are not assumed commutative. Let  $k$  be a ring. A  $k$ -coalgebra is a  $k$ -module  $C$  together with a coassociative comultiplication  $C \rightarrow C \otimes_k C$  possessing a counit  $C \rightarrow k$ . Equivalently,  $C$  is a  $k$ -module together with the structure of a monad on the functor  $\text{Hom}_{k\text{-Mod}}(C, -)$  that it represents.

The tensor algebra  $T(M)$  and symmetric algebra  $S(M)$  of a  $k$ -module  $M$  are graded  $k$ -algebras and are, respectively, the free  $k$ -algebra on  $M$  and the free commutative  $k$ -algebra on  $M$ . If  $A$  is a  $k$ -algebra, then there is a graded  $k$ -algebra homomorphism  $T(A) \rightarrow k + XA[X]$  induced by the multiplication maps  $A^{\otimes_k n} \rightarrow A$  for  $n \geq 1$ . This homomorphism factors through the homomorphism  $T(A) \rightarrow S(A)$  if and only if  $A$  is commutative, in which case the homomorphism  $S(A) \rightarrow k + XA[X]$  is an isomorphism.

If  $C$  is a cocommutative  $k$ -coalgebra, then, by [5, 2.2], the  $k$ -algebra  $S(C)$  has a natural  $k$ - $k$ -birings structure. A  $k$ -bialgebra is a monoid object in the category of  $k$ -coalgebras, or equivalently a comonoid object in the category of  $k$ -algebras. If  $C$  and  $C'$  are cocommutative  $k$ -coalgebras, then by [5, 2.3] there is an isomorphism  $S(C) \odot S(C') \cong S(C \otimes_k C')$  of  $k$ - $k$ -birings, where  $a \odot b$  corresponds to  $a \otimes b$  for all  $a \in C$  and  $b \in C'$ . Thus, if  $A$  is a cocommutative  $k$ -bialgebra, then the multiplication map  $A \otimes_k A \rightarrow A$  and unit map  $k \rightarrow A$  of  $k$ -coalgebras induce maps

$$\begin{aligned} S(A) \odot S(A) &\cong S(A \otimes_k A) \rightarrow S(A) \\ k[X] &\cong S(k) \rightarrow S(A) \end{aligned}$$

of  $k$ - $k$ -birings that give  $S(A)$  the structure of a  $k$ -plethory. A  $k$ -plethory isomorphic to one of the form  $S(A)$  for a cocommutative  $k$ -bialgebra  $A$  is said to be *linear*. By [5, 2.2–6], the functor  $S(-)$  induces an equivalence between the category of cocommutative  $k$ -bialgebras and the category of linear  $k$ -plethories, and for any  $k$ -bialgebra  $A$  there is an equivalence between the Eilenberg–Moore category of the monad  $A \otimes_k -$  and that of the monad  $S(A) \odot -$ .

#### 4. Idempotence and linearity

Let  $k$  be a ring. We say that a  $k$ -plethory  $P$  is *idempotent* if the comonad  $\mathbb{W}_P$  is idempotent, or equivalently if the monad  $P \odot -$  is idempotent. Proposition 2.5, which characterizes the idempotent plethories, follows from the adjunction  $(P \odot -) \dashv \mathbb{W}_P$ , Yoneda’s lemma, and the fact that  $P$  represents the comonad  $\mathbb{W}_P$ .

**Proof of Proposition 2.5.** Statements (1) and (2) are equivalent by the adjunction  $(P \odot -) \dashv \mathbb{W}_P$ . Statements (1) and (4) are equivalent because  $P$  represents the functor  $\mathbb{W}_P$  and  $P \odot P$  represents the functor  $\mathbb{W}_P \circ \mathbb{W}_P$ . (Also, statements (2) and (4) are equivalent because the map  $P \odot k[e] \rightarrow P$  is an isomorphism.) By the adjunction  $(P \odot -) \dashv \mathbb{W}_P$  we have a natural bijection  $\text{Hom}_{k\text{-Alg}}(P \odot P, A) \cong \text{Hom}_{k\text{-Alg}}(P, \mathbb{W}_P(A))$  for all  $k$ -algebras  $A$ . If (3) holds, then there is also a natural bijection  $\text{Hom}_{k\text{-Alg}}(P, \mathbb{W}_P(A)) \cong \text{Hom}_{k\text{-Alg}}(P, A)$ , and therefore (4) holds by Yoneda's lemma. Thus (3) implies (4). Finally, we show that (1) implies (3). If (1) holds, then the comonad  $\mathbb{W}_P$  is idempotent, and therefore by Proposition 3.1 the map  $A \rightarrow \mathbb{W}_P(A)$  is an isomorphism for every coalgebra  $A$  over the comonad  $\mathbb{W}_P$ , that is, for every  $P$ -ring  $A$ . In particular, since  $P$  is a  $P$ -ring, the map  $P \rightarrow \mathbb{W}_P(P)$  is an isomorphism, that is, (4) holds.  $\square$

We will see that the idempotent  $k$ -plethories are the plethystic analogue of the  $k$ -epimorphs, which are the  $k$ -algebras defined by the following proposition.

**Proposition 4.1** ([4, Proposition 4.2.3], [38, Theorem 1]). *Let  $k$  be a commutative ring, and let  $A$  be a  $k$ -algebra, not necessarily commutative. The following conditions are equivalent.*

1. *The multiplication map  $A \otimes_k A \rightarrow A$  is a  $k$ -algebra isomorphism.*
2. *The monad  $A \otimes_k -$  on  $k\text{-Mod}$  is idempotent.*
3. *The map  $A \rightarrow \text{Hom}_{k\text{-Mod}}(A, A)$  is a  $k$ -algebra isomorphism.*
4. *The comonad  $\text{Hom}_{k\text{-Mod}}(A, -)$  on  $k\text{-Mod}$  is idempotent.*
5. *The forgetful functor from  $A\text{-Mod}$  to  $k\text{-Mod}$  is full and faithful.*
6. *Either of the  $k$ -algebra homomorphisms  $A \rightarrow A \otimes_k A$  is an isomorphism.*
7. *The two  $k$ -algebra homomorphisms  $A \rightarrow A \otimes_k A$  are equal.*
8. *One has  $a \otimes b = b \otimes a$  in  $A \otimes_k A$  for all  $a, b \in A$ .*
9. *The tensor algebra  $T(A)$  of the  $k$ -module  $A$  is commutative.*
10. *The graded  $k$ -algebra homomorphism  $T(A) \rightarrow S(A)$ , where  $S(A)$  is the symmetric algebra of the  $k$ -module  $A$ , is an isomorphism.*
11. *The graded  $k$ -algebra homomorphism  $T(A) \rightarrow k + XA[X]$  is an isomorphism.*
12. *One has  $A \otimes_k \text{coker}(k \rightarrow A) = 0$  as  $k$ -modules.*
13. *The map  $k \rightarrow A$  is an epimorphism of  $\mathbb{Z}$ -algebras.*
14. *The map  $k \rightarrow A$  is an epimorphism of commutative rings.*

**Proof.** The equivalence of the first five conditions follows from Proposition 3.1 and the adjunction  $(A \otimes_k -) \dashv \text{Hom}_{k\text{-Mod}}(A, -)$ , and the equivalence of the last nine conditions and condition (1) follows from [38, Theorem 1].  $\square$

**Example 4.2.** Let  $k$  be an integral domain with quotient field  $K$ . A  $k$ -algebra  $A$  is a  $k$ -torsion-free  $k$ -epimorph if and only if  $A$  is isomorphic to a  $k$ -subalgebra of  $K$  and

$A \otimes_k A$  is  $k$ -torsion-free. In particular, if  $k \subseteq A \subseteq K$  and  $A$  is flat as a  $k$ -algebra, then  $A$  is a  $k$ -epimorph.

The following result provides some analogous characterizations of the idempotent plethories.

**Theorem 4.3.** *Let  $k$  be a ring and  $P$  a  $k$ -plethory. The following conditions are equivalent.*

1.  $P$  is idempotent.
2. Either of the  $k$ -algebra homomorphisms  $P \rightarrow P \odot P$  is an isomorphism.
3. The two  $k$ -algebra homomorphisms  $P \rightarrow P \odot P$  are equal.
4. The  $k$ - $k$ -biring homomorphism  $Q(P) \rightarrow k[X] \otimes_k \bigotimes_{n=1}^{\infty} P \cong k[X] \otimes_k (P \odot k[X_1, X_2, \dots])$  induced by composition, where  $Q(R) = \bigotimes_{n=0}^{\infty} R^{\odot n}$  for any  $k$ - $k$ -biring  $R$  denotes the free  $k$ -plethory on  $R$  [5, 2.1], is an isomorphism.
5. For every  $k$ -plethory  $Q$  there is at most one  $k$ -plethory homomorphism  $P \rightarrow Q$ .
6. The map  $k[e] \rightarrow P$  is an epimorphism of  $k$ -plethories.

**Proof.** Both homomorphisms  $\text{id} \odot e$  and  $e \odot \text{id}$  from  $P$  to  $P \odot P$  (acting by  $a \mapsto a \odot e$  and  $a \mapsto e \odot a$ , respectively) are sections of the composition map  $P \odot P \rightarrow P$ . It follows that (1)  $\Leftrightarrow$  (2)  $\Rightarrow$  (3). Suppose that (3) holds, and let  $\varphi, \psi : P \rightarrow Q$  be  $k$ -plethory homomorphisms. Then the commutative diagram

$$\begin{array}{ccccc}
 P \odot P & \xleftarrow{\text{id} \odot e} & P & \xrightarrow{e \odot \text{id}} & P \odot P \\
 \varphi \odot \psi \downarrow & & \varphi \downarrow & \psi \downarrow & \varphi \odot \psi \downarrow \\
 Q \odot Q & \longrightarrow & Q & \longleftarrow & Q \odot Q
 \end{array}$$

shows that  $\varphi = \psi$ . Therefore (3)  $\Rightarrow$  (5), and clearly (5)  $\Rightarrow$  (3). Since  $k[e]$  is the initial  $k$ -plethory, one also has (5)  $\Leftrightarrow$  (6). Moreover, (1)  $\Leftrightarrow$  (4) follows by projecting all tensor coordinates, besides that for  $n = 2$ , onto  $k$  using the cozero  $P \rightarrow k$ .

Thus it remains only to show that (3)  $\Rightarrow$  (1). Suppose that (3) holds, so that  $a \odot e = e \odot a$  for all  $a \in P$ . Consider two maps  $P \odot P \rightarrow P \odot P \odot P$ . One is the map

$$P \odot P \rightarrow P \odot (P \odot P), \quad a \odot b \mapsto a \odot (b \odot e) = a \odot (e \odot b),$$

and the other is the map

$$P \odot P \rightarrow (P \odot P) \odot P, \quad a \odot b \mapsto (a \odot e) \odot b = (e \odot a) \odot b.$$

It follows that, as maps from  $P \odot P$  to  $P \odot P \odot P$ , they are identical. Therefore  $a \odot b \odot e = e \odot a \odot b$  in  $P \odot P \odot P$  for all  $a, b \in P$ . Composing the first two coordinates we see that  $(a \odot b) \odot e = a \odot b$ . Therefore the composition  $P \odot P \rightarrow P \rightarrow P \odot P$  is the identity, whence both maps are isomorphisms and so  $P$  is idempotent. Therefore (3)  $\Rightarrow$  (1).  $\square$

We now show that the trivial  $k$ -plethory is the only linear idempotent  $k$ -plethory.

**Proposition 4.4.** *Let  $k$  be a ring. The following conditions are equivalent for any cocommutative  $k$ -bialgebra  $A$ .*

1. *The linear  $k$ -plethory  $S(A)$  is idempotent.*
2. *The map  $k \rightarrow A$  is a ring epimorphism, or equivalently, the underlying  $k$ -algebra of  $A$  is a  $k$ -epimorph.*
3. *The map  $k \rightarrow A$  is a ring isomorphism.*

*In particular, the trivial  $k$ -plethory  $k[e] \cong S(k)$  is the only linear idempotent  $k$ -plethory.*

**Proof.** If (1) holds, then the map  $S(A \otimes_k A) \cong S(A) \odot S(A) \rightarrow S(A)$  is an isomorphism of graded  $k$ -algebras and therefore induces a  $k$ -module isomorphism  $A \otimes_k A \rightarrow A$  of the graded one components, whence (2) holds. Thus (1)  $\Rightarrow$  (2). Suppose that (2) holds. The ring epimorphism  $\varphi : k \rightarrow A$  possesses a retraction  $\psi : A \rightarrow k$ , so, since  $\varphi \circ \psi \circ \varphi = \varphi = \text{id}_A \circ \varphi$ , one has  $\varphi \circ \psi = \text{id}_A$ . Therefore  $\varphi$  is a ring isomorphism. Thus (2)  $\Rightarrow$  (3), and that (3)  $\Rightarrow$  (1) is clear.  $\square$

**Corollary 4.5.** *Let  $k$  be a ring. If every  $k$ -plethory is linear, then  $k[e]$  is the only idempotent  $k$ -plethory.*

Recently Magnus Carlson has shown that every  $k$ -plethory is linear if  $k$  is a field of characteristic zero [13, Theorem 1.1], answering a question posed in [3, p. 336]. It follows in this case that  $k[e]$  is the only idempotent  $k$ -plethory.

Finally, we mention two natural conditions on plethories that are stronger than idempotence. First, we say that a  $k$ -plethory  $P$  is *strongly idempotent* if  $P$  is  $k$ -torsion-free and the map  $\mathbb{W}_P(A) \rightarrow A$  is injective for every  $k$ -torsion-free  $k$ -algebra  $A$ . Examples of strongly idempotent plethories, besides the trivial  $k$ -plethory and  $\text{Int}(\mathbb{Z})$ , include the plethories discussed in Proposition 2.6 and in Sections 7 and 8 (e.g., Theorems 7.9 and 7.11). Example 6.5 in Section 6 is an example of an idempotent  $\mathbb{F}_p$ -plethory that is not strongly idempotent.

**Proposition 4.6.** *Let  $k$  be a ring and  $P$  a  $k$ -plethory. Then  $P$  is strongly idempotent if and only if  $k[e] \rightarrow P$  is an epimorphism in the category of  $k$ -torsion-free  $k$ -algebras. Moreover, if either condition holds, then  $P$  is idempotent.*

**Proof.** The map  $\mathbb{W}_P(A) \rightarrow A$  is injective if and only if for every  $a \in A$  there is at most one  $k$ -algebra homomorphism  $P \rightarrow A$  sending  $e$  to  $a$ . The equivalence of the two conditions then follows. From the two conditions it follows that the surjective map  $\mathbb{W}_P(P) \rightarrow P$  is also injective, whence  $P$  is idempotent.  $\square$



By Theorem 8.9, the binomial plethory  $\text{Int}(\mathbb{Z})$ , and more generally the  $D$ -plethory  $\text{Int}(D)$  for any Dedekind domain  $D$  with finite residue fields, also satisfies the conditions in the following proposition.

**Proposition 4.7.** *Let  $k$  be a ring and  $P$  a  $k$ -plethory. The following conditions are equivalent.*

1. Every  $P$ -ring is  $k$ -torsion-free.
2.  $\mathbb{W}_P(A)$  is  $k$ -torsion-free for every  $k$ -algebra  $A$ .
3.  $P \odot A$  is  $k$ -torsion-free for every  $k$ -algebra  $A$ .

Moreover, if  $P$  is idempotent, then the above conditions hold if and only if every  $P$ -reflective  $k$ -algebra is  $k$ -torsion-free.

**Proof.** Clearly (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3). Let  $A$  be a  $P$ -ring, so there are inclusions  $A \rightarrow \mathbb{W}_P(A)$  and  $A \rightarrow P \odot A$ , whence  $A$  is  $k$ -torsion-free if either  $\mathbb{W}_P(A)$  or  $P \odot A$  are. Therefore (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1). Finally, the last statement of the proposition follows from Corollary 5.3 of the next section.  $\square$

## 5. Eilenberg–Moore category

Let  $T : \mathcal{C} \rightarrow \mathcal{D}$  be a natural transformation from a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  to a functor  $G : \mathcal{C} \rightarrow \mathcal{D}$ , where  $\mathcal{C}$  and  $\mathcal{D}$  are categories. We say that an object  $A$  of  $\mathcal{C}$  is a *fixed component* of  $T$  if  $T(A) : F(A) \rightarrow G(A)$  is an isomorphism in  $\mathcal{D}$ , and we call the full subcategory of  $\mathcal{C}$  whose objects are the fixed components of  $T$  the *fixed category* of  $T$ . This terminology is borrowed from [18,33].

Let  $k$  be a ring, and let  $\eta : S \rightarrow R$  be a  $k$ -algebra homomorphism. We will say that a  $k$ -algebra  $A$  is  $\eta$ -reflective if  $A$  is a fixed component of the natural transformation  $-\circ\eta : \text{Hom}_{k\text{-Alg}}(R, -) \rightarrow \text{Hom}_{k\text{-Alg}}(S, -)$ , where the given hom functors are from  $k\text{-Alg}$  to **Sets**. Equivalently,  $A$  is  $\eta$ -reflective if and only if  $-\circ\eta : \text{Hom}_{k\text{-Alg}}(R, A) \rightarrow \text{Hom}_{k\text{-Alg}}(S, A)$  is a bijection, if and only if every  $k$ -algebra homomorphism  $S \rightarrow A$  factors uniquely through  $\eta$ . If  $R$  itself is  $\eta$ -reflective, then one says that  $\eta$  is a *reflection map* in  $k\text{-Alg}$  [9]. We denote by  $\eta\text{-Refl}$ , or  $R\text{-Refl}$ , the full subcategory of  $k\text{-Alg}$  with the  $\eta$ -reflective  $k$ -algebras as objects.

