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Hopf algebras for matroids over hyperfields

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Recently, M. Baker and N. Bowler introduced the notion of *matroids over hyperfields* as a unifying theory of various generalizations of matroids. In this paper we generalize the notion of minors and direct sums from ordinary matroids to matroids over hyperfields. Using this we generalize the classical construction of matroid-minor Hopf algebras to the case of matroids over hyperfields.

1. Introduction

A basic result in algebraic geometry is that the category $\mathbf{Aff}_k$ of affine schemes over a field k is equivalent to the opposite category $\mathbf{Alg}_k^{\mathrm{op}}$ of the category of commutative

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k-algebras. When one enhances $\mathbf{Aff}_k$ to affine group schemes over k, one obtains *Hopf algebras* as an enrichment of commutative k-algebras. In fact, Hopf algebras naturally appear in algebraic geometry, algebraic topology, representation theory, quantum field theory, and combinatorics. For a brief historical background for Hopf algebras, we refer readers to [3]. Our main interest in this paper is in Hopf algebras arising in combinatorics, namely those obtained from *matroids*, and more generally from *matroids over hyperfields*. A comprehensive introduction to Hopf algebras in combinatorics can be found in [8].

Matroids and their generalizations arise naturally in many areas of mathematics; this rich interplay with other areas of mathematics attests to their importance. For instance, N. Mnëv's universality theorem [12] roughly states that any semi-algebraic set in $\mathbb{R}^n$ is the moduli space of realizations of an oriented matroid up to homotopy equivalence. There is an analogue in algebraic geometry known as Murphy's Law by R. Vakil [17] for ordinary matroids. Valuated matroids are analogous to "linear spaces" in the setting of tropical geometry. Dressians, i.e. moduli spaces of valuated matroids, have received much attention.

Hopf algebras arising in combinatorics are usually created to encode the basic operations of an interesting class of combinatorial objects. The basic operations on matroids are *deletion*, *contraction*, and *direct sum*; an iterated sequence of deletions and contractions on a matroid results in a *minor* of the matroid. The Hopf algebra associated to a set of isomorphism classes of matroids closed under taking minors and direct sums is called a *matroid-minor* Hopf algebra. In this paper, we generalize the construction of the matroid-minor Hopf algebra to the setting of *matroids over hyperfields*, first introduced by M. Baker and N. Bowler in [4].

Remark 1.1. In fact, many combinatorial objects posses notions of "deletion" and "contraction" and hence one can associate Hopf algebras. In [7], C. Dupont, A. Fink, and L. Moci associate a *universal Tutte character* to such combinatorial objects specializing to Tutte polynomials in the case of matroids and graphs generalizing the work [10] of T. Krajewski, I. Moffatt, and A. Tanasa. See §6.2 in connection with our work.

M. Krasner first introduced *Hyperfields* in his work [11] on an approximation of a local field of positive characteristic by using local fields of characteristic zero. Krasner's motivation was to impose, for a given multiplicative subgroup G of a commutative ring A, a "ring-like" structure on the set of equivalence classes A/G, where G acts on A by left multiplication. Krasner abstracted algebraic properties of A/G, ultimately defining hyperfields. Roughly speaking, hyperfields are fields with *multi-valued* addition. After Krasner's work, hyperfields have been studied mainly in applied mathematics. Recently several authors have investigated hyperstructures in the context of algebraic geometry and number theory. Recently M. Baker (later with N. Bowler) employed hyperfields in combinatorics: Baker and Bowler found a beautiful framework which simultaneously generalizes the notion of linear subspaces, matroids, oriented matroids, and valuated matroids. In light of various fruitful applications of Hopf algebra methods in combinatorics, we ask the following.

Question. *Can we generalize matroid-minor Hopf algebras to matroids over hyperfields?*

We answer this question in the affirmative, following [4] definitions of minors for matroids over hyperfields which generalize the definition of minors for ordinary matroids.

Theorem A *(§3). Let H be a hyperfield. There are two cryptomorphic definitions, circuits and Grassmann-Plücker functions, of minors of matroids over H. Furthermore, if M is a weak (resp. strong) matroid over H, then all minors of M are weak (resp. strong).*

Next, we generalize the notion of direct sums to matroids over hyperfields and prove that direct sums preserve the type (weak or strong) of matroids over hyperfields in the following sense.

Theorem B *(§3). Let H be a hyperfield. There are two cryptomorphic definitions, circuits and Grassmann-Plücker functions, of direct sums of matroids over H. Furthermore, if M_1 and M_2 are (weak or strong) matroids over H, then the direct sum $M = M_1 \oplus M_2$ is always a weak matroid over H and M is strong if and only if both M_1 and M_2 are strong.*

Appealing to the above results, we define Hopf algebras for matroids over hyperfields in §5. Finally, in §6, we list some future directions for our work.

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2. Preliminaries

In this section, we recall definitions and basic properties of the key players in the paper. Throughout, we let $\mathbb{N} = \mathbb{Z}_{\geq 0}$ denote the natural numbers.

2.1. Matroids

This section is intended as a brief refresher on the basic notions and operations in matroid theory. We refer readers to [13] and [18] for further details and proofs of the facts from this section.

Matroids encode and generalize the combinatorics of linear independence in a finite dimensional vector space; similarly, matroids generalize properties of cycles in finite graphs. While there are several "cryptomorphic" definitions for matroids, chief among these are the notions of *bases* and *circuits*.

Let E be a finite set, the *ground set* of a matroid. A nonempty collection $\mathcal{B} \subseteq \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the power set of E, is a set of *bases* of a matroid when $\mathcal{B}$ satisfies the *basis exchange axiom*, given below.

(B) For all $X, Y \in \mathcal{B}$ and all $x \in X \setminus Y$ there is an element $y \in Y \setminus X$ such that $(X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Every pair of bases B_1, B_2 of a finite dimensional vector space satisfies this property by the Steinitz Exchange Lemma. In the context of finite graphs, this is a corollary of the Tree Exchange Property satisfied by the edge sets of spanning forests.

A collection $\mathcal{C} \subseteq \mathcal{P}(E)$ is a set of *circuits* of a matroid on E when $\mathcal{C}$ satisfies the following three axioms.

(1) (*Nondegeneracy*) $\emptyset \notin \mathcal{C}$.
(2) (*Incomparability*) If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $X = Y$.
(3) (*Circuit elimination*) For all $X, Y \in \mathcal{C}$ and all $e \in X \cap Y$, there is a $Z \in \mathcal{C}$ such that $Z \subseteq (X \cup Y) \setminus \{e\}$.

For finite graphs, circuits are precisely the edge sets of cycles in the graph. Circuits correspond with minimal dependence relations on a finite set of vectors in a vector space.

Remark 2.1. There is a natural correspondence between sets of circuits of a matroid on E and sets of bases of a matroid on E. Given a set $\mathcal{C}$ of circuits of a matroid, the set of maximal subsets of E not containing any element of $\mathcal{C}$ is the set of bases of a matroid. Likewise, given a set $\mathcal{B}$ of bases of a matroid, the set of minimal nonempty subsets of E which are not contained in any element of $\mathcal{B}$ is the set of circuits of a matroid. Moreover, the constructions above are inverse to one another. In this sense a set of bases $\mathcal{B}$ and the corresponding set of circuits $\mathcal{C}$ carry the same information; these are said to determine the same matroid on E "cryptomorphically."

Example 2.2. The motivating examples of matroids are given as follows.

(1) Let V be a finite dimensional vector space and $E \subseteq V$ a spanning set of vectors. The bases of V contained in E form the bases of a matroid on E, and the minimal dependent subsets of E form the circuits of a matroid on E. Furthermore, these are the same matroid.

(2) Let Γ be a finite, undirected graph with edge set E (loops and parallel edges are allowed). The sets of edges of spanning forests in Γ form the bases of a matroid on E, and the sets of edges of cycles form the circuits of a matroid on E. Furthermore, these are the same matroid.

Definition 2.3. Let M_1 (resp. M_2) be a matroid on E_1 (resp. E_2) defined by a set $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) of bases. We say that M_1 is *isomorphic* to M_2 if there exists a bijection $f\colon E_1 \to E_2$ such that $f(B) \in \mathcal{B}_2$ if and only if $B \in \mathcal{B}_1$. In this case, f is said to be an *isomorphism*.

Example 2.4. Let Γ_1 and Γ_2 be finite graphs and M_1 and M_2 be the corresponding matroids. Every graph isomorphism between Γ_1 and Γ_2 gives rise to a matroid isomorphism between M_1 and M_2, but the converse need not hold.

Given any base $B \in \mathcal{B}(M)$ and any element $e \in E \setminus B$, there is a unique *fundamental circuit* $C_{B,e}$ of e with respect to B such that $C_{B,e} \subseteq B \cup \{e\}$.

Definition 2.5 *(Direct sum of matroids).* Let M_1 and M_2 be matroids on E_1 and E_2 given by bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. The *direct sum* $M_1 \oplus M_2$ is the matroid on $E_1 \sqcup E_2$ given by bases $\mathcal{B} = \{B_1 \sqcup B_2 \mid B_i \in \mathcal{B}_i \text{ for } i = 1, 2\}$.

Definition 2.6 *(Dual, Restriction, Deletion, and Contraction).* Let M be a matroid on a finite set E_M with the set $\mathcal{B}_M$ of bases and the set $\mathcal{C}_M$ of circuits. Let S be a subset of E_M.

(1) The *dual* M^* of M is a matroid on E_M given by bases

$$\mathcal{B}_{M^*} = \{E_M - B \mid B \in \mathcal{B}_M\}.$$

(2) The *restriction* $M|S$ of M to S is a matroid on S given by circuits

$$\mathcal{C}_{M|S} = \{D \subseteq S \mid D \in \mathcal{C}_M\}.$$

(3) The *deletion* $M \setminus S$ of S is the matroid $M \setminus S = M|(E \setminus S)$.
(4) The *contraction* of M by S is $M/S = (M^* \setminus S)^*$.

Each operation above results in a matroid. A *minor* of a matroid M is any matroid obtained from M by a series of deletions and/or contractions. Basic properties relating these operations are given below.

Proposition 2.7. *Let M be a matroid on E. We have the following for all disjoint subsets S and T of E.*

(1) $M/\emptyset = M = M \setminus \emptyset$.
(2) $(M \setminus S) \setminus T = M \setminus (S \cup T)$.
(3) $(M/S)/T = M/(S \cup T)$.
(4) $(M \setminus S)/T = (M/T) \setminus S$.

