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Polynomial extensions of modules with the quasi-Baer property

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ABSTRACT

In this paper it is shown that, for a module M over a ring R with $S = End_R(M)$, the endomorphism ring of the $R[x]$-module $M[x]$ is isomorphic to a subring of $S[[x]]$. Also the endomorphism ring of the $R[[x]]$-module $M[[x]]$ is isomorphic to $S[[x]]$. As a consequence, we show that for a module M_R and an arbitrary nonempty set of not necessarily commuting indeterminates X, M_R is quasi-Baer if and only if $M[X]_{R[X]}$ is quasi-Baer if and only if $M[[X]]_{R[[X]]}$ is quasi-Baer if and only if $M[x]_{R[x]}$ is quasi-Baer if and only if $M[[x]]_{R[[x]]}$ is quasi-Baer. Moreover, a module M_R with *IFP*, is Baer if and only if $M[x]_{R[x]}$ is Baer if and only if $M[[x]]_{R[[x]]}$ is Baer. It is also shown that, when M_R is a finitely generated module, and every semicentral idempotent in S is central, then $M[[X]]_{R[[X]]}$ is endo-p.q.-Baer if and only if $M[[x]]_{R[[x]]}$ is endo-p.q.-Baer if and only if M_R is endo-p.q.-Baer and every countable family of fully invariant direct summand of M has a generalized countable join. Our results extend several existing results.

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1. Introduction

Throughout this paper R denotes a ring with unity and M is a unital right R-module. The endomorphism ring of M is denoted by $S = End_R(M)$. Thus M can be viewed as a left S-right R-bimodule. Recall that a ring R is (quasi-)Baer if the right annihilator of every nonempty subset (resp. right ideal) of R is generated (as a right ideal) by an idempotent of R. These definitions are left-right symmetric. In [9] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras, von Neumann algebras, and complete $*$-regular rings.

In [7] Clark defined quasi-Baer rings, and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Large classes of rings satisfy the Baer property - examples include right self-injective von Neumann regular rings, von Neumann algebras, and the endomorphisms rings of semisimple modules. Examples of quasi-Baer rings also include large classes, such as all prime rings and rings of matrices over Baer rings.

In [14] Pollingher and Zaks have shown that the class of quasi-Baer rings is closed under $n \times n$ matrix rings and under $n \times n$ upper (resp. lower) triangular matrix rings. Furthermore, it can be followed from their results that the quasi-Baer condition is a Morita invariant property. Thus an $n \times n$ ($n > 1$) matrix ring over a non-Prüfer commutative domain is a prime PI quasi-Baer ring which is not Baer ([9], p. 17). Also an $n \times n$ ($n > 1$) upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not Baer ([9], p. 16). Thus the class of quasi-Baer rings seems to behave better than the class of Baer rings under various extensions.

According to Birkenmeier, Kim and Park [5], a ring is said to be right (resp. left) *principally quasi-Baer* if the right (resp. left) annihilator of a principal right (resp. left) ideal is generated (as a right (resp. left) ideal) by an idempotent. This definition is not left-right symmetric. The class of p.q.-Baer rings includes all biregular rings and all quasi-Baer rings and is closed under direct products and Morita invariance.

Birkenmeier, Kim and Park [6] have shown that a number of polynomial extensions such as formal power series, Ore extensions of endomorphism type, and Laurent series do not preserve the Baer condition. However all is not lost for, in spite of these examples, some "Baerness" remains. They have also shown that for many polynomial extensions (including formal power series, Laurent polynomials, and Laurent series), a ring R is quasi-Baer if and only if the polynomial extension over R is quasi-Baer.

In [4] Birkenmeier, Kim and Park have shown that R is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring. Moreover, there is a p.q.-Baer ring R such that the ring $R[[x]]$ is not p.q.-Baer. In [12], Liu has proved that R is right p.q.-Baer if and only if $R[[x]]$ is right p.q.-Baer and any countable family of idempotents has generalized join, when all the left semicentral idempotents are central. For a right p.q.-Baer ring, the condition that left semicentral idempotents are central is equivalent to assume R is semiprime, ([5], Proposition 1.17]). Huang [8] showed that, in Liu's result, the condition requiring all left semicentral idempotents being central, is redundant.

In 2004, the notion of Baer rings was placed in the general module-theoretic setting by Rizvi and Roman utilizing the endomorphism ring of the module for the first time [15]. It was shown that many results for Baer rings can be proved in the general setting of modules including a type theoretic decomposition similar to the one provided for Baer rings by Kaplansky in [9]. Considering an R-module M as an (S, R)-bimodule where $S = End_R(M)$, a module M is said to be Baer if the right annihilator in M of any nonempty subset of S is generated by an idempotent of S (see also [16], [17], [18]). Some examples of Baer modules include Baer rings R viewed as right R-modules, semisimple modules, nonsingular (K-nonsingular) extending modules, free modules of countable rank over a PID. The module M_R is said to be a *quasi-Baer* module if the right annihilator in M of any two-sided ideal of $S = End_R(M)$ is a direct summand of M. It is easy to see that, when $M = R_R$, the two notions coincide with the existing definitions of Baer and quasi-Baer rings, respectively.

In [1] we introduced the notion of endo-principally quasi-Baer modules as a generalization of quasi-Baer modules. Let M be a right R-module and $S = End_R(M)$, M is called an *endo-principally quasi-Baer* (or simply, *endo-p.q.-Baer*) module if, for every $m \in M$, $l_S(Sm) = Se$, for some $e^2 = e \in S$.

Alternative definitions of Baer (quasi-Baer) modules appear in [10] by Lee and Zhou. Extending the notion of a reduced ring (for which $a^2 = 0$ implies $a = 0$, for each $a \in R$), they introduced the notion of a reduced module. By their definition, a right R-module M is a reduced module if, for every $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. Knowing that reduced rings have been used to obtain results on various annihilator conditions, such as Baer and quasi-Baer properties of the (Laurent) polynomial extensions and (Laurent) power series extensions of rings, they extended some of the results from reduced rings to reduced modules. Extending the notion of Baer and quasi-Baer rings, Lee and Zhou define an R-module M to be Baer (resp. quasi-Baer) if $r_R(X) = eR$, for every subset X of M (resp. for every submodule X of M), where $e^2 = e \in R$.

In [10], the authors prove that a module M_R is quasi-Baer if and only if $M[x]_{R[x]}$ is quasi-Baer if and only if $M[[x]]_{R[[x]]}$ is quasi-Baer if and only if $M[x, x^{-1}]_{R[x,x^{-1}]}$ is quasi-Baer if and only if $M[[x, x^{-1}]]_{R[[x,x^{-1}]]}$ is quasi-Baer.

Throughout this paper, Baer, quasi-Baer and endo-p.q.-Baer modules are applied based on their definitions in [15] and [1], respectively. We first determine endomorphism rings of polynomial and power series extensions of any R-module M and then we characterize the Baer and quasi-Baer properties of these extensions, in terms of those of M.

Let M be an R-module, $S = End_R(M)$, X an arbitrary nonempty set of not necessarily commuting indeterminates and $\mathcal{A}$ be the set of all monomials in $R[X]$. Note that we suppose that every element of R commutes with every indeterminate. We define R-modules $M[x]$ and $M[[x]]$ as:

$M[x] = \left\{ \sum_{i=0}^{r} m_i x^i : r \geq 0, m_i \in M \right\}$, $M[[x]] = \left\{ \sum_{i=0}^{\infty} m_i x^i : m_i \in M \right\}$. Then $M[x]$ is a right $R[x]$-module and $M[[x]]$ is a right $R[[x]]$-module, by usual scalar products (clearly every element of R commutes with x).

In particular $M[x]$ is a left $S[x]$-module by the following:
for every $p[x] = \sum_{i=0}^{r} m_i x^i \in M[x]$ and $f[x] = \sum_{j=0}^{s} f_j x^j \in S[x]$,

$$f[x]p[x] = \sum_{j=0}^{s} \sum_{i=0}^{r} f_j(m_i)x^{i+j}.$$

Similarly, $M[[x]]$ is a left $S[[x]]$-module.