Applying this to  $S = k[\mathbf{X}]$ , where  $\mathbf{X}$  is a set, we see that, if  $\eta : k[\mathbf{X}] \rightarrow R$  is a  $k$ -algebra homomorphism, then a  $k$ -algebra  $A$  is  $\eta$ -reflective if and only if for every  $(a_X)_{X \in \mathbf{X}} \in A^{\mathbf{X}}$  there is a unique  $k$ -algebra homomorphism  $R \rightarrow A$  sending  $\eta(X)$  to  $a_X$  for all  $X \in \mathbf{X}$ .

**Example 5.1.** For any integral domain  $D$  with quotient field  $K$  and any set  $\mathbf{X}$ , if  $i_{\mathbf{X}} : D[\mathbf{X}] \rightarrow \text{Int}(D^{\mathbf{X}})$  denotes the natural inclusion, then  $\text{Int}(\mathbf{E}, D)$  is an  $i_{\mathbf{X}}$ -reflective  $D$ -algebra for any set  $\mathbf{Y}$  and any subset  $\mathbf{E}$  of  $K^{\mathbf{Y}}$ .

If  $\eta : S \rightarrow R$  a homomorphism of  $k$ - $k$ -birings, then a  $k$ -algebra  $A$  is  $\eta$ -reflective if and only if  $A$  is a fixed component of the natural transformation  $-\circ\eta : \mathbb{W}_R \rightarrow \mathbb{W}_S$ . In particular, if  $P$  is an idempotent  $k$ -plethory, then  $P$ -Refl is just the fixed category of the natural transformation  $\mathbb{W}_P \rightarrow \text{id}_{k\text{-Alg}}$ . For example,  $k[e]$ -Refl is the category  $k\text{-Alg}$ , and, as we will see in Section 8,  $\text{Int}(\mathbb{Z})$ -Refl is the category of binomial rings. By the corollary to the following proposition, the forgetful functor from  $P$ -Rings to  $k\text{-Alg}$  is an isomorphism onto  $P$ -Refl.

**Proposition 5.2.** *The following are equivalent for any ring  $k$  and any  $k$ -plethory  $P$ .*

1.  $P$  is idempotent.
2.  $P$  is  $P$ -reflective.
3. Every  $P$ -ring is  $P$ -reflective.
4. The map  $A \rightarrow \mathbb{W}_P(A)$  is an isomorphism for every  $P$ -ring  $A$ .
5. The map  $P \odot A \rightarrow A$  is an isomorphism for every  $P$ -ring  $A$ .
6. The forgetful functor from  $P$ -Rings to  $k\text{-Alg}$  is full and faithful.
7.  $\mathbb{W}_P(\epsilon) = \epsilon(\mathbb{W}_P)$  as natural transformations  $\mathbb{W}_P \circ \mathbb{W}_P \rightarrow \mathbb{W}_P$ , where  $\epsilon$  is the natural transformation  $\mathbb{W}_P \rightarrow \text{id}_{k\text{-Alg}}$ .
8.  $\mathbb{W}_P(\delta) = \delta(\mathbb{W}_P)$  as natural transformations  $\mathbb{W}_P \circ \mathbb{W}_P \rightarrow \mathbb{W}_P \circ \mathbb{W}_P \circ \mathbb{W}_P$ , where  $\delta : \mathbb{W}_P \rightarrow \mathbb{W}_P \circ \mathbb{W}_P$  is the comonad structure on  $\mathbb{W}_P$ .
9.  $(P \odot -)(\eta) = \eta(P \odot -)$  as natural transformations  $P \odot - \rightarrow P \odot P \odot -$ , where  $\eta$  is the natural transformation  $\text{id}_{k\text{-Alg}} \rightarrow P \odot -$ .
10.  $(P \odot -)(\mu) = \mu(P \odot -)$  as natural transformations  $P \odot P \odot P \odot - \rightarrow P \odot P \odot -$ , where  $\mu : P \odot P \odot - \rightarrow P \odot -$  is the monad structure on  $P \odot -$ .

**Proof.** The equivalence of conditions (1) and (2) follows immediately from Proposition 2.5, and statements (3) and (4) are trivially equivalent. By Proposition 3.1, then, it follows that statements (1) through (6) are equivalent. Finally, the equivalences (1)  $\Leftrightarrow$  (9)  $\Leftrightarrow$  (10) follow from the corresponding equivalences (i)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v) of [34, Proposition] for idempotent monads in general, and the equivalences (1)  $\Leftrightarrow$  (7)  $\Leftrightarrow$  (8) follow from the corresponding dual statements for idempotent comonads.  $\square$

**Corollary 5.3.** *Let  $k$  be a ring and  $P$  an idempotent  $k$ -plethory, and let  $A$  be a  $k$ -algebra. The following conditions are equivalent.*

1.  $A$  is  $P$ -reflective.
2. There is a unique  $P$ -ring structure on  $A$ .
3. There is a  $P$ -ring structure on  $A$ .

Moreover, if  $A$  and  $A'$  are  $P$ -reflective  $k$ -algebras, then  $\text{Hom}_{k\text{-Alg}}(A, A') = \text{Hom}_{P\text{-Rings}}(A, A')$ . Therefore, the forgetful functor from  $P$ -Rings to  $k\text{-Alg}$  is an isomorphism onto  $P$ -Refl.

**Proof.** Suppose that (1) holds. Let  $\epsilon : \mathbb{W}_P \rightarrow \text{id}_{k\text{-Alg}}$  denote the counit of the comonad  $\mathbb{W}_P$ . Reversing the arrows in the commutative diagram

$$\begin{array}{ccc} \mathbb{W}_P(\mathbb{W}_P(A)) & \xrightarrow{\mathbb{W}_P(\epsilon(A))} & \mathbb{W}_P(A) \\ \epsilon(\mathbb{W}_P(A)) \downarrow & & \downarrow \epsilon(A) \\ \mathbb{W}_P(A) & \xrightarrow{\epsilon(A)} & A \end{array}$$

of  $k$ -algebra isomorphisms shows that  $A$  has the structure of a  $P$ -ring. Uniqueness follows from the fact that any  $P$ -ring structure  $A \rightarrow \mathbb{W}_P(A)$  is a section of the isomorphism  $\mathbb{W}_P(A) \rightarrow A$ . Thus (1)  $\Rightarrow$  (2). That (2)  $\Rightarrow$  (3) is clear, and (3)  $\Rightarrow$  (1) by [Proposition 5.2](#). Finally, the last statement of the corollary follows since the forgetful functor from  $P$ -Rings to  $k$ -Alg is full and faithful.  $\square$

**Corollary 5.4.** *Let  $k$  be a ring and  $P$  an idempotent  $k$ -plethory. Then  $P \odot A \in P\text{-Refl}$  and  $\mathbb{W}_P(A) \in P\text{-Refl}$  for every  $k$ -algebra  $A$ , so that  $P \odot -$  and  $\mathbb{W}_P$  define functors from  $k\text{-Alg}$  to  $P\text{-Refl}$ . Moreover, we have the following.*

1. *The functor  $P \odot -$  from  $k\text{-Alg}$  to  $P\text{-Refl}$  is a left adjoint to the inclusion from  $P\text{-Refl}$  to  $k\text{-Alg}$ .*
2. *The functor  $\mathbb{W}_P$  from  $k\text{-Alg}$  to  $P\text{-Refl}$  is a right adjoint to the inclusion from  $P\text{-Refl}$  to  $k\text{-Alg}$ .*
3. *A  $k$ -algebra  $A$  is  $P$ -reflective if and only if the  $k$ -algebra homomorphism  $A \rightarrow P \odot A$  is an isomorphism, if and only if  $A$  is a fixed component of the natural transformation  $\text{id}_{k\text{-Alg}} \rightarrow P \odot -$ . In that case, the inverse map  $P \odot A \rightarrow A$  acts by  $r \odot a \mapsto i_a(r)$ , where  $i_a$  is the unique map  $P \rightarrow A$  sending  $e$  to  $a$ .*
4. *In particular,  $P\text{-Refl}$  is the fixed category of both natural transformations  $\mathbb{W}_P \rightarrow \text{id}_{k\text{-Alg}}$  and  $\text{id}_{k\text{-Alg}} \rightarrow P \odot -$ .*

**Proof.** The  $k$ -algebras  $P \odot A$  and  $\mathbb{W}_P(A)$  have natural  $P$ -ring structures, so they are  $P$ -reflective by [Corollary 5.3](#). The functors  $P \odot -$  and  $\mathbb{W}_P$  from  $k\text{-Alg}$  to  $P\text{-Rings}$  are left and right adjoints, respectively, to the forgetful functor from  $P\text{-Rings}$  to  $k\text{-Alg}$ . Therefore (1) and (2) follow from [Corollary 5.3](#). Finally, statements (3) and (4) follow from [Proposition 5.2](#) and [Corollary 5.3](#). Alternatively, to prove (3), note first that if  $A \rightarrow P \odot A$  is an isomorphism, then  $A$  is  $P$ -reflective since  $P \odot A$  is. Conversely, suppose that  $A$  is  $P$ -reflective. Then we have natural bijections

$$\begin{aligned} \text{Hom}_{k\text{-Alg}}(P \odot A, -) &\cong \text{Hom}_{k\text{-Alg}}(A, \mathbb{W}_P(-)) \\ &= \text{Hom}_{P\text{-Rings}}(A, \mathbb{W}_P(-)) \\ &\cong \text{Hom}_{k\text{-Alg}}(A, -). \end{aligned}$$

Therefore we have an isomorphism  $P \odot A \longrightarrow A$  (corresponding to  $\text{id}_A \in \text{Hom}_{k\text{-Alg}}(A, A)$ ) acting by  $r \odot a \longmapsto i_a(r)$  that is an inverse of the map  $A \longrightarrow P \odot A$ .  $\square$

If  $\mathcal{C}$  is a subcategory of a category  $\mathcal{D}$ , then we denote by  $\overline{\mathcal{C}}$  the *isomorphic closure* of  $\mathcal{C}$  in  $\mathcal{D}$ , that is, the full subcategory of  $\mathcal{D}$  whose objects are the objects of  $\mathcal{D}$  that are isomorphic to some object in  $\mathcal{C}$ . (Below we assume  $\mathcal{D} = k\text{-Alg}$ .)

**Corollary 5.5.** *Let  $k$  be a ring and  $P$  an idempotent  $k$ -plethory.*

1. *Let  $\mathcal{C}$  be a full subcategory of  $k\text{-Alg}$  with  $P \odot A \in \mathcal{C}$  for all  $A \in k\text{-Alg}$ . Then  $P \odot -$  defines a left adjoint to the inclusion from  $\mathcal{C}$  to  $k\text{-Alg}$  if and only if  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ , if and only if  $\overline{\mathcal{C}} = P\text{-Refl}$ .*
2. *Let  $\mathcal{C}$  be a full subcategory of  $k\text{-Alg}$  with  $\mathbb{W}_P(A) \in \mathcal{C}$  for all  $A \in k\text{-Alg}$ . Then  $\mathbb{W}_P$  defines a right adjoint to the inclusion from  $\mathcal{C}$  to  $k\text{-Alg}$  if and only if  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ , if and only if  $\overline{\mathcal{C}} = P\text{-Refl}$ .*

**Proof.** We prove (2), and then (1) follows by adjunction.

Let the category  $\mathcal{C}$  be as in (2). Suppose that  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ . Then by [Corollary 5.4\(2\)](#) we have for  $B \in \mathcal{C}$  and  $A \in k\text{-Alg}$  natural bijections

$$\text{Hom}_{\mathcal{C}}(B, \mathbb{W}_P(A)) = \text{Hom}_{k\text{-Alg}}(B, \mathbb{W}_P(A)) \cong \text{Hom}_{k\text{-Alg}}(B, A).$$

Therefore  $\mathbb{W}_P$  defines a right adjoint to the inclusion  $\mathcal{C} \longrightarrow k\text{-Alg}$ . Conversely, suppose that  $\mathbb{W}_P$  defines such a right adjoint. Let  $A \in \mathcal{C}$ . Then we have natural bijections

$$\text{Hom}_{\mathcal{C}}(-, \mathbb{W}_P(A)) \cong \text{Hom}_{k\text{-Alg}}(-, A) = \text{Hom}_{\mathcal{C}}(-, A),$$

so the map  $\mathbb{W}_P(A) \longrightarrow A$  is an isomorphism, whence  $A \in P\text{-Refl}$ . Thus  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ .

Suppose now that  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ . Let  $A \in P\text{-Refl}$ . Then, since  $\mathbb{W}_P(A)$  is in  $\mathcal{C}$  and is isomorphic to  $A$ , it follows  $A \in \overline{\mathcal{C}}$ . Therefore  $P\text{-Refl}$  is a subcategory of  $\overline{\mathcal{C}}$ . Since  $\mathcal{C} \subseteq P\text{-Refl} \subseteq \overline{\mathcal{C}}$ , it follows that  $\overline{\mathcal{C}} = P\text{-Refl}$ . Conversely, if  $\overline{\mathcal{C}} = P\text{-Refl}$ , then  $\mathcal{C}$  is a subcategory of  $P\text{-Refl}$ .  $\square$

## 6. Idempotent plethory structures

In this section we address issues surrounding the existence and uniqueness of idempotent plethory structures.

**Theorem 6.1.** *Let  $k$  be a ring and  $\eta : k[X] \longrightarrow R$  a  $k[X]$ -algebra.*

1. *An idempotent  $k$ -plethory structure on  $R$  with unit  $\eta$ , if it exists, is unique.*

2. If  $R$  is an  $\eta$ -reflective  $k$ -algebra, then any  $k$ - $k$ -biring structure on  $R$  such that  $\eta$  is a homomorphism of  $k$ - $k$ -birings extends uniquely to a (necessarily idempotent)  $k$ -plethory structure on  $R$  with unit  $\eta$ .

**Proof.**

1. Let  $P$  and  $P'$  be idempotent  $k$ -plethory structures on  $R$  with unit  $\eta$ . Consider the categories  $P$ -Refl and  $P'$ -Refl. Both of these categories are equal to the category  $\mathbf{C}$  of  $k$ -algebras  $A$  such that every  $k$ -algebra homomorphism  $k[X] \rightarrow A$  factors uniquely through  $\eta$ . Since the inclusion from  $\mathbf{C}$  to  $k$ -Alg has a left and a right adjoint and  $\mathbf{C}$  is bicomplete, by the reconstruction theorem of [5, Introduction],  $\mathbf{C}$  is the category of  $Q$ -rings for a  $k$ -plethory  $Q$  that is unique up to isomorphism. Thus there must exist an isomorphism  $P \rightarrow P'$  of  $k$ -plethories, which is necessarily induced by a  $k[X]$ -automorphism of  $R$ . But  $\text{Hom}_{k[X]\text{-Alg}}(R, R) \cong \text{Hom}_{k[X]\text{-Alg}}(P, P')$  is trivial since  $R$  is  $\eta$ -reflective, whence  $P = P'$ .
2. Suppose that  $R$  is  $\eta$ -reflective and has a  $k$ - $k$ -biring structure such that  $\eta$  is a homomorphism of  $k$ - $k$ -birings. The map  $\mathbb{W}_R(R) \rightarrow R$  acting by  $\varphi \mapsto \varphi(\eta(X))$  is a bijection, hence a  $k$ -algebra isomorphism. By adjunction, its inverse  $R \rightarrow \mathbb{W}_R(R)$  induces a  $k$ -algebra homomorphism  $\circ : R \odot R \rightarrow R$  sending  $a \odot b$  to  $a \circ b = i_b(a)$ , where  $i_b$  is the unique  $k$ -algebra endomorphism of  $R$  sending  $e = \eta(X)$  to  $b$ . As in the proof in Section 4 of Proposition 2.5—specifically the proof that condition (3) of the proposition implies condition (4)—it follows from the fact that the map  $R \rightarrow \mathbb{W}_R(R)$  is an isomorphism that the map  $\circ : R \odot R \rightarrow R$  is also an isomorphism. Now, since the map  $\eta$  is by assumption a homomorphism of  $k$ - $k$ -birings, the map  $R \cong R \odot k[X] \rightarrow R \odot R$  given by  $r \mapsto r \odot e$  is also a homomorphism of  $k$ - $k$ -birings, and therefore its inverse  $\circ$  is also a homomorphism of  $k$ - $k$ -birings. We claim that the map  $\circ$  is associative. Indeed, one has  $a \circ (b \circ c) = i_{i_c(b)}(a)$  while  $(a \circ b) \circ c = i_c(i_b(a))$ , and since  $i_{i_c(b)}(\eta(X)) = i_c(b) = (i_c \circ i_b)(\eta(X))$ , one has  $i_{i_c(b)} = i_c \circ i_b$  and therefore  $a \circ (b \circ c) = (a \circ b) \circ c$ . Likewise, one easily checks that  $a \circ e = a = e \circ a$ . Therefore  $R$  has the structure of a  $k$ -plethory with composition  $\circ$  and unit  $\eta(X)$ . Finally, since  $\circ$  is an isomorphism, the  $k$ -plethory  $R$  is idempotent.  $\square$

**Corollary 6.2.** *Let  $k$  be a ring and  $\eta : k[X] \rightarrow R$  a  $k[X]$ -algebra. Then there exists a (necessarily unique) idempotent  $k$ -plethory structure on  $R$  with unit  $\eta$  if and only if  $R$  is an  $\eta$ -reflective  $k$ -algebra and there is a  $k$ - $k$ -biring structure on  $R$  such that  $\eta$  is a homomorphism of  $k$ - $k$ -birings.*

Let us say that a  $k[X]$ -algebra  $\eta : k[X] \rightarrow R$  (or  $R$ , if the  $k[X]$ -algebra structure is clear) is *(plethystic) idempotent* if the equivalent conditions of Corollary 6.2 hold, that is, if there exists a (necessarily unique) idempotent  $k$ -plethory structure on  $R$  with unit  $\eta$ . An idempotent  $k$ -plethory may be thought of as a property—namely, plethystic

idempotence—of the underlying  $k[X]$ -algebra rather than as a structure in and of itself. Specifically, we have the following.