2.2. Matroids over hyperfields

In this section, we review basic definitions and properties for matroids over hyperfields first introduced by Baker and Bowler in [4]. Let's first recall the definition of a hyperfield. By a *hyperaddition* on a nonempty set H, we mean a function $+\colon H \times H \to \mathcal{P}^*(H)$ such that $+(a,b) = +(b,a)$ for all $a,b \in H$, where $\mathcal{P}^*(H)$ is the set of nonempty subsets of H. We will simply write $a+b$ for $+(a,b)$. A hyperaddition $+$ on H is *associative* if the following condition holds: for all $a,b,c \in H$,

$$a + (b + c) = (a + b) + c. \tag{1}$$

For subsets A and B of H, we write $A + B := \bigcup_{a\in A, b\in B} a + b$; thus the notation in (1) makes sense. We also write singleton $\{a\}$ as a when confusion is unlikely.

Definition 2.8. Let H be a nonempty set with an associative hyperaddition $+$. We say that $(H,+)$ is a *canonical hypergroup* when the following conditions hold:

(1) (*Identity*) $\exists!\ 0 \in H$ such that $a + 0 = a$ for all $a \in H$.
(2) (*Inverse*) $\forall\ a \in H$, $\exists!\ b\ (=: -a) \in H$ such that $0 \in a + b$.
(3) (*Reversibility*) $\forall\ a,b,c \in H$, if $a \in b + c$, then $c \in a + (-b)$.

We will write $a - b$ instead of $a + (-b)$ for brevity of notation.

Definition 2.9. By a *hyperring*, we mean a nonempty set H with a binary operation $\cdot$ and hyperaddition $+$ such that $(H,+,0)$ is a canonical hypergroup and $(H,\cdot,1)$ is a commutative monoid satisfying the following conditions: for all $a,b,c \in H$,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad 0 \cdot a = 0, \quad \text{and } 1 \neq 0.$$

When $(H - \{0\}, \cdot, 1)$ is a group, we call H a *hyperfield*.

Definition 2.10. Let H_1 and H_2 be hyperrings. A *homomorphism* of hyperrings from H_1 to H_2 is a function $f\colon H_1 \to H_2$ such that f is a monoid morphism with respect to multiplication satisfying the following conditions:

$$f(0) = 0 \ \text{ and } \ f(a + b) \subseteq f(a) + f(b),\ \forall a,b \in H_1.$$

The following are some typical examples of hyperfields found in the literature.

Example 2.11 *($\mathbb{K}$; Krasner hyperfield).* Let $\mathbb{K} := \{0, 1\}$ and impose the usual multiplication $0 \cdot 0 = 0$, $1 \cdot 1 = 1$, and $0 \cdot 1 = 0$. Hyperaddition is defined as $0 + 1 = 1$, $0 + 0 = 0$, and $1 + 1 = \mathbb{K}$. The structure $\mathbb{K}$ is the *Krasner hyperfield*.

Remark 2.12. For every hyperfield H there is a unique homomorphism $\kappa\colon H \to \mathbb{K}$, sending every nonzero element to 1 and 0 to 0. Thus $\mathbb{K}$ is final in the category of hyperfields.

Example 2.13 *($\mathbb{S}$; hyperfield of signs).* Let $\mathbb{S} := \{-1, 0, 1\}$ and impose multiplication in a usual way following the rule of signs; $1 \cdot 1 = 1$, $(-1) \cdot 1 = (-1)$, $(-1) \cdot (-1) = 1$, and $1 \cdot 0 = (-1) \cdot 0 = 0 \cdot 0 = 0$. Hyperaddition also follows the rule of signs as follows:

$$1 + 1 = 1,\ (-1) + (-1) = (-1),\ 1 + 0 = 1,\ (-1) + 0 = (-1),\ 0 + 0 = 0,\ 1 + (-1) = \mathbb{S}.$$

The structure $\mathbb{S}$ is the *hyperfield of signs*.

Example 2.14 *($\mathbb{P}$; phase hyperfield).* Let $\mathbb{P} := S^1 \cup \{0\}$, where S^1 is the unit circle in the complex plane. The multiplication on $\mathbb{P}$ is the usual multiplication of complex numbers. Hyperaddition is defined by:

$$a+b = \begin{cases} \{-a, 0, a\} & \text{if } a = -b (-b \text{ as a complex number}), \\ \text{the shorter open arc connecting} a \text{ and } b & \text{if } a \neq -b. \end{cases}$$

The structure $\mathbb{P}$ is the *phase hyperfield*.

Example 2.15 *($\mathbb{T}$; tropical hyperfield).* Let G be a (multiplicative) totally ordered abelian group. Then one can enrich the structure of G to define a hyperfield. To be precise, let $G_{hyp} := G \cup \{-\infty\}$ and define multiplication via the multiplication of G together with the rule $g \cdot (-\infty) = -\infty$ for all $g \in G$. Hyperaddition is defined as follows:

$$a + b = \begin{cases} \max\{a, b\} & \text{if } a \neq b \\ [-\infty, a] & \text{if } a = b, \end{cases}$$

where $[-\infty, a] := \{g \in G_{hyp} \mid g \leq a\}$ with $-\infty$ the smallest element. Then G_{hyp} is a hyperfield. For $\mathbb{R}$ the set of real numbers with the usual addition and ordering, we obtain the *tropical hyperfield* $\mathbb{T} := \mathbb{R}_{hyp}$.

In what follows, let $(H, \boxplus, \odot)$ be a hyperfield, $H^\times = H - \{0_H\}$, r a positive integer, $[r] = \{1, \ldots, r\}$, $\mathbf{x}$ an element of E^r such that $\mathbf{x}(i) \in E$ is the ith coordinate of $\mathbf{x}$. Now we recall the two notions (weak and strong) of matroids over hyperfields introduced by Baker and Bowler. These notions are given cryptomorphically by structures analogous to the bases and circuits of ordinary matroids. Their definition simultaneously generalizes several existing theories of "matroids with extra structure," evidenced by the following examples:

Example 2.16. Matroids over the following hyperfields have been studied in the past:

- A (strong or weak) matroid over a field k is a linear subspace of k^r.
- A (strong or weak) matroid over the Krasner hyperfield $\mathbb{K}$ is an ordinary matroid.
- A (strong or weak) matroid over the hyperfield of signs $\mathbb{S}$ is an oriented matroid.
- A (strong or weak) matroid over the tropical hyperfield $\mathbb{T}$ is a valuated matroid.

We first recall the generalization of bases to the setting of matroids over hyperfields. This is done via *Grassmann-Plücker functions*.

Definition 2.17. Let H be a hyperfield, E a finite set, r a nonnegative integer, and Σ_r the symmetric group on r letters with a canonical action on E^r (acting on indices).

(1) A function $\varphi\colon E^r \to H$ is a *nontrivial H-alternating function* when:
  (G1) The function φ is not identically zero.
  (G2) For all $\mathbf{x} \in E^r$ and all $\sigma \in \Sigma_r$ we have $\varphi(\sigma \cdot \mathbf{x}) = \operatorname{sgn}(\sigma)\varphi(\mathbf{x})$.
  (G3) If $\mathbf{x} \in E^r$ has $\mathbf{x}(i) = \mathbf{x}(j)$ for some $i < j$, then $\varphi(\mathbf{x}) = 0_H$.
(2) A nontrivial H-alternating function $\varphi\colon E^r \to H$ is a *weak-type Grassmann-Plücker function* over H when:
  (WG) For all $a, b, c, d \in E$ and all $\mathbf{x} \in E^{r-2}$ we have

$$0_H \in \varphi(a, b, \mathbf{x})\varphi(c, d, \mathbf{x}) - \varphi(a, c, \mathbf{x})\varphi(b, d, \mathbf{x}) + \varphi(b, c, \mathbf{x})\varphi(a, d, \mathbf{x}).$$

(3) A nontrivial H-alternating function $\varphi\colon E^r \to H$ is a *strong-type Grassmann-Plücker function* over H when:
  (SG) For all $\mathbf{x} \in E^{r+1}$ and all $\mathbf{y} \in E^{r-1}$ we have

$$0_H \in \sum_{k=1}^{r+1} (-1)^k \varphi(\mathbf{x}|_{[r+1]\setminus\{k\}})\varphi(\mathbf{x}(k), \mathbf{y}).$$

(4) The *rank* of a Grassmann-Plücker function $\varphi\colon E^r \to H$ is r.
(5) Two Grassmann-Plücker functions $\varphi, \psi\colon E^r \to H$ are *equivalent* when there is an element $a \in H^\times$ with $\psi = a \odot \varphi$.

A *matroid over H* is an $H^\times$-equivalence class $[\varphi]$ of a Grassmann-Plücker function φ. Before presenting circuits of matroids over hyperfields, we need a technical definition.

Definition 2.18. Let $\mathcal{S}$ be a collection of inclusion-incomparable subsets of a set E. A *modular pair* in $\mathcal{S}$ is a pair of distinct elements $X, Y \in \mathcal{S}$ such that for all $A, B \in \mathcal{S}$, if $A \cup B \subseteq X \cup Y$, then $A \cup B = X \cup Y$.

Having Definition 2.18, we can now give definitions of collections of circuits for matroids over hyperfields. In what follows, we will simply write $\sum$ instead of $\boxplus$ if the context is clear.

Definition 2.19. Let E be a finite set, $(H, \boxplus, \odot)$ a hyperfield, H^E the set of functions from E to H. For any $X \in H^E$, we define $\operatorname{supp}(X) := \{a \in E \mid X(a) \neq 0_H\}$.