We define $M[X]$ as the set of all elements of the form $\sum_{\alpha\in\mathcal{A}} m_\alpha \alpha$ such that, for every $\alpha \in \mathcal{A}$, $m_\alpha \in M$ and there are only finitely many $\alpha \in \mathcal{A}$, in which $m_\alpha \neq 0$. $M[[X]]$ is the set of all elements of the form $\sum_{\alpha\in\mathcal{A}} m_\alpha \alpha$. By these definitions $M[X]$ is a right $R[X]$-module by the following:
for every $\sum_{\alpha\in\mathrm{A}} m_\alpha \alpha \in M[X]$ and $\sum_{\beta\in\mathrm{A}} r_\beta \beta \in R[X]$,

$$(\sum_{\alpha\in\mathrm{A}} m_\alpha \alpha)(\sum_{\beta\in\mathrm{A}} r_\beta \beta) = \sum_{\alpha\in\mathrm{A}} \sum_{\beta\in\mathrm{A}} m_\alpha r_\beta \alpha\beta.$$

Similarly, $M[[X]]$ is a right $R[[X]]$-module. In particular, every element of $S[[X]]$ can be represented as $\sum_{\beta\in\mathcal{A}} f_\beta \beta$ in which, for some $\beta \in \mathcal{A}$, $f_\beta \in S$. By these notations $M[[X]]$ is a left $S[[X]]$-module, by the following scalar multiplication:

$$(\sum_{\beta\in\mathrm{A}} f_\beta \beta)(\sum_{\alpha\in\mathrm{A}} m_\alpha \alpha) = \sum_{\beta\in\mathrm{A}} \sum_{\alpha\in\mathrm{A}} f_\beta(m_\alpha)\beta\alpha.$$

In this paper we prove that, the endomorphism ring of the module $M[x]_{R[x]}$ is isomorphic to a subring of $S[[x]]$ containing power series $\sum_{i=0}^{\infty} f_i x^i$ in which, for every $m \in M$, there are only finitely many indices $i \geq 0$ such that $f_i(m) \neq 0$. Obviously, this ring contains $S[x]$. We also show that the endomorphism ring of the module $M[[x]]_{R[[x]]}$ is isomorphic to $S[[x]]$. As a consequence, we show that, for a module M_R and an arbitrary nonempty set of not necessarily commuting indeterminates X, M_R is quasi-Baer if and only if $M[X]_{R[X]}$ is quasi-Baer if and only if $M[[X]]_{R[[X]]}$ is quasi-Baer if and only if $M[x]_{R[x]}$ is quasi-Baer if and only if $M[[x]]_{R[[x]]}$ is quasi-Baer. It is also shown that, $M[[x]]$ is an endo-p.q.-Baer right $R[[x]]$-module if and only if M_R is an endo-p.q.-Baer module and, for every countable subset $N = \{m_0, m_1, \ldots\}$ of M, $l_S(\sum_{i=0,1,\ldots} Sm_i)$ is generated by an idempotent in S, as a left ideal of S. As a corollary we show that, the power series ring $R[[x]]$ is left p.q.-Baer if and only if R is left p.q.-Baer and the left annihilator of every countably generated left ideal of R is generated by an idempotent. This result shows that in the Liu's result [12, Theorem 3], the semiprime condition is redundant. Our results also extend several other existing results.

2. Polynomial extensions of modules with Baer property

The notations $N \leq M$, $N \leq^{\oplus} M$ or $N \trianglelefteq M$ mean that N is a submodule, a direct summand of M or a fully invariant submodule (i.e. $\forall \varphi \in End_R(M), \varphi(N) \subseteq N$). We also denote $l_S(N) = \{\varphi \in S | \varphi N = 0\}$ for $N \leq M$. $I \trianglelefteq S$ means I is an ideal in S and we denote $r_M(I) = \{m \in M | Im = 0\}$ and $r_S(I) = \{\varphi \in S | I\varphi = 0\}$.

According to [3], an idempotent $e \in R$ is called right (left) semicentral, if for each $r \in R$, $er = ere$ (resp. $re = ere$). First, we recall some of the basic properties of idempotents, where $\mathcal{S}_l(R)$ and $\mathcal{S}_r(R)$ denote the set of left semicentral idempotents and right semicentral idempotents of R, respectively.

Lemma 2.1 *([5], Lemma 1.1). For an idempotent $e \in R$, the following conditions are equivalent:*

(i) $e \in \mathcal{S}_l(R)$;
(ii) $1 - e \in \mathcal{S}_r(R)$;
(iii) $(1 - e)Re = 0$;
(iv) *eR is an ideal of R;*
(v) *$R(1 - e)$ is an ideal of R;*
(vi) *$eR(1 - e)$ is an ideal of R and $eR = eR(1 - e) \bigoplus Re$, as a direct sum of left ideals.*

According to Liu [12], a countable family of idempotents $E = \{e_0, e_1, \ldots\}$ of R is said to have a *generalized join* e if there exists $e^2 = e \in R$ such that

(1) $e_i R(1 - e) = 0$;
(2) If d is an idempotent and $e_i R(1 - d) = 0$ then $eR(1 - d) = 0$.

By Huang [8], a countable subset $E = \{e_0, e_1, \ldots\}$ of $\mathcal{S}_r(R)$ has a *generalized countable join* e if there exists $e \in \mathcal{S}_r(R)$ such that, given $a \in R$:

(1) $e_i e = e_i$, for all positive integer i;
(2) $e_i a = e_i$, for all positive integer i, then $ea = e$.

Since Baer and quasi-Baer modules are defined by exploring the connections between a module M and its endomorphism ring, we investigate the structure of the endomorphism rings of $M[x]_{R[x]}$ and $M[[x]]_{R[[x]]}$.

Proposition 2.2. *Let M be an R-module with $S = End_R(M)$ and consider the modules $M[x]_{R[x]}$ and $M[[x]]_{R[[x]]}$. Then*

(1) *Every endomorphism of $M[x]_{R[x]}$ has a representation as a power series in $S[[x]]$.*
(2) *Every endomorphism of $M[[x]]_{R[[x]]}$ has a representation as a power series in $S[[x]]$.*

Proof. (1) For every $\varphi[x] \in End_{R[x]}(M[x])$ and $m \in M$, we will have $\varphi[x](m) \in M[x]$. Take $\varphi[x](m) = m_0 + m_1 x + m_2 x^2 + \ldots + m_t x^t$. Now, for every $i \geq 0$, define the map $f_i : M \to M$ as $f_i(m) = m_i$. Since $\varphi[x] \in End_{R[x]}(M[x])$, for every i, f_i is a well defined endomorphism of M and so $f_i \in S$. Hence, for every $m \in M$, $\varphi[x](m) = \sum_{i=0}^{\infty} f_i(m)x^i$. Clearly, for every $m \in M$, there are only finitely many indices i such that $f_i(m) \neq 0$. By a routine computation, it is easy to show that, for every $\sum_{j=0}^{p} n_j x^j \in M[x]$ and $\varphi[x] \in End_{R[x]}(M[x])$, we have

$$\varphi[x](\sum_{j=0}^{p} n_j x^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} f_i(n_j))x^k = (\sum_{i=0}^{\infty} f_i x^i)(\sum_{j=0}^{p} n_j x^j).$$

Therefore, every $\varphi[x] \in End_{R[x]}(M[x])$ is represented by $\sum_{i=0}^{\infty} f_i x^i$ where $f_i \in S$ and, for every $m \in M$, there are only finitely many indices $i \geq 0$ such that $f_i(m) \neq 0$.

(2) For every $\varphi[x] \in End_{R[[x]]}(M[[x]])$ and $m \in M$, we will have $\varphi[x](m) \in M[[x]]$. Take $\varphi[x](m) = m_0 + m_1 x + m_2 x^2 + \ldots$. Now, for every $i \geq 0$, define the map $f_i : M \to M$ as $f_i(m) = m_i$. Since $\varphi[x] \in End_{R[[x]]}(M[[x]])$, for every i, f_i is a well defined endomorphism of M. So, for every i, $f_i \in S$ and, for every $m \in M$, $\varphi[x](m) = \sum_{i=0}^{\infty} f_i(m)x^i$. By a routine computation, we can show that, for every $\sum_{j=0}^{\infty} n_j x^j \in M[[x]]$ and $\varphi[x] \in End_{R[[x]]}(M[[x]])$, we have

$$\varphi[x](\sum_{j=0}^{\infty} n_j x^j) = \sum_{k=0}^{\infty}(\sum_{i+j=k} f_i(n_j))x^k = (\sum_{i=0}^{\infty} f_i x^i)(\sum_{j=0}^{\infty} n_j x^j).$$

Therefore, every $\varphi[x] \in End_{R[[x]]}(M[[x]])$ is represented by $\sum_{i=0}^{\infty} f_i x^i$. $\square$

In the next theorem we characterize the structure of endomorphism rings of the modules $M[x]_{R[x]}$ and $M[[x]]_{R[[x]]}$.