**Corollary 6.3.** *Let  $k$  be a ring. The forgetful functor from the category of idempotent  $k$ -plethories to the category of idempotent  $k[X]$ -algebras (both with the obvious morphisms) is an isomorphism.*

The following theorem provides a useful characterization of the idempotent  $k[X]$ -algebras.

**Theorem 6.4.** *Let  $\eta : k[X] \rightarrow R$  be a  $k[X]$ -algebra. Then  $R$  is an idempotent  $k[X]$ -algebra if and only if  $R^{\otimes n}$  is an  $\eta$ -reflective  $k$ -algebra for  $0 \leq n \leq 3$ .*

**Proof.** If  $R$  is an idempotent  $k[X]$ -algebra, then  $R$ , and therefore every tensor power of  $R$ , is  $\eta$ -reflective. Therefore it remains only to prove sufficiency. Let  $e = \eta(X)$ . By Lemma 3.2, a  $k$ - $k$ -biring structure on  $R$  compatible with  $\eta$  (in the sense that  $\eta$  is a  $k$ - $k$ -biring homomorphism) exists if and only if there exist  $k$ -algebra homomorphisms

$$\begin{aligned} \Delta^+ : R &\rightarrow R^{\otimes 2} \\ \epsilon^+ : R &\rightarrow k \\ \sigma : R &\rightarrow R \\ \Delta^\times : R &\rightarrow R^{\otimes 2} \\ \epsilon^\times : R &\rightarrow k \end{aligned}$$

sending  $X$ , respectively, to  $\Delta^+(e) = e \otimes 1 + 1 \otimes e$ ,  $\epsilon^+(e) = 0$ ,  $\sigma(e) = -e$ ,  $\Delta^\times(e) = e \otimes e$ , and  $\epsilon^\times(e) = 1$ , together satisfying the appropriate commutative diagrams, as well as a ring homomorphism

$$\beta : k \rightarrow \mathbb{W}_R(k)$$

which when composed with the map  $\mathbb{W}_R(k) \rightarrow k$  is the identity. These homomorphisms are, respectively, the coaddition, cozero, coadditive coinverse, comultiplication, counit, and co- $k$ -linear structure of a  $k$ - $k$ -biring structure on  $R$  compatible with  $\eta$ .

Suppose that, for  $0 \leq n \leq 3$ , the  $n$ th tensor power  $R^{\otimes n}$  of  $R$  is  $\eta$ -reflective, so that, for any  $a \in R^{\otimes n}$  there is a unique  $k$ -algebra homomorphism  $\varphi : R \rightarrow R^{\otimes n}$  sending  $\eta(X)$  to  $a$ . The existence and uniqueness of the homomorphisms  $\Delta^+$ ,  $\epsilon^+$ ,  $\sigma$ ,  $\Delta^\times$ , and  $\epsilon^\times$  thus follow. In other words, the  $k$ - $\mathbb{Z}$ -biring co-operations on  $k[X]$  extend uniquely to the given co-operations on  $R$ . Moreover, since  $R^{\otimes n}$  is  $\eta$ -reflective for  $0 \leq n \leq 3$ , all of the commutative diagrams (as listed in [40, Appendix A], for example) required of the co-operations on  $k[X]$  to make  $k[X]$  into a  $K$ - $\mathbb{Z}$ -biring lift uniquely to the same commutative diagrams for the co-operations on  $R$ . Therefore, the extended

co-operations on  $R$  make  $R$  into a  $k$ - $\mathbb{Z}$ -biring. Finally, since  $k = R^{\otimes 0}$  is  $\eta$ -reflective, the map  $\mathbb{W}_R(k) \rightarrow k$  acting by  $\varphi \mapsto \varphi(e)$  is an isomorphism of  $\mathbb{Z}$ -algebras, and therefore its inverse  $\beta$  is a co- $k$ -linear structure on the  $k$ - $\mathbb{Z}$ -biring  $R$ . Therefore  $R$  has a  $k$ - $k$ -biring structure compatible with  $\eta$ .  $\square$

The following example provides an application of [Theorem 6.4](#) to the construction of the perfect closure of a ring of prime characteristic.

**Example 6.5** (*Perfect closure and perfection*). Let  $p$  be a prime. A ring  $A$  of characteristic  $p$  is said to be *perfect* if the Frobenius endomorphism  $f = (-)^p$  of  $A$  is an isomorphism, that is, if every element of  $A$  has a unique  $p$ -th root. The inclusion from the category  $\mathbb{F}_p$ -Perf of perfect rings of characteristic  $p$  to the category  $\mathbb{F}_p$ -Alg has both a left adjoint  $l$  and right adjoint  $r$ . The ring  $l(A) = A^{p^{-\infty}}$  is known as the *perfect closure* of  $A$ . The ring  $r(A)$  is known as the *perfection* of  $A$  and is the inverse limit of the inverse system  $\cdots \xrightarrow{f} A \xrightarrow{f} A \xrightarrow{f} A$ . By [Theorem 2.9](#), it follows that there is an  $\mathbb{F}_p$ -plethory  $P$ , unique up to isomorphism, for which  $\mathbb{F}_p$ -Perf =  $P$ -Refl. Using [Theorem 6.4](#), we may construct the plethory  $P$  without assuming the existence of  $l$  and  $r$ , as follows. First, note that if  $P$  is to exist then one must have  $P = l(\mathbb{F}_p[X])$ , so as a ring  $P$  must be equal to the perfect closure  $\mathbb{F}_p[X, X^{1/p}, X^{1/p^2}, \dots]$  of  $\mathbb{F}_p[X]$ . Let  $P$  be this ring. For any  $\mathbb{F}_p$ -algebra  $A$  there is a natural bijection

$$\Phi : \text{Hom}_{\mathbb{F}_p\text{-Alg}}(P, A) \rightarrow r(A) := \{(a_0, a_1, \dots) : a_n \in A, a_0 = a_1^p, a_1 = a_2^p, \dots\}.$$

Thus an  $\mathbb{F}_p$ -algebra  $A$  is  $P$ -reflective, that is,  $\text{Hom}_{\mathbb{F}_p\text{-Alg}}(P, A) \rightarrow \text{Hom}_{\mathbb{F}_p\text{-Alg}}(\mathbb{F}_p[X], A)$  is a bijection, if and only if  $A$  is perfect. Thus, since  $P^{\otimes n}$  is perfect, hence  $P$ -reflective, for all  $n$ , it follows from [Theorem 6.4](#) that  $P$  is an idempotent  $\mathbb{F}_p[X]$ -algebra. Therefore  $P$  has a unique (idempotent) plethory structure with unit given by the inclusion  $\mathbb{F}_p[X] \rightarrow P$ . Moreover,  $\Phi : \mathbb{W}_P \rightarrow r(-)$  is a natural isomorphism, and therefore by its universal property the functor  $P \odot -$  is isomorphic to the functor  $l = (-)^{p^{-\infty}}$ . This therefore provides an alternative construction of the perfect closure.

The following result provides another characterization of the idempotent  $k[X]$ -algebras.

**Proposition 6.6.** *Let  $k$  be a ring and  $R$  a  $k$ -algebra. Then  $R$  has the structure of an idempotent  $k[X]$ -algebra for some ring homomorphism  $\eta : k[X] \rightarrow R$  if and only if there exists a full subcategory  $\mathcal{C}$  of  $k$ -Alg such that  $R$  represents a right adjoint  $F$  to the inclusion  $I$  from  $\mathcal{C}$  to  $k$ -Alg such that the corresponding map  $F(R) \rightarrow R$  is an isomorphism. Moreover, if both of these conditions hold, then  $\overline{\mathcal{C}} = R$ -Refl and  $I \circ F = \mathbb{W}_R$ , where  $R$  has the unique induced  $k$ -plethory structure with unit  $\eta$ .*

**Proof.** The forward direction of the equivalence is clear. Suppose that the second condition holds. Then, since  $F$  is represented by  $R$ , the  $k$ -algebra  $R$  has the unique structure of a  $k$ - $k$ -biring for which  $\mathbb{W}_R = I \circ F$ . Moreover, by [Lemma 3.2](#), the counit

$\mathbb{W}_R = I \circ F \rightarrow \text{id}_{k\text{-Alg}}$  of the given adjunction is given by evaluation at  $e$  for a unique ring-like element  $e$  of  $R$ . It follows, again from Lemma 3.2, that the unique map  $\eta : k[X] \rightarrow R$  of  $k$ -algebras sending  $X$  to  $e$  is a homomorphism of  $k$ - $k$ -birings. Since the map  $\mathbb{W}_R(R) \rightarrow R$  is evaluation at  $e = \eta(X)$  and is by assumption an isomorphism, it follows that  $R$  is  $\eta$ -reflective. Therefore, by Corollary 6.2,  $\eta : k[X] \rightarrow R$  is an idempotent  $k[X]$ -algebra, and by Corollary 5.5 one has  $\overline{\mathcal{C}} = R\text{-Refl}$ .  $\square$

By the above proposition, if  $\eta : k[X] \rightarrow R$  is an idempotent  $k[X]$ -algebra, then we may say unambiguously that a  $k$ -algebra is  $R$ -reflective if it is  $\eta$ -reflective. In particular, another  $k[X]$ -algebra structure  $\theta : k[X] \rightarrow R$  on  $R$  is idempotent if and only if  $R$  is  $\theta$ -reflective, if and only if there is a (necessarily unique) automorphism of  $R$  sending  $\eta(X)$  to  $\theta(X)$ . Thus, an idempotent  $k[X]$ -algebra  $\eta : k[X] \rightarrow R$  may be loosely identified with the  $k$ -algebra  $R$ . Such a  $k$ -algebra  $R$  has a set of distinguished elements, namely, the set of universal elements of the functor  $\text{Hom}_{k\text{-Alg}}(R, -)$ , or, equivalently, the orbit in  $R$  of  $e = \eta(X)$  under the action of the group  $\text{Aut}_{k\text{-Alg}}(R)$ , which is anti-isomorphic via the map  $\varphi \mapsto \varphi(e)$  to the group of *plethystic units* of  $R$ , that is, the group of units of the monoid  $R, \circ$ . For example, the group of plethystic units of  $k[X]$  is the group  $\{aX + b : a \in k^*, b \in k\}$  under  $\circ$ , which is isomorphic to  $k \rtimes k^*$ .

The following theorem, which immediately implies Theorem 2.10 of Section 2, summarizes the results of this section.

**Theorem 6.7.** *Let  $k$  be a ring and  $\eta : k[X] \rightarrow R$  a  $k[X]$ -algebra, and suppose that  $R$  is  $\eta$ -reflective. Then the following conditions are equivalent.*

1.  $\eta : k[X] \rightarrow R$  is an idempotent  $k[X]$ -algebra, that is,  $R$  has the structure, necessarily unique, of an idempotent  $k$ -plethory with unit  $\eta$ .
2.  $R$  has the structure, necessarily unique, of a  $k$ - $k$ -biring such that  $\eta$  is a homomorphism of  $k$ - $k$ -birings.
3.  $R$  represents an endofunctor  $F$  of  $k\text{-Alg}$  for which evaluation at  $\eta(X) \in R$  defines a natural transformation from  $F$  to  $\text{id}_{k\text{-Alg}}$ .
4. There exists a full subcategory  $\mathcal{D}$  of  $k\text{-Alg}$  such that  $R$  represents a right adjoint  $F_{\mathcal{D}}$  to the inclusion  $I$  from  $\mathcal{D}$  to  $k\text{-Alg}$  for which the counit  $I_{\mathcal{D}} \circ F_{\mathcal{D}} \rightarrow \text{id}_{k\text{-Alg}}$  acts by evaluation at  $\eta(X) \in R$ .
5. There exists a full subcategory  $\mathcal{C}$  of  $k\text{-Alg}$  such that  $R$  represents a right adjoint  $F_{\mathcal{C}}$  to the inclusion  $I_{\mathcal{C}}$  from  $\mathcal{C}$  to  $k\text{-Alg}$  such that the map  $F_{\mathcal{C}}(R) \rightarrow R$  is an isomorphism.
6.  $R$  has the structure of an idempotent  $k[X]$ -algebra for some ring homomorphism  $\theta : k[X] \rightarrow R$ .
7. The category of  $\eta$ -reflective  $k$ -algebras is a full, bicomplete, and bireflective subcategory of  $k\text{-Alg}$ .
8.  $R^{\otimes n}$  is an  $\eta$ -reflective  $k$ -algebra for all  $n$ .
9.  $R^{\otimes n}$  is an  $\eta$ -reflective  $k$ -algebra for  $0 \leq n \leq 3$ .



Suppose that the above conditions hold. Then  $\overline{C} = \overline{D} = \eta\text{-Refl}$  and  $I_C \circ F_C \cong F = \mathbb{W}_R = I_D \circ F_D$ . In particular,  $\eta\text{-Refl}$  is the largest subcategory  $\mathcal{D}$  of  $k\text{-Alg}$  satisfying (4) or  $\mathcal{C}$  of  $k\text{-Alg}$  satisfying (5). Moreover, there is a unique  $k$ -algebra automorphism  $\iota$  of  $R$  such that the correspondence  $- \circ \iota : I_C \circ F_C \longrightarrow I_D \circ F_D$  is a natural isomorphism, or alternatively such that  $\theta = \iota \circ \eta$ , and one has  $\iota = - \circ a$  for a unique plethystic unit  $a$  of  $R$ .

Let  $P$  be a  $k$ -plethory. If  $k'$  a  $P$ -ring, then it follows from the base change of plethories [5, 1.13] that  $k' \otimes_k P$  has the structure of a  $k'$ -plethory with unit given by  $k'[e] = k' \otimes_k k[e] \longrightarrow k' \otimes_k P$ . By the following proposition, whose proof is clear, plethory base changes respect idempotence.

**Proposition 6.8.** *Let  $k$  be a ring,  $R$  an idempotent  $k[X]$ -algebra, and  $k'$  an  $R$ -reflective  $k$ -algebra. Then  $R' = k' \otimes_k R$  is an idempotent  $k'[X]$ -algebra. Moreover, if  $A$  is an  $R$ -reflective  $k$ -algebra, then  $k' \otimes_k A$  is an  $R'$ -reflective  $k'$ -algebra; dually, if  $A'$  is an  $R'$ -reflective  $k'$ -algebra, then  $A'$  is  $R$ -reflective as a  $k$ -algebra; and, furthermore, the functor  $k' \otimes_k - : R\text{-Refl} \longrightarrow R'\text{-Refl}$  is a left adjoint to the restriction of scalars functor  $R'\text{-Refl} \longrightarrow R\text{-Refl}$ .*

## 7. Plethories of univariate polynomials

Recall that  $\text{Int}(k)$  for any ring  $k$  with total quotient ring  $K$  denotes the subring  $\{f \in K[X] : f(k) \subseteq k\}$  of  $K[X]$ . If  $\text{Int}(k)$  has the structure of a  $k$ -plethory with unit given by the canonical inclusion  $k[X] \longrightarrow \text{Int}(k)$ , then we denote by  $e$  the image of  $X$  in  $\text{Int}(k)$  and  $K[X]$ , so that  $\text{Int}(k) = \{f \in K[e] : f(k) \subseteq k\}$  as a  $k$ -plethory, and  $K[e] \cong K \otimes_k \text{Int}(k)$  is the trivial  $K$ -plethory. In this section we study the  $k$ -plethories contained in  $K[e]$ , which is the situation described in Proposition 2.6.

**Proof of Proposition 2.6.** Write  $\diamond$  for composition in  $P$  and  $\circ$  for composition of polynomials in  $K[e]$ . Let  $f, g \in P$ . There exists a non-zerodivisor  $c \in k$  so that  $cf \in k[e]$ . Then

$$c(f \diamond g) = (cf) \diamond g = (cf) \circ g = c(f \circ g)$$

in  $K[e]$ , and therefore  $f \diamond g = f \circ g$ . Now let  $\varphi : P \longrightarrow P$  be any  $k$ -algebra homomorphism with  $\varphi(e) = 0$ . Then  $\varphi(f) = f(0) = f \circ 0$  for all  $f \in k[X]$ . Let  $f \in P$ , so there exists a non-zerodivisor  $c \in k$  so that  $cf \in k[e]$ . Then

$$c\varphi(f) = \varphi(cf) = (cf)(0) = cf(0)$$

in  $K$ , whence  $\varphi(f) = f(0) = f \circ 0$ . Thus  $\varphi = 0$  in  $\mathbb{W}_P(P)$ . Therefore the  $k$ -algebra homomorphism  $\mathbb{W}_P(P) \longrightarrow P$  is injective, hence an isomorphism. Thus  $P$  is idempotent.