(1) A collection $\mathcal{C} \subseteq H^E$ is a family of *pre-circuits* over H when it satisfies the following axioms:
 (C1) $\mathbf{0} \notin \mathcal{C}$
 (C2) $H^\times \odot \mathcal{C} = \mathcal{C}$
 (C3) For all $X, Y \in \mathcal{C}$, if $\operatorname{supp}(X) \subseteq \operatorname{supp}(Y)$, then $Y = a \odot X$ for some $a \in H^\times$.
(2) A pre-circuit set $\mathcal{C}$ over H is a *weak-type circuit set* when it satisfies the following additional axiom:
 (WC) For all X, Y in $\mathcal{C}$ such that $\{\operatorname{supp}(X), \operatorname{supp}(Y)\}$ forms a modular pair in $\operatorname{supp}(\mathcal{C}) := \{\operatorname{supp}(X) \mid X \in \mathcal{C}\}$ and for all $e \in \operatorname{supp}(X) \cap \operatorname{supp}(Y)$, there is a $Z \in \mathcal{C}$ such that

$$Z(e) = 0_H \text{ and } Z \in X(e) \odot Y - Y(e) \odot X,$$

 i.e., for all $f \in E$, $Z(f) \in X(e) \odot Y(f) - Y(e) \odot X(f)$.
(3) A pre-circuit set $\mathcal{C}$ over H is a *strong-type circuit set* when it satisfies the following additional axioms:
 (SC1) The set $\operatorname{supp}(\mathcal{C})$ is the set of circuits of an ordinary matroid $M_{\mathcal{C}}$.
 (SC2) For all bases $B \in \mathcal{B}_{\mathcal{C}}$ and all $X \in \mathcal{C}$ we have

$$X \in \sum_{e \in E \setminus B} X(e) \odot Y_{B,e},$$

 where $Y_{B,e}$ is the (unique) element of $\mathcal{C}$ such that $Y_{B,e}(e) = 1$ and $\operatorname{supp}(Y_{B,e})$ is the fundamental circuit of e with respect to B.

Remark 2.20. The definition of strong-type circuit sets given above is not the original definition. This is equivalent to the original definition by [4, Theorem 3.8, Remark 3.9]. For our purposes, we shall use the definition given above.

The following result is proved in [4]:

Proposition 2.21. *Let H be a hyperfield. The $H^\times$-orbits of Grassmann-Plücker functions over H are in natural one-to-one correspondence with the H-circuits of a matroid, preserving both ranks and types (weak and strong).*

The correspondence is described as follows:
Given a Grassmann-Plücker function φ over H, one first shows that the collection of

subsets $B \subseteq E$ for which an ordering $\mathbf{B}$ has $\varphi(\mathbf{B}) \neq 0_H$ forms a set of bases for an ordinary matroid M_φ. Next, one can define a set of H-circuits by defining for all ordered bases $\mathbf{B}$ of M_φ and all $e \in E \setminus B$ a function $X = X_{\mathbf{B},e}$ supported on the fundamental circuit for e by B via the equality

$$X(\mathbf{B}(i))X(e)^{-1} = (-1)^i \varphi(e, \mathbf{B}|_{[r]\setminus\{i\}})\varphi(\mathbf{B})^{-1} \tag{2}$$

This equality uniquely determines $X\colon E \to H$, up to the multiplicative action of $H^\times$. The collection of all such X is a collection of H-circuits of the same type as φ.

Constructing a Grassmann-Plücker function from circuits is more difficult to describe, and requires the additional notion of dual pairs. An explicit description of this construction is unnecessary for our purposes; the interested reader is referred to [4].

We now describe the duality operation for matroids over hyperfields in terms of Grassmann-Plücker functions and subsequently in terms of circuits. It should be noted that the duality described in [4] incorporates a notion of conjugation generalizing the complex conjugation. This changes the duality operation, but the change is equally well described by another operation (called "pushforward through a morphism") as noted in a footnote in [4, §6]. Our treatment will also assume that the conjugation is trivial.

Fix a total ordering $\leq$ of E. A *dual* of a Grassmann-Plücker function φ over H is defined by the equation $\varphi^*(\mathbf{B}) := \mathrm{sgn}_{\leq}(\mathbf{B}, \mathbf{E} \setminus \mathbf{B})\varphi(\mathbf{E} \setminus \mathbf{B})$ for all cobases B of the underlying matroid of φ, using the convention that $\mathbf{S}$ denotes the ordered tuple with coordinates the elements of S arranged according to our fixed total ordering on E and $\mathrm{sgn}_{\leq}(\mathbf{B}, \mathbf{E} \setminus \mathbf{B})$ denotes the sign of the permutation given by the word $(\mathbf{B}, \mathbf{E} \setminus \mathbf{B})$ with respect to the ordering $\leq$. The definition can be uniquely extended to the set $E^{\#E-r}$ by alternation and the degeneracy conditions for Grassmann-Plücker functions over H. It is relatively easy to see that if φ is a Grassmann-Plücker function, then φ^* is a Grassmann-Plücker function of the same type. Notice that this duality is well-defined up to the chosen ordering; a different ordering will induce a Grassmann-Plücker function which is multiplied by the sign of the permutation used to translate between the two orderings. In particular, this notion of duality is constant on the level of $H^\times$-orbits of Grassmann-Plücker functions, and thus sends an H-matroid M to an H-matroid M^* of the same type despite the fact that there is no canonical dual to the original Grassmann-Plücker function.

The dual of a circuit set requires some more care to define; it is here that the contrast between weak and strong H-matroids is most stark.

Definition 2.22. Let H be a hyperfield and E a finite set.

(1) The *dot product* of two functions $X, Y\colon E \to H$ is the following subset of H:

$$X \cdot Y := \sum_{e \in E} X(e)Y(e).$$

(2) Two functions X and Y are *strong orthogonal*, denoted $X \perp_s Y$, when

$$0_H \in X \cdot Y.$$

(3) Two functions X and Y are *weak orthogonal*, denoted $X \perp_w Y$, when either $X \perp_s Y$ or the following condition holds:

$$\#(\mathrm{supp}(X) \cap \mathrm{supp}(Y)) > 3.$$

(4) Let $\mathcal{C}$ be a set of strong H-circuits on E, we define the following subset of H^E:

$$\mathcal{C}^{\perp_s} := \{X : E \to H \mid X \perp_s Y \text{ for all } Y \in \mathcal{C}\}.$$

(5) Let $\mathcal{C}$ be a set of weak H-circuits, we define the following subset of H^E:

$$\mathcal{C}^{\perp_w} := \{X : E \to H \mid X \perp_w Y \text{ for all } Y \in \mathcal{C}\}.$$

For ease of notation the symbol $\perp$ is to be understood in context as either $\perp_s$ or $\perp_w$.

Definition 2.23. Let H be a hyperfield and M be a weak (resp. strong) H-matroid with the set $\mathcal{C}$ of weak type (resp. strong type) H-circuits. The *cocircuits* of $\mathcal{C}$, denoted by $\mathcal{C}^*$, are the elements of the perpendicular set $\mathcal{C}^{\perp_w}$ (resp. $\mathcal{C}^{\perp_s}$) with minimal support.

Remark 2.24. In [4, §6.6], Baker and Bowler show that this determines an H-matroid with the properties $\mathrm{supp}(\mathcal{C}^*) = (\mathrm{supp}(\mathcal{C}))^*$ and $\mathcal{C}^{**} = \mathcal{C}$; in other words, the underlying matroid of the dual is the dual of the underlying matroid and the double dual is identical to the original H-matroid.

Remark 2.25. From an algebraic geometric view point, Baker and Bowler's definition of matroids over hyperfields can be considered as points of a Grassmannian over a hyperfield H. Motivated by this observation, in [9], the second author proves that certain topological spaces (the underlying spaces of a scheme, Berkovich analytification of schemes, real schemes) are homeomorphic to sets of rational points of a scheme over a hyperfield. Also, recently L. Anderson and J. Davis defined and investigated hyperfield Grassmannians in connection to the MacPhersonian (from oriented matroid theory) in [1].

2.3. Matroid-minor Hopf algebras

In this subsection we recall the definition of matroid-minor Hopf algebras. First we briefly recall the definition of commutative Hopf algebras; interested readers are referred to [6] for more details.

Definition 2.26. Let k be a field. A commutative k-algebra A is a *Hopf algebra* if A is equipped with maps

(1) (Comultiplication) $\Delta\colon A \to A \otimes_k A$,
(2) (Counit) $\varepsilon\colon A \to k$,
(3) (Antipode) $S\colon A \to A$

such that the following diagrams commute:

$$\begin{tikzcd} A \otimes_k A \arrow[r, "\Delta\otimes\mathrm{id}"] & A \otimes_k A \otimes_k A \\ A \arrow[u, "\Delta"] \arrow[r, "\Delta"] & A \otimes_k A, \arrow[u, "\mathrm{id}\otimes\Delta"'] \end{tikzcd} \quad \begin{tikzcd} A \otimes_k A \arrow[r, "\varepsilon\otimes\mathrm{id}"] & k \otimes_k A \\ A \arrow[u, "\Delta"] \arrow[r, "\mathrm{id}"] & A, \arrow[u, "\simeq"'] \end{tikzcd} \quad \begin{tikzcd} A \otimes_k A \arrow[r, "\mu\circ(S\otimes\mathrm{id})"] & A \\ A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & k, \arrow[u, "i"'] \end{tikzcd}$$

where $\mu\colon A \otimes_k A \to A$ is the multiplication of A and $\eta\colon k \to A$ is the unit map. If A is only equipped with Δ and ε satisfying the first two commutative diagrams, then A is a *bialgebra*.

Definition 2.27. Let $(A, \mu, \Delta, \eta, \varepsilon)$ be a bialgebra over a field k.

(1) A is *graded* if there is a grading $A = \bigoplus_{i\in\mathbb{N}} A_i$ which is compatible with the bialgebra structure of A, i.e., μ, Δ, η, and ε are graded k-linear maps.
(2) A is *connected* if A is graded and $A_0 = k$.

Definition 2.28. Let A_1 and A_2 be Hopf algebras over a field k. A *homomorphism* of Hopf algebras is a k-bialgebra map $\alpha\colon A_1 \to A_2$ which preserves the antipodes, i.e., $S_{A_1}\alpha = \alpha S_{A_2}$.

The following theorem shows that indeed there is no difference between bialgebra maps and Hopf algebra maps.

Theorem 2.29. [6, Proposition 4.2.5.] Let A_1 and A_2 be Hopf algebras over a field k. Let $\alpha\colon A_1 \to A_2$ be a morphism of k-bialgebras. Then α is a homomorphism of Hopf algebras.

We also introduce the following notation:

Definition 2.30. Let A be a Hopf algebra over a field k and $i \in \mathbb{Z}_{\geq 1}$.