Theorem 2.3. *Let M be an R-module and $S = End_R(M)$. Then*

(1) *the endomorphism ring of the module $M[x]_{R[x]}$ is isomorphic to a subring of $S[[x]]$ containing power series $\sum_{i=0}^{\infty} f_i x^i$ in which, for every $m \in M$, there are only finitely many indices i such that $f_i(m) \neq 0$. Moreover, this ring contains $S[x]$.*
(2) *the endomorphism ring of the module $M[[x]]_{R[[x]]}$ is isomorphic to $S[[x]]$.*

Proof. (1) By Proposition 2.2, every $\varphi[x] \in End_{R[x]}(M[x])$ is represented by a power series $\sum_{i=0}^{\infty} f_i x^i \in S[[x]]$ in which, for every $m \in M$, there are only finitely many indices i such that $f_i(m) \neq 0$. Conversely, consider $\sum_{i=0}^{\infty} f_i x^i \in S[[x]]$ in which, for every $m \in M$, there are only finitely many indices $i \geq 0$ such that $f_i(m) \neq 0$. Now define $\varphi[x] : M[x]_{R[x]} \to M[x]_{R[x]}$, with $\varphi[x](p[x]) = \sum_{t=0}^{\infty}(\sum_{i+j=t} f_i(m_j))x^t$, for $p[x] = m_0 + m_1 x + \ldots + m_t x^t$. Then, it is easy to check that, φ is well-defined and $\varphi[x] \in End_{R[x]}(M[x])$. Define $\mathcal{S}$ as the set of all $\sum_{i=0}^{\infty} f_i x^i \in S[[x]]$ in which, for every $m \in M$, there are only finitely many indices i such that $f_i(m) \neq 0$. Clearly, $\mathcal{S}$ is a subring of $S[[x]]$. Now, define $g : End_{R[x]}(M[x]) \to \mathcal{S}$ with, $g(\varphi[x]) = \sum_{i=0}^{\infty} f_i x^i$, for every $\varphi[x] \in End_{R[x]}(M[x])$. Then by a routine computation, g is a well-defined group isomorphism. Now, let $\varphi[x]$

and $\psi[x]$ be two arbitrary elements of $End_{R[x]}(M[x])$ with $g(\varphi[x]) = \sum_{i=0}^{\infty} f_i x^i$ and $g(\psi[x]) = \sum_{i=0}^{\infty} h_i x^i$. Then, for every $p[x] = m_0 + m_1 x + \ldots + m_t x^t \in M[x]$,

$$(\psi[x]\varphi[x])(p[x]) = \psi[x](\varphi[x](p[x])) = \psi(\sum_{t=0}^{\infty}(\sum_{i+j=t} f_i m_j)x^t)$$

$$= \sum_{t=0}^{\infty}(\sum_{(i+j)+k=t} h_k(f_i m_j))x^t = \sum_{t=0}^{\infty}(\sum_{(i+k)+j=t} (h_k f_i)m_j)x^t$$

$$= \sum_{t=0}^{\infty}(\sum_{i+j=t} (g(\psi[x])g(\varphi[x]))_i m_j)x^t = (g(\psi[x])g(\varphi[x]))(p[x]).$$

Hence, for every $\varphi[x], \psi[x] \in End_{R[x]}(M[x])$, $g(\psi[x]\varphi[x]) = g(\psi[x])g(\varphi[x])$. Thus g is a ring isomorphism and so $End_{R[x]}(M[x])$ is isomorphic to a subring of $S[[x]]$ containing power series $\sum_{i=0}^{\infty} f_i x^i$ such that, for every $m \in M$, there are only finitely many indices $i \geq 0$ such that $f_i(m) \neq 0$. Clearly, this ring contains $S[x]$.

(2) By Proposition 2.3, every $\varphi \in End_{R[[x]]}(M[[x]])$ can be represented by a power series $\sum_{i=0}^{\infty} f_i x^i \in S[[x]]$. Conversely, consider power series $\sum_{i=0}^{\infty} f_i x^i \in S[[x]]$. Now we define a map $\varphi[x] : M[[x]]_{R[[x]]} \to M[[x]]_{R[[x]]}$, with $\varphi[x](p[x]) = \sum_{t=0}^{\infty}(\sum_{i+j=t} f_i m_j)x^t$, for $p[x] = m_0 + m_1 x + \ldots \in M[[x]]$. Then, it is easy to check that, φ is well-defined and $\varphi \in End_{R[[x]]}(M[[x]])$. Now, define $g : End_{R[[x]]}(M[[x]]) \to S[[x]]$, with $g(\varphi[x]) = \sum_{i=0}^{\infty} f_i x^i$, for every $\varphi[x] \in End_{R[[x]]}(M[[x]])$. Clearly g is surjective and similar to (1), we can show that g is a ring monomorphism. So $End_{R[[x]]}(M[[x]])$ is isomorphic to $S[[x]]$. $\square$

Example 2.4. Consider $\mathbb{Q}$ as a right $\mathbb{Z}$-module. Then $End_{\mathbb{Z}}(\mathbb{Q}) \simeq \mathbb{Q}$ and so, by Theorem 2.3, we will have:

$$End_{\mathbb{Z}[[x]]}(\mathbb{Q}[[x]]) \simeq \mathbb{Q}[[x]].$$

Example 2.5. Let R be a ring and consider $\bigoplus_1^{\infty} R$ as a right $\prod_1^{\infty} R$-module. Then, obviously $End_{(\prod_1^{\infty} R)}(\bigoplus_1^{\infty} R) \simeq \prod_1^{\infty} R$ and hence, by Theorem 2.3, we will have:

$$End_{(\prod_1^{\infty} R)[[x]]}((\bigoplus_1^{\infty} R)[[x]]) \simeq \prod_1^{\infty} R[[x]].$$

Proposition 2.6. *Let M be an R-module with $S = End_R(M)$ and X an arbitrary nonempty set of not necessarily commuting indeterminates. Consider the modules $M[X]_{R[X]}$ and $M[[X]]_{R[[X]]}$. Then*

(1) *every endomorphism of $M[X]_{R[X]}$ has a representation as a power series in $S[[X]]$.*
(2) *every endomorphism of $M[[X]]_{R[[X]]}$ has a representation as a power series in $S[[X]]$.*

Proof. (1) For every $\varphi \in End_{R[X]}(M[X])$ and $m \in M$, obviously $\varphi(m) \in M[X]$. Define $\mathcal{A}$ as the set of all monomials in $R[X]$ and take $\varphi(m) = \sum_{\alpha\in\mathcal{A}} m_\alpha \alpha$ in which there are only finitely many indices α such that $m_\alpha \neq 0$. For every $\alpha \in \mathcal{A}$, define the map $f_\alpha : M \to M$ as $f_\alpha(m) = m_\alpha$. Since $\varphi \in End_{R[x]}(M[x])$, for every α, f_α is a well defined endomorphism of M. So, for every α, $f_\alpha \in S$ and, for every $m \in M$, $\varphi(m) = \sum_{\alpha\in\mathcal{A}} f_\alpha(m)\alpha$. Clearly, for every $m \in M$, there are only finitely many α such that $f_\alpha(m) \neq 0$. By a routine computation, it is easy to show that, for every $\sum_{\beta\in\mathcal{A}} n_\beta \beta \in M[X]$ and $\varphi \in End_{R[X]}(M[X])$, we have

$$\varphi(\sum_{\beta\in\mathcal{A}} n_\beta \beta) = \sum_{\beta\in\mathcal{A}} \varphi(n_\beta)\beta = \sum_{\beta\in\mathcal{A}}\sum_{\alpha\in\mathcal{A}} f_\alpha(n_\beta)\alpha\beta = (\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha)(\sum_{\beta\in\mathcal{A}} n_\beta\beta).$$

Therefore, every $\varphi \in End_{R[X]}(M[X])$ is represented by $\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha$, where $f_\alpha \in S$ and, for every $m \in M$, there are only finitely many indices $\alpha \in \mathcal{A}$ such that $f_\alpha(m) \neq 0$.