Finally, for all  $f \in P$  one has  $f(c) = f \circ c = \beta(c)(f) \in k$  for all  $c \in k$ , whence  $P$  is a subring of  $\text{Int}(k)$ .  $\square$

The following is a weak converse to [Proposition 2.6](#).

**Proposition 7.1.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $P$  be a  $k$ -plethory. Each of the following conditions implies the next.*

1.  $P$  is isomorphic to a  $k[e]$ -subalgebra of  $K[e]$ .
2.  $P$  is strongly idempotent and  $K$  is  $P$ -reflective.
3.  $P$  is idempotent and  $k$ -torsion-free and  $K$  is  $P$ -reflective.

Moreover, the three conditions are equivalent if  $K[e]$  is, up to isomorphism, the only idempotent  $K$ -plethory (which holds, for example, if  $k$  is a domain of characteristic zero).

**Proof.** Suppose that (1) holds. Then  $P$  is  $k$ -torsion-free, and if  $\varphi(e) = 0$  for some  $\varphi \in \mathbb{W}_P(A)$ , where  $A$  is a  $k$ -torsion-free  $k$ -algebra, then  $\varphi$  extends to the unique map  $K[e] \cong P \otimes_k K \rightarrow A \otimes_k K$  sending  $e$  to 0, which restricts to  $\varphi$  since  $P$  and  $A$  are  $k$ -torsion-free, and so  $\varphi = 0$  in  $\mathbb{W}_P(A)$ . Thus  $P$  is strongly idempotent. That  $K$  is  $P$ -reflective follows from [Lemma 7.3\(2\)](#) below. Thus, (1) implies (2). That (2) implies (3) follows from [Proposition 4.6](#). Finally, suppose that (3) holds and  $K[e]$  is the only trivial  $K$ -plethory. By [Proposition 6.8](#),  $K \otimes_k P$  has the structure of an idempotent  $K$ -plethory, and therefore  $K \otimes_k P$  is isomorphic as a  $K$ -plethory to  $K[e]$ . Then, since  $P$  is  $k$ -torsion-free, it follows that  $P$  is isomorphic to a  $k[e]$ -subalgebra of  $K[e]$ .  $\square$

**Remark 7.2.** The total quotient ring of  $k$  need not have the structure of a  $P$ -ring for every  $k$ -plethory  $P$ , even if  $k = \mathbb{Q}[X]$ . Let  $Q = \mathbb{Q}[e, f, f \circ f, \dots]$  be the  $\mathbb{Q}$ -plethory generated by a ring-like element  $f$  [[5, Example 2.7](#)]. A  $Q$ -ring is equivalently a  $\mathbb{Q}$ -algebra  $A$  together with an endomorphism  $f$  of  $A$ . Let  $k$  be the  $Q$ -ring  $\mathbb{Q}[X]$  with the endomorphism  $f$  sending  $X$  to 0. Consider the  $k$ -plethory  $P = k \otimes_{\mathbb{Q}} Q$ . A  $P$ -ring is equivalently a  $k$ -algebra  $A$  together with an endomorphism of  $A$  that is compatible with  $f$ , that is, that sends  $X$  to 0. Thus, the total quotient ring  $\mathbb{Q}(X)$  of  $k$  is not a  $P$ -ring since there is no endomorphism of  $\mathbb{Q}(X)$  sending  $X$  to 0.

Rings  $R$  between  $k[X]$  and  $K[X]$ , or  $k[X]$ -subalgebras of  $K[X]$ , are called *polynomial overrings of  $k[X]$* . For such  $k[X]$ -algebras  $R$  we have the following elementary characterizations of the  $k$ -torsion-free  $R$ -reflective  $k$ -algebras.

**Lemma 7.3.** *Let  $k$  be a ring with total quotient ring  $K$  and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ .*

1. A  $k$ -torsion-free  $k$ -algebra  $A$  is  $R$ -reflective if and only if for every  $a \in A$  there is a  $k$ -algebra homomorphism  $R \rightarrow A$  sending  $X$  to  $a$ , if and only if  $R \subseteq \text{Int}(A)$ .
2.  $A$  is  $R$ -reflective for any  $K$ -algebra  $A$ .
3.  $R$  is  $R$ -reflective if and only if  $R$  is closed under composition.
4.  $k$  is  $R$ -reflective if and only if  $R \subseteq \text{Int}(k)$ .
5. If  $R$  is  $R$ -reflective, then  $k$  is  $R$ -reflective if and only if  $R \cap K = k$ .

**Proof.** Clear.  $\square$

By the above lemma and [Theorem 6.7](#) we have the following.

**Proposition 7.4.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ . The following conditions are equivalent.*

1.  $R$  has the (necessarily unique) structure of a  $k$ -plethory such that the unit  $k[e] \rightarrow R$  is the natural inclusion.
2.  $R$  is closed under composition, and  $R$  has the (necessarily unique) structure of a  $k$ - $k$ -biring such that the inclusion  $k[X] \rightarrow R$  is a homomorphism of  $k$ - $k$ -birings.
3.  $R^{\otimes n}$  is  $R$ -reflective for  $0 \leq n \leq 3$ .
4.  $R$  is contained in  $\text{Int}(k)$  and is closed under composition and  $R^{\otimes 2}$  and  $R^{\otimes 3}$  are  $R$ -reflective.

Moreover, if these conditions hold for  $R = \text{Int}_l(k)$  for some overring  $l$  of  $k$ , then  $R = \text{Int}_l(k)$  is the largest  $k$ -plethory contained in  $l[X]$ .

There is for any  $k[X]$ -subalgebra  $R$  of  $K[X]$  and for any set  $\mathbf{X}$  a canonical  $k$ -algebra homomorphism  $\theta_{\mathbf{X}} : R^{\otimes \mathbf{X}} \rightarrow K[\mathbf{X}]$ , where the tensor power is over  $k$ . We write  $R^{(\otimes \mathbf{X})} = \text{im } \theta_{\mathbf{X}}$ , and if  $\mathbf{X}$  is of finite cardinality  $n$  we write  $R^{(\otimes n)} = R^{(\otimes \mathbf{X})}$ .

**Proposition 7.5.** *Let  $k$  be a ring with total quotient ring  $K$  and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$  with  $R \cap K = k$ . The following conditions are equivalent.*

1.  $R^{(\otimes \mathbf{X})}$  is closed under pre-composition by any element of  $R$  for any set  $\mathbf{X}$ .
2.  $R^{(\otimes \mathbf{X})}$  is  $R$ -reflective for every set  $\mathbf{X}$ .
3.  $R^{(\otimes n)}$  is  $R$ -reflective for some integer  $n > 1$ .
4.  $R^{(\otimes 2)}$  is  $R$ -reflective.
5.  $R$  is  $R$ -reflective, and for all  $f \in R$ , the polynomials  $f(X + Y)$  and  $f(XY)$  lie in  $R^{(\otimes 2)}$ , that is, they can be written as sums of polynomials of the form  $g(X)h(Y)$  for  $g, h \in R$ .
6.  $R$  is  $R$ -reflective, and the compositum of any collection of  $R$ -reflective  $k$ -algebras contained in some  $k$ -torsion-free  $k$ -algebra is again  $R$ -reflective.

Moreover, the above conditions hold if  $R$  is idempotent; and, conversely, if the above conditions hold and  $R^{\otimes n}$  is  $k$ -torsion-free for  $n = 2, 3$ , then  $R$  is idempotent.

**Proof.** Clearly we have  $(1) \Leftrightarrow (2) \Rightarrow (3)$  and  $(6) \Rightarrow (2)$ . Moreover, the last statement of the proposition is clear. Thus we need only show that  $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$ . Suppose that statement (3) holds, and let  $f(X, Y) \in R^{(\otimes 2)}$ . We may assume without loss of generality that the variables in  $R^{(\otimes n)}$  are  $X, Y, X_3, X_4, \dots, X_n$ , whence  $R^{(\otimes 2)}$  is a subring of  $R^{(\otimes n)}$  and  $f(X, Y) \in R^{(\otimes n)}$ . Now, let  $g \in R$ . Then by (3)  $g(f(X, Y))$  lies in  $R^{(\otimes n)}$ . Thus we can write

$$g(f(X, Y)) = \sum_i f_{i1}(X)f_{i2}(Y)f_{i3}(X_3) \cdots f_{in}(X_n),$$

where  $f_{ij} \in R$  for all  $i, j$ . Setting  $X_i = 0$  for all  $i > 2$ , we see that

$$g(f(X, Y)) = \sum_i f_{i1}(X)f_{i2}(Y)f_{i3}(0) \cdots f_{in}(0),$$

whence  $g(f(X, Y)) \in R^{(\otimes 2)}$ . Thus  $R^{(\otimes 2)}$  is an  $R$ -reflective  $k$ -algebra. Therefore we have  $(3) \Rightarrow (4)$ . The proof that  $(4) \Rightarrow (5)$  is similar. Suppose that statement (5) holds. To prove (6), it suffices to show that the compositum  $C$  of two  $R$ -reflective  $k$ -algebras  $B$  and  $B'$  of  $k$  contained some  $k$ -torsion-free  $k$ -algebra is again an  $R$ -reflective  $k$ -algebra. Let  $f \in R$ , and let  $b \in B$  and  $b' \in B'$ . By (5) the polynomials  $f(X + Y)$  and  $f(XY)$  can be written in the form  $\sum_{i=1}^n g_i(X)h_i(Y)$ , where the  $g_i$  and  $h_i$  are in  $R$ . It follows that  $f(b + b')$  and  $f(bb')$  lie in the compositum  $C$ . Since this holds for all  $b \in B$  and  $b' \in B'$ , we have  $f(C) \subseteq C$ . Therefore  $C$  is an  $R$ -reflective  $k$ -algebra.  $\square$

**Corollary 7.6.** *Let  $k$  be a ring with total quotient ring  $K$  and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ . If  $R$  is an idempotent  $k[X]$ -algebra, then  $R$  is closed under composition,  $R \subseteq \text{Int}(k)$ , and for all  $f \in R$  the polynomials  $f(X + Y)$  and  $f(XY)$  in  $K[X, Y]$  can be written as sums of polynomials of the form  $g(X)h(Y)$  for  $g, h \in R$ . Moreover, the converse holds if  $R^{\otimes n}$  is  $k$ -torsion-free for  $n = 2, 3$ .*

The above corollary provides rather explicit criteria for  $R$  to be an idempotent  $k[X]$ -algebra in the case where  $R^{\otimes 2}$  and  $R^{\otimes 3}$  are  $k$ -torsion-free (e.g., when  $R$  is flat as a  $k$ -module). It will be exploited later in this section to construct various  $k$ -plethories contained in  $K[e]$ .

Next we investigate the functors  $P \odot -$  and  $\mathbb{W}_P$  restricted to the category of  $k$ -torsion-free  $k$ -algebras for the idempotent  $k$ -plethories  $P$  contained in  $K[e]$ .

**Proposition 7.7.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ . Let  $A$  be a  $k$ -torsion-free  $k$ -algebra.*

1.  $A$  is contained in a smallest  $k$ -torsion-free  $R$ -reflective  $k$ -algebra  $w^R(A)$ , equal to the intersection of all  $R$ -reflective  $k$ -algebras containing  $A$  and contained in  $K \otimes_k A$ .
2. One has  $w^R(A) = A$  if and only if  $A$  is  $R$ -reflective.
3. One has  $w^R(A) \cong w^R(k[\mathbf{X}]) / (K \ker \varphi \cap w^R(k[\mathbf{X}]))$  for any surjective  $k$ -algebra homomorphism  $\varphi : k[\mathbf{X}] \rightarrow A$ .
4. The association  $A \mapsto w^R(A)$  defines a functor from the category of  $k$ -torsion-free  $k$ -algebras to the category of  $k$ -torsion-free  $R$ -reflective  $k$ -algebras that is a left adjoint to the inclusion functor.

**Proof.** The proof is similar to the proof of [21, Proposition 8.6] and the proof of Proposition 7.8 below.  $\square$

**Proposition 7.8.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ . Assume that  $R \cap K = k$  and that  $R^{(\otimes 2)}$  is an  $R$ -reflective  $k$ -algebra. (Equivalently, assume that the equivalent conditions of Proposition 7.5 hold.) Let  $A$  be a  $k$ -torsion-free  $k$ -algebra.*

1.  $A$  contains a largest  $R$ -reflective  $k$ -algebra  $w_R(A)$ , equal to the compositum of all  $R$ -reflective  $k$ -algebras contained in  $A$ .
2. One has  $w_R(A) = A$  if and only if  $A$  is  $R$ -reflective.
3. One has  $w_R(A) = \{a \in A : a = \varphi(X) \text{ for some } \varphi \in \text{Hom}_{k\text{-Alg}}(R, A)\}$ .
4. The association  $A \mapsto w_R(A)$  defines a functor from the category of  $k$ -torsion-free  $k$ -algebras to the category of  $k$ -torsion-free  $R$ -reflective  $k$ -algebras that is a right adjoint to the inclusion functor.

**Proof.**

1. This follows from Proposition 7.5 and the fact that  $k$  itself is an  $R$ -reflective  $k$ -algebra.
2. This is clear from (1).
3. Let  $a \in A$ . Suppose that  $a \in w_R(A)$ . Then there is a  $k$ -algebra homomorphism  $\psi : K[X] \rightarrow K$  sending  $f$  to  $f(a)$  for all  $f \in K[X]$ , where  $K$  is the quotient field of  $k$ , and  $\psi$  restricts to a  $k$ -algebra homomorphism  $\varphi : R \rightarrow w_R(A) \subseteq A$  sending  $X$  to  $a$ . Conversely, suppose that there exists a  $k$ -algebra homomorphism  $\varphi : R \rightarrow A$  sending  $X$  to  $a$ . Tensoring with  $K$  we see that  $\varphi$  is evaluation at  $a$ , that is,  $\varphi(f) = f(a) \in K \otimes_k A$  for all  $f \in R$ . Since  $\text{im } \varphi \subseteq A$  it follows that  $f(a) \in A$  for all  $f \in R$ . Thus we also have  $g(\varphi(f)) = g(f(a)) = \varphi(g \circ f) \in A$  for all  $f, g \in R$ . It follows that  $\text{im } \varphi \subseteq A$  is an  $R$ -reflective  $k$ -algebra and therefore  $a \in \text{im } \varphi \subseteq w_R(A)$ .
4. Functoriality follows easily from (3). To prove adjointness, we must show that the natural map

$$\text{Hom}_{k\text{-Alg}}(A, w_R(B)) \rightarrow \text{Hom}_{k\text{-Alg}}(A, B)$$

is a bijection for any  $k$ -torsion-free  $k$ -algebras  $A$  and  $B$ , where  $A$  is  $R$ -reflective. But this is clear from functoriality and (2).  $\square$

As a corollary of [Propositions 7.7 and 7.8](#), we obtain the following.

**Theorem 7.9.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $R$  be any  $k$ - $k$ -biring with  $k[X] \subseteq R \subseteq K[X]$  such that the inclusion  $k[X] \rightarrow R$  is a  $k$ - $k$ -biring homomorphism.*

1. *For any  $k$ -torsion-free  $k$ -algebra  $A$ , the  $k$ -algebra homomorphism  $R \odot A \rightarrow K \otimes_k A \cong K \otimes_k (R \odot A)$  acting by  $f \odot a \mapsto f(a)$  has image equal to  $w^R(A)$ . Therefore the functor  $w^R$  is isomorphic to the functor  $\mathbf{T}\text{-free}_k(R \odot -)$  restricted to the category of  $k$ -torsion-free  $k$ -algebras, where  $\mathbf{T}\text{-free}_k = \text{im}(- \rightarrow K \otimes_k -)$  denotes the left adjoint to the inclusion from the category of  $k$ -torsion-free  $k$ -algebras to  $k\text{-Alg}$ .*
2. *The map  $R \cong R \odot k[X] \rightarrow w^R(k[X])$  is an isomorphism. In particular,  $R$  is  $R$ -reflective and is therefore an idempotent  $k[X]$ -algebra. Moreover, the map  $R^{\otimes \mathbf{X}} \cong R \odot k[\mathbf{X}] \rightarrow w^R(k[\mathbf{X}]) = R^{(\otimes \mathbf{X})}$  is surjective for any set  $\mathbf{X}$  and is an isomorphism if and only if  $R^{\otimes \mathbf{X}}$  is  $k$ -torsion-free. For any surjective  $k$ -algebra homomorphism  $\varphi : k[\mathbf{X}] \rightarrow A$ , one has  $w^R(A) \cong R^{(\otimes \mathbf{X})} / K \ker \varphi \cap R^{(\otimes \mathbf{X})}$ .*
3. *For any  $k$ -torsion-free  $k$ -algebra  $A$ , the  $k$ -algebra homomorphism  $\mathbb{W}_R(A) \rightarrow A$  acting by  $\varphi \mapsto \varphi(X)$  is an inclusion with image equal to  $w_R(A)$ . In particular, the unique  $k$ -plethory structure on  $R$  with unit given by the inclusion  $k[X] \rightarrow R$  is strongly idempotent, and the functor  $w_R$  is isomorphic to the functor  $\mathbb{W}_R$  restricted to the category of  $k$ -torsion-free  $k$ -algebras.*

**Proof.** Statement (1) follows from [Proposition 7.7](#). The isomorphism  $R^{\otimes \mathbf{X}} \cong R \odot k[\mathbf{X}]$  of statement (2) follows from [\[5, Example 1.5\(1\)\]](#), from which it follows that the map  $R \cong R \odot k[X] \rightarrow w^R(k[X])$  is an isomorphism and therefore  $R$  is  $R$ -reflective and hence an idempotent  $k[X]$ -algebra by [Corollary 6.2](#). The rest of statement (2) then follows from statement (1) and [Proposition 7.7](#). Finally, statement (3) follows from [Proposition 7.8](#) and the definition of strong idempotence.  $\square$

Statement (2) of the theorem implies the following (cf., [Corollary 6.2](#) and [Proposition 7.4](#)).