(1) (Iterated multiplication): $\mu^i\colon A^{\otimes(i+1)} \to A$ is defined inductively as

$$\mu^i := \mu \circ (\mathrm{id}\otimes\mu^{(i-1)}).$$

(2) (Iterated comultiplication): $\Delta^i\colon A \to A^{\otimes(i+1)}$ is defined inductively as

$$\Delta^i := (\mathrm{id}\otimes\Delta^{(i-1)}) \circ \Delta.$$

Now, let's recall the definition of matroid-minor Hopf algebras, first introduced by W. R. Schmitt in [14]. Let $\mathcal{M}$ be a collection of matroids which is closed under taking minors and direct sums. Let $\mathcal{M}_{\text{iso}}$ be the set of isomorphism classes of matroids in $\mathcal{M}$. For a matroid M in $\mathcal{M}$, we write $[M]$ for the isomorphism class of M in $\mathcal{M}_{\text{iso}}$. One can enrich $\mathcal{M}_{\text{iso}}$ to a commutative monoid with the direct sum

$$[M_1] \cdot [M_2] \coloneqq [M_1 \oplus M_2],$$

under which the identity is $[\emptyset]$, the equivalence class of the empty matroid. Let A be the monoid algebra $k[\mathcal{M}_{\text{iso}}]$ over a field k.

For any matroid M, let E_M denote the ground set of M. Consider the following maps:

- (Comultiplication)

$$\Delta\colon k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}] \otimes_k k[\mathcal{M}_{\text{iso}}], \quad [M] \mapsto \sum_{S \subseteq E_M} [M|_S] \otimes [M/S].$$

- (Counit)

$$\varepsilon\colon k[\mathcal{M}_{\text{iso}}] \to k, \quad [M] \mapsto \begin{cases} 1 & \text{if } E_M = \emptyset \\ 0 & \text{if } E_M \neq \emptyset. \end{cases}$$

Under the above maps, $k[\mathcal{M}_{\text{iso}}]$ becomes a connected bialgebra; $k[\mathcal{M}_{\text{iso}}]$ is graded by cardinalities of ground sets. It follows from the result of M. Takeuchi [16] that $k[\mathcal{M}_{\text{iso}}]$ has a unique Hopf algebra structure with a unique antipode S given by

$$S = \sum_{i \in \mathbb{N}} (-1)^i \mu^{i-1} \circ \pi^{\otimes i} \circ \Delta^{i-1}, \tag{3}$$

where μ^{-1} is the unit map $\eta\colon k \to k[\mathcal{M}_{\text{iso}}]$, $\Delta^{-1} \coloneqq \varepsilon$, and $\pi\colon k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}]$ is the projection map defined by

$$\pi|_{A_n} \begin{cases} \text{id} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0, \end{cases}$$

and extended linearly to $k[\mathcal{M}_{\text{iso}}]$, where A_n is the nth graded piece of A.

3. Minors and sums of matroids over hyperfields

In this section we explicitly write out the constructions of restriction, deletion, contraction, and direct sums for matroids over hyperfields. We do this cryptomorphically via both circuits and Grassmann-Plücker functions in both the weak and strong cases. Primarily, we define the restriction, and subsequently use our characterization to derive the other cryptomorphic descriptions of minors. It should be noted that formulas for deletion and contraction in the case of phirotopes are given for phased matroids in [2] and for general Grassmann-Plücker functions without proof in [4].[1] For completeness, we give full proofs and expand the previous work by giving formulas for the circuits of these objects as well.

3.1. Circuits of H-matroid restrictions

Let H be a hyperfield, E be a finite set, $\mathcal{C}$ be a set of (either weak-type or strong-type) H-circuits on E, and $S \subseteq E$. Recall that H^S is the set of functions from S to H. We make the following notation.

$$\mathcal{C}\,|S := \left\{ X|_S \in H^S \;\middle|\; X \in \mathcal{C} \text{ and } \operatorname{supp}(X) \subseteq S \right\}. \tag{4}$$

Proposition 3.1. *Let $\mathcal{C}$ be a set of weak-type (resp. strong-type) H-circuits of a matroid M over H on a ground set E.*

(1) *For all $S \subseteq E$, the set $\mathcal{C}\,|S$ is a set of weak-type (resp. strong-type) H-circuits on S.*
(2) *The underlying matroid of the H-matroid M determined by $\mathcal{C}\,|S$ is precisely the restriction of the underlying matroid* $\operatorname{supp}(M)|S$. *In other words, the restriction commutes with the* push-forward operation *to the Krasner hyperfield $\mathbb{K}$.*

Proof. If $\mathcal{C}$ is a set of circuits of an H-matroid, then $\operatorname{supp}(\mathcal{C}\,|S) = \operatorname{supp}(\mathcal{C})|S$ and hence $\operatorname{supp}(\mathcal{C}\,|S)$ is the set of circuits of the restriction of the ordinary matroid; in particular, the second statement follows immediately from the first statement.

Now we prove the first statement. Let $X, Y \in \mathcal{C}$ have $\operatorname{supp}(X)$ and $\operatorname{supp}(Y) \subseteq S$. Thus

$$\operatorname{supp}(X|_S) = \operatorname{supp}(X) \text{ and } \operatorname{supp}(Y|_S) = \operatorname{supp}(Y). \tag{5}$$

We first show that if $\mathcal{C}$ is a set of pre-circuits over H on E, then $\mathcal{C}\,|S$ is also a set of pre-circuits over H on S. Indeed, since $\operatorname{supp}(X) \subseteq S$ and $X \neq \mathbf{0}$, we have that $X|_S \neq \mathbf{0}$ and $(a \odot X|_S) \in \mathcal{C}\,|S$ for all $a \in H^\times$. Finally, if $\operatorname{supp}(X|_S) \subseteq \operatorname{supp}(Y|_S)$, then

$$\operatorname{supp}(X) = \operatorname{supp}(X|_S) \subseteq \operatorname{supp}(Y|_S) = \operatorname{supp}(Y)$$

yields $Y = a \odot X$ for some $a \in H^\times$ and hence $Y|_S = a \odot X|_S$ as desired. This proves $\mathcal{C}|_S$ is a set of pre-circuits over H on S.

[1] Note that [2] accidentally assumes Axiom (WG) implies Axiom (SG); as a result, the results therein fail to take account of differences in the weak and strong cases. While [4] fixes this issue, many results refer their proofs back to [2] without presenting the required adjustments.

Next we prove that if $\mathcal{C}$ is a set of weak-type H-circuits on E, then $\mathcal{C}\,|S$ is also a set of weak-type H-circuits on S. In fact, if $X|_S$ and $Y|_S$ form a modular pair in $\mathcal{C}\,|S$, then (5) implies X and Y are a modular pair in $\mathcal{C}$. More precisely, in this case, the condition

$$A \cup B \subseteq \mathrm{supp}(X) \cup \mathrm{supp}(Y) \subseteq S, \quad A, B \subset \mathrm{supp}(\mathcal{C})$$

implies $A, B \subseteq S$. Thus for all $e \in \mathrm{supp}(X) \cap \mathrm{supp}(Y)$, there is a $Z \in \mathcal{C}$ such that

$$Z(e) = 0 \text{ and } Z \in X(e)Y - Y(e)X. \tag{6}$$

On the other hand, if $a \notin \mathrm{supp}(X) \cup \mathrm{supp}(Y)$, then $X(e)Y(a) - Y(e)X(a) = \{0\}$. Thus, for $A \in \mathcal{C}$, $A \in X(e)Y - Y(e)X$ implies $\mathrm{supp}(A) \subseteq \mathrm{supp}(X) \cup \mathrm{supp}(Y) \subseteq S$. Hence $\mathrm{supp}(Z) \subseteq S$ and $Z|_S \in \mathcal{C}|_S$, and $\mathcal{C}\,|S$ inherits (WC) from $\mathcal{C}$. Thus $\mathcal{C}|_S$ is a set of weak-type H-circuits.

Finally, we show that if $\mathcal{C}$ is a set of strong-type H-circuits on E, then $\mathcal{C}\,|S$ is also a set of strong-type H-circuits on S. As we mentioned before, $\mathrm{supp}(\mathcal{C}\,|S) = \mathrm{supp}(\mathcal{C})|S$ is a set of circuits of a matroid as these are given by the same formula and $\mathrm{supp}(\mathcal{C})$ is a set of circuits of an ordinary matroid; in particular $\mathcal{C}\,|S$ satisfies axiom (SC1). Let $\mathcal{B}_{\mathcal{C}\,|S}$ (resp. $\mathcal{B}_{\mathcal{C}}$) be the set of bases of an underlying matroid $M_{\mathcal{C}\,|S}$ (resp. $M_{\mathcal{C}}$) given by the set $\mathrm{supp}(\mathcal{C}\,|S)$ (resp. $\mathrm{supp}(\mathcal{C})$) of circuits. If $B \in \mathcal{B}_{\mathcal{C}\,|S}$, then we have that $B = \widetilde{B} \cap S$ for some $\widetilde{B} \in \mathcal{B}_{\mathcal{C}}$. Applying (SC2) to $\mathcal{C}$ with $\widetilde{B}$ and X, we obtain

$$X \in \sum_{e \in E \setminus \widetilde{B}} X(e) \odot Y_{\widetilde{B},e}. \tag{7}$$

Now $Y_{\widetilde{B},e}|_S = Y_{B,e}$ by incomparability of circuits in ordinary matroids, and thus (7) implies

$$X|_S \in \sum_{e \in E \setminus B} X|_S(e) \odot Y_{B,e}.$$

Thus Axiom (SC2) holds for $\mathcal{C}\,|S$ and hence $\mathcal{C}\,|S$ is a strong-type H-circuit set, as claimed. $\square$

Now, thanks to Proposition 3.1, the following definition makes sense.

Definition 3.2. Let M be a matroid over hyperfield H on a ground set E given by weak (resp. strong) H-circuits $\mathcal{C}$, and let S be a subset of E. The *restriction* of matroid M to S is the matroid $M|S$ over H given by weak (resp. strong) H-circuits $\mathcal{C}\,|S$.