(2) The proof is similar to that of (1). □

Corollary 2.7. *Let M be an R-module with $S = End_R(M)$, X an arbitrary nonempty set of not necessarily commuting indeterminates and $\mathcal{A}$ be the set of all monomials in $R[X]$. Then*

(1) *the endomorphism ring of the module $M[X]_{R[X]}$ is isomorphic to a subring of $S[[X]]$ containing power series $\sum_{\alpha\in \mathrm{A}} f_\alpha\alpha$, where $f_\alpha \in S$ and, for every $m \in M$, there are only finitely many indices $\alpha \in \mathrm{A}$ such that $f_\alpha(m) \neq 0$.*
(2) *the endomorphism ring of the module $M[[X]]_{R[[X]]}$ is isomorphic to $S[[X]]$.*

Proof. (1) By Proposition 2.6, every $\varphi \in End_{R[X]}(M[X])$ is represented by a power series $\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha \in S[[X]]$ in which, for every $m \in M$, there are only finitely many indices $\alpha \in \mathcal{A}$ such that $f_\alpha(m) \neq 0$. Define $\mathcal{S}$ as the set of all power series $\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha \in S[[X]]$ in which, for every $m \in M$, there are only finitely many indices $\alpha \in \mathcal{A}$ such that $f_\alpha(m) \neq 0$. Clearly, $\mathcal{S}$ is a subring of $S[[X]]$. Now define $g : End_{R[x]}(M[x]) \to \mathcal{S}$ as, for every $\varphi \in End_{R[X]}(M[x])$, $g(\varphi) = \sum_{\alpha\in\mathcal{A}} f_\alpha\alpha$. Then by a routine computation, g is a well-defined group monomorphism. Let $\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha \in \mathcal{S}$ and define $\varphi : M[X]_{R[X]} \to M[X]_{R[X]}$ as, for every $p[X] = \sum_{\beta\in\mathcal{A}} m_\beta\beta$, $\varphi(p[X]) = \sum_{\beta\in\mathcal{A}}\sum_{\alpha\in\mathcal{A}} f_\alpha(m_\beta)\alpha\beta$. Then, it is easy to check that, $\varphi \in End_{R[X]}(M[X])$ is well-defined and, by the definition, $g(\varphi) = \sum_{\alpha\in\mathcal{A}} f_\alpha\alpha$. Thus g is a group isomorphism. Now, assume that φ and ψ are two arbitrary elements of $End_{R[X]}(M[X])$ such that $g(\varphi) = \sum_{\alpha\in\mathcal{A}} f_\alpha\alpha$ and $g(\psi) = \sum_{\gamma\in\mathrm{A}} h_\gamma\gamma$ and $p[X] = \sum_{\beta\in\mathcal{A}} m_\beta\beta$ is an arbitrary element of $M[X]$. Then

$$(\psi\varphi)(p[X]) = \psi(\varphi(p[X])) = \psi(\sum_{\beta\in\mathcal{A}}\sum_{\alpha\in\mathcal{A}} f_\alpha(m_\beta)\alpha\beta)$$

$$= \sum_{\beta\in\mathcal{A}}\sum_{\gamma\in\mathcal{A}}\sum_{\alpha\in\mathcal{A}} h_\gamma f_\alpha(m_\beta)\gamma\alpha\beta = [(\sum_{\gamma\in\mathcal{A}} h_\gamma\gamma)(\sum_{\alpha\in\mathcal{A}} f_\alpha\alpha)](\sum_{\beta\in\mathcal{A}} m_\beta\beta)$$

$$= [(\sum_{\gamma \in \mathcal{A}} h_\gamma \gamma)(\sum_{\alpha \in \mathcal{A}} f_\alpha \alpha)](p[X]).$$

Hence, for every $\varphi, \psi \in End_{R[X]}(M[X])$, $g(\psi\varphi) = g(\psi)g(\varphi)$. Therefore, g is a ring isomorphism and so $End_{R[x]}(M[X])$ is isomorphic to $\mathcal{S}$.

(2) By Proposition 2.6, every $\varphi \in End_{R[[X]]}(M[[X]])$ is represented by a power series $\sum_{\alpha \in \mathcal{A}} f_\alpha \alpha \in S[[X]]$. Conversely, consider $\sum_{\alpha \in \mathcal{A}} f_\alpha \alpha \in S[[X]]$ and $p[X] = \sum_{\beta \in \mathcal{A}} m_\beta \beta$ define the map $\varphi : M[[X]]_{R[[X]]} \to M[[X]]_{R[[X]]}$ with $\varphi(p[X]) = \sum_{\beta \in \mathcal{A}} \sum_{\alpha \in \mathcal{A}} f_\alpha(m_\beta)\alpha\beta$. Then, it is easy to check that, φ is well-defined and $\varphi \in End_{R[[X]]}(M[[X]])$. Now, we define a map $g : End_{R[[X]]}(M[[X]]) \to S[[X]]$ as, for every $\varphi \in End_{R[[X]]}(M[[X]])$, $g(\varphi) = \sum_{\alpha \in \mathcal{A}} f_\alpha \alpha$. Then g is surjective and similar to (1), we can show that g is a ring monomorphism. So $End_{R[[X]]}(M[[X]])$ is isomorphic to $S[[X]]$. $\square$

In [6], Birkenmeier, Kim and Park show that for many polynomial extensions (including formal power series, Laurent polynomials, and Laurent series), a ring R is quasi-Baer if and only if the polynomial extension over R is quasi-Baer. In the following we provide a module-theoretic analogue of these results.

Theorem 2.8. *If M_R is a quasi-Baer module, then $M[X]_{R[X]}$ and $M[[X]]_{R[[X]]}$ are quasi-Baer modules, where X is an arbitrary nonempty set of not necessarily commuting indeterminates.*

Proof. We will prove that $M[x]_{R[x]}$ is a quasi-Baer module. The remaining cases are similar. Let $S = End_R(M)$, $E = End_{R[x]}(M[x])$ and $\mathcal{N}$ be a fully invariant submodule of $M[x]_{R[x]}$. We claim that $l_E(\mathcal{N}) = Ee$, for some idempotent $e \in E$. If $\mathcal{N} = 0$, we are finished. Suppose $\mathcal{N} \neq 0$ and C be the subset of M containing all coefficients of terms of elements in $\mathcal{N}$ with minimal degree. Clearly $0 \in \mathrm{C}$. We claim that C is a fully invariant submodule of M. Let $c \in \mathrm{C}$. Then there is some $p[x] = cx^t + m_1x^{t+1} + \ldots + m_rx^{t+r} \in \mathcal{N}$, for some integers $(r, t \geq 0)$. Since $S \subset E$, for every $f \in S$, $f(p[x]) = f(c)x^t + f(m_1)x^{t+1} + \ldots + f(m_r)x^{t+r} \in \mathcal{N}$. Thus $f(c) \in \mathrm{C}$ and so C is a fully invariant submodule of M. Since M_R is a quasi-Baer module, there exists an idempotent $e \in S$ such that $l_S(\mathrm{C}) = Se$. First, to see that $Ee \subseteq l_E(\mathcal{N})$, take $n[x] \in \mathcal{N}$. If $n[x] = 0$, then $en[x] = 0$. So assume $n[x] = n_0 + n_1x + n_2x^2 + \ldots + n_rx^r \neq 0$. Since $n_0 \in \mathrm{C}$, $en_0 = 0$. Now $en[x] = en_1x + en_2x^2 + \ldots + en_rx^r \in \mathcal{N}$ and so $en_1 \in \mathrm{C}$. Thus $en_1 = e(en_1) = 0$. Similarly, we can get $en_2 = \ldots = en_r = 0$. So $en[x] = 0$ and $e \in l_E(\mathcal{N})$. Hence $Ee \subseteq l_E(\mathcal{N})$. Now, we claim that $l_E(\mathcal{N}) \subseteq Ee$. Let $g[x] = \sum_{i=0}^{\infty} g_ix^i \in l_E(\mathcal{N})$. Then, for every $p[x] = p_0x^k + p_1x^{k+1} + p_2x^{k+2} + \ldots + p_tx^{k+t} \in \mathcal{N}$, where $p_0 \neq 0$ and k is a nonnegative integer, $g[x]p[x] = 0$ and so $g_0p_0 = 0$. So $g_0 \in l_S(\mathrm{C}) = Se \subseteq l_E(\mathcal{N})$. Thus $g_0e = g_0$ and similarly, $g_0p_i = 0$, for $(i = 0, \ldots, t)$. Now $g_1p_0 = 0$. Hence similarly, $g_1e = g_1$ and $g_1p_i = 0$, for $(i = 0, \ldots, t)$. This process can be continued to yield that $g_ie = g_i$, for $(i = 0, \ldots, t)$. So $g[x] = g[x]e$. Therefore $l_E(\mathcal{N}) = Ee$ and the result follows. $\square$

Remark 2.9. Consider $p[X] = \sum_{\alpha \in \mathcal{A}} m_\alpha \alpha \in M[[X]]$. For every $\alpha \in \mathcal{A}$, define the degree of α as the number of indeterminates that are in α. Then $p[X]$ may have more than one term of minimal degree. In the proof of the previous theorem, in the general case of the (non-commuting) set of indeterminates ($M[X]_{R[X]}$ and $M[[X]]_{R[[X]]}$), without loss of generality, we define C as a subset of M containing all coefficients of terms in $\mathcal{N}$ with minimal degree.