**Corollary 7.10.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $R$  be a  $k[X]$ -subalgebra of  $K[X]$ . Then  $R$  is an idempotent  $k[X]$ -algebra if and only if  $R$  has the structure, necessarily unique, of a  $k$ - $k$ -biring such that the inclusion  $k[X] \rightarrow R$  is a  $k$ - $k$ -biring homomorphism, in which case  $R \cong w^R(k[X])$  is  $R$ -reflective, hence closed under composition.*

Thus, the  $k$ -plethories contained in  $K[e]$  are equivalently the  $k$ - $k$ -birings containing  $k[X]$  and contained in  $K[X]$ .

The remaining results of this section provide examples of  $k$ -plethories contained in  $K[e]$ . Let  $l$  be any overring of  $k$ , that is, a ring  $l$  with  $k \subseteq l \subseteq K$ . We define  $\text{Int}_l(k) =$

$\text{Int}(k) \cap l[X] = \{f \in l[X] : f(k) \subseteq k\}$  and  $\text{Int}_l(k^n) = \text{Int}(k^n) \cap l[X_1, X_2, \dots, X_n]$  for any positive integer  $n$ . Also, for any set  $\mathbf{X}$  we let  $\text{Int}_l(k^{\mathbf{X}}) = \{f \in l[\mathbf{X}] : f(k^{\mathbf{X}}) \subseteq k\}$ . The following result generalizes [Theorem 2.4](#).

**Theorem 7.11.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Each of the following conditions implies the next.*

1.  $k$  is a Krull domain.
2.  $k$  is a domain of Krull type.
3.  $k$  is a PVMD and  $\text{Int}(k_{\mathfrak{p}}) = \text{Int}(k)_{\mathfrak{p}}$  for every maximal ideal  $\mathfrak{p}$  of  $k$ .
4.  $\text{Int}_l(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
5.  $\text{Int}_l(k)$  is free as a  $k$ -module, or  $\text{Int}_l(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
6. For every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l(k)^{\otimes_{k^n}} \rightarrow \text{Int}_l(k^n)$  is an isomorphism.
7. The canonical  $k$ -algebra homomorphism  $\text{Int}_l(k)^{\otimes_{k^n}} \rightarrow \text{Int}_l(k^n)$  is an isomorphism for  $n = 2$  and an inclusion for  $n = 3$ .
8.  $\text{Int}_l(k)$  is an idempotent  $k[X]$ -algebra, that is, it has a unique  $k$ -plethory structure with unit given by the inclusion  $k[X] \rightarrow \text{Int}_l(k)$ .

**Proof.** (1)  $\Rightarrow$  (2). This is clear.

(2)  $\Rightarrow$  (3). Since a domain  $k$  of Krull type has finite  $t$ -character, one has  $\text{Int}(k)_{\mathfrak{p}} = \text{Int}(k_{\mathfrak{p}})$  for all  $\mathfrak{p}$  by [\[24, Proposition 2.4\]](#).

(3)  $\Rightarrow$  (4). Let  $\mathfrak{p}$  be a maximal ideal of  $k$ . Then

$$\text{Int}_l(k)_{\mathfrak{p}} = \text{Int}(k)_{\mathfrak{p}} \cap l_{\mathfrak{p}}[X] = \text{Int}(k_{\mathfrak{p}}) \cap l_{\mathfrak{p}}[X] = \text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}}).$$

If  $\text{Int}(k)_{\mathfrak{p}} = k_{\mathfrak{p}}[X]$ , then  $\text{Int}_l(k)_{\mathfrak{p}} = k_{\mathfrak{p}}[X]$  is free as a  $k_{\mathfrak{p}}$ -module. Suppose, on the other hand, that  $\text{Int}(k)_{\mathfrak{p}} \neq k_{\mathfrak{p}}[X]$ . Then  $\mathfrak{p}$  is  $t$ -maximal by [\[22, Proposition 3.3\]](#), so, since  $k$  is a PVMD,  $k_{\mathfrak{p}}$  is a valuation domain. Therefore, since  $\text{Int}(k_{\mathfrak{p}}) \neq k_{\mathfrak{p}}[X]$  and  $k_{\mathfrak{p}}$  is a valuation domain, by [\[10, Proposition I.3.16\]](#) the ideal  $\mathfrak{p}k_{\mathfrak{p}}$  of  $k_{\mathfrak{p}}$  is principal with finite residue field, generated, say, by  $\pi \in k_{\mathfrak{p}}$ . Since  $k_{\mathfrak{p}}$  is a valuation domain and  $k_{\mathfrak{p}} \subseteq l_{\mathfrak{p}} \subseteq K$ , where  $K$  is the quotient field of  $k$ , either (i)  $l_{\mathfrak{p}} = k_{\mathfrak{p}}$  or (ii)  $l_{\mathfrak{p}}$  is the localization of  $k_{\mathfrak{p}}$  at a prime ideal  $\mathfrak{q} \subsetneq (\pi)$  of  $k_{\mathfrak{p}}$  and so  $\frac{1}{\pi} \in l_{\mathfrak{p}}$ . In case (i),  $\text{Int}_l(k)_{\mathfrak{p}} = k_{\mathfrak{p}}[X]$  is free as a  $k_{\mathfrak{p}}$ -module. In case (ii), since  $k_{\mathfrak{p}}$  is a local domain with principal maximal ideal  $(\pi)$ , it follows from [\[10, Exercise II.16\]](#) that  $\text{Int}(k_{\mathfrak{p}})$  is freely generated as a  $k_{\mathfrak{p}}$ -module by polynomials with coefficients in  $k_{\mathfrak{p}}[\frac{1}{\pi}] \subseteq l_{\mathfrak{p}}$ , so  $\text{Int}(k_{\mathfrak{p}}) \subseteq l_{\mathfrak{p}}[X]$  and therefore  $\text{Int}_l(k)_{\mathfrak{p}} = \text{Int}(k_{\mathfrak{p}})$  is free as a  $k_{\mathfrak{p}}$ -module in that case as well.

(4)  $\Rightarrow$  (5). This is clear.

(5)  $\Rightarrow$  (6). If  $\text{Int}_l(k)$  is free as a  $k$ -module, then the argument in the proof of [\[10, Proposition XI.1.13\]](#) and also [\[21, Lemma 6.7\]](#), for example, shows that the canonical

map  $\theta_n : \text{Int}_l(k)^{\otimes_{k^n}} \rightarrow \text{Int}_l(k^n)$  is an isomorphism for all  $n$ . Suppose, on the other hand, that  $\text{Int}_l(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ . This implies that  $\text{Int}_l(k)$  is locally free, hence flat, as a  $k$ -module. Therefore the map  $\theta_n$  is injective and so induces an isomorphism onto its image,  $\text{Int}_l(k)^{(\otimes_{k^n})}$ . Given that  $\text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})$  is free as a  $k_{\mathfrak{p}}$ -module, we have  $\text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}}^n) = \text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})^{(\otimes_{k_{\mathfrak{p}}} n)}$ , and therefore

$$\text{Int}_l(k^n)_{\mathfrak{p}} \subseteq \text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}}^n) = \text{Int}_{l_{\mathfrak{p}}}(k_{\mathfrak{p}})^{(\otimes_{k_{\mathfrak{p}}} n)} = (\text{Int}_l(k)^{(\otimes_{k^n})})_{\mathfrak{p}} \subseteq \text{Int}_l(k^n)_{\mathfrak{p}},$$

for every maximal ideal  $\mathfrak{p}$  of  $k$ , whence  $\text{Int}_l(k^n) = \text{Int}_l(k)^{(\otimes_{k^n})}$ . It follows that  $\theta_n$  is an isomorphism for all  $n$  in this case as well.

(6)  $\Rightarrow$  (7). This is clear.

(7)  $\Rightarrow$  (8). Since by Lemma 7.3(1) the  $k$ -algebra  $\text{Int}_l(k^n)$  is  $\text{Int}_l(k)$ -reflective for any  $n$ , it follows from (7) that  $\text{Int}_l(k)^{(\otimes_{k^2})} = \text{Int}_l(k^2)$  is  $\text{Int}_l(k)$ -reflective. Condition (8) therefore follows from Corollary 7.6.  $\square$

Domains  $D$  such that  $\text{Int}(D_{\mathfrak{p}}) = \text{Int}(D)_{\mathfrak{p}}$  for all maximal ideals  $\mathfrak{p}$  of  $D$  are studied in [15,24], for example, and are said to be *polynomially L-regular*. (The “L” stands for “localization.”) A domain  $D$  satisfies condition (3) of Theorem 7.11 if and only if it is a polynomially L-regular PVMD. If a domain  $D$  is not polynomially L-regular, then the technique of localization is of limited use in studying  $\text{Int}(D)$ . However, any local domain is automatically polynomially L-regular, and for domains one has the implications Noetherian  $\Rightarrow$  Mori  $\Rightarrow$  TV  $\Rightarrow$  of finite  $t$ -character  $\Rightarrow$  polynomially L-regular [24], so the class of polynomially L-regular domains is substantial. Nevertheless, there exist *almost Dedekind domains*, that is, domains that are locally DVRs, that are not polynomially L-regular, or alternatively that are polynomially L-regular but not Dedekind and therefore not of Krull type [15]. In particular, the implications Krull type domain  $\Rightarrow$  polynomial L-regular PVMD  $\Rightarrow$  PVMD are not reversible. Moreover, the polynomially L-regular domains, the polynomially L-regular PVMDs, or even just the polynomially L-regular almost Dedekind domains [15], are not easily characterized.

**Corollary 7.12.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Suppose that for every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l(k)^{\otimes_{k^n}} \rightarrow \text{Int}_l(k^n)$  is an isomorphism (which holds, for example, if  $k$  is a domain of Krull type). Then  $\text{Int}_l(k^{\mathbf{X}})$  for any set  $\mathbf{X}$  has the unique structure of a  $k$ - $k$ -biring such that the inclusion  $k[\mathbf{X}] \rightarrow \text{Int}_l(k^{\mathbf{X}})$  is a homomorphism of  $k$ - $k$ -birings.*

**Remark 7.13.** Let  $k$  be a ring. Suppose that  $\text{Int}(k)$  is flat as a  $k$ -module and  $\text{Int}(k, k') = k' \text{Int}(k)$  for all flat  $k$ -algebras  $k'$ , which holds, for example, if  $k$  is a TV PVMD, by [24, Theorem 1.2]. By Theorem 7.11, Proposition 6.8, and the proof of [21, Theorem 3.12], one has the following.

1. For every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}(k)^{\otimes_{k^n}} \rightarrow \text{Int}(k^n)$  is an isomorphism, and therefore  $\text{Int}(k)$  is an idempotent  $k[X]$ -algebra.



2. Let  $k'$  be a flat  $\text{Int}(k)$ -reflective  $k$ -algebra. Then  $\text{Int}(k') = k'\text{Int}(k) = k' \otimes_k \text{Int}(k)$  is an idempotent  $k'[X]$ -algebra,  $\text{Int}(k')$  is flat as a  $k'$ -module, and  $\text{Int}(k', k'') = k''\text{Int}(k')$  for all flat  $k'$ -algebras  $k''$ . Moreover, a  $k'$ -algebra is  $\text{Int}(k')$ -reflective if and only if it is  $\text{Int}(k)$ -reflective as a  $k$ -algebra.

In the literature on integer-valued polynomial rings, no attention has been given to rings with zerodivisors. Using [Theorem 7.11](#) and the following proposition, we may construct idempotent  $k$ -plethories, even on  $\text{Int}_l(k)$ , for certain rings  $k$  with zerodivisors.

**Proposition 7.14.** *Let  $k = \prod_{i=1}^n k_i$ , where  $k_1, k_2, \dots, k_n$  are rings, and let  $R_i$  be a  $k_i[X]$ -algebra for all  $i$ . Then  $\prod_{i=1}^n R_i$  is an idempotent  $k[X]$ -algebra if and only if  $R_i$  is an idempotent  $k_i[X]$ -algebra for all  $i$ . In particular, the idempotent  $k[X]$ -algebras are precisely those isomorphic to  $\prod_{i=1}^n R_i$ , where each  $R_i$  is an idempotent  $k_i[X]$ -algebra.*

**Proof.** This follows from [Theorem 6.4](#) and the fact that  $(\prod_{i=1}^n R_i)^{\otimes m} \cong \prod_{i=1}^n R_i^{\otimes m}$  for all  $m$ .  $\square$

**Corollary 7.15.** *Let  $k = \prod_{i=1}^n k_i$  and  $l = \prod_{i=1}^n l_i$ , where each  $k_i$  is a ring and  $l_i$  is an overring of  $k_i$ . Then  $\text{Int}_l(k)$  is isomorphic as a  $k[X]$ -algebra to  $\prod_{i=1}^n \text{Int}_{l_i}(k_i)$ . Moreover,  $\text{Int}_l(k)$  is an idempotent  $k[X]$ -algebra if and only if  $\text{Int}_{l_i}(k_i)$  is an idempotent  $k_i[X]$ -algebra for all  $i$ . In particular, both conditions hold if  $k_i$  is a domain of Krull type, or more generally a polynomially  $L$ -regular PVMD, for all  $i$ .*

Let  $k$  be a ring with total quotient ring  $K$ . Let  $n$  be a positive integer, and let  $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^n$ , and let  $\infty = (\infty, \dots, \infty)$ . Let

$$\text{Int}^{(\mathbf{r})}(k^n) = \left\{ f \in K[X_1, \dots, X_n] : \frac{\partial^{i_1+\dots+i_n} f}{\partial X_1^{i_1} \dots \partial X_n^{i_n}} \in \text{Int}(k^n) \text{ if } i_k \leq r_k \text{ for all } k \right\},$$

and for any overring  $l$  of  $k$ , we let  $\text{Int}_l^{(\mathbf{r})}(k^n) = \text{Int}^{(\mathbf{r})}(k^n) \cap l[X_1, \dots, X_n]$ . Note that the rings  $\text{Int}_l^{(r)}(k)$  are closed under composition for all  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ .

**Example 7.16.**  $\text{Int}^{(\infty)}(\mathbb{Z})$  is free as a  $\mathbb{Z}$ -module with  $\mathbb{Z}$ -basis consisting of the polynomials  $c_0, c_1X, c_2\binom{X}{2}, c_3\binom{X}{3}, \dots$ , where  $c_n = \prod_{p \leq n} p^{\lfloor n/p \rfloor}$  for all  $n$  [[10, Corollary IX.3.6](#)], [[10, Remarks IX.4.9\(ii\)](#)].

**Lemma 7.17.** *Let  $k$  be a ring, and let  $l$  be an overring of  $k$  such that  $\text{Int}_l^{(\infty)}(k)$  is free as a  $k$ -module. Then the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{(\infty)}(k)^{\otimes_{k^n}} \rightarrow \text{Int}_l^{(\infty)}(k^n)$  is an isomorphism for all positive integers  $n$ .*

**Proof.** We prove the lemma for  $n = 2$ . The general case is similar. Let  $f_0, f_1, f_2, \dots$  be a  $k$ -basis of  $\text{Int}_l^{(\infty)}(k)$ . Then it is also an  $l$ -basis of  $l[X]$  and an  $l[Y]$ -basis of  $l[X, Y]$ . Let

$F \in \text{Int}_l^{(\infty)}(k^2) \subseteq l[X, Y]$ . Then there exist unique polynomials  $g_j(Y) \in l[Y]$  such that  $F = \sum_i f_i(X)g_i(Y)$ . Let  $a \in k$ . One has

$$\sum_i f_i^{(n)}(X)g_i^{(m)}(a) = \frac{\partial^{n+m} F}{\partial X^n \partial Y^m}(X, a) \in \text{Int}_l(k)$$

for all  $n, m$ , and therefore

$$\sum_i f_i(X)g_i^{(m)}(a) \in \text{Int}_l^{(\infty)}(k)$$

for all  $m$ . Since the  $f_i$  form a  $k$ -basis for  $\text{Int}_l^{(\infty)}(k)$  and an  $l$ -basis for  $l[X]$ , it follows that  $g_i^{(m)}(a) \in k$  for all  $i$ . Therefore  $g_i(Y) \in \text{Int}_l^{(\infty)}(k)$  for all  $i$ . Thus for all  $i$  we may write  $g_i(Y) = \sum_j a_{ij}f_j(Y)$  for some  $a_{ij} \in k$ , and therefore  $F = \sum_{i,j} a_{ij}f_i(X)f_j(Y)$ . It follows, then, that the polynomials  $f_i(X)f_j(Y)$  for all pairs  $i, j$  form a  $k$ -basis for  $\text{Int}_l^{(\infty)}(k^2)$ . Thus, the map  $\text{Int}_l^{(\infty)}(k)^{\otimes_{k^2}} \rightarrow \text{Int}_l^{(\infty)}(k^2)$  is onto. Moreover, it is injective since  $\text{Int}_l^{(\infty)}(k)$  is free, hence flat, as a  $k$ -module.  $\square$

**Lemma 7.18.** *Let  $k$  be a ring, and let  $l$  be an overring of  $k$  such that the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{(\infty)}(k)^{\otimes_{k^2}} \rightarrow \text{Int}_l^{(\infty)}(k^2)$  is an isomorphism for  $n = 2$  and an inclusion for  $n = 3$ . Then  $\text{Int}_l^{(\infty)}(k)$  is an idempotent  $k[X]$ -algebra.*

**Proof.** Let  $K$  be the total quotient ring of  $k$ , and let  $R = \text{Int}_l^{(\infty)}(k)$ . Note that  $R \cap K = k$ , so  $k$  is  $R$ -reflective, and  $R$  is  $R$ -reflective because it is closed under composition of polynomials. Let  $f \in R$ . Then clearly  $f(X+Y), f(XY) \in \text{Int}_l^{(\infty)}(k^2)$ , so both  $f(X+Y)$  and  $f(XY)$  are of the form  $\sum_i g_i(X)h_i(Y)$  for some  $g_i, h_i \in R$ . Moreover,  $R^{\otimes_{k^2} n}$  is  $k$ -torsion-free for  $n = 2, 3$ . Therefore, by Proposition 7.5,  $R$  is an idempotent  $k[X]$ -algebra.  $\square$

**Lemma 7.19.** *Let  $D$  be an integral domain and  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . If  $\text{Int}^{(r)}(D)_{\mathfrak{p}} \neq D_{\mathfrak{p}}[X]$  for some prime ideal  $\mathfrak{p}$  of  $D$ , then  $\mathfrak{p}$  is  $t$ -maximal with finite residue field. Moreover, if  $D$  is polynomially  $L$ -regular, then  $\text{Int}^{(r)}(S^{-1}D) = S^{-1}\text{Int}^{(r)}(D)$  for every multiplicative subset  $S$  of  $D$ .*

**Proof.** This follows from the corresponding well-known results for  $\text{Int}(D)$  (e.g., [22, Proposition 3.3] and [10, Proposition IX.1.4]).  $\square$

Using the preceding three lemmas, one can readily adapt the proof of Theorem 7.11 to yield the following.