3.2. Grassmann-Plücker functions of H-matroid restrictions

We now describe restriction of H-matroids via Grassmann-Plücker functions. Let H be a hyperfield, E a finite set, r a positive integer, and φ a (weak-type or strong-type)

Grassmann-Plücker function over H on E of rank r. Let M_φ denote the underlying matroid of φ given by bases

$$\mathcal{B}_\varphi = \{\{b_1, \ldots, b_r\} \subseteq E \mid \varphi(b_1, \ldots, b_r) \neq 0\}.$$

Recall that for any ordered basis $B = \{b_1, b_2, \ldots, b_k\}$ of M_φ/S, we let

$$\mathbf{B} = (b_1, b_2, \ldots, b_k) \in E^k.$$

For any subset $S \subseteq E$ and any (ordered) basis $B = \{b_1, b_2, \ldots, b_k\}$ of M_φ/S, we define

$$\varphi^{\mathbf{B}}\colon S^{r-k} \longrightarrow H, \quad \mathbf{A} \mapsto \varphi(\mathbf{A}, \mathbf{B}).$$

Proposition 3.3. *Let φ be a weak-type (resp. strong-type) Grassmann-Plücker function over H on E of rank r and let $S \subseteq E$. For all ordered bases $\mathbf{B}$ of M_φ/S, the function $\varphi^{\mathbf{B}}$ is a weak-type (resp. strong-type) Grassmann-Plücker function. Moreover, all such $\varphi^{\mathbf{B}}$ determine the H-circuits $\mathcal{C}|S$ of $M|S$.*

Proof. For notational convenience, let $[n] = \{1, 2, \ldots, n\}$ and regard $\mathbf{B}$ as a function $\mathbf{B}\colon [k] \to E$. First, one can observe that $\varphi^{\mathbf{B}}$ is a nontrivial H-alternating function as $\varphi^{\mathbf{B}}$ is a restriction of a nontrivial H-alternating function to a subset containing a base of M_φ. We claim that if φ is a weak-type Grassmann-Plücker function over H, then $\varphi^{\mathbf{B}}$ is also a weak-type Grassmann-Plücker function over H. To see this, let $a, b, c, d \in E$ and $\mathbf{Y}\colon [r-k-1] \to E$ be given. Applying Axiom (WG) to $a, b, c, d \in E$ and $\mathbf{x} = (\mathbf{Y}, \mathbf{B}) \in E^{r-2}$, we obtain

$$\begin{aligned} 0_H &\in \varphi(a, b, \mathbf{Y}, \mathbf{B})\varphi(c, d, \mathbf{Y}, \mathbf{B}) - \varphi(a, c, \mathbf{Y}, \mathbf{B})\varphi(b, d, \mathbf{Y}, \mathbf{B}) + \varphi(a, d, \mathbf{Y}, \mathbf{B})\varphi(b, c, \mathbf{Y}, \mathbf{B}) \\ &= \varphi^{\mathbf{B}}(a, b, \mathbf{Y})\varphi^{\mathbf{B}}(c, d, \mathbf{Y}) - \varphi^{\mathbf{B}}(a, c, \mathbf{Y})\varphi^{\mathbf{B}}(b, d, \mathbf{Y}) + \varphi^{\mathbf{B}}(a, d, \mathbf{Y})\varphi^{\mathbf{B}}(b, c, \mathbf{Y}). \end{aligned}$$

This shows that Axiom (WG) holds for $\varphi^{\mathbf{B}}$ and hence $\varphi^{\mathbf{B}}$ is a weak-type Grassmann-Plücker function over H.

We next show that if φ is a strong-type Grassmann-Plücker function over H, then $\varphi^{\mathbf{B}}$ is also a strong-type Grassmann-Plücker function over H. Indeed, let $\mathbf{X}\colon [r-k+1] \to S$ and $\mathbf{Y}\colon [r-k-1] \to S$ be given. Applying Axiom (SG) to $\mathbf{x} := (\mathbf{X}, \mathbf{B})$ and $\mathbf{y} := (\mathbf{Y}, \mathbf{B})$, we obtain the following:

$$\begin{aligned} 0_H \in &\sum_{j \in [r-k+1]} (-1)^j \varphi(\mathbf{X}|_{[r-k+1]\setminus\{j\}}, \mathbf{B})\varphi(\mathbf{X}(j), \mathbf{Y}, \mathbf{B}) \\ &\quad + \sum_{j \in [k]} (-1)^{r-k+1+j} \varphi(\mathbf{X}, \mathbf{B}|_{[k]\setminus\{j\}})\varphi(\mathbf{B}(j), \mathbf{Y}, \mathbf{B}) \\ = &\sum_{j \in [r-k+1]} (-1)^j \varphi(\mathbf{X}|_{[r-k+1]\setminus\{j\}}, \mathbf{B})\varphi(\mathbf{X}(j), \mathbf{Y}, \mathbf{B}) + \sum_{j \in [k]} (-1)^{r-k+1+j} 0_H \end{aligned}$$

$$= \sum_{j\in[r-k+1]} (-1)^j \varphi^{\mathbf{B}}(\mathbf{X}|_{[r-k+1]\setminus\{j\}})\varphi^{\mathbf{B}}(\mathbf{X}(j),\mathbf{Y})$$

This shows that Axiom (SG) holds for $\varphi^{\mathbf{B}}$ and hence $\varphi^{\mathbf{B}}$ is a strong-type Grassmann-Plücker function over H.

Finally, we show $\varphi^{\mathbf{B}'}$ determines the same set of circuits as $\varphi^{\mathbf{B}}$ for all ordered bases $\mathbf{B}$ and $\mathbf{B}'$ of M_φ/S. Indeed, we show the circuits determined by $\varphi^{\mathbf{B}}$ are precisely $\mathcal{C}\,|S$. Fix an ordered base $\mathbf{A}$ of $M_\varphi|S$. Now, $\mathbf{y} \coloneqq (\mathbf{A},\mathbf{B})$ is an ordered base of M_φ. Moreover, for all $e \in S \setminus A$, the fundamental H-circuit $X = X_{A\cup B,e}$ satisfies

$$\begin{aligned} X|_S(\mathbf{A}(i))X|_S(e)^{-1} &= X(\mathbf{y}(i))X(e)^{-1} \\ &= (-1)^i \varphi(e,\mathbf{A}|_{[r-k]\setminus\{i\}},\mathbf{B})\varphi(\mathbf{A},\mathbf{B})^{-1} \\ &= (-1)^i \varphi^{\mathbf{B}}(e,\mathbf{A}|_{[r]\setminus\{i\}})\varphi^{\mathbf{B}}(\mathbf{A})^{-1}, \end{aligned}$$

for all $i \in [r-k]$ by the cryptomorphism relating $\mathcal{C}$ and φ. On the other hand, $X|_S = X_{A\cup B,e}|_S$ is the fundamental H-circuit for e by the basis A in $\mathcal{C}\,|S$. Hence $\mathcal{C}\,|S$ is the set of H-circuits determined by the Grassmann-Plücker function $\varphi^{\mathbf{B}}$ for all ordered bases $\mathbf{B}$ of M_φ/S. $\square$

We summarize our results from this section as follows:

Proposition 3.4. *The restriction of an H-matroid to a subset is well-defined, and admits cryptomorphic description in terms of Grassmann-Plücker functions over H and H-circuits. Furthermore, this correspondence preserves types and all such restrictions have underlying matroid the ordinary restriction. Finally, we have the following:*

(1) *The restriction $M|S$ is given by H-circuits*

$$\mathcal{C}\,|S = \{X|_S \mid X \in \mathcal{C} \text{ and } \operatorname{supp}(X) \subseteq S\}.$$

(2) *The restriction $M|S$ is obtained by fixing any base $\mathbf{B} = (b_1, b_2, \ldots, b_k)$ of M_φ/S and defining*

$$\varphi^{\mathbf{B}} \colon S^{r-k} \longrightarrow H, \quad \mathbf{x} \mapsto \varphi(\mathbf{x},\mathbf{B}).$$

In particular, the H-matroid $M|S$ is determined by the $H^\times$-class $[\varphi^{\mathbf{B}}]$ of any such $\mathbf{B}$.

3.3. Deletion and contraction

As noted previously, deletion and contraction for H-matroids were defined by Baker and Bowler in [4] by using Grassmann-Plücker functions. In this section, we also provide a cryptomorphic definition for deletion and contraction via H-circuits by appealing to the definitions of dual H-matroids and restrictions. Throughout let H be a hyperfield, E a finite set, r a positive integer, and M be a matroid over H on ground set E of rank r with circuits $\mathcal{C}$ and a Grassmann-Plücker function φ.

Definition 3.5. Let S be a subset of E.

(1) The *deletion* $M \setminus S$ of S from M is the H-matroid $M|(E \setminus S)$.
(2) The *contraction* M/S of S from M is the H-matroid $(M^* \setminus S)^*$.

Remark 3.6. It follows from Definition 3.5 that if M is weak type (resp. strong type), then the deletion $M \setminus S$ and the contraction M/S are also weak type (resp. strong type).

Proposition 3.7. *Let S be a subset of E.*

(1) *The deletion $M \setminus S$ is given by H-circuits*

$$\mathcal{C}\,|(E \setminus S) = \left\{ X|_{E\setminus S} \;\middle|\; X \in \mathcal{C} \ \text{ and } S \cap \operatorname{supp}(X) = \emptyset \right\}. \tag{8}$$

(2) *The deletion $M \setminus S$ is obtained by fixing base $\mathbf{B} = (b_1, b_2, \ldots, b_{r-k})$ of $M_\varphi/(E \setminus S)$ and letting $\varphi^{\mathbf{B}} \colon (E \setminus S)^k \to H, \quad \mathbf{x} \mapsto \varphi(\mathbf{x}, \mathbf{B})$. The H-matroid $M \setminus S$ is determined by the $H^\times$-class $[\varphi^{\mathbf{B}}]$ for any such $\mathbf{B}$.*

Proof. The first statement is immediate from the definition of the deletion and the second statement directly follows from Proposition 3.4. $\square$

A description of contractions is slightly more complicated.

Proposition 3.8. *Let S be a subset of E.*

(1) *The contraction M/S is given by H-circuits $\mathcal{C}\,/S = (\mathcal{C}^*\,|(E \setminus S))^*$. More explicitly*

$$\mathcal{C}\,/S = \min \left\{ Z \in H^{E\setminus S} \setminus \{\mathbf{0}\} \;\middle|\; \begin{array}{c} Z \perp X|_{E\setminus S} \text{ for all } X \in H^E \setminus \{\mathbf{0}\} \\ \text{with } \operatorname{supp}(X) \cap S = \emptyset \text{ and } X \perp Y \text{ for all } Y \in \mathcal{C} \end{array} \right\}. \tag{9}$$

(2) *The contraction M/S is given by the class of Grassmann-Plücker functions $((\varphi^*)|_{E\setminus S})^*$. More explicitly, let $\mathbf{B} = (b_1, \ldots, b_k)$ be an ordered basis of $M_\varphi|S$. A representative of the $H^\times$-orbit of Grassmann-Plücker functions determining M/S is given by*

$$\varphi_{\mathbf{B}} \colon (E \setminus S)^{r-k} \longrightarrow H, \quad \mathbf{x} \mapsto \varphi(\mathbf{B}, \mathbf{x}). \tag{10}$$

The formula for $\varphi_{\mathbf{B}}$ in (10) is given in [4], with proof deferred to [2]; we give a full proof.