Theorem 2.10. *A module M_R is quasi-Baer if and only if $M[X]_{R[X]}$ is quasi-Baer if and only if $M[[X]]_{R[[X]]}$ is quasi-Baer, where X is a nonempty set of not necessarily commuting indeterminates.*

Proof. We prove that M_R is quasi-Baer when $M[x]_{R[x]}$ is quasi-Baer. The other cases can be shown similarly. Let N be a fully invariant submodule of M. Then, it is easy to see that, $N[x]$ is a fully invariant submodule of $M[x]$. Since $M[x]_{R[x]}$ is quasi-Baer, there exists an idempotent $e[x] \in E = End_{R[x]}(M[x])$ such that $l_E(N[x]) = Ee[x]$. Since $e[x]$ is an idempotent, constant term of $e[x]$ is an idempotent $e_0 \in S$. We claim that $l_S(N) = Se_0$. Since $e[x]N = 0$, $e_0N = 0$ and so $Se_0 \subseteq l_S(N)$. Conversely, let $f \in l_S(N)$. Then $fN[x] = 0$ and so $f \in l_E(N[x]) = Ee[x]$. Thus there is some $h[x] \in E$ such that $f = h[x]e[x]$. Since $f \in S$, $f = h_0e_0$, where h_0 is the constant term of $h[x]$. Hence $f \in Se_0$ and so $l_S(N) \subseteq Se_0$. Therefore $l_S(N) = Se_0$ and M is a quasi-Baer module. The remainder of the proof follows from Theorem 2.8. □

In 1974, Armendariz has shown that, for a reduced ring R, $R[x]$ is a Baer ring if and only if R is a Baer ring ([2], Theorem B). In the following we provide a module-theoretic analogue of Armendariz's result.

Definition 2.11. ([11], Definition 3.2) A right R-module M is said to satisfy the *IFP* (insertion of factors property) if, for all $\varphi \in S$, $r_M(\varphi)$ is a fully invariant submodule of M (or, equivalently, for all $m \in M$, $l_S(m) \trianglelefteq S$).

Proposition 2.12. *An R-module with IFP is Baer if and only if it is quasi-Baer.*

Proof. Assume that M is a quasi-Baer R-module and I is a left ideal of $S = End_R(M)$. Since M satisfies the *IFP*, for every $\varphi \in I$, $r_M(\varphi) \trianglelefteq M$. Thus $r_M(I) = \cap_{\varphi \in I} r_M(\varphi)$ is a fully invariant submodule of M and so M is Baer. The converse is trivial. □

Proposition 2.13. *Let M_R be a quasi-Baer module. The following conditions are equivalent:*

(1) *M_R satisfies IFP;*
(2) *$M[x]_{R[x]}$ satisfies IFP;*
(3) *$M[[x]]_{R[[x]]}$ satisfies IFP.*

Proof. (1) $\Rightarrow$ (2) Let $p[x] = m_0x^k + m_1x^{k+1} + \ldots + m_rx^{k+r} \in M[x]$, with $m_0 \neq 0$. Then, for every $\varphi[x] = f_0 + f_1x + f_2x^2 + \ldots \in End_{R[x]}M[x]$, such that $\varphi[x](p[x]) = 0$ we have

(1) $f_0(m_0) = 0$,
(2) $f_0(m_1) + f_1(m_0) = 0$,
(3) $f_0(m_2) + f_1(m_1) + f_2(m_0) = 0$,
$\vdots$

By hypothesis and Proposition 2.12, M is a Baer module. Let $S = End_R(M)$. Since M satisfies the *IFP*, for every idempotent $e \in S$, $r_M(e) = (1-e)M \trianglelefteq M$ and so $1-e$ is a left semicentral idempotent and e is a right semicentral idempotent. Hence, for every idempotent $e \in S$, $1-e$ is a left semicentral idempotent and consequently e is a left semicentral idempotent. Therefore every idempotent $e \in S$ is left and right semicentral. So every idempotent $e \in S$ is central and S is an abelian ring. So there are central idempotents $e_0, \ldots, e_r \in S$ such that $l_S(m_i) = Se_i$. By (1), $f_0e_0 = e_0f_0 = f_0$. Left multiplying (2) by e_0 we obtain $f_0(m_1) = 0$. Hence $f_1(m_0) = 0$. So $f_0e_1 = e_1f_0 = f_0$ and $f_1e_0 = e_0f_1 = f_1$. Left multiplying (3) by e_0 we obtain $f_0(m_2) + f_1(m_1) = 0$. Multiply this equality by e_1 from the left we have $f_2(m_0) = 0$. Hence $f_1(m_1) = 0$. Continuing in this way we may obtain $f_i(m_j) = 0$, for all i and j. Hence $f_iS(m_j) = 0$. Now, let $\psi[x] = g_0 + g_1x + g_2x^2 + \ldots \in E = End_{R[x]}M[x]$. Then $\varphi[x]\psi[x] = \sum_{t=0}^{\infty}(\sum_{i+j=t} f_ig_j)x^t$. So $\varphi[x]\psi[x](p[x]) = 0$. Hence $l_E(p[x])$ is an ideal of E. Therefore $M[x]_{R[x]}$ satisfies the IFP.
(2) $\Rightarrow$ (1) It is clear.
(1) $\Leftrightarrow$ (3) The proof is similar. $\square$

Theorem 2.14. *A module M_R with IFP, is a Baer module if and only if $M[x]_{R[x]}$ is a Baer module if and only if $M[[x]]_{R[[x]]}$ is a Baer module.*

Proof. The result follows from Propositions 2.12, 2.13 and Theorem 2.10. $\square$

The next example shows that, the "*IFP*" condition in Corollary 2.14 is not superfluous. There exists an example of a Baer module M_R such that $M[[x]]_{R[[x]]}$ is not a Baer module.

Example 2.15. By ([15], Proposition 2.19), the $\mathbb{Z}$-module $M = \mathbb{Z} \bigoplus \mathbb{Z}$ is a Baer module. Let $S = End_{\mathbb{Z}}(M) = M_2(\mathbb{Z})$. Then, by ([6], Example 1.1), $End_{\mathbb{Z}[[x]]}(M[[x]]) \simeq S[[x]] = M_2(\mathbb{Z})[[x]]$ is not Baer. Hence, by ([15], Theorem 4.1), $M[[x]]_{\mathbb{Z}[[x]]}$ is not a Baer module. Note that, since $l_{M_2(\mathbb{Z})}(1,0)$ is not an ideal, M does not satisfy "*IFP*".

In ([4], Theorem 2.1), the authors proved that R is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring. Also, by ([4], Example 2.6), there exists a commutative von Neumann regular (hence p.q.-Baer) ring R such that $R[[x]]$ is not p.q.-Baer. In [12], Liu has shown that $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotent has generalized join when all left semicentral idempotents are central. For a right p.q.-Baer ring, the condition left semicentral idempotents are central is equivalent to assume that R is semiprime ([5], Proposition 1.17). Huang [8] showed that, in Liu's result, the condition requiring all left semicentral idempotents being central, is redundant.

Definition 2.16. [1] Let M be a right R-module and $S = End_R(M)$. M is called an *endo-principally quasi-Baer* (or simply *endo-p.q.-Baer*) module if, for every $m \in M$, $l_S(Sm) = Se$, for some $e^2 = e \in S$.