**Theorem 7.20.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Each of the following conditions implies the next.*

1.  $k$  is a Krull domain.
2.  $k$  is a domain of Krull type such that  $\text{Int}^{(\infty)}(k_{\mathfrak{p}})$  is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
3.  $k$  is a PVMD and  $\text{Int}^{(\infty)}(k_{\mathfrak{p}}) = \text{Int}^{(\infty)}(k)_{\mathfrak{p}}$  is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
4.  $\text{Int}_l^{(\infty)}(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}^{(\infty)}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
5.  $\text{Int}_l^{(\infty)}(k)$  is free as a  $k$ -module, or  $\text{Int}_l^{(\infty)}(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}^{(\infty)}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
6. For every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{(\infty)}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{(\infty)}(k^n)$  is an isomorphism.
7. The canonical  $k$ -algebra homomorphism  $\text{Int}_l^{(\infty)}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{(\infty)}(k^n)$  is an isomorphism for  $n = 2$  and an inclusion for  $n = 3$ .
8.  $\text{Int}_l^{(\infty)}(k)$  is an idempotent  $k[X]$ -algebra.

Let  $l$  be an overring of a ring  $k$ . Let  $\mathbf{X}$  be a set,  $\mathbf{r} \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^{\mathbf{X}}$ , and  $\mathbf{E}$  a subset of  $K^{\mathbf{X}}$ . We define  $\text{Int}_l^{(\mathbf{r})}(\mathbf{E}, k) \subseteq K[\mathbf{X}]$  in the obvious way, and we set  $\text{Int}_l^{(\mathbf{r})}(k^{\mathbf{X}}) = \text{Int}_l^{(\mathbf{r})}(k^{\mathbf{X}}, k)$ . Then  $\text{Int}_l^{(\mathbf{r})}(\mathbf{E}, k)$  is an  $i$ -reflective  $k$ -algebra, where  $i : k[X] \longrightarrow \text{Int}_l^{(\infty)}(k)$  is the canonical inclusion. In particular, if  $\text{Int}_l^{(\infty)}(k)$  is a  $k$ -plethory with unit  $i$ , then  $\text{Int}_l^{(\mathbf{r})}(\mathbf{E}, k)$  is  $\text{Int}_l^{(\infty)}(k)$ -reflective.

**Corollary 7.21.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Suppose that for every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{(\infty)}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{(\infty)}(k^n)$  is an isomorphism (which holds, for example, if  $k$  is a Krull domain). Then  $\text{Int}_l^{(\infty)}(k^{\mathbf{X}})$  for any set  $\mathbf{X}$  has the unique structure of a  $k$ - $k$ -biring such that the inclusion  $k[X] \longrightarrow \text{Int}_l^{(\infty)}(k^{\mathbf{X}})$  is a homomorphism of  $k$ - $k$ -birings.*

**Proposition 7.22.** *Let  $k$  be a ring and  $r$  a positive integer.*

1.  $\text{Int}^{(r)}(k)^{(\otimes_k n)}$  is  $\text{Int}^{(r)}(k)$ -reflective for  $n = 0, 1$ .
2.  $\text{Int}^{(r)}(k) = \text{Int}^{(\infty)}(k)$  if and only if  $\text{Int}^{(r)}(k) = \text{Int}^{(r+1)}(k)$ , if and only if  $\text{Int}^{(r)}(k) = \text{Int}^{(s)}(k)$  for some integer  $s > r$ .
3. If  $\text{Int}^{(r)}(k)$  is an idempotent  $k[X]$ -algebra, then  $\text{Int}^{(r)}(k) = \text{Int}^{(\infty)}(k)$ .

**Proof.** Statements (1) and (2) are clear. Suppose  $\text{Int}^{(r)}(k) \neq \text{Int}^{(\infty)}(k)$ . By (2) we may choose  $f \in \text{Int}^{(r)}(k) - \text{Int}^{(2r)}(k)$ . Suppose to obtain a contradiction that  $\text{Int}^{(r)}(k)^{(\otimes_k 2)}$  is  $\text{Int}^{(r)}(k)$ -reflective. Then  $f(X + Y) \in \text{Int}^{(r)}(k)^{(\otimes_k 2)}$ . Thus we can write  $f(X + Y) = \sum_i g_i(X)h_i(Y)$  for some  $g_i, h_i \in \text{Int}^{(r)}(k)$ . Then  $f^{(2r)}(X + Y) = \sum_i g_i^{(r)}(X)h_i^{(r)}(Y)$ , whence  $f^{(2r)}(X) = \sum_i g_i^{(r)}(X)h_i^{(r)}(0) \in \text{Int}(k)$ , which is a contradiction. Therefore  $\text{Int}^{(r)}(k)^{(\otimes_k 2)}$  is not  $\text{Int}^{(r)}(k)$ -reflective, so by [Proposition 7.5](#) the  $k[X]$ -algebra  $\text{Int}^{(r)}(k)$  is not idempotent.  $\square$

**Corollary 7.23.** *Let  $D$  be a Krull domain of characteristic zero such that  $D_{\mathfrak{p}}$  has a finite residue field for some height one prime ideal  $\mathfrak{p}$  of  $D$ . Then  $\text{Int}^{(r)}(D)$  properly contains  $\text{Int}^{(r+1)}(k)$  and is therefore not an idempotent  $D[X]$ -algebra, for any positive integer  $r$ .*

**Proof.** Since  $D_{\mathfrak{p}}$  is a characteristic zero DVR with finite residue field, by [10, Lemma IX.2.12] the domain  $\text{Int}^{(r)}(D)_{\mathfrak{p}} = \text{Int}^{(r)}(D_{\mathfrak{p}})$  properly contains  $\text{Int}^{(r+1)}(D)_{\mathfrak{p}} = \text{Int}^{(r+1)}(D_{\mathfrak{p}})$ , and therefore  $\text{Int}^{(r)}(D)$  properly contains  $\text{Int}^{(r+1)}(D)$ , for all positive integers  $r$ .  $\square$

Note that if a ring  $k$  with total quotient ring  $K$  is of characteristic  $n > 0$ , then  $f^{(n)} = 0$  for all  $f \in K[X]$ , and so  $\text{Int}^{(n-1)}(k) = \text{Int}^{(\infty)}(k)$ .

We now provide examples of rings  $k$  such that  $\text{Int}(k)$  is not a  $k$ -plethory.

**Proposition 7.24.** *Let  $k$  be a ring, let  $k[\varepsilon] = k[T]/(T^2)$ , where  $\varepsilon$  denotes the image of  $T$  in  $k[T]/(T^2)$ , and let  $r$  be a nonnegative integer. Then  $\text{Int}^{(r)}(k[\varepsilon]) = \text{Int}^{(r+1)}(k) + \text{Int}^{(r)}(k)\varepsilon$ . Suppose that  $\text{Int}^{(r+1)}(k) \neq \text{Int}^{(\infty)}(k)$ . Then the ring  $\text{Int}^{(r)}(k[\varepsilon])$  is not an idempotent  $k[\varepsilon][X]$ -algebra.*

**Proof.** Let  $R = \text{Int}^{(r)}(k[\varepsilon])$ . The total quotient ring of  $k[\varepsilon]$  is the ring  $K[\varepsilon]$ , where  $K$  is the total quotient ring of  $k$ . The equality  $\text{Int}^{(r)}(k[\varepsilon]) = \text{Int}^{(r+1)}(k) + \text{Int}^{(r)}(k)\varepsilon$  as subrings of  $K[\varepsilon][X] = K[X][\varepsilon]$  is proved in [26] and is straightforward to verify from the fact that  $f(a + b\varepsilon) = f(a) + f'(a)b\varepsilon$  for all  $f \in K[X]$  and all  $a, b \in K$ . It follows from this that  $R^{(\otimes_k 2)} = \text{Int}^{(r+1)}(k)^{(\otimes_k 2)} + M\varepsilon$  for some  $k[X, Y]$ -submodule  $M$  of  $K[X, Y]$ . We may choose  $f \in \text{Int}^{(r+1)}(k) - \text{Int}^{(2(r+1))}(k)$ . Suppose to obtain a contradiction that  $R^{(\otimes_k 2)}$  is  $R$ -reflective. Then, since  $f \in R$ , one has  $f(X + Y) \in R^{(\otimes_k 2)}$ , whence  $f(X + Y) \in \text{Int}^{(r+1)}(k)^{(\otimes_k 2)}$ . Thus we can write  $f(X + Y) = \sum_i g_i(X)h_i(Y)$  for some  $g_i, h_i \in \text{Int}^{(r+1)}(k)$ , which implies  $f^{(2(r+1))}(X + Y) = \sum_i g_i^{(r+1)}(X)h_i^{(r+1)}(Y)$  and therefore  $f^{(2(r+1))}(X) = \sum_i g_i^{(r+1)}(X)h_i^{(r+1)}(0) \in \text{Int}(k)$ , a contradiction.  $\square$

**Corollary 7.25.** *For any ring  $k$ , the  $k$ -algebra  $k[\varepsilon]$  is  $\text{Int}^{(r)}(k)$ -reflective for any  $r \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ , but it is  $\text{Int}(k)$ -reflective if and only if  $\text{Int}(k) = \text{Int}^{(\infty)}(k)$ .*

**Corollary 7.26.** *Let  $D$  be a Krull domain of characteristic zero such that  $D_{\mathfrak{p}}$  has a finite residue field for some height one prime ideal  $\mathfrak{p}$  of  $D$ . Then  $\text{Int}(D[\varepsilon])$  is not an idempotent  $D[\varepsilon][X]$ -algebra.*

**Conjecture 7.27.** *There exists an integral domain  $D$  such that  $\text{Int}(D)$  is not an (idempotent)  $D$ -plethory.*

As a corollary of Proposition 7.24, for any ring  $k$  one has

$$\text{Int}^{(\infty)}(k[\varepsilon]) = \text{Int}^{(\infty)}(k)[\varepsilon] \cong k[\varepsilon] \otimes_k \text{Int}^{(\infty)}(k).$$

Moreover,  $k[\varepsilon]$  is an  $\text{Int}^{(\infty)}(k)$ -reflective  $k$ -algebra since  $\text{Int}^{(\infty)}(k) \subseteq \text{Int}(k[\varepsilon])$ . Therefore, by [Proposition 6.8](#), we have the following.

**Proposition 7.28.** *Let  $k$  be a ring. If  $\text{Int}^{(\infty)}(k)$  is an idempotent  $k[X]$ -algebra, then  $\text{Int}^{(\infty)}(k[\varepsilon]) = \text{Int}^{(\infty)}(k)[\varepsilon]$  is an idempotent  $k[\varepsilon][X]$ -algebra.*

For any ring  $k$  with total quotient ring  $K$  and integral closure  $\bar{k}$  (in  $K$ ), the ring  $k[\varepsilon]$  has total quotient ring  $K[\varepsilon]$  and integral closure  $\bar{k} + \varepsilon K[\varepsilon]$ . One has

$$\begin{aligned} \text{Int}^{(r)}(k + \varepsilon K[\varepsilon]) &= \text{Int}^{(r)}(k) + \varepsilon K[X] \\ &= (k + \varepsilon K[\varepsilon])\text{Int}^{(r)}(k) \\ &\cong (k + \varepsilon K[\varepsilon]) \otimes_k \text{Int}^{(r)}(k) \end{aligned}$$

for any  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , where the given isomorphism holds since  $k + \varepsilon K[\varepsilon] \cong k \oplus K$  is flat as a  $k$ -module. Moreover,  $k + \varepsilon K[\varepsilon]$  is an  $\text{Int}^{(r)}(k)$ -reflective  $k$ -algebra since  $\text{Int}^{(r)}(k) \subseteq \text{Int}(k + \varepsilon K[\varepsilon])$ . Therefore, by [Proposition 6.8](#), we also have the following.

**Proposition 7.29.** *Let  $k$  be a ring with total quotient ring  $K$ , and let  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . If  $\text{Int}^{(r)}(k)$  is an idempotent  $k[X]$ -algebra, then  $\text{Int}^{(r)}(k + \varepsilon K[\varepsilon]) = \text{Int}^{(r)}(k) + \varepsilon K[X]$  is an idempotent  $(k + \varepsilon K[\varepsilon])[X]$ -algebra.*

One can show that the ring  $k + \varepsilon K[\varepsilon]$  for any Krull domain  $k$  is an example of a Krull ring with zerodivisors, in the sense of [\[31\]](#). Given the above proposition, it is reasonable to conjecture the following.

**Conjecture 7.30.**  *$\text{Int}(k)$  is an idempotent  $k[X]$ -algebra for any Krull ring  $k$ .*

Finally, we provide analogues of [Theorem 7.20](#) through [Corollary 7.23](#) for the rings  $\text{Int}^{[r]}(k)$  of polynomials in  $K[X]$  that along with all of their *finite differences of order up to  $r$*  are integer-valued on  $k$ , for  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , as studied for integral domains in [\[10, Chapter IX\]](#), [\[16\]](#) and defined for general rings  $k$  as follows. For all  $f \in k[X]$ , we may write  $f(X + Y) - f(X) = Yg(X, Y)$  for a unique  $g \in k[X, Y]$ . We write  $\Delta_Y f(X) = g(X, Y) \in k[Y][X]$ , so that  $\Delta_Y f(X)$  denotes  $\frac{f(X+Y) - f(X)}{Y}$  but is a polynomial in  $X$  and  $Y$ . We may then define  $\Delta_h f(X) = g(X, h) \in k[X]$  for all  $h \in k$ . One has  $\Delta_Y f(X) = f'(X) + YG(X, Y)$  for some  $G \in k[X, Y]$ , and therefore  $\Delta_0 f(X) = f'(X)$ . One has the following generalization of the product and chain rules for derivatives:

$$\Delta_Y(f \cdot g)(X) = \Delta_Y f(X) \cdot g(X + Y) + f(X) \cdot \Delta_Y g(X)$$

and

$$\Delta_Y(f \circ g)(X) = (\Delta_{g(X+Y)-g(X)} f)(g(X)) \Delta_Y g(X)$$

for all  $f, g \in k[X]$ . We let

$$\text{Int}^{[1]}(k) = \{f \in \text{Int}(k) : \Delta_h f \in \text{Int}(k) \text{ for all } h \in k\}.$$

More generally, we let  $\text{Int}^{[0]}(k) = \text{Int}(k)$  and for all positive integers  $r$  we let

$$\begin{aligned} \text{Int}^{[r]}(k) &= \{f \in \text{Int}(k) : \Delta_h f \in \text{Int}^{[r-1]}(k) \text{ for all } h \in k\} \\ &= \{f \in K[X] : \Delta_{h_1} \cdots \Delta_{h_s} f \in \text{Int}(k) \text{ for all } s \leq r \text{ and } h_1, \dots, h_s \in k\}, \end{aligned}$$

and we let  $\text{Int}^{[\infty]}(k) = \bigcap_{r=0}^{\infty} \text{Int}^{[r]}(k)$ . The generalized product and chain rules allow one to show that  $\text{Int}^{[r]}(k)$  is a  $k[X]$ -subalgebra of  $\text{Int}^{(r)}(k)$  that is also closed under composition, for all  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . Moreover, these rings generalize to analogues  $\text{Int}_l^{[r]}(k^n)$  of the rings  $\text{Int}_l^{(r)}(k^n)$  for all  $\mathbf{r} \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^n$  in the obvious way.