Proof. *Proof of (1)*: Since $M/S := (M^* \setminus S)^*$, the formula $\mathcal{C}/S = (\mathcal{C}^* |(E \setminus S))^*$ follows from the duality cryptomorphism and the restriction constructions of Propositions 3.4 and 3.7. Recall

$$\mathcal{C}^* = \min \left\{ X \in H^E \setminus \{\mathbf{0}\} \mid X \perp Y \text{ for all } Y \in \mathcal{C} \right\},$$

where "min" takes elements of minimal support. The following shows (9).

$$\begin{aligned}
&(\mathcal{C}^* |(E \setminus S))^* \\
&\quad = (\min \left\{ X \in H^E \setminus \{\mathbf{0}\} \mid X \perp Y \text{ for all } Y \in \mathcal{C}^* \right\} |(E \setminus S))^{\perp} \\
&\quad = \min \left\{ X|_{E\setminus S} \in H^{E\setminus S} \;\middle|\; \begin{array}{c} X \in H^E \setminus \{\mathbf{0}\} \text{ and } \operatorname{supp}(X) \cap S = \emptyset \\ \text{and } X \perp Y \text{ for all } Y \in \mathcal{C}^* \end{array} \right\}^{\perp} \\
&\quad = \min \left\{ Z \in H^{E\setminus S} \setminus \{\mathbf{0}\} \;\middle|\; \begin{array}{c} Z \perp X|_{E\setminus S} \text{ for all } X \in H^E \setminus \{\mathbf{0}\} \\ \text{with } \operatorname{supp}(X) \cap S = \emptyset \text{ and } X \perp Y \text{ for all } Y \in \mathcal{C} \end{array} \right\}
\end{aligned}$$

Proof of (2): We prove that $\varphi_{\mathbf{B}}$ is the Grassmann-Plücker function determined by $\mathcal{C}/S$. Let B be a base of the ordinary matroid $M_\varphi|S$. Then, for any base A of the ordinary matroid M_φ/S, $A \cup B$ is a base of M_φ. Let $e \in (E \setminus S) \setminus A$ be given, and let $\widetilde{C}_{A\cup B,e}$ be the fundamental circuit of e with respect to $A \cup B$ in M (see (2) and the paragraph before it). Now suppose $X \in H^E \setminus \{\mathbf{0}\}$ satisfies $\operatorname{supp}(X) \cap S = \emptyset$ and $X \perp Y$ for all $Y \in \mathcal{C}$. In particular, $X \perp \widetilde{C}_{A\cup B,e}$ since $\widetilde{C}_{A\cup B,e} \in \mathcal{C}$. On the other hand, as $X(s) = \{0\}$ for all $s \in S$, we have

$$X|_{E\setminus S} \perp \widetilde{C}_{A\cup B,e}|_{E\setminus S}.$$

Hence $\widetilde{C}_{A\cup B,e}|_{E\setminus S} = C_{A,e}$ is the fundamental H-circuit of $e \in E \setminus S$ with respect to A in M/S by incomparability of supports of elements in $\mathcal{C}/S$ and the fact that $\operatorname{supp}(\widetilde{C}_{A\cup B,e}|_{E\setminus S})$ is precisely the fundamental circuit of e by A in the underlying matroid of the contraction. The following computation completes the proof.

$$\begin{aligned}
(-1)^i \varphi_{\mathbf{B}}(e, \mathbf{A}|_{[r-k]\setminus\{i\}})\varphi_{\mathbf{B}}(\mathbf{A})^{-1} &= (-1)^i \varphi(\mathbf{B}, e, \mathbf{A}|_{[r-k]\setminus\{i\}})\varphi(\mathbf{B}, \mathbf{A})^{-1} \\
&= \widetilde{C}_{A\cup B,e}(\mathbf{A}(i))\widetilde{C}_{A\cup B,e}(e)^{-1} \\
&= C_{A,e}(\mathbf{A}(i))C_{A,e}(e)^{-1} \quad \square
\end{aligned}$$

3.4. Elementary properties of minors

We summarize the constructions of the preceding sections below for easy reference.

Proposition 3.9. *Let H be a hyperfield, E a finite set, r a positive integer, and M be a matroid over H of rank r with circuits $\mathcal{C}$ and a Grassmann-Plücker function φ. Let S be a subset of E.*

(1) *The restriction $M|S$ is given by H-circuits:*

$$\mathcal{C}_S = \{X|_S \mid X \in \mathcal{C} \ \textit{and} \ \operatorname{supp}(X) \subseteq S\}.$$

(2) *The restriction $M|S$ is obtained by fixing an ordered base $\mathbf{B} = (b_1, b_2, \ldots, b_k)$ of the underlying matroid $M_\varphi/(E \setminus S)$ and defining:*

$$\varphi^{\mathbf{B}} \colon S^{r-k} \longrightarrow H, \quad \mathbf{x} \mapsto \varphi(\mathbf{x}, \mathbf{B}).$$

In particular, the H-matroid $M|S$ is determined by the $H^\times$-class $[\varphi^{\mathbf{B}}]$ of any such $\mathbf{B}$.

(3) *The deletion $M \setminus S$ is given by H-circuits:*

$$\mathcal{C}\,|(E \setminus S) = \left\{X|_{E\setminus S} \;\middle|\; X \in \mathcal{C} \ \textit{and} \ S \cap \operatorname{supp}(X) = \emptyset\right\}.$$

(4) *The deletion $M \setminus S$ is obtained by fixing an ordered base $\mathbf{B} = (b_1, b_2, \ldots, b_{r-k})$ of M_φ/S and defining*

$$\varphi^{\mathbf{B}} \colon (E \setminus S)^k \longrightarrow H, \quad \mathbf{x} \mapsto \varphi(\mathbf{x}, \mathbf{B}).$$

In particular, the H-matroid $M \setminus S$ is determined by the $H^\times$-class $[\varphi^{\mathbf{B}}]$ of any $\mathbf{B}$.

(5) *The contraction M/S is given by H-circuits $\mathcal{C} = (\mathcal{C}^*\,|(E \setminus S))^*$. More explicitly,*

$$\mathcal{C}\,/S = \left\{Z \in H^{E\setminus S} \setminus \{\mathbf{0}\} \;\middle|\; \begin{array}{c} Z \perp X|_{E\setminus S} \textit{ for all } X \in H^E \setminus \{\mathbf{0}\} \\ \textit{with } \operatorname{supp}(X) \cap S = \emptyset \textit{ and } X \perp Y \textit{ for all } Y \in \mathcal{C} \end{array}\right\}$$

(6) *The contraction M/S is given by the class of Grassmann-Plücker functions $((\varphi^*)|_{E\setminus S})^*$. More explicitly, let $\mathbf{B} = (b_1, \ldots, b_k)$ be an ordered basis of $M_\varphi|S$. A representative of the $H^\times$-orbit of Grassmann-Plücker functions determining M/S is given by*

$$\varphi_{\mathbf{B}} \colon (E \setminus S)^{r-k} \longrightarrow H, \quad \mathbf{x} \mapsto \varphi(\mathbf{B}, \mathbf{x}).$$

Via the above constructions we obtain simple proofs of the following properties of minors, extending those of ordinary matroids.

Corollary 3.10. *Let M be an H-matroid on E with $S, T \subseteq E$ disjoint. We have*

(1) $M/\emptyset = M = M \setminus \emptyset$.
(2) $(M \setminus S) \setminus T = M \setminus (S \cup T)$.
(3) $(M/S)/T = M/(S \cup T)$.
(4) $(M \setminus S)/T = (M/T) \setminus S$.

Proof. Let $\mathcal{C}$ and φ denote the circuits and a Grassmann-Plücker function of M respectively. Preserve the notation from Proposition 3.9, with some self-explanatory adjustments.

Proof of Part (1): This follows immediately from the corresponding fact on M_φ.

Proof of Part (2): Let $\mathbf{B}_S = (a_1, a_2, \dots, a_k)$ and $\mathbf{B}_T = (b_1, b_2, \dots, b_l)$ be ordered bases of M_φ/S and M_φ/T respectively, and let $m = r - k - l$. Note that the concatenation $\mathbf{B}_T\mathbf{B}_S$ is an ordered basis of $M_\varphi/(S \cup T)$. We complete the proof via the following computation for $\mathbf{x} \in ((E \setminus S) \setminus T)^m = (E \setminus (S \cup T))^m$.

$$(\varphi^S)^T(\mathbf{x}) = (\varphi^S)(\mathbf{x}, \mathbf{B}_T) = \varphi(\mathbf{x}, \mathbf{B}_T, \mathbf{B}_S) = \varphi(\mathbf{x}, \mathbf{B}_T\mathbf{B}_S) = \varphi^{TS}(\mathbf{x})$$

Proof of Part (3): Let $\mathbf{B}_S = (a_1, a_2, \dots, a_k)$ be an ordered basis of $M_\varphi|S$, let $\mathbf{B}_T = (b_1, b_2, \dots, b_l)$ be an ordered basis of $M_\varphi|T$ such that the concatenation $\mathbf{B}_S\mathbf{B}_T$ is an ordered basis of $M_\varphi|(S \cup T)$, and let $m = r - k - l$. We complete the proof via the following computation for $\mathbf{x} \in ((E \setminus S) \setminus T)^m = (E \setminus (S \cup T))^m$.

$$(\varphi_S)_T(\mathbf{x}) = (\varphi_S)(\mathbf{B}_T, \mathbf{x}) = \varphi(\mathbf{B}_S, \mathbf{B}_T, \mathbf{x}) = \varphi(\mathbf{B}_S\mathbf{B}_T, \mathbf{x}) = \varphi_{ST}(\mathbf{x})$$

Proof of Part (4): Let $\mathbf{B}_S = (a_1, a_2, \dots, a_k)$ and $\mathbf{B}_T = (b_1, b_2, \dots, b_l)$ be ordered bases of M_φ/S and $M_\varphi|T$ respectively such that the concatenation $\mathbf{B}_S\mathbf{B}_T$ is an independent $(k+l)$-tuple of M_φ, and let $m = r - k - l$. We complete the proof via the following computation for $\mathbf{x} \in (E \setminus (S \cup T))^m$.

$$(\varphi^S)_T(\mathbf{x}) = \varphi^S(\mathbf{B}_T, \mathbf{x}) = \varphi(\mathbf{B}_T, \mathbf{x}, \mathbf{B}_S) = \varphi_T(\mathbf{x}, \mathbf{B}_S) = (\varphi_T)^S(\mathbf{x}) \quad \square$$

Remark 3.11. The proofs of parts (2) and (3) above are very similar; having proved either, one may apply the duality cryptomorphism to the dual H-matroid M^* to obtain a proof of the other.

