



# Rank 3 arithmetically Cohen–Macaulay bundles over hypersurfaces



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## ABSTRACT

Let  $X$  be a smooth projective hypersurface of dimension  $\geq 5$  and let  $E$  be an arithmetically Cohen–Macaulay bundle on  $X$  of any rank. We prove that  $E$  splits as a direct sum of line bundles if and only if  $H_*^i(X, \wedge^2 E) = 0$  for  $i = 1, 2, 3, 4$ . As a corollary this result proves a conjecture of Buchweitz, Greuel and Schreyer for the case of rank 3 arithmetically Cohen–Macaulay bundles.

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## 1. Introduction

We work over an algebraically closed field of characteristic 0. Let  $\{X, \mathcal{O}_X(1)\} \subset \mathbb{P}^{n+1}$  be a smooth projective hypersurface of degree  $d$ . We say a vector bundle on  $X$  is *split* if it can be written as a direct sum of line bundles. We say that it is *indecomposable* if it can not be written as a direct sum of vector bundles of strictly smaller rank.

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An *arithmetically Cohen–Macaulay (ACM)* vector bundle  $E$  on  $X$  is a locally free sheaf satisfying

$$H_*^i(X, E) := \bigoplus_{k \in \mathbb{Z}} H^i(X, E(k)) = 0 \quad \text{for } i = 1, \dots, n-1$$

Some of the reasons why the study of ACM bundles is important are:

- On projective space, ACM bundles are precisely the bundles which are direct sum of line bundles [12].
- By semicontinuity, ACM bundles form an open set in any flat family of vector bundles over  $X$ .
- The  $n$ 'th syzygy of a resolution of any vector bundle on  $X$  by split bundles, is an arithmetically Cohen–Macaulay bundle [10].
- These sheaves correspond to maximal Cohen–Macaulay modules over the associated coordinate ring [1].

For  $d > 1$  there always exist indecomposable arithmetically Cohen–Macaulay bundles of rank  $> 1$  – see e.g. [18] for a low dimensional construction and [4] for a construction for higher dimensional hypersurfaces. The following conjecture forms the basis of research done in the direction of investigating the splitting behaviour of ACM bundles over hypersurfaces:

**Conjecture** (Buchweitz, Greuel and Schreyer [4]). *Let  $X \subset \mathbb{P}^n$  be a hypersurface. Let  $E$  be an ACM bundle on  $X$ . If  $\text{rank } E < 2^e$ , where  $e = \left\lfloor \frac{n-2}{2} \right\rfloor$ , then  $E$  splits. (Here  $[q]$  denotes the largest integer  $\leq q$ .)*  $\square$

This conjecture can not be strengthened further as the authors constructed an indecomposable ACM bundle of rank  $2^e$  in [4].

For rank 2 ACM bundles, the conjecture follows from [15]. Generic behaviour for rank 2 case is also well understood when  $n \geq 4$  and we refer the reader to [7–9, 18–20] and to the reference cited in these articles. For lower dimensional case, we refer the reader to [16, 17, 11, 5] and [6]. The result for rank 2 bundles was generalized to complete intersections in [2].

For rank 3 ACM bundles the conjecture predicts splitting for  $n \geq 5$  dimensional hypersurfaces. We proved a weaker version in [21]. In this article, we prove the conjecture for rank 3 arithmetically Cohen–Macaulay bundles.

**Theorem 1.1.** *Let  $X$  be a smooth hypersurface of dimension  $\geq 5$ . Let  $E$  be a rank 3 arithmetically Cohen–Macaulay bundle over  $X$ . Then  $E$  is a split bundle.*

This result follows as a corollary from the main result of this article – a splitting criterion for ACM bundles of any rank.