Although one has  $\text{Int}^{[\infty]}(k) = \text{Int}^{(\infty)}(k)$  for  $k = \mathbb{Z}$ , the equality fails for many number rings  $k$ , including  $k = \mathbb{Z}[i]$ . For example, equality holds for  $k = \mathbb{Z}[\zeta]$ , where  $\zeta$  is a primitive  $n$ th root of unity, if and only if  $n$  is squarefree; and equality holds for  $k = \mathcal{O}_K$ , where  $K = \mathbb{Q}(\sqrt{d})$  and  $d$  is a squarefree integer, if and only if  $d$  is congruent to 1 modulo 4 [10, Exercise IX.15]. Dedekind domains (resp., number rings)  $k$  for which  $\text{Int}^{[\infty]}(k) = \text{Int}^{(\infty)}(k)$  are characterized in [10, Theorem IX.2.16] (resp., [10, Corollary IX.2.17 and Remark IX.2.18]).

The proofs of Theorems 7.11 and 7.20 can be adapted to yield the following.

**Theorem 7.31.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Each of the following conditions implies the next.*

1.  $k$  is a Krull domain.
2.  $k$  is a domain of Krull type such that  $\text{Int}^{[\infty]}(k_{\mathfrak{p}})$  is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
3.  $k$  is a PVMD and  $\text{Int}^{[\infty]}(k_{\mathfrak{p}}) = \text{Int}^{[\infty]}(k)_{\mathfrak{p}}$  is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
4.  $\text{Int}_l^{[\infty]}(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}^{[\infty]}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
5.  $\text{Int}_l^{[\infty]}(k)$  is free as a  $k$ -module, or  $\text{Int}_l^{[\infty]}(k)_{\mathfrak{p}}$  is equal to  $\text{Int}_{l_{\mathfrak{p}}}^{[\infty]}(k_{\mathfrak{p}})$  and is free as a  $k_{\mathfrak{p}}$ -module for every maximal ideal  $\mathfrak{p}$  of  $k$ .
6. For every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{[\infty]}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{[\infty]}(k^n)$  is an isomorphism.
7. The canonical  $k$ -algebra homomorphism  $\text{Int}_l^{[\infty]}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{[\infty]}(k^n)$  is an isomorphism for  $n = 2$  and an inclusion for  $n = 3$ .
8.  $\text{Int}_l^{[\infty]}(k)$  is an idempotent  $k[X]$ -algebra.

**Corollary 7.32.** *Let  $k$  be a ring and  $l$  an overring of  $k$ . Suppose that for every positive integer  $n$  the canonical  $k$ -algebra homomorphism  $\text{Int}_l^{[\infty]}(k)^{\otimes_k n} \longrightarrow \text{Int}_l^{[\infty]}(k^n)$  is an iso-*

morphism (which holds, for example, if  $k$  is a Krull domain). Then  $\text{Int}_t^{[\infty]}(k^{\mathbf{X}})$  for any set  $\mathbf{X}$  has the unique structure of a  $k$ - $k$ -biring such that the inclusion  $k[\mathbf{X}] \longrightarrow \text{Int}_t^{[\infty]}(k^{\mathbf{X}})$  is a homomorphism of  $k$ - $k$ -birings.

**Proposition 7.33.** *Let  $k$  be a ring and  $r$  a positive integer.*

1.  $\text{Int}^{[r]}(k)^{(\otimes_k n)}$  is  $\text{Int}^{[r]}(k)$ -reflective for  $n = 0, 1$ .
2.  $\text{Int}^{[r]}(k) = \text{Int}^{[\infty]}(k)$  if and only if  $\text{Int}^{[r]}(k) = \text{Int}^{[r+1]}(k)$ , if and only if  $\text{Int}^{[r]}(k) = \text{Int}^{[s]}(k)$  for some integer  $s > r$ .
3. If  $\text{Int}^{[r]}(k)$  is an idempotent  $k[X]$ -algebra, then  $\text{Int}^{[r]}(k) = \text{Int}^{[\infty]}(k)$ .

**Corollary 7.34.** *Let  $D$  be a Krull domain such that  $D_{\mathfrak{p}}$  has a finite residue field, say, of characteristic  $p$ , for some height one prime ideal  $\mathfrak{p}$  of  $D$ . Then  $\text{Int}^{[r]}(D)$  is not an idempotent  $D[X]$ -algebra for any positive integer  $r$  less than  $p - 1$ .*

**Proof.** If  $1 \leq r < p - 1$ , then  $\text{Int}^{[r+1]}(D_{\mathfrak{p}}) \neq \text{Int}^{[r]}(D_{\mathfrak{p}})$  by [10, Lemma IX.2.12]. The corollary follows, then, as in the proof of Corollary 7.23.  $\square$

**Corollary 7.35.** *Let  $D$  be a Krull domain that has finite residue fields of arbitrarily large characteristic at the height one primes of  $D$ . Then  $\text{Int}^{[r]}(D)$  is not an idempotent  $D[X]$ -algebra for any positive integer  $r$ .*

If  $D$  is a DVR, or more generally a UFD, then  $\text{Int}^{(r)}(D)$  and  $\text{Int}^{[r]}(D)$  are free as  $D$ -modules for all  $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  [10, Exercise IX.18]. Moreover, if  $D$  is a valuation domain, then  $\text{Int}(D) \neq D[X]$  if and only if the maximal ideal of  $D$  is principal with finite residue field.

**Problem 7.36.** For which valuation domains  $D$  is  $\text{Int}^{(\infty)}(D)$  (resp.,  $\text{Int}^{[\infty]}(D)$ ) free as a  $D$ -module? (They are both necessarily free if  $D$  is a DVR, if  $D$  has a non-principal maximal ideal, or if  $D$  has an infinite residue field.)

If  $\text{Int}^{(\infty)}(D)$  (resp.,  $\text{Int}^{[\infty]}(D)$ ) is in fact free as a  $D$ -module for all valuation domains  $D$ , then, like Theorem 7.11, the final conclusion of Theorem 7.20 (resp., Theorem 7.31) holds for all polynomially L-regular PVMDs, hence all domains of Krull type.

## 8. Integer-valued polynomial rings

In this section we apply the results of Sections 4–7 to the ring  $\text{Int}(D)$ , where  $D$  is any integral domain. As we noted already, Theorem 6.7 immediately implies Theorem 2.10, and this to a certain degree solves Problem 2.3. Moreover, since  $D[X] \subseteq \text{Int}(D) \subseteq K[X]$ , where  $K$  is the quotient field of  $D$ , and since  $\text{Int}(D)^{\otimes n}$  is  $\text{Int}(D)$ -reflective for  $n = 0, 1$ , all of the results in the previous section on rings of univariate polynomials apply.

In [25] we defined a  $D$ -algebra  $A$  to be *weakly polynomially complete*, or *WPC*, if for every  $a \in A$  there exists a  $D$ -algebra homomorphism  $\text{Int}(D) \rightarrow A$  sending  $X$  to  $a$ . A  $D$ -torsion-free  $D$ -algebra  $A$  is WPC if and only if  $A$  is  $\text{Int}(D)$ -reflective, if and only if  $\text{Int}(D) \subseteq \text{Int}(A)$ . A ring is *quasi-binomial* if it is a WPC  $\mathbb{Z}$ -algebra, or, equivalently, if it is a quotient of a binomial ring. In [25, Sections 6–8] we proved a number of generalizations of the results in [20] on binomial and quasi-binomial rings to the WPC  $D$ -algebras. However, the following problem is still open.

**Problem 8.1** ([21, Section 7], [25, Section 6]). Let  $D$  be an integral domain. Is every WPC  $D$ -algebra a quotient of some  $\text{Int}(D)$ -reflective  $D$ -algebra?

The term “WPC,” though unfortunate, was motivated as follows. A subset  $S$  of an integral domain  $A$  such that  $\text{Int}(S, A) = \text{Int}(A)$  is said to be *polynomially dense* in  $A$ . Equivalently,  $S$  is polynomially dense in  $A$  if any polynomial with coefficients in the quotient field of  $A$  that maps  $S$  to  $A$  also maps  $A$  to  $A$ . If  $D$  is polynomially dense in an extension  $A$ , then, dually, and for lack of a better term,  $A$  is in some sense polynomially “complete” over  $D$ . Thus we defined, in [21], a domain extension  $A$  of a domain  $D$  to be *polynomially complete*, or *PC*, if  $D$  is polynomially dense in  $A$ . The WPC condition is in turn a relaxation of the PC condition. Since these terms relate to several similarly defined notions defined elsewhere, we will continue to use them here.

If  $D$  is not finite, then for any set  $\mathbf{X}$  the domain  $\text{Int}(D^{\mathbf{X}})$  is the free PC extension of  $D$  generated by  $\mathbf{X}$  [21, Proposition 2.4], and it is also the *polynomial completion with respect to  $D$*  of  $D[\mathbf{X}]$  [21, Proposition 8.2]. In [21], the smallest subring of  $\text{Int}(D^{\mathbf{X}})$  containing  $D[\mathbf{X}]$  that is closed under pre-composition by elements of  $\text{Int}(D)$  was denoted  $\text{Int}_w(D^{\mathbf{X}})$ . In the notation of Proposition 7.7, this is just  $w^{\text{Int}(D)}(D[\mathbf{X}])$ . For any domain  $D$  (finite or infinite), the domain  $w^{\text{Int}(D)}(D[\mathbf{X}])$  is the free WPC extension of  $D$  generated by  $\mathbf{X}$  [21, Proposition 7.2]. It is also the *weak polynomial completion with respect to  $D$*  of  $D[\mathbf{X}]$ , as defined in [21, Section 8].

A  $D$ -algebra  $A$  is said to be *almost polynomially complete*, or *APC*, if for every set  $\mathbf{X}$  and for any  $(a_X)_{X \in \mathbf{X}} \in A^{\mathbf{X}}$  there exists a  $D$ -algebra homomorphism  $\text{Int}(D^{\mathbf{X}}) \rightarrow A$  sending  $X$  to  $a_X$  for all  $X \in \mathbf{X}$ . Equivalently,  $A$  is APC if and only if  $A$  is a  $D$ -algebra quotient of  $\text{Int}(D^{\mathbf{X}})$  for some set  $\mathbf{X}$ , if and only if  $A$  is a quotient of some  $\text{Int}(D)$ -reflective  $D$ -algebra. By [21, Propositions 7.4 and 7.7], if  $A$  is a domain extension of  $D$ , then  $A$  is APC if and only if  $\text{Int}(D^n) \subseteq \text{Int}(A^n)$  for all positive integers  $n$ . Any PC domain extension of  $D$  is APC, but the converse is false since the extension  $\mathbb{Z}[T/2]$  of  $\mathbb{Z}[T]$  is APC but not PC [21, Proposition 7.2 and Example 7.3]. Clearly any APC  $D$ -algebra is WPC. We suspect that the converse does not hold but do not know a counterexample. Also,  $\text{Int}(D^{\mathbf{X}})$  is the free APC extension of  $D$  generated by  $\mathbf{X}$  [21, Proposition 7.7] (whether or not  $D$  is infinite). It is also the *almost polynomial completion with respect to  $D$*  of  $D[\mathbf{X}]$  [21, Section 8].

In analogy with ordinary polynomial rings, there is for any set  $\mathbf{X}$  a canonical  $D$ -algebra homomorphism  $\theta_{\mathbf{X}} : \text{Int}(D)^{\otimes \mathbf{X}} \rightarrow \text{Int}(D^{\mathbf{X}})$ , where the tensor power is over  $D$ . There



are several classes of domains for which  $\theta_{\mathbf{X}}$  is an isomorphism for all  $\mathbf{X}$ , such as the Krull domains, the almost Newtonian domains [21, Section 5], and the polynomially L-regular PVMDs, hence the domains of Krull type as well. However, we do not know whether or not  $\theta_{\mathbf{X}}$  is an isomorphism for all  $\mathbf{X}$  for every domain  $D$ , in that neither a proof nor a counterexample is known. As in [22–24] we say that a domain  $D$  is *polynomially composite* if  $\theta_{\mathbf{X}}$  is an isomorphism for all  $\mathbf{X}$ . [21, Section 6], [22, Section 4], and [24, Section 3.3] provide several known classes of polynomially composite domains. Most notably, the condition holds if  $\text{Int}(D)$  is free as a  $D$ -module, or if  $D$  is polynomially L-regular and  $\text{Int}(D)$  is locally free as a  $D$ -module, or if  $D$  is polynomially F-regular [24] and  $\text{Int}(D)$  is flat as a  $D$ -module. By Theorem 2.4, if  $D$  is polynomially composite, then  $\text{Int}(D)$  is an idempotent  $D[X]$ -algebra. However, an *a priori* weaker condition is relevant here. If  $\text{Int}(D)^{(\otimes \mathbf{X})}$  denotes the image of  $\theta_{\mathbf{X}}$ , then we have  $\text{Int}(D)^{(\otimes \mathbf{X})} \subseteq w^{\text{Int}(D)}(D[\mathbf{X}])$ , and equality holds for a given set  $\mathbf{X}$  if and only if  $\text{Int}(D)^{(\otimes \mathbf{X})}$  is a WPC extension of  $D$  (or equivalently, is  $\text{Int}(D)$ -reflective). If equality holds for all  $\mathbf{X}$  then we will say that  $D$  is *weakly polynomially composite*. Proposition 7.5 implies the following.

**Proposition 8.2.** *The following conditions are equivalent for any domain  $D$ .*

1.  $D$  is weakly polynomially composite.
2.  $\text{Int}(D)^{(\otimes \mathbf{X})}$  is a WPC extension of  $D$  for every set  $\mathbf{X}$ .
3.  $\text{Int}(D)^{(\otimes n)}$  is a WPC extension of  $D$  for some integer  $n > 1$ .
4.  $\text{Int}(D)^{(\otimes 2)}$  is a WPC extension of  $D$ .
5. For all  $f \in \text{Int}(D)$ , the polynomials  $f(X + Y)$  and  $f(XY)$  lie  $\text{Int}(D)^{(\otimes 2)}$ , that is, they can be written as sums of polynomials of the form  $g(X)h(Y)$  for  $g, h \in \text{Int}(D)$ .
6. The compositum of any collection of WPC  $D$ -algebras contained in some  $D$ -torsion-free  $D$ -algebra is again WPC.

Moreover,  $D$  is weakly polynomially composite if  $\text{Int}(D)$  is an idempotent  $D[X]$ -algebra, and the converse holds if  $\text{Int}(D)^{\otimes n}$  is  $D$ -torsion-free for  $n = 2, 3$ .

**Corollary 8.3.** *Let  $D$  be an integral domain. Suppose that  $\text{Int}(D)^{\otimes n}$  is  $D$ -torsion-free for  $n = 2, 3$ . Then  $\text{Int}(D)$  is an idempotent  $D[X]$ -algebra if and only if  $\text{Int}(D)^{(\otimes 2)}$  is a WPC extension of  $D$  (or equivalently, for all  $f \in \text{Int}(D)$  the polynomials  $f(X + Y)$  and  $f(XY)$  can be written as sums of polynomials of the form  $g(X)h(Y)$  for  $g, h \in \text{Int}(D)$ ).*

By the following proposition, if  $D$  is weakly polynomially composite, then the map  $\theta_{\mathbf{X}} : \text{Int}(D)^{\otimes \mathbf{X}} \rightarrow \text{Int}(D^{\mathbf{X}})$  is “almost” surjective. More precisely,  $\theta_{\mathbf{X}}$  is surjective for every set  $\mathbf{X}$  if and only if every WPC extension of  $D$  is APC and  $D$  is weakly polynomially composite. By [24, Theorem 3.11], surjectivity follows if  $D$  is polynomially  $t$ -regular or polynomially L- $t$ -regular [24, Section 3.1].

**Proposition 8.4.** *The following conditions are equivalent for any infinite integral domain  $D$  with quotient field  $K$ .*

1.  $\theta_{\mathbf{X}}$  is surjective for every set  $\mathbf{X}$ .
2.  $\theta_{\mathbf{X}}$  is surjective for some infinite set  $\mathbf{X}$ .
3.  $\theta_{\mathbf{X}}$  is surjective for every finite set  $\mathbf{X}$ .
4.  $\text{Int}(\text{Int}(D^{\mathbf{X}}))$  is the  $\text{Int}(D^{\mathbf{X}})$ -module generated by  $\text{Int}(D)$  for every (finite) set  $\mathbf{X}$ .
5.  $\text{Int}(D)^{(\otimes \mathbf{X})}$  is a PC extension of  $D$  for every (finite) set  $\mathbf{X}$ .
6. One has  $\text{Int}(D)^{(\otimes \mathbf{X})} = w^{\text{Int}(D)}(D[\mathbf{X}]) = \text{Int}(D^{\mathbf{X}})$  for every (finite) set  $\mathbf{X}$ .
7. For any element  $f$  of  $\text{Int}(D)$ , the polynomials  $f(X+Y)$  and  $f(XY)$  lie in  $\text{Int}(D)^{\otimes 2}$ , and for any  $n$  the domain  $\text{Int}(D^n)$  is the smallest subring of  $K[X_1, X_2, \dots, X_n]$  containing  $D[X_1, X_2, \dots, X_n]$  that is closed under pre-composition by elements of  $\text{Int}(D)$ .
8.  $D$  is weakly polynomially composite, and every WPC domain extension of  $D$  is APC.