Finally, we obtain that restriction, deletion, and contraction commute with pushforwards.

Corollary 3.12. *The following all hold.*

(1) *The pushforward of a restriction is the restriction of the pushforward.*
(2) *The pushforward of a deletion is the deletion of the pushforward.*
(3) *The pushforward of a contraction is the contraction of the pushforward.*

Proof. Let $f\colon H_1 \to H_2$ be a hyperfield morphism, let φ be a Grassmann-Plücker function for H_1-matroid M, and let $S \subseteq E$.

Proof of Part (1): Let $\mathbf{B}$ be an ordered basis of $M_\varphi/(E \setminus S)$. We compute

$$(f\varphi)^{\mathbf{B}}(\mathbf{x}) = (f\varphi)(\mathbf{x}, \mathbf{B}) = f(\varphi(\mathbf{x}, \mathbf{B})) = f(\varphi^{\mathbf{B}}(\mathbf{x})) = (f\varphi^{\mathbf{B}})(\mathbf{x}).$$

Proof of Part (2): This follows from part (1) as deletion is restriction to a complement.

Proof of Part (3): Let $\mathbf{B}$ be an ordered basis of $M_\varphi|S$. We compute

$$(f\varphi)_{\mathbf{B}}(\mathbf{x}) = (f\varphi)(\mathbf{B}, \mathbf{x}) = f(\varphi(\mathbf{B}, \mathbf{x})) = f(\varphi_{\mathbf{B}}(\mathbf{x})) = (f\varphi_{\mathbf{B}})(\mathbf{x}). \quad \square$$

Example 3.13. Let H be a hyperfield and $\kappa\colon H \to \mathbb{K}$ be the canonical map. For any H-matroid M, the pushforward $\kappa_* M$ is the underlying matroid of M. In this case, Corollary 3.12 states that each of these operations corresponds with the operation of the same name on the underlying matroid.

3.5. Direct sums of matroids over hyperfields

For matroids over hyperfields, we provide two cryptomorphic definitions (Grassmann-Plücker functions and circuits) for direct sum. Let M and N be H-matroids. To state a precise formula for a sum of Grassmann-Plücker functions, we will need some additional notation. Fix a total order $\leq$ on $E_M \sqcup E_N$ such that $x < y$ whenever $x \in E_M$ and $y \in E_N$. Now for every $\mathbf{x} \in (E_M \sqcup E_N)^{r_M + r_N}$ either $\mathbf{x}$ has exactly r_M components in E_M or not. If so, we let $\sigma_{\mathbf{x}}$ denote the unique permutation of $[r_M + r_N]$ such that $\sigma_{\mathbf{x}} \cdot \mathbf{x}$ is monotone increasing with respect to $\leq$.

Proposition 3.14. *Let H be a hyperfield and M (resp. N) be an H-matroid of rank r_M (resp. r_N) given by a Grassmann-Plücker function φ_M (resp. φ_N). The function $\varphi_M \oplus \varphi_N$, defined by formula (a) is a weak-type Grassmann-Plücker function.*

$$\begin{aligned} &\varphi_M \oplus \varphi_N \colon (E_M \sqcup E_N)^{r_M + r_N} \to H, \\ &\quad \mathbf{x} \mapsto \begin{cases} 0, & \textit{if } \mathbf{x} \textit{ does not have precisely } r_M \textit{ components in } E_M \\ \operatorname{sgn}(\sigma_{\mathbf{x}})\varphi_M((\sigma_{\mathbf{x}} \cdot \mathbf{x})|_{[r_M]})\varphi_N((\sigma_{\mathbf{x}} \cdot \mathbf{x})|_{[r_M + r_N] \setminus [r_M]}), & \textit{otherwise.} \end{cases} \end{aligned} \tag{a}$$

Moreover, $\varphi_M \oplus \varphi_N$ is strong type precisely when both φ_M and φ_N are strong type. Furthermore, the $H^\times$-class of $\varphi_M \oplus \varphi_N$ depends only on M and N.

We verify this result via several lemmas. Let $\varphi := \varphi_M \oplus \varphi_N$ in what follows.

Lemma 3.15. *The function defined in (a) is nondegenerate and H-alternating.*

Proof of Lemma 3.15. Note φ is clearly nontrivial. To see φ is H-alternating, we let τ be an arbitrary permutation of $[r_M + r_N]$; note that $\mathbf{x}$ has precisely r_M components in E_M if and only if $\tau \cdot \mathbf{x}$ has precisely r_M components in E_M. If $\mathbf{x}$ has precisely r_M components in E_M, then $\sigma_{\tau \cdot \mathbf{x}} = \sigma_{\mathbf{x}} \tau^{-1}$ and the following computation completes the proof of our claim.

$$\begin{aligned}\varphi(\tau\cdot\mathbf{x}) &= \operatorname{sgn}(\sigma_{\tau\cdot\mathbf{x}})\varphi_M((\sigma_{\tau\cdot\mathbf{x}}\cdot\tau\cdot\mathbf{x})|_{[r_M]})\varphi_N((\sigma_{\tau\cdot\mathbf{x}}\cdot\tau\cdot\mathbf{x})|_{[r_M+r_N]\setminus[r_M]})\\ &= \operatorname{sgn}(\sigma_{\mathbf{x}}\cdot\tau^{-1})\varphi_M((\sigma_{\mathbf{x}}\cdot\tau^{-1}\cdot\tau\cdot\mathbf{x})|_{[r_M]})\varphi_N((\sigma_{\mathbf{x}}\cdot\tau^{-1}\cdot\tau\cdot\mathbf{x})|_{[r_M+r_N]\setminus[r_M]})\\ &= \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma_{\mathbf{x}})\varphi_M((\sigma_{\mathbf{x}}\cdot\mathbf{x})|_{[r_M]})\varphi_N((\sigma_{\mathbf{x}}\cdot\mathbf{x})|_{[r_M+r_N]\setminus[r_M]})\\ &= \operatorname{sgn}(\tau)(\varphi_M\oplus\varphi_N)(\mathbf{x}) \quad \square\end{aligned}$$

To prove φ satisfies Grassmann-Plücker relations, we need a technical lemma.

Lemma 3.16. *Suppose $\mathbf{x}\in E^{r+1}$ and $\mathbf{y}\in E^{r-1}$ are nondecreasing where $r = r_M + r_N$ and $E = E_M \sqcup E_N$. The Grassmann-Plücker sum corresponding to the pair $(\mathbf{x},\mathbf{y})$ can be expressed as a constant times a Grassmann-Plücker sum in either M or N.*

This lemma takes care of all Grassmann-Plücker relations for $M\oplus N$ (weak and strong where applicable) up to permuting coordinates.

Proof. Let $\varphi_P(\mathbf{z}) = 0$ if $\mathbf{z}$ does not have all components in E_P. Fix the following notations.

$$\begin{aligned}\mathbf{x}_k &:= \mathbf{x}|_{[r+1]\setminus\{k\}}, & \mathbf{y}_k &:= (\mathbf{x}(k),\mathbf{y}),\\ \mathbf{z}^M &:= \mathbf{z}|_{[r_M]}, & \mathbf{z}^N &:= \mathbf{z}|_{[r_M+r_N]\setminus[r_M]}.\end{aligned}$$

We may assume $\mathbf{x}$ has either exactly r_M+1 components in E_M or exactly r_N+1 components in E_N; otherwise $\varphi(\mathbf{x}_k) = 0$ for all k. Without loss (i.e. up to switching the roles of M and N below), we assume $\mathbf{x}$ has exactly r_M+1 components in E_M. Similarly, we assume $\mathbf{y}$ has exactly r_M-1 components in E_M lest the corresponding Grassmann-Plücker sum is uniformly 0.

Next make several observations. Note $\sigma_{\mathbf{x}_k}$ is the identity for all k, and $(\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^N = y_k^N$ for all $k\le r_M+1$; thus $(\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^M = \sigma_{\mathbf{y}_k^M}\cdot\mathbf{y}_k^M$. Moreover, $\mathbf{x}_k^N$ and $\mathbf{y}_k^N$ are constant for $k\le r_M+1$, and $\mathbf{x}_k$ has r_M+1 components in E_M for $k > r_M+1$. We now compute as follows using formula (a).

$$\begin{aligned}&\sum_{k=1}^{r+1}(-1)^k\varphi(\mathbf{x}_{\hat{k}})\varphi(\mathbf{y}_k)\\ &\quad=\sum_{k=1}^{r+1}(-1)^k\varphi_M(\mathbf{x}_k^M)\varphi_N(\mathbf{x}_k^N)\operatorname{sgn}(\sigma_{\mathbf{y}_k})\varphi_M((\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^M)\varphi_N((\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^N)\\ &\quad=\sum_{k=1}^{r_M+1}(-1)^k\operatorname{sgn}(\sigma_{\mathbf{y}_k})\varphi_M(\mathbf{x}_k^M)\varphi_M((\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^M)\varphi_N(\mathbf{x}_k^N)\varphi_N(\mathbf{y}_k^N)\end{aligned}$$

$$+\sum_{k=r_M+2}^{r_M+r_N+1}(-1)^k\varphi_M(\mathbf{x}_k^M)\operatorname{sgn}(\sigma_{\mathbf{y}_k})\varphi_M((\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^M)\varphi_N(\mathbf{x}_k^N)\varphi_N((\sigma_{\mathbf{y}_k}\cdot\mathbf{y}_k)^N)$$

$$=\varphi_N(\mathbf{x}_k^N)\varphi_N(\mathbf{y}_k^N)\sum_{k=1}^{r_M+1}(-1)^k\varphi_M(\mathbf{x}_k^M)\operatorname{sgn}(\sigma_{\mathbf{y}_k^M})\varphi_M(\sigma_{\mathbf{y}_k^M}\cdot\mathbf{y}_k^M)$$

$$=\varphi_N(\mathbf{x}_k^N)\varphi_N(\mathbf{y}_k^N)\sum_{k=1}^{r_M+1}(-1)^k\varphi_M(\mathbf{x}_k^M)\varphi_M(\mathbf{y}_k^M)$$