**Theorem 1.2.** *Let  $X$  be a smooth hypersurface of dimension  $\geq 5$ . Let  $E$  be an arithmetically Cohen–Macaulay vector bundle on  $X$  of any rank. Then  $E$  splits if and only if  $H_*^i(X, \wedge^2 E) = 0$  for  $i = 1, 2, 3, 4$ .*

## 2. Preliminaries

In this section, we will recall some standard facts about arithmetically Cohen–Macaulay bundles over hypersurfaces.

Let  $X \subset \mathbb{P}^{n+1}$  be a degree  $d$  smooth hypersurface given by homogeneous polynomial  $f = 0$ . Let  $E$  be an ACM bundle of rank  $r$  on  $X$ . By Serre’s duality,  $E^\vee$  is also ACM.

For notational ease, we will use  $\widetilde{\phantom{x}}$  to denote a vector bundle on  $\mathbb{P}^{n+1}$ . By Hilbert’s syzygy theorem, being a coherent sheaf on  $\mathbb{P}^{n+1}$ ,  $E$  admits a finite length minimal free resolution

$$0 \rightarrow \widetilde{F}_t \rightarrow \widetilde{F}_{t-1} \rightarrow \dots \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow E \rightarrow 0$$

where  $\widetilde{F}_i$  are direct sums of the form  $\oplus_j \mathcal{O}_{\mathbb{P}^{n+1}}(a_j)$ . By minimality of the resolution and the ACM condition on  $E$ , the first syzygy  $\widetilde{K} = \text{Ker}(\widetilde{F}_0 \rightarrow E)$  is an ACM bundle on  $\mathbb{P}^{n+1}$  and therefore is a split bundle by Horrocks’s criterion. Thus the minimal free resolution of  $E$  on  $\mathbb{P}^{n+1}$  is of the form

$$0 \rightarrow \widetilde{F}_1 \xrightarrow{\phi} \widetilde{F}_0 \rightarrow E \rightarrow 0 \quad (1)$$

Localizing at the generic point, one can verify that the ranks of  $\widetilde{F}_1$  and  $\widetilde{F}_0$  are same. Restricting the above resolution to  $X$  gives

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(E, \mathcal{O}_X) \rightarrow \bar{F}_1 \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0$$

where one computes the  $\text{Tor}$  term by tensoring  $0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \xrightarrow{\times f} \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_X \rightarrow 0$  with  $E$  to get  $\text{Tor}_{\mathbb{P}^{n+1}}^1(E, \mathcal{O}_X) = E(-d)$  as multiplication by  $f$  vanishes on  $X$ . Thus the above four term sequence breaks up as

$$0 \rightarrow E^\sigma \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0 \quad (2)$$

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0 \quad (3)$$

where  $\bar{F}_i = \widetilde{F}_i \otimes \mathcal{O}_X$  are split bundles over  $X$  of rank  $m$  and  $E^\sigma := \text{Ker}(\bar{F}_0 \twoheadrightarrow E)$  is an arithmetically Cohen–Macaulay bundle on  $X$ .

We state the following facts (without proof) about matrix factorization theory of Eisenbud and the connection between  $E$  and  $E^\sigma$ . We choose a matrix (with homogeneous polynomial entries) to represent the map  $\phi : \widetilde{F}_1 \rightarrow \widetilde{F}_0$  and henceforth we will use the symbol  $\phi$  interchangeably to represent either the matrix or the map. Then

- (1) There exists an injective map  $\psi : \widetilde{F}_0(-d) \rightarrow \widetilde{F}_1$  such that  $\phi\psi = \psi\phi = f\mathbb{I}$  where  $\mathbb{I}$  denotes the identity matrix.
- (2)  $\text{Coker}(\psi) = E^\sigma$  and  $E$  is indecomposable if and only if  $E^\sigma$  is indecomposable.
- (3)  $0 \rightarrow \widetilde{F}_0(-d) \rightarrow \widetilde{F}_1 \rightarrow E^\sigma \rightarrow 0$  is a minimal free resolution of  $E^\sigma$ .

For details, we refer to section 6 of [10] and section 2 of [6].