Theorem 2.17. *Let M_R be a module and X an arbitrary nonempty set of not necessarily commuting indeterminates. If one of the extension modules $M[X]_{R[X]}$, $M[[X]]_{R[[X]]}$, $M[x]_{R[x]}$ or $M[[x]]_{R[[x]]}$ of M_R is an endo-p.q.-Baer module, then so is M_R.*

Proof. We will prove that M_R is an endo-p.q.-Baer module when the module $M[x]_{R[x]}$ is endo-p.q.-Baer. The other cases can be shown similarly. Assume that $m \in M$ and $E = End_{R[x]}M[x]$ then $l_E(Em) = Ee[x]$, for some idempotent $e[x] = e_0 + e_1x + \ldots \in E$. Clearly e_0 is an idempotent in S. We claim that $l_S(Sm) = Se_0$ and so M is an endo-p.q.-Baer module. Since $e[x]Em = 0$, $e[x]Sm = 0$ and so $e_0Sm = 0$. Thus $Se_0 \subseteq l_SSm$. On the other hand, assume that $f \in l_S(Sm)$ then $f \in l_E(Em) \bigcap S = Ee[x] \bigcap S$. Hence there is some $h[x] \in E$ such that $f = h[x]e[x]$ and so $f = h_0e_0$. Thus $l_SSm \subseteq Se_0$. Therefore $l_S(Sm) = Se_0$. $\square$

Corollary 2.18. *Let R be a ring. If either of the extension rings $R[x]$ or $R[[x]]$ of R is a left p.q.-Baer ring, then so is R.*

Theorem 2.19. *If M_R is an endo-p.q.-Baer module, then $M[X]_{R[X]}$ is endo-p.q.-Baer, where X is a nonempty set of not necessarily commuting indeterminates.*

Proof. We will prove $M[x]_{R[x]}$ is an endo-p.q.-Baer module. The other cases are similar. Let $S = End_R(M)$ and $E = End_{R[x]}(M[x])$. Consider $p[x] = m_0 + m_1x + \ldots + m_nx^n$ as an arbitrary element of $M[x]$, $n \in \mathbb{N}$. It is clear that $Ep[x] \subseteq (\sum_{1\le i\le n} Sm_i)[x]$. Since M is an endo-p.q.-Baer module, for every $1 \le i \le n$, $l_S(Sm_i) = Se_i$, for some $e_i \in \mathcal{S}_r(S)$. So there is some $e \in \mathcal{S}_r(S)$ such that $\bigcap_{1\le i\le n} Se_i = Se$. Hence $Se = \bigcap_{1\le i\le n} Se_i = l_E((\sum_{1\le i\le n} Sm_i)[x]) \subseteq l_E(Ep[x])$. Thus $Ee \subseteq l_E(Ep[x])$. Now, let $f[x] = f_0 + f_1x + f_2x^2 \ldots \in l_E(Ep[x])$. For every $g \in S$, $f[x]gp[x] = 0$. So

(1) $f_0(gm_0) = 0$,
(2) $f_0(gm_1) + f_1(gm_0) = 0$,
(3) $f_0(gm_2) + f_1(gm_1) + f_2(gm_0) = 0$,
$\vdots$

By (1), $f_0 \in l_S Sm_0 = Se_0$ and so $f_0e_0 = f_0$. By (2), $f_0(e_0gm_1)+f_1(e_0gm_0) = 0$. Since $egm_0 = 0$, $f_0(gm_1) = 0$. Hence $f_0 \in l_S Sm_1 = Se_1$ and so $f_0e_1 = f_1$. Take $e' = e_0e_1$, by (3), $f_0(e'gm_2)+f_1(e'gm_1)+f_2(e'gm_0) = 0$. Since $e'gm_0 = 0$ and $e'gm_1 = 0$, $f_0(gm_2) = 0$. Thus $f_0 \in l_S Sm_2 = Se_2$. Continuing in this way we may obtain, for every $1 \leq i \leq n$, $f_0 \in l_S Sm_i = Se_i$. Hence $f_0 \in Se$. Similarly, we may obtain, for every j, $f_j \in Se$. Thus, $f[x] \in Ee$ and so $l_E(Ep[x]) = Ee$. Therefore, $M[x]_{R[x]}$ is an endo-p.q.-Baer module. $\square$

In ([4], Theorem 2.1) Birkenmeier et al. have shown that R is a right p.q.-Baer ring if and only if $R[x]$ is a right p.q.-Baer ring. As a consequence of Theorem 2.19, we obtain the following corollary that is similar to the result for a ring to be left p.q.-Baer. Notice that the left p.q.-Baer property of $R[x]$ is not equivalent to its right p.q.-Baer property.

Corollary 2.20. *A ring R is left p.q.-Baer if and only if $R[x]$ is left p.q.-Baer.*

By ([4], Example 2.6), there is a p.q.-Baer ring R such that $R[[x]]$ is not a p.q.-Baer ring. Therefore, as illustrated in the following example, for an endo-p.q.-Baer module M_R, the power series extension $M[[X]]_{R[[X]]}$ is not necessarily endo-p.q.-Baer.

Example 2.21. By ([15], Proposition 2.19), the $\mathbb{Z}$-module $M = \mathbb{Z} \bigoplus \mathbb{Z}$ is a Baer module and so M is an endo-p.q.-Baer module. Let $S = End_{\mathbb{Z}}(M) = M_2(\mathbb{Z})$. Then, by Theorem 2.3, $End_{\mathbb{Z}[[x]]}(M[[x]]) = S[[x]]$. Assume to the contrary that the finitely generated $\mathbb{Z}[[x]]$-module $M[[x]]_{\mathbb{Z}[[x]]}$ is an endo-p.q.-Baer $\mathbb{Z}[[x]]$-module. Then, by ([1], Proposition 3.5), $S[[x]] = M_2(\mathbb{Z})[[x]]$ is a left p.q.-Baer ring. But, similar to ([6], Example 1.1), no nonzero idempotent element in $S[[x]] = M_2(\mathbb{Z})[[x]]$ is contained in $l(\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x)$, which is a contradiction. Thus $M[[x]]$ is not an endo-p.q.-Baer $\mathbb{Z}[[x]]$-module.

In the following we investigate conditions in which a power series extension of an endo-p.q.-Baer module is endo-p.q.-Baer.

Similar to the Liu's definition of the generalized join [12], we give the following.

Definition 2.22. A countable family of left semicentral idempotents $\{e_0, e_1, \ldots\}$ of a ring R is said to have a generalized join e if there exists $e \in \mathcal{S}_l(R)$ such that:

(1) $(1 - e)Re_i = 0$.
(2) If d is an idempotent and $(1 - d)Re_i = 0$ then $(1 - d)Re = 0$.

Definition 2.23. Let M be a right R-module and $S = End_R(M)$. Consider the countable family of fully invariant direct summands $\mathcal{E} = \{e_0M, e_1M, \ldots\}$ of M. We say $\mathcal{E}$ has a generalized countable join if there exists a fully invariant direct summand eM of M such that:

(1) $\sum_{i\in\mathbb{N}} e_iM \subseteq eM$.
(2) If fM is a direct summand of M and $\sum_{i\in\mathbb{N}} e_iM \subseteq fM$, then $eM \subseteq fM$.

By ([1], Proposition 3.17), if M is a quasi-Baer module then the sum of every family of direct summands is a direct summand of M, so, in this case, every countable family of fully invariant direct summands has a generalized countable join.

Proposition 2.24. *Let M be a right R-module and $S = End_R(M)$. Every countable family of fully invariant direct summands of M has a generalized countable join if and only if every countable family of left semicentral idempotents of S has a generalized join.*

Proof. Assume that every countable family of fully invariant direct summands of M has a generalized countable join and $\{e_1, e_2, \dots\}$ is a countable family of left semicentral idempotents of S. Then $\mathcal{E} = \{e_1M, e_2M, \dots\}$ is a countable family of fully invariant direct summands of M. So $\mathcal{E}$ has a generalized countable join, as, eM. Hence, for every $i \in \mathbb{N}$, $e_iM \subseteq eM$ and so $(1-e)Se_i = 0$. Assume that there is some $f \in \mathcal{S}_l(S)$ such that $(1-f)Se_i = 0$ then, for every $i \in \mathbb{N}$, $e_iM \subseteq fM$. Hence, by part (2) in Definition 2.23, $eM \subseteq fM$ and so $(1-f)Se = 0$. As a result $\{e_1, e_2, \dots\}$ has a generalized countable join e. The converse is obtained by reversing all above arguments. $\square$

The next result shows that, when left semicentral idempotents of a ring R are central then the set of left semicentral idempotents of $R[[x]]$ is $\mathcal{S}_l(R)$.