**Proof.** The first four conditions are equivalent by [21, Proposition 6.3]. Conditions (1) and (5) are equivalent because  $\text{Int}(D^{\mathbf{X}})$  is the polynomial completion of  $D[\mathbf{X}]$  with respect to  $\mathbf{X}$  [21, Example 8.3]. Conditions (1) and (6) are equivalent because  $\text{im } \theta_{\mathbf{X}} = \text{Int}(D)^{(\otimes \mathbf{X})} \subseteq w^{\text{Int}(D)}(D[\mathbf{X}]) \subseteq \text{Int}(D^{\mathbf{X}})$ . Conditions (6) and (7) are equivalent by Proposition 8.2 and the definition of  $w^{\text{Int}(D)}(D[\mathbf{X}])$ . Finally, conditions (6) and (8) are equivalent by [21, Proposition 7.9].  $\square$

Whether or not  $\theta_{\mathbf{X}}$  is injective for every set  $\mathbf{X}$  depends on properties of the  $D$ -module  $\text{Int}(D)$ . In particular, injectivity certainly follows if  $\text{Int}(D)$  is assumed flat as a  $D$ -module. The flatness hypothesis has been shown useful for studying integer-valued polynomial rings. (See, for example, [21, Propositions 6.8 and 7.10 and Corollaries 6.2 and 6.9] and [24, Theorems 3.6, 3.7, and 3.11].) Moreover, under a number of conditions, including [24, Theorems 1.2 and 3.8 and Lemma 2.8] and [25, Theorem 4.2],  $\text{Int}(D)$  is locally free, hence flat, as a  $D$ -module. These include the cases where  $D$  is a domain of Krull type or more generally polynomially L-regular PVMD. Remarkably, however, there is no known example of an integral domain  $D$  such that  $\text{Int}(D)$  is not free as a  $D$ -module.

**Problem 8.5** ([22]). Do there exist integral domains  $D$  such that:

1.  $\text{Int}(D)$  is not free as a  $D$ -module?
2.  $\text{Int}(D)$  is not flat as a  $D$ -module?
3.  $\text{Int}(D)^{\otimes n}$  is not  $D$ -torsion-free for  $n = 2$  or  $n = 3$ ?

If  $M$  is a flat  $D$ -module, then every tensor power of  $M$  is  $D$ -torsion-free, and the converse holds if  $M$  is finitely generated [14, 39]; however,  $\text{Int}(D)$  is not finitely generated. We are thus also led naturally to the following problems.

**Problem 8.6.** Let  $D$  be an integral domain.

1. Classify the domains  $D$  for which  $\text{Int}(D)^{\otimes n}$  is an  $\text{Int}(D)$ -reflective  $D$ -algebra for all  $n$  (or equivalently, for which  $D$  is  $\text{Int}(D)$  an idempotent  $D[X]$ -algebra).

2. If  $\text{Int}(D)^{\otimes n}$  is an  $\text{Int}(D)$ -reflective  $D$ -algebra, then is it necessarily  $D$ -torsion-free?
3. Is every tensor power of  $\text{Int}(D)$   $D$ -torsion free? If not, then for which domains  $D$  does this hold?
4. If  $\text{Int}(D)^{\otimes n}$  is  $D$ -torsion free for  $n = 2, 3$ , then is every tensor power of  $\text{Int}(D)$  necessarily  $D$ -torsion free?
5. If every tensor power of  $\text{Int}(D)$  is  $D$ -torsion free, then is  $\text{Int}(D)$  necessarily flat as a  $D$ -module?
6. In general, if  $A$  is a  $D$ -algebra such that every tensor power of  $A$  over  $D$  is  $D$ -torsion-free, then is  $A$  necessarily flat as a  $D$ -module?

Finally, in the remainder of this section we examine the  $D$ -plethory  $\text{Int}(D)$  in the case where  $D$  is a Dedekind domain, where it is known that  $\text{Int}(D)$  is free as a  $D$ -module and therefore  $\theta_{\mathbf{X}}$  is an isomorphism for all  $\mathbf{X}$  [10, Remark II.3.7(iii)], [21, Proposition 6.8].

By [20, Proposition 9.3] one has  $\text{Bin}(A) \cong \mathbb{Z}_p$  for any integral domain  $A$  of characteristic  $p$ , where  $\mathbb{Z}_p$  denotes the ring of  $p$ -adic integers, and in particular one has  $\text{Bin}(\mathbb{F}_p) \cong \mathbb{Z}_p$ . This generalizes as follows.

**Proposition 8.7.** *Let  $D$  be a Dedekind domain, and let  $\mathfrak{p}$  be a maximal ideal of  $D$  with finite residue field. Then the map  $\widehat{D}_{\mathfrak{p}} \rightarrow \mathbb{W}_{\text{Int}(D)}(D/\mathfrak{p})$  acting by  $\alpha \mapsto (f \mapsto f(\alpha) \bmod \mathfrak{p}\widehat{D}_{\mathfrak{p}})$  is a  $D$ -algebra isomorphism. More generally, for any  $D$ -algebra  $A$  that is a domain with  $\mathfrak{p}A = 0$ , the diagram*

$$\begin{array}{ccc} \widehat{D}_{\mathfrak{p}} & \xrightarrow{\quad} & \mathbb{W}_{\text{Int}(D)}(D/\mathfrak{p}) \\ & \searrow & \downarrow \\ & & \mathbb{W}_{\text{Int}(D)}(A) \end{array}$$

*is a commutative diagram of  $D$ -algebra isomorphisms.*

**Proof.** By [10, Theorem V.2.10], the prime ideals of  $\text{Int}(D)$  lying above  $\mathfrak{p}$  are maximal and are in bijective correspondence with  $\widehat{D}_{\mathfrak{p}}$ , where  $\alpha \in \widehat{D}_{\mathfrak{p}}$  corresponds to the maximal ideal  $\mathfrak{p}_{\alpha} = \{f \in \text{Int}(D) : f(\alpha) \in \mathfrak{p}\widehat{D}_{\mathfrak{p}}\}$  of  $\text{Int}(D)$ . Given any such  $\alpha$ , the  $D$ -algebra homomorphism  $\text{eval}_{\alpha} : \text{Int}(D) \rightarrow \widehat{D}_{\mathfrak{p}}$  acting by  $f \mapsto f(\alpha)$  induces  $D$ -algebra isomorphisms  $\text{Int}(D)/\mathfrak{p}_{\alpha} \cong \widehat{D}_{\mathfrak{p}}/\mathfrak{p}\widehat{D}_{\mathfrak{p}} \cong D/\mathfrak{p}$ . It follows that the map  $\widehat{D}_{\mathfrak{p}} \rightarrow \mathbb{W}_{\text{Int}(D)}(D/\mathfrak{p})$  given in the statement of the proposition is a well-defined bijection. Moreover, this bijection is  $D$ -linear, and one checks that it also preserves multiplication and unity and is therefore an isomorphism of  $D$ -algebras. Finally, if  $A$  is any  $D$ -algebra that is a domain with  $\mathfrak{p}A = 0$ , then the kernel of any  $\varphi \in \mathbb{W}_{\text{Int}(D)}(A)$  is a prime ideal of  $\text{Int}(D)$  lying over  $\mathfrak{p}$  and therefore is of the form  $\mathfrak{p}_{\alpha}$  for some  $\alpha \in \widehat{D}_{\mathfrak{p}}$ , whence  $\varphi$  factors through  $\text{eval}_{\alpha}$ . It follows, then, that the  $D$ -algebra homomorphism  $\mathbb{W}_{\text{Int}(D)}(D/\mathfrak{p}) \rightarrow \mathbb{W}_{\text{Int}(D)}(A)$  is a bijection and therefore an isomorphism.  $\square$

To verify, as claimed throughout this paper, that the binomial rings coincide with the  $\text{Int}(\mathbb{Z})$ -reflective  $\mathbb{Z}$ -algebras, we must show that the latter are  $\mathbb{Z}$ -torsion-free. In the next theorem we show more generally that, if  $D$  is a Dedekind domain with finite residue fields, then every  $\text{Int}(D)$ -reflective  $D$ -algebra is  $D$ -torsion-free. (Recall [Proposition 4.7.](#)) Given this fact, it follows that  $\mathbb{W}_{\text{Int}(\mathbb{Z})}$  is isomorphic to the functor  $\text{Bin}$  since both functors are right adjoints to the inclusion from binomial rings to rings.

**Lemma 8.8.** *Let  $D$  be an integral domain and  $S$  a multiplicative subset of  $D$ . Then  $S^{-1}D$  is  $\text{Int}(D)$ -reflective.*

**Proof.** A domain extension  $A$  of  $D$  is  $\text{Int}(D)$ -reflective if and only if  $A$  is a WPC extension of  $D$ , if and only if  $\text{Int}(D) \subseteq \text{Int}(A)$ . By [\[10, Proposition I.2.5\]](#), one has  $\text{Int}(D) \subseteq \text{Int}(D, S^{-1}D) = \text{Int}(S^{-1}D)$ , and in particular  $\text{Int}(D) \subseteq \text{Int}(S^{-1}D)$ . Therefore  $S^{-1}D$  is an  $\text{Int}(D)$ -reflective extension of  $D$ .  $\square$

**Theorem 8.9.** *If  $D$  is a Dedekind domain with finite residue fields, then every  $\text{Int}(D)$ -reflective  $D$ -algebra is  $D$ -torsion-free.*

**Proof.** Note first that, by [Theorem 2.4](#), there exists a unique  $D$ -plethory structure on  $\text{Int}(D)$  with unit given by the inclusion  $D[X] \rightarrow \text{Int}(D)$ .

We first reduce to the case where  $D$  is a DVR. Let  $A$  be an  $\text{Int}(D)$ -reflective  $D$ -algebra and  $\mathfrak{p}$  a maximal ideal of  $D$ . By the lemma,  $D_{\mathfrak{p}}$  is  $\text{Int}(D)$ -reflective. It follows, then, from [Proposition 6.8](#) that  $D_{\mathfrak{p}} \otimes_D A = A_{\mathfrak{p}}$  is  $\text{Int}(D)_{\mathfrak{p}}$ -reflective, hence  $\text{Int}(D_{\mathfrak{p}})$ -reflective. Thus, since  $A$  is  $D$ -torsion-free if and only if  $A_{\mathfrak{p}}$  is  $D_{\mathfrak{p}}$ -torsion-free for all  $\mathfrak{p}$ , we may therefore assume that  $D$  is a DVR.

Since the  $D$ -plethory  $\text{Int}(D)$  is idempotent, an  $\text{Int}(D)$ -reflective  $D$ -algebra is equivalently an  $\text{Int}(D)$ -ring. Let  $A$  be an  $\text{Int}(D)$ -ring. To show that  $A$  is  $D$ -torsion-free, it suffices to show that  $A$  has no  $\pi$ -torsion, where  $\pi$  is a generator of the maximal ideal of  $D$ . Now,  $D/(\pi)$  is by hypothesis a finite field, say, having  $q$  elements. The polynomial  $F = (X^q - X)/\pi$  is then an element of  $\text{Int}(D)$ . Note that  $f(X, Y) = F(X + Y) - F(X) - F(Y)$  lies in  $(X, Y)D[X, Y]$ , and therefore

$$F \circ 0 = F \circ (0 + 0) = F \circ 0 + F \circ 0 + f(0, 0) = F \circ 0 + F \circ 0$$

in  $A$ , so  $F \circ 0 = 0$ . Note also that  $F(\pi X) = \pi^{q-1}X^q - X$ . Therefore, if  $\pi a = 0$  for some  $a \in A$ , then

$$0 = F \circ (\pi a) = F \circ ((\pi X) \circ a) = F(\pi X) \circ a = \pi^{q-1}a^q - a = -a,$$

whence  $a = 0$ .  $\square$

**Corollary 8.10.** *A ring is binomial if and only if it is an  $\text{Int}(\mathbb{Z})$ -reflective  $\mathbb{Z}$ -algebra.*

**Example 8.11.** Let  $D$  be an integral domain and  $D'$  an overring of  $D$ .

1. If  $D$  has only infinite residue fields, or more generally (by [10, Corollary 1.3.7]) if  $\text{Int}(D) = D[X]$ , then every  $D$ -algebra is  $\text{Int}(D)$ -reflective, and, in particular, not every  $\text{Int}(D)$ -reflective  $D$ -algebra is  $D$ -torsion-free.
2. If  $D$  is a Krull domain and  $\mathfrak{p}$  is a nonzero prime ideal of  $D$  such that  $D' \subseteq D_{\mathfrak{p}}$ , then  $D_{\mathfrak{p}}/\mathfrak{p}D_{\mathfrak{p}}$  is  $\text{Int}_{D'}(D)$ -reflective but not  $D$ -torsion-free.
3. If  $D$  is a Krull domain and  $\mathfrak{p}$  is a nonzero prime ideal of  $D$ , then  $D[\varepsilon]/(\mathfrak{p}\varepsilon) = D[T]/((T)^2 + \mathfrak{p}(T))$  is  $\text{Int}_{D_{\mathfrak{p}}}^{(\infty)}(D)$ -reflective since  $\text{Int}_{D_{\mathfrak{p}}}^{(\infty)}(D) \subseteq \text{Int}(D[\varepsilon]/(\mathfrak{p}\varepsilon))$  [26], but is not  $D$ -torsion-free.

**Problem 8.12.**

1. Determine necessary and sufficient conditions on an integral domain  $D$  so that every  $\text{Int}(D)$ -reflective (resp.,  $\text{Int}^{(\infty)}(D)$ -reflective)  $D$ -algebra is  $D$ -torsion-free.
2. Determine necessary and sufficient conditions (beyond those of Proposition 4.7) on a  $k$ -plethory  $P$  so that every  $P$ -ring is  $k$ -torsion-free.

To further emphasize the connection with binomial rings, we may combine Theorem 8.9 with [21, Theorem 1.2 and Proposition 4.1], which are generalizations of the corresponding results for binomial and quasi-binomial rings (namely, [20, Theorem 4.2]), as follows.

**Proposition 8.13.** *Let  $D$  be a Dedekind domain with finite residue fields and  $A$  a  $D$ -algebra. Then  $A$  is  $\text{Int}(D)$ -reflective if and only if  $A$  is  $D$ -torsion-free and  $A$  satisfies any of the following equivalent conditions.*

1.  $A$  is a  $D$ -algebra quotient of  $\text{Int}(D^{\mathbf{X}})$  for some set  $\mathbf{X}$ .
2.  $A$  is a  $D$ -algebra quotient of an  $\text{Int}(D)$ -reflective  $D$ -algebra.
3.  $a^{|D/\mathfrak{p}|} \equiv a \pmod{\mathfrak{p}A}$  for all  $a \in A$  (or equivalently, the endomorphism  $a \mapsto a^{|D/\mathfrak{p}|}$  of  $A/\mathfrak{p}A$  is the identity) for every maximal ideal  $\mathfrak{p}$  of  $D$ .
4. For every maximal ideal  $\mathfrak{p}$  of  $D$ , the  $D$ -algebra  $A/\mathfrak{p}A$  is locally isomorphic to  $D/\mathfrak{p}$ .
5. For every maximal ideal  $\mathfrak{p}$  of  $D$ , the  $D$ -algebra  $A/\mathfrak{p}A$  is reduced and its residue fields are all isomorphic to  $D/\mathfrak{p}$ .
6. For every maximal ideal  $\mathfrak{p}$  of  $D$ , the  $D$ -algebra  $A/\mathfrak{p}A$  is isomorphic to a  $D$ -subalgebra of  $(D/\mathfrak{p})^{\mathbf{X}}$  for some set  $\mathbf{X}$ .
7.  $A$  is unramified, with trivial residue field extensions, at every maximal ideal of  $D$ .

**Corollary 8.14.** *Let  $D = R$  be a Dedekind domain with finite residue fields. The  $R$ -plethory  $\text{Int}(R)$  coincides with the  $R$ -plethory  $\Lambda_{R,E}$  of [6] modulo the relations  $\psi_{\mathfrak{m}} - \text{id}$ , where  $E$  is the set of all maximal ideals of  $R$  and where the  $\psi_{\mathfrak{m}}$  for  $\mathfrak{m} \in E$  are the*

analogues of the Adams operations  $\psi_p$  of  $\Lambda = \Lambda_{\mathbb{Z}, E}$ . In particular, an  $\text{Int}(R)$ -reflective  $R$ -algebra is equivalently a  $\Lambda_{R, E}$ -ring on which the  $\psi_{\mathfrak{m}}$  act trivially.

**Proof.** The  $\text{Int}(R)$ -reflective  $R$ -algebras coincide with the  $P$ -rings, where  $P$  is the  $R$ -plethory  $\Lambda_{R, E}/(\psi_{\mathfrak{m}} - \text{id})$ , so the result follows from the reconstruction theorem of [5, Introduction].  $\square$

One can generalize Proposition 8.13 and Corollary 8.14 to show that, for any Dedekind domain  $R$  with finite residue fields and any overring  $R'$  of  $R$  (which is necessarily a localization of  $R$  at a saturated multiplicative subset of  $R$ ), the  $R$ -plethory  $\text{Int}_{R'}(R)$  coincides with the  $R$ -plethory  $\Lambda_{R, E}$  modulo the relations  $\psi_{\mathfrak{m}} - \text{id}$  for  $\mathfrak{m} \in E$ , where  $E$  is the set of all maximal ideals  $\mathfrak{m}$  of  $R$  such that  $\mathfrak{m}R' = R'$ .

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