**Lemma 2.1.** *Let  $f$  be any homogeneous (perhaps reducible) polynomial of degree  $d$ . Let  $X = V(f) \subset \mathbb{P}^{n+1}$  be the vanishing set. Suppose  $\mathcal{F}$  is any coherent sheaf on  $X$  which admits a free resolution on  $\mathbb{P}^{n+1}$  of the form*

$$0 \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow \mathcal{F} \rightarrow 0$$

where  $\widetilde{F}_i$  are direct sum of line bundles on  $\mathbb{P}^{n+1}$ . Then  $\mathcal{F}$  is a reflexive sheaf on  $X$ .

**Proof.** We apply  $\text{Hom}(-, \mathcal{O}_{\mathbb{P}^{n+1}})$  on the resolution of  $\mathcal{F}$  to get

$$0 \rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow \widetilde{F}_0^\vee \rightarrow \widetilde{F}_1^\vee \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow 0$$

First term vanishes. To compute the  $\mathcal{E}xt$  term, we apply  $\text{Hom}(\mathcal{F}, -)$  on

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \rightarrow \mathcal{O}_X \rightarrow 0$$

to get

$$0 \rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \rightarrow \text{Hom}(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}})(-d) \xrightarrow{\times f}$$

Here the first term vanishes as before and the last map (multiplication by  $f$ ) vanishes as the sheaves are supported on  $X$ . Thus we get  $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}_{\mathbb{P}^{n+1}}) \cong \mathcal{F}^\vee(d)$  and a resolution of  $\mathcal{F}^\vee$  on  $\mathbb{P}^{n+1}$  as

$$0 \rightarrow \widetilde{F}_0^\vee(-d) \rightarrow \widetilde{F}_1^\vee(-d) \rightarrow \mathcal{F}^\vee \rightarrow 0 \quad (4)$$

Applying the whole process once again to the above resolution of  $\mathcal{F}^\vee$  we get the following resolution of  $\mathcal{F}^{\vee\vee}$

$$0 \rightarrow \widetilde{F}_1 \rightarrow \widetilde{F}_0 \rightarrow \mathcal{F}^{\vee\vee} \rightarrow 0$$

Comparing with the resolution of  $\mathcal{F}$ , one gets the claim.  $\square$

Given a short exact sequence of vector bundles  $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$  on a variety  $X$ , there exists a resolution of the  $k$ 'th exterior power  $\wedge^k E_3$ ,

$$0 \rightarrow S^k E_1 \rightarrow S^{k-1} E_1 \otimes \wedge^1 E_2 \rightarrow \dots \wedge^k E_2 \rightarrow \wedge^k E_3 \rightarrow 0 \quad (5)$$

Dually, we also have a resolution of  $k$ 'th symmetric power,

$$0 \rightarrow \wedge^k E_1 \rightarrow \wedge^k E_2 \rightarrow \wedge^{k-1} E_2 \otimes S^1 E_3 \rightarrow \dots \wedge^1 E_2 \otimes S^{k-1} E_3 \rightarrow S^k E_3 \rightarrow 0 \quad (6)$$

For details we refer the reader to [3].

### 3. A cokernel sheaf

Suppose  $\text{rank } \widetilde{F}_0 = \text{rank } \widetilde{F}_1 = m$ . Fix any integer  $k \leq \min\{\text{rank}(E), \text{rank}(E^\sigma)\}$ . Let  $X_k = V(f^k)$  denote the scheme-theoretic  $k$ 'th thickening of  $X \subset \mathbb{P}^{n+1}$ .

We consider the  $k$ 'th exterior power of the map  $\phi : \widetilde{F}_1 \rightarrow \widetilde{F}_0$  in equation (1) and denote the cokernel sheaf by  $\mathcal{F}_k$

$$0 \rightarrow \wedge^k \widetilde{F}_1 \xrightarrow{\wedge^k \phi} \wedge^k \widetilde{F}_0 \rightarrow \mathcal{F}_k \rightarrow 0 \quad (7)$$

The following lemma states some properties of the sheaf  $\mathcal{F}_k$ . Our proof is similar to that in section 2 of [18] where the case when  $E$  is a rank 2 ACM bundle and  $k = 2$  was studied.