Lemma 2.25. *If R is a ring with $\mathcal{S}_l(R) \subseteq C(R)$, then $\mathcal{S}_l(R[[x]]) = \mathcal{S}_l(R)$.*

Proof. Clearly $\mathcal{S}_l(R) \subseteq \mathcal{S}_l(R[[x]])$. If $e[x] = e_0 + e_1x + e_2x^2 + \dots \in \mathcal{S}_l(R[[x]])$ then $e[x]^2 = e[x]$. So

(1) $e_0^2 = e_0$
(2) $e_0e_1 + e_1e_0 = e_1$,
(3) $e_0e_2 + e_1e_1 + e_2e_0 = e_2$,
$\vdots$

By (1), e_0 is an idempotent in R. Since $e[x] \in \mathcal{S}_l(R[[x]])$, $e_0 \in \mathcal{S}_l(R)$. So e_0 is central and by (2), $2e_0e_1 = e_1$ and $2e_0e_1 = e_0e_1$. Hence $e_0e_1 = 0$ and so $e_1 = 0$. From left multiplication (3) by e_0 we may obtain that $e_0e_2 = 0$ and $e_2 = 0$. Continuing in this way we may obtain, for every i, $e_0e_i = 0$ and $e_i = 0$. Thus, $e[x] = e_0 \in \mathcal{S}_l(R)$. Therefore, $\mathcal{S}_l(R[[x]]) \subseteq \mathcal{S}_l(R)$ and finally $\mathcal{S}_l(R[[x]]) = \mathcal{S}_l(R)$. $\square$

Theorem 2.26. *Let M_R be a finitely generated module with $S = End_R(M)$. If every semicentral idempotent in S is central then the following statements are equivalent:*

(1) *$M[[X]]_{R[[X]]}$ is an endo-p.q.-Baer module;*
(2) *$M[[x]]_{R[[x]]}$ is an endo-p.q.-Baer module;*
(3) *M_R is an endo-p.q.-Baer module and every countable family of fully invariant direct summand of M has a generalized countable join.*

Proof. We will prove part (1) $\Leftrightarrow$ (3), for $X = \{x\}$. The remaining cases are similar. (1) $\Rightarrow$ (3) By Theorem 2.17, M_R is an endo-p.q.-Baer module. Assume that $\{e_0, e_1, \dots\}$ be a countable set of left semicentral idempotents of S. By Corollary 2.7, $E = End_{R[[x]]}M[[x]] = S[[x]]$. Take $e[x] = e_0 + e_1 x + e_2 x^2 + \dots$. So $e[x] \in E$. Since M is finitely generated, $M[[x]]_{R[[x]]}$ is a finitely generated endo-p.q.-Baer module and so, by ([1], Proposition 3.5), $E = S[[x]]$ is a left p.q.-Baer ring. Thus there is some $f[x] \in \mathcal{S}_l(E)$ such that $l_E(Ee[x]) = Ef[x]$. By the previous lemma, there is some $f_0 \in \mathcal{S}_l(S)$ such that $f[x] = f_0$. Hence, for every $g \in S$, $f_0 g e[x] = 0$ and so, for every i, $f_0 g e_i = 0$. Take $e = 1 - f_0$ then, for every i, $(1-e)Se_i = 0$. Suppose that there is some $f \in \mathcal{S}_l(S)$ such that, for every i, $(1-f)Se_i = 0$. Then, for every $g \in S$ and $h[x] = h_0 + h_1 x + h_2 x^2 + \dots \in E$,

$$(1-f)g(h[x]e[x]) = (1-f)g(\sum_k^{\infty} \sum_{i+j=k} h_i e_j) = 0.$$

Thus $(1-f)g \subseteq l_E Ee[x] = Ef_0$. Hence $(1-f)gf_0 = (1-f)g$ and so we have $(1-f)ge = 0$. Therefore, $\{e_0, e_1, \dots\}$ has generalized join e and so, by Proposition 2.24, every countable family of fully invariant direct summand of M has a generalized countable join.

Conversely, assume that M_R is an endo-p.q.-Baer module and every countable family of fully invariant direct summand of M has a generalized countable join. Consider an arbitrary element of $M[[x]]$ as $p[x] = m_0 + m_1 x + m_2 x^2 \dots$. Then $Ep[x] \subseteq (\sum_{i=0,1,\dots} Sm_i)[x]$. Since M is an endo-p.q.-Baer module, for every i, there is $e_i \in \mathcal{S}_r(S)$ such that $l_S Sm_i = Se_i$ and $Sm_i \subseteq (1-e_i)M$. So the countable family of fully invariant direct summands $\{(1-e_0)M, (1-e_1)M, \dots\}$ has a generalized countable join $(1-e)M$. Thus

$$Se = l_S(1-e)M \subseteq l_S \sum_{i=0,1,\dots} (1-e_i)M \subseteq l_S(\sum_{i=0,1,\dots} Sm_i).$$

Therefore $Ee \subseteq l_E(Ep[x])$. Now, let $\varphi[x] = f_0 + f_1 x + f_2 x^2 \dots \in l_E Ep[x]$, for every $g \in S$, $f[x]gp[x] = 0$. So

(1) $f_0(gm_0) = 0$,
(2) $f_0(gm_1) + f_1(gm_0) = 0$,
(3) $f_0(gm_2) + f_1(gm_1) + f_2(gm_0) = 0$,
$\vdots$

By (1), $f_0 \in l_S Sm_0 = Se_0$ and so $f_0e_0 = f_0$. By (2), we will have $f_0(e_0gm_1) + f_1(e_0gm_0) = 0$. Since $e_0gm_0 = 0$, $f_0(gm_1) = 0$. Hence $f_0 \in l_S Sm_1 = Se_1$ and so $f_0e_1 = f_1$. If $e' = e_0e_1$ then $f_0(e'gm_2) + f_1(e'gm_1) + f_2(e'gm_0) = 0$, by (3). Since $e'gm_0 = 0$ and $e'gm_1 = 0$, $f_0(gm_2) = 0$ and so $f_0 \in l_S Sm_2 = Se_2$. Continuing in this way we may obtain, for every $(i = 0, 1, \ldots)$, $f_0 \in l_S Sm_i = Se_i$. Similarly, we may obtain, for every j, $f_j \in \bigcap Se_i$. Hence, for every $(i, j = 0, 1, \ldots)$, $f_je_i = f_j$. Since M_R is a finitely generated module, by ([1], Proposition 3.5), $S = End_R(M\varphi)$ is a left p.q.-Baer ring. So, for every $(j = 0, 1, \ldots)$, there is some idempotent $h_j \in S$ such that $l_S Sf_j = Sh_j$. Since e_j is right semicentral, by the hypothesis, e_j is central. Thus, for every $g \in S$, $gf_j = gf_je_i = e_igf_j$ and so $(1-e_i)gf_j = 0$ holds, for every $(i = 0, 1, \ldots)$. Hence, for every $(i = 0, 1, \ldots)$, $(1-e_i) \in l_S Sf_j = Sh_j$ and so $h_j(1-e_i) = (1-e_i)h_j = (1-e_i)$. Let $1-e$ is a generalized join for $\{(1-e_0), (1-e_1), \ldots\}$, for every $(j = 0, 1, \ldots)$, $h_j(1-e) = (1-e)$ and so $(1-h_j)e = (1-h_j)$. Hence $f_j = (1-h_j)f_j = f_j(1-h_j) = f_j(1-h_j)e = f_je$. Thus $\varphi[x] = \varphi[x]e \in Ee[x]$. So $l_E Ep[x] = Ee$. Therefore $M[[x]]_{R[[x]]}$ is an endo-p.q.-Baer module. $\square$

By ([12], Theorem 3), if R is a ring with $\mathcal{S}_l(R) \subseteq C(R)$ then $R[[x]]$ is a right p.q.-Baer ring if and only if R is a right p.q.-Baer ring and any countable idempotent in R has a generalized join. As a consequence of the previous theorem we obtain the following corollary that is similar to the result for the left p.q.-Baer rings. Note that, for a ring R with $\mathcal{S}_l(R) \subseteq C(R)$, $\mathcal{S}_l(R) = \mathcal{S}_r(R) = C(R)$.

Corollary 2.27. *Let R be a ring with $\mathcal{S}_l(R) \subseteq C(R)$. Then $R[[x]]$ is a left p.q.-Baer ring if and only if R is a left p.q.-Baer ring and any countable family of left semicentral idempotent has a generalized join.*

Similar to the Huang's definition [8], we give the following.

Definition 2.28. A countable subset $E = \{e_0, e_1, \ldots\}$ of $\mathcal{S}_l(R)$ is said to have a generalized countable join e if there exists $e \in \mathcal{S}_l(R)$ such that, given $a \in R$:

(1) $ee_i = e_i$, for all positive integers i.
(2) If $ae_i = e_i$, for all positive integers i, then $ae = e$.