#### Lemma 3.1.

- (1)  $\mathcal{F}_k$  is a coherent  $\mathcal{O}_{X_k}$ -module where  $X_k$  is the thickened hypersurface defined scheme theoretically by  $f^k$ .
- (2)  $\bar{\mathcal{F}}_k := \mathcal{F}_k \otimes \mathcal{O}_X$  is a vector bundle on  $X$  of rank  $\binom{m}{k} - \binom{m-r}{k}$ .
- (3)  $\mathcal{F}_k$  is an ACM and reflexive sheaf on  $X_k$ .

**Proof.** First two claims can be verified locally. By localizing on  $X$ , one can assume that equation (1) looks like

$$0 \rightarrow \mathcal{O}_p^{\oplus m} \xrightarrow{\phi} \mathcal{O}_p^{\oplus m} \rightarrow E_p \rightarrow 0$$

and the matrix  $\phi$  is given by the  $m \times m$  diagonal matrix

$$\{f, \dots, f, 1, \dots, 1\}$$

where  $f$  appears  $r = \text{rank}(E)$  times and 1 appears  $m - r$  times on the diagonal. Then locally the matrix  $\wedge^k \phi$  is the diagonal matrix

$$\{f^k, \dots, f^k, f^{k-1} \dots f^{k-1}, f^{k-2}, \dots, f, 1, 1, \dots, 1\}$$

where  $f^{k-i}$  appears  $\binom{r}{k-i} \binom{m-r}{i}$  times on the diagonal. In particular, locally  $\mathcal{F}_k$  is of the form

$$\mathcal{O}_{X_k}^{\oplus \binom{r}{k}} \oplus \mathcal{O}_{X_{k-1}}^{\oplus \binom{r}{k-1} \cdot \binom{m-r}{1}} \oplus \dots \oplus \mathcal{O}_{X_{k-i}}^{\oplus \binom{r}{k-i} \cdot \binom{m-r}{i}} \dots \oplus \mathcal{O}_X^{\oplus \binom{r}{1} \cdot \binom{m-r}{k-1}}$$

This proves the first claim and also that  $\bar{\mathcal{F}}_k = \mathcal{F}_k \otimes \mathcal{O}_X$  is a vector bundle on  $X$ . Claim about the rank is verified by the above local description of  $\mathcal{F}_k$  and the combinatorial identity

$$\binom{m}{k} = \sum_i \binom{r}{i} \binom{m-r}{k-i}$$

By equation (7), one easily sees that  $\mathcal{F}_k$  is an ACM sheaf on  $X_k$ . Lemma 2.1 completes the proof by showing that  $\mathcal{F}_k$  is a reflexive sheaf.  $\square$

We now restrict sequence (7) to  $X$

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{\mathcal{F}}_k \rightarrow 0 \quad (8)$$

This is a sequence of vector bundles and the  $\text{Tor}$  term is a vector bundle of same rank as  $\bar{\mathcal{F}}_k$ . In fact, the map  $F_1 \rightarrow F_0$  factors via  $E^\sigma$ , therefore by functoriality of exterior product, the map  $\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0$  factors via  $\wedge^k E^\sigma$  and the sequence (8) breaks up as

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k E^\sigma \rightarrow 0 \quad (9)$$

and

$$0 \rightarrow \wedge^k E^\sigma \rightarrow \wedge^k \bar{F}_0 \rightarrow \bar{\mathcal{F}}_k \rightarrow 0 \quad (10)$$

Thus the  $\text{Tor}$  term appears as the first term in the filtration of  $k$ 'th exterior power of  $\bar{F}_1$  derived from the sequence  $0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0$ . We can say more,

**Lemma 3.2.**  $\text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \overline{\mathcal{F}}_k^{\vee}(-kd)$ .

**Proof.** We consider the  $k$ 'th exterior power of the minimal resolution of  $E^\vee$  given by sequence (4)

$$0 \rightarrow (\wedge^k \widetilde{F}_0^\vee)(-kd) \rightarrow (\wedge^k \widetilde{F}_1^\vee)(-kd) \rightarrow \mathcal{F}'_k \rightarrow 0 \quad (11)$$

where  $\mathcal{F}'_k$  is defined by the sequence. Restricting to  $X$  gives

$$0 \rightarrow \text{Tor}_{\mathbb{P}^{n+1}}^1(\mathcal{F}'_k, \mathcal{O}_X) \rightarrow (\wedge^k \bar{F}_0^\vee)(-kd) \rightarrow (\wedge^k \bar{F}_1^\vee)(-kd) \rightarrow \bar{\mathcal{F}}'_k \rightarrow 0$$

As in Lemma 3.1 one can verify (by looking at the exterior power matrix locally) that  $\bar{\mathcal{F}}'_k$  is a vector bundle and thus above is a exact sequence of vector bundles. So we can dualize (and then twist by  $-kd$ ) to get:

$$0 \rightarrow \bar{\mathcal{F}}'^{\vee}_k(-kd) \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \text{Tor}^1(\mathcal{O}_X, \mathcal{F}'_k)^\vee(-kd) \rightarrow 0 \quad (12)$$

Comparing with equation (8), we get

$$\mathrm{Tor}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \bar{\mathcal{F}}_k^\vee(-kd) \quad (13)$$

We complete the proof by showing that  $\mathcal{F}'_k \cong \mathcal{F}_k^\vee$ . Applying  $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^{n+1}})$  to sequence (11) and simplifying as in the proof of Lemma 2.1, we get

$$0 \rightarrow \wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0 \rightarrow \mathcal{F}'_k{}^\vee \rightarrow 0 \quad (14)$$

Comparing this with the sequence (7) and using the fact that by Lemma 2.1,  $\mathcal{F}_k, \mathcal{F}'_k$  are both reflexive sheaves, we get that  $\mathcal{F}_k^\vee \cong \mathcal{F}'_k$ .  $\square$

**Lemma 3.3.** *There exists a short exact sequence*

$$0 \rightarrow \wedge^k E(-kd) \rightarrow \mathrm{Tor}_{\mathbb{P}}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow \mathrm{Tor}_{X_k}^1(\mathcal{F}_k, \mathcal{O}_X) \rightarrow 0$$

**Proof.** We restrict the sequence (7) to  $X_k$  to get a free  $\mathcal{O}_{X_k}$ -resolution of  $\mathcal{F}_k$

$$\cdots \rightarrow \wedge^k F_1(-kd) \rightarrow \wedge^k F_0(-kd) \rightarrow \wedge^k F_1 \rightarrow \wedge^k F_0 \rightarrow \mathcal{F}_k \rightarrow 0$$

Tensoring this resolution with  $\mathcal{O}_X$  gives a complex from which we get

$$\mathrm{Tor}_{X_k}^1(\mathcal{F}_k, \mathcal{O}_X) \cong \frac{\mathrm{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0)}{\mathrm{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1)} \quad (15)$$

To compute  $\mathrm{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0)$ , we tensor the sequence (7) with  $\mathcal{O}_X$  to get

$$\mathrm{Ker}(\wedge^k \bar{F}_1 \rightarrow \wedge^k \bar{F}_0) \cong \mathrm{Tor}_{\mathbb{P}}^1(\mathcal{F}_k, \mathcal{O}_X)$$

For the  $\mathrm{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1)$  term, we note that the map  $\bar{F}_0(-d) \rightarrow \bar{F}_1$  factors via  $E(-d)$  so by functoriality of wedge power,

$$\mathrm{Im}(\wedge^k \bar{F}_0(-kd) \rightarrow \wedge^k \bar{F}_1) \cong \wedge^k E(-kd)$$