Proposition 2.29. *Let M_R be an endo-p.q.-Baer module with $S = End_R(M)$. If every countable subset of left semicentral idempotents in S has a generalized countable join in $\mathcal{S}_l(R)$ then, for every countable subset $N = \{m_1, m_2, \ldots\}$ of M, $l_S(\sum_{i\in\mathbb{N}} Sm_i)$ is generated by an idempotent, as a left ideal of S. The converse holds if M contains a copy of S as a left S-module.*

Proof. Assume that $N = \{m_1, m_2, \ldots\} \subseteq M$ and $I = l_S(\sum_{i\in\mathbb{N}} Sm_i)$. We get $l_S(Sm_i) = Se_i$, for each $m_i \in N$. Suppose that e is a generalized countable join for the set of left semicentral idempotents $\{(1-e_1),(1-e_2),\ldots\}$. Then we have $e(1-e_i)=(1-e_i)$ and so $e-ee_i=1-e_i$. Hence $1-e=(1-e)e_i$. Thus, for each $m_i\in N$, $(1-e)Sm_i=(1-e)e_iSm_i=0$. Therefore $S(1-e)\subseteq I$. On the other hand, suppose $g\in I=\bigcap_{i\in\mathbb{N}}Se_i$ then, for every $i\in\mathbb{N}$, $ge_i=g$. Hence $g(1-e_i)=0$ and so $(1-g)(1-e_i)=(1-e_i)$. Thus $g(1-e)=g$ and so $g\in S(1-e)$. Therefore $I=S(1-e)$. Conversely, assume that M contains a copy of S as a left S-module and $X=\{e_1,e_2,\ldots\}$ is a subset of left semicentral idempotents in S. Now according to assumption, $l_S(\sum_{i\in\mathbb{N}}Se_i)$ is generated by an idempotent $e\in S$, as a left ideal of S. So, for every i, $(1-e)e_i=e_i$. If, for every i, $ae_i=e_i$ then $(1-a)e_i=0$. So $(1-a)fe_i=(1-a)e_ife_i=0$, for every $f\in S$. Hence $(1-a)\in l_SSe_i$, for every i. Thus $(1-a)\in Se$ and so $(1-a)e=e$. Therefore $1-e$ is a countable generalized join for X. $\square$

Corollary 2.30. *A ring R is left p.q.-Baer and every countable subset of left semicentral idempotents in R has a generalized countable join if and only if the left annihilator of every countably generated left ideal of R is generated by an idempotent.*

Theorem 2.31. *For an R-module M, $M[[x]]$ is an endo-p.q.-Baer right $R[[x]]$-module if and only if M_R is an endo-p.q.-Baer module and for every countable subset $N=\{m_0,m_1,\ldots\}$ of M, $l_S(\sum_{i=0,1,\ldots}Sm_i)$ is generated by an idempotent in S, as a left ideal of S.*

Proof. Let M be an R-module, $S=End_R(M)$ and $E=End_{R[[x]]}(M[[x]])$. If $M[[x]]_{R[[x]]}$ is an endo-p.q.-Baer module then, by Theorem 2.17, M_R is an endo-p.q.-Baer module. Let $N=\{m_0,m_1,\ldots\}$ be a countable subset of M. Since M_R is an endo-p.q.-Baer module, for every $(i=0,1,\ldots)$, there is some idempotent f_i in S such that $l_S(Sm_i)=Sf_i$. Since $M[[x]]_{R[[x]]}$ is an endo-p.q.-Baer module, for every $p[x]=m_0+m_1x+\ldots\in M[[x]]$, there is some idempotent $e[x]=e_0+e_1x+\ldots\in E$ such that $l_E(Ep[x])=Ee[x]$. So, for every $g\in S$, $e[x]gp[x]=0$ and we have:

(1) $e_0(gm_0)=0$,
(2) $e_0(gm_1)+e_1(gm_0)=0$,
(3) $e_0(gm_2)+e_1(gm_1)+e_2(gm_0)=0$,
$\vdots$

By (1), $e_0\in l_SSm_0=Sf_0$ so $e_0f_0=e_0$. In (2), $e_0(f_0gm_1)+e_1(f_0gm_0)=0$. Since $f_0gm_0=0$, $e_0(gm_1)=0$ and we obtain that $e_0\in l_SSm_1=Sf_1$ and so $e_0f_1=e_0$. Take $e=f_0f_1$, by (3), we have $e_0(egm_2)+e_1(egm_1)+e_2(egm_0)=0$. Since $egm_0=0$ and $egm_1=0$, $e_0(gm_2)=0$. Hence $e_0\in l_SSm_2=Sf_2$. Continuing in this way we may obtain, for every i, $e_0Sm_i=0$. Thus, clearly e_0 is an idempotent in S $e_0\in andl_S(\sum_{i=0,1,\ldots}Sm_i)$. So $Se_0\subseteq l_S(\sum_{i=0,1,\ldots}Sm_i)$. For the revers inclusion, take $g\in l_S(\sum_{i=0,1,\ldots}Sm_i)$. Since $Ep[x]\subseteq(\sum_{i=0,1,\ldots}Sm_i)[x]$, $g\in l_E(Ep[x])=Ee[x]$. So $g=ge[x]$. Hence $g=g_0e_0$ and so $g\in Se_0$. Therefore $l_S(\sum_{i=0,1,\ldots}Sm_i)=Se_0$.

Conversely, assume that M_R is an endo-p.q.-Baer module and, for every countable subset $\{m_0, m_1, \ldots\}$ of M, $l_S(\sum_{i=0,1,\ldots} Sm_i)$ is generated by an idempotent. Hence there is some $e \in \mathcal{S}_r(S)$ such that $l_S(\sum_{i=0,1,\ldots} Sm_i) = Se$. If we take $p[x] = m_0 + m_1x + m_2x^2 + \ldots \in M[[x]]$ then $Ep[x] \subseteq (\sum_{i=0,1,\ldots} Sm_i)[x]$. Hence $Ee \subseteq l_E(Ep[x])$. Now, let $e[x] = e_0+e_1x+e_2x^2 \ldots$ be an element in $l_E(Ep[x])$. Then, for every $g \in S$, $e[x]gp[x] = 0$. So

(1) $e_0(gm_0) = 0$,
(2) $e_0(gm_1) + e_1(gm_0) = 0$,
(3) $e_0(gm_2) + e_1(gm_1) + e_2(gm_0)$,
$\vdots$

Since M_R is an endo-p.q.-Baer module, for every $(i = 0, 1, \ldots)$, there is some idempotent f_i in S such that $l_SSm_i = Sf_i$. By (1), $e_0 \in l_SSm_0 = Sf_0$ so $e_0f_0 = e_0$. In (2), $e_0(f_0e_0gm_1) + e_1(f_0e_0gm_0) = 0$. Since $f_0e_0gm_0 = 0$, $e_0(gm_1) = 0$. Hence $e_0 \in l_SSm_1 = Sf_1$ and so $e_0f_1 = e_0$. Take $e = f_0f_1$, by (3), $e_0(egm_2)+e_1(egm_1)+e_2(egm_0) = 0$. Since $egm_0 = 0$ and $egm_1 = 0$, $e_0(gm_2) = 0$. Thus $e_0 \in l_SSm_2 = Sf_2$. Continuing in this way we may obtain, for every i, $e_0 \in l_SSm_i = Sf_i$. Hence $e_0 \in Se$. Similarly, we may obtain, for every i, $e_i \in Se$. Thus, $e[x] \in Ee$ and so $l_EEp[x] = Ee$. Therefore, $M[[x]]_{R[[x]]}$ is an endo-p.q.-Baer module. □

In [13], Liu has shown that $R[[x]]$ is left p.q.-Baer if and only if R is left p.q.-Baer and the left annihilator of the left ideal generated by any countable family of idempotents in R is generated by an idempotent. By applying Theorem 2.31 we obtain the following corollaries.

Corollary 2.32. *$R[[x]]$ is left p.q.-Baer if and only if R is left p.q.-Baer and the left annihilator of every countably generated left ideal of R is generated by an idempotent.*

In [12], Liu has shown that R is right p.q.-Baer if and only if $R[[x]]$ is right p.q.-Baer and any countable family of idempotent has generalized join when all the left semicentral idempotent are central. We also recall from [5, Proposition 1.17] that, for a right p.q.-Baer ring, the condition left semicentral idempotents are central is equivalent to assume R is semiprime. In the following corollary we see that the semiprime condition in Liu's result is not necessary.

Corollary 2.33. *A ring R is left p.q.-Baer and every countable subset of left semicentral idempotents in R has a generalized countable join in R if and only if $R[[x]]$ is a left p.q.-Baer ring.*

Proof. By Corollary 2.32 and Corollary 2.30, the result follows. □

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