This completes the proof of the lemma.  $\square$

### 3.1. A short exact sequence

Let  $\mathcal{F}$  be any coherent  $\mathcal{O}_{X_k}$ -module. The inclusions  $X_{k-1} \hookrightarrow \mathbb{P}^{n+1}$  and  $X \hookrightarrow X_k$  induce following short exact sequences

$$0 \rightarrow \mathcal{O}_{X_{k-1}}(-d) \rightarrow \mathcal{O}_{X_k} \rightarrow \mathcal{O}_X \rightarrow 0 \quad (16)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-(k-1)d) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{X_{k-1}} \rightarrow 0 \quad (17)$$

Tensoring both sequences with  $\otimes_{\mathbb{P}} \mathcal{F}$ , we get

$$0 \rightarrow \operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}}(-d)) \rightarrow \mathcal{F}(-kd) \rightarrow \operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{F}|_{X_{k-1}}(-d) \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{F}} \rightarrow 0 \quad (18)$$

$$0 \rightarrow \operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}}) \rightarrow \mathcal{F}(-(k-1)d) \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_{X_{k-1}} \rightarrow 0 \quad (19)$$

Similarly, tensoring sequence (16) with  $\otimes_{X_k} \mathcal{F}$ , we get

$$0 \rightarrow \operatorname{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{F}|_{X_{k-1}}(-d) \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{F}} \rightarrow 0 \quad (20)$$

Comparing sequences (18) and (20) gives

$$0 \rightarrow \operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_{X_{k-1}})(-d) \rightarrow \mathcal{F}(-kd) \rightarrow \operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \operatorname{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow 0 \quad (21)$$

**Lemma 3.4.** *With notations as above,*

$$\operatorname{Ker}[\operatorname{Tor}_{\mathbb{P}}^1(\mathcal{F}, \mathcal{O}_X) \rightarrow \operatorname{Tor}_{X_k}^1(\mathcal{F}, \mathcal{O}_X)] \cong \operatorname{Ker}[\mathcal{F}(-d) \rightarrow \mathcal{F}|_{X_{k-1}}(-d)]$$

**Proof.** Twist the sequence (19) by  $-d$  and compare it with the sequence (21).  $\square$

**Proposition 3.5.** *There exists a short exact sequence*

$$0 \rightarrow \wedge^k E(-(k-1)d) \rightarrow \mathcal{F}_k \rightarrow \mathcal{F}_k|_{X_{k-1}} \rightarrow 0$$

**Proof.** Follows from Lemma 3.3 and by putting  $\mathcal{F} = \mathcal{F}_k$  in Lemma 3.4.  $\square$

#### 4. $E \otimes E$ is ACM implies $E$ is split

Let  $f \in R = k[x_0, x_1, \dots, x_{n+1}]$  be a homogeneous irreducible polynomial of positive degree. Let  $S = R/(f)$  and  $X = \operatorname{Proj}(S)$  be the corresponding hypersurface.

We state the following result without proof.

**Lemma 4.1.** *Let  $E$  be a vector bundle on  $X$ . Let  $M = H_*^0(X, E)$  be corresponding graded  $S$ -module. Then  $E$  splits if  $M$  is a free  $S$ -module.*

For a proof we refer the reader to Lemma 5.4 of [14]. Following result is Theorem 3.1 in [13].

**Theorem 4.2** (Huneke–Wiegand). *Let  $(R, m)$  be an abstract hypersurface and let  $M, N$  be  $R$ -modules, at least one of which has constant rank. If  $M \otimes_R N$  is a maximal Cohen–Macaulay  $R$ -module then either  $M$  or  $N$  is free.*

The corresponding version for vector bundles is of course not true as every vector bundle on a planar curve is ACM (vacuously) and there exists indecomposable vector bundles on various planar curves. Though for our need, the following corollary suffices.



**Theorem 4.3** (Corollary to [Theorem 4.2](#)). Let  $X = \text{Proj}(S)$  be a hypersurface of dimension  $\geq 3$ . Let  $E$  be an ACM vector bundle on  $X$ . Further assume that  $E \otimes E$  is ACM. Then  $E$  splits.

**Proof.** We consider a minimal resolution of  $E$  on  $X$

$$0 \rightarrow E^\sigma \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0 \quad (22)$$

and

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow E^\sigma \rightarrow 0 \quad (23)$$

Where  $\bar{F}_0, \bar{F}_1$  are direct sum of line bundles. Tensoring sequence (22) with  $E$  and sequence (23) with  $E^\sigma$  and using the fact that  $E \otimes E$  is ACM, we deduce that  $E \otimes E^\sigma$  is ACM. Thus there exists a short exact sequence of graded  $S$ -modules:

$$0 \rightarrow H_*^0(E^\sigma \otimes E) \rightarrow H_*^0(\bar{F}_0 \otimes E) \rightarrow H_*^0(E \otimes E) \rightarrow 0$$

Here we are using the fact that  $\dim(X) \geq 3$ . Sequence (22) yields the following right exact sequence

$$H_*^0(E^\sigma) \otimes H_*^0(E) \rightarrow H_*^0(\bar{F}_0) \otimes H_*^0(E) \rightarrow H_*^0(E) \otimes H_*^0(E) \rightarrow 0$$

Thus we get the following commutative diagram

$$\begin{array}{ccccccc} H_*^0(E^\sigma) \otimes H_*^0(E) & \longrightarrow & H_*^0(\bar{F}_0) \otimes H_*^0(E) & \longrightarrow & H_*^0(E) \otimes H_*^0(E) & \longrightarrow & 0 \\ \downarrow \phi_2 & & \parallel & & \downarrow \phi_1 & & \\ 0 \longrightarrow & H_*^0(E^\sigma \otimes E) & \longrightarrow & H_*^0(\bar{F}_0 \otimes E) & \longrightarrow & H_*^0(E \otimes E) & \longrightarrow 0 \end{array}$$

where all the vertical maps are naturally defined. Middle map is an equality because  $\bar{F}_0$  is a split bundle. By snake lemma,  $\phi_1$  is a surjective map.

Similarly we get following commutative diagram from the sequence (23)

$$\begin{array}{ccccccc} H_*^0(E(-d)) \otimes H_*^0(E) & \longrightarrow & H_*^0(\bar{F}_1) \otimes H_*^0(E) & \longrightarrow & H_*^0(E^\sigma) \otimes H_*^0(E) & \longrightarrow & 0 \\ \downarrow & & \parallel & & \downarrow \phi_2 & & \\ 0 \longrightarrow & H_*^0(E(-d) \otimes E) & \longrightarrow & H_*^0(\bar{F}_1 \otimes E) & \longrightarrow & H_*^0(E^\sigma \otimes E) & \longrightarrow 0 \end{array}$$

By snake lemma  $\phi_2$  is surjective. In turn this implies that  $\phi_1$  is injective and hence  $H_*^0(E) \otimes H_*^0(E) \rightarrow H_*^0(E \otimes E)$  is an isomorphism. Thus  $H_*^0(E) \otimes H_*^0(E)$  is a maximal Cohen–Macaulay module and we can apply [Theorem 4.2](#) to conclude that  $H_*^0(E)$  is free and therefore  $E$  splits.  $\square$

## 5. Proof of the main theorem

We now apply above results for  $k = 2$ .

**Proposition 5.1.** *Let  $E$  be an ACM bundle on a smooth hypersurface of dimension  $\geq 3$ . Then  $\wedge^2 E$  is ACM if and only if  $\wedge^2 E^\sigma$  is ACM.*

**Proof.** Assume that  $\wedge^2 E$  is ACM. For  $k = 2$ , we get following short exact sequences for  $E$  (sequence (10) and the sequence from Lemma 3.5)

$$0 \rightarrow \wedge^2 E^\sigma \rightarrow \wedge^2 \bar{F}_0 \rightarrow \bar{\mathcal{F}}_2 \rightarrow 0 \quad (24)$$

$$0 \rightarrow \wedge^2 E(-d) \rightarrow \mathcal{F}_2 \rightarrow \bar{\mathcal{F}}_2 \rightarrow 0 \quad (25)$$

Comparing sequences (24), (25) and using the fact that  $\wedge^2 \bar{F}_0, \mathcal{F}_2$  are all ACM, we get  $H_*^i(\wedge^2 E^\sigma) = 0$  when  $i = 2, \dots, n-1$  where  $n = \dim(X)$ .

To prove the vanishing for  $i = 1$ , we note that  $E^\vee$  is also ACM and  $E^{\vee\sigma} \cong E^{\sigma\vee}(-d)$ , e.g. by lemma 2.5 of [6]. Therefore the same proof shows that  $H_*^i(\wedge^2 (E^{\sigma\vee})) = 0$  when  $i = 2, \dots, n-1$ . Applying Serre's duality completes the proof.  $\square$

We now prove our main result,

**Proof of Theorem 1.2.** Suffices to show one direction. Assume  $H_*^i(X, \wedge^2 E) = 0$  for  $i = 1, 2, 3, 4$ . Consider the composition of sequences (5) and (6):

$$0 \rightarrow \wedge^2 E(-2d) \rightarrow \wedge^2 \bar{F}_1 \rightarrow \bar{F}_1 \otimes E^\sigma \rightarrow E^\sigma \otimes \bar{F}_0 \rightarrow \wedge^2 \bar{F}_0 \rightarrow \wedge^2 E \rightarrow 0$$

One concludes that  $H^i(X, \wedge^2 E(k)) = H^{i+4}(X, \wedge^2 E(k-2d))$  for  $i = 1, \dots, n-5$ . Thus  $\wedge^2 E$  is ACM. By Lemma 5.1,  $\wedge^2 E^\sigma$  is also ACM. We consider sequence (5)

$$0 \rightarrow S^2 E(-d) \rightarrow E(-d) \otimes \bar{F}_1 \rightarrow \wedge^2 \bar{F}_1 \rightarrow \wedge^2 E^\sigma \rightarrow 0$$

This gives  $H_*^i(S^2 E) = 0$  when  $i = 3, \dots, n-1$ . Since  $\wedge^2 E$  is ACM implies  $\wedge^2 E^\vee$  is also ACM, we do a dual analysis to get  $H_*^i(S^2 E^\vee) = 0$  when  $i = 3, \dots, n-1$ . Applying Serre's duality and combining this with the vanishing for  $S^2 E$ , we get that when  $n-3 \geq 2$  then  $S^2 E$  is also ACM.

Thus when  $\dim(X) \geq 5$ ,  $E \otimes E = \wedge^2 E \oplus S^2 E$  is ACM which by Theorem 4.3 implies that  $E$  is split.  $\square$

**Remark 5.2.** We note that the statement  $\wedge^2 E$  is ACM implies  $E \otimes E$  is ACM – is tight in dimension. Consider any rank 2 indecomposable ACM vector bundle on a hypersurface of dimension 4. Then  $\wedge^2 E$  is ACM but  $E \otimes E \cong E \otimes E^\vee(t)$  can not be ACM for otherwise  $H_*^2(X, \mathcal{E}nd(E)) = 0$  and hence by lemma 2.2 of [18],  $E$  is split which contradicts the indecomposability of  $E$ .

**Proof of Theorem 1.1.** The perfect pairing  $E \times \wedge^2 E \mapsto \wedge^3 E = \mathcal{O}_X(e)$  induces an isomorphism  $\wedge^2 E \cong E^\vee(e)$ . By Serre’s duality then  $\wedge^2 E$  is ACM and hence we can apply Theorem 1.2.  $\square$

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