The last equality above yields a scalar times a Grassmann-Plücker sum in M as desired. $\square$

Proof of Proposition 3.14. By Lemma 3.15 we have that the φ of formula (a) is nontrivial and H-alternating. By Lemma 3.16 we have φ satisfies the same type of Grassmann-Plücker relations satisfied by both M and N. Finally, $\alpha\varphi_M \oplus \beta\varphi_N = \alpha\beta(\varphi_M \oplus \varphi_N)$ for all $\alpha, \beta \in H^\times$ yields $H^\times$-class invariance. $\square$

Proposition 3.17. *Let M and N be H-matroids of rank r_M and r_N on disjoint ground sets E_M and E_N given by H-circuits $\mathcal{C}_M$ and $\mathcal{C}_N$ respectively. Define*

$$\mathcal{C}_M \oplus \mathcal{C}_N = \left\{ X\colon E_M \sqcup E_N \to H \;\middle|\; \begin{array}{r} \text{either both } X|_{E_M} \in \mathcal{C}_M \text{ and } X|_{E_N} = \mathbf{0} \\ \text{or both } X|_{E_M} = \mathbf{0} \text{ and } X|_{E_N} \in \mathcal{C}_N \end{array} \right\}$$

Then, $\mathcal{C}_M \oplus \mathcal{C}_N$ is a set of H-circuits. Furthermore, $\mathcal{C}_M \oplus \mathcal{C}_N$ is strong type exactly when both $\mathcal{C}_M$ and $\mathcal{C}_N$ are strong type.

Proof. That $\mathcal{C}_M \oplus \mathcal{C}_N$ is a set of pre-circuits over H follows trivially from its definition. Moreover, we observe

$$\operatorname{supp}(\mathcal{C}_M \oplus \mathcal{C}_N) = \operatorname{supp}(\mathcal{C}_M) \sqcup \operatorname{supp}(\mathcal{C}_N) \tag{11}$$

and hence the underlying matroid of the H-matroid determined thereby is the direct sum of the underlying matroids of the summands. It follows that every modular pair in $\mathcal{C}_M \oplus \mathcal{C}_N$ reduces to modular pair in either $\mathcal{C}_M$ or $\mathcal{C}_N$. Thus (WC) holds by noting that any modular pair with nontrivial intersection is either a modular pair in $\mathcal{C}_M$ or a modular pair in $\mathcal{C}_N$. If $\mathcal{C}_M$ and $\mathcal{C}_N$ are both strong, then by (11), (SC1) holds. Moreover (SC2) holds by noting that the computation reduces to a computation in precisely one of $\mathcal{C}_M$ or $\mathcal{C}_N$. $\square$

The next result shows that the direct sum of H-matroids admits the cryptomorphic descriptions given in this section.

Proposition 3.18. *Let M be an H-matroid given by H-circuits $\mathcal{C}_M$ and Grassmann-Plücker function φ_M on E_M and N be an H-matroid given by H-circuits $\mathcal{C}_N$ and*

Grassmann-Plücker function φ_N on E_N such that $E_M \cap E_N = \emptyset$. Then, $\varphi_M \oplus \varphi_N$ and $\mathcal{C}_M \oplus \mathcal{C}_N$ both determine the same H-matroid under cryptomorphism. Furthermore, this H-matroid has underlying matroid the direct sum of the underlying matroids of M and N.

Proof. We must verify that $\mathcal{C}_M \oplus \mathcal{C}_N$ is cryptomorphically determined by $\varphi_M \oplus \varphi_N$. Let B_M and B_N be any bases of the underlying matroids of M and N, respectively. Notice that for all $e \in (E_M \sqcup E_N) \setminus (B_M \sqcup B_N)$, the fundamental circuit $X_{B_M \cup B_N, e}$ has support contained in E_M or in E_N. Thus, the cryptomorphism relation required reduces to the relation on the fundamental circuit of the part containing $e \in E_M \sqcup E_N$. Hence $\varphi_M \oplus \varphi_N$ and $\mathcal{C}_M \oplus \mathcal{C}_N$ determine the same H-matroid as desired. $\square$

Corollary 3.19. *The pushforward of a direct sum of H-matroids is the direct sum of the pushforwards. In other words, direct sum commutes with pushforward.*

Proof. Let M_1 and M_2 be H_1-matroids with H_1-circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, and let $f\colon H_1 \to H_2$ be a hyperfield morphism. The following computation completes the proof.

$$
\begin{aligned}
&f(\mathcal{C}_M \oplus \mathcal{C}_N)\\
&\quad = H_2^\times \left\{ fX\colon E_M \sqcup E_N \to H \;\middle|\; \begin{array}{c} \text{either both } X|_{E_M} \in \mathcal{C}_M \text{ and } X|_{E_N} = \mathbf{0} \\ \text{or both } X|_{E_M} = \mathbf{0} \text{ and } X|_{E_N} \in \mathcal{C}_N \end{array} \right\}\\
&\quad = H_2^\times \left\{ fX\colon E_M \sqcup E_N \to H \;\middle|\; \begin{array}{c} \text{either both } (fX)|_{E_M} \in f\,\mathcal{C}_M \text{ and } (fX)|_{E_N} = \mathbf{0} \\ \text{or both } (fX)|_{E_M} = \mathbf{0} \text{ and } (fX)|_{E_N} \in f\,\mathcal{C}_N \end{array} \right\}\\
&\quad = (f\,\mathcal{C}_1) \oplus (f\,\mathcal{C}_2) \quad \square
\end{aligned}
$$

Remark 3.20. Part of the proof of Proposition 3.18 was to notice this property holds for the pushforward to the Krasner hyperfield $\mathbb{K}$, i.e., taking underlying matroids.

4. Isomorphisms of matroids over hyperfields

In this section, we introduce a notion of isomorphisms of matroids over hyperfields which generalizes the definition of isomorphisms of ordinary matroids. We will subsequently use this definition to construct matroid-minor Hopf algebras for matroids over hyperfields in §5.

Definition 4.1 *(Isomorphism via Grassmann-Plücker function).* Let E_1 and E_2 be finite sets, r be a positive integer, and H be a hyperfield. Let M_1 (resp. M_2) be a matroid on E_1 (resp. E_2) of rank r over H which is represented by a Grassmann-Plücker function φ_1 (resp. φ_2). We say that M_1 and M_2 are *isomorphic* if there is a bijection $f\colon E_1 \to E_2$ and an element $\alpha \in H^\times$ such that the following diagram commutes.

$$\begin{tikzcd} E_1^r \arrow[r, "\varphi_1"] \arrow[d, "f^r"'] & H \arrow[d, "\odot\alpha"] \\ E_2^r \arrow[r, "\varphi_2"] & H \end{tikzcd} \tag{12}$$

Proposition 4.2. *Definition 4.1 is well-defined.*

Proof. Let φ_1' and φ_2' be different representatives of M_1 and M_2. In other words, there exist $\beta, \gamma \in H^\times$ such that $\varphi_1' = \beta \odot \varphi_1$ and $\varphi_2' = \gamma \odot \varphi_2$. In this case we have

$$\gamma^{-1} \odot \varphi_2' \circ f^r = \varphi_2 \circ f^r = \alpha \odot \varphi_1 = (\alpha \odot \beta^{-1}) \odot \varphi_1'.$$

It follows that $\varphi_2' \circ f^r = (\gamma \odot \alpha \odot \beta^{-1}) \odot \varphi_1'$ and hence Definition 4.1 is well-defined. $\square$

Proposition 4.3. *Let H and K be hyperfields and $g\colon H \to K$ be a morphism of hyperfields. If M_1 and M_2 are matroids over H which are isomorphic, then the pushforwards g_*M_1 and g_*M_2 are isomorphic as well.*

Proof. Let M_1 (resp. M_2) be represented by a Grassmann-Plücker function φ_1 (resp. φ_2). Since M_1 and M_2 are isomorphic, there exist $a \in H^\times$ and a bijection $f\colon E_1 \to E_2$ such that $\varphi_2 \circ f^r = a \odot \varphi_1$. Notice that the pushforward g_*M_1 (resp. g_*M_2) is represented by the Grassmann-Plücker function $g \circ \varphi_1$ (resp. $g \circ \varphi_2$), we obtain

$$(g \circ \varphi_2) \circ f^r = g \circ (\varphi_2 \circ f^r) = g \circ (a \odot \varphi_1) = g(a) \odot (g \circ \varphi_1). \quad \square$$

Note that in the special case $K = \mathbb{K}$, the underlying matroids of two isomorphic matroids are isomorphic in the classical sense. Therefore, our definition of isomorphisms generalizes the definition of isomorphisms of ordinary matroids.

Proposition 4.4. *If M and M' (resp. N and N') are isomorphic H-matroids, then $M \oplus N$ and $M' \oplus N'$ are isomorphic H-matroids.*

Proof. Consider Grassmann-Plücker functions φ_M, $\varphi_{M'}$, φ_N, and $\varphi_{N'}$. By assumption there are bijections $f_M\colon E_M \to E_{M'}$ and $f_N\colon E_N \to E_N'$ and constants $\alpha_M, \alpha_N \in H^\times$ such that $\alpha_M \odot \varphi_M = \varphi_{M'} \circ f_M^{r_M}$ and $\alpha_N \odot \varphi_N = \varphi_{N'} \circ f_N^{r_N}$. Let $f_M \sqcup f_N\colon E_M \sqcup E_N \to E_{M'} \sqcup E_{N'}$ denote the obvious bijection. Then, we have

$$\begin{aligned} \alpha_M\alpha_N \odot (\varphi_M \oplus \varphi_N) &= (\alpha_M \odot \varphi_M) \oplus (\alpha_N \odot \varphi_N) \\ &= (\varphi_{M'} \circ f_M^{r_M}) \oplus (\varphi_{N'} \circ f_N^{r_N}) \\ &= (\varphi_{M'} \oplus \varphi_{N'}) \circ (f_M \sqcup f_N)^{r_M + r_N}. \quad \square \end{aligned}$$

5. Matroid-minor Hopf algebra associated to matroids over a hyperfield

In this section, by appealing to Definition 4.1, Propositions 3.14 and 3.17, we generalize the classical construction of matroid-minor Hopf algebras to the case of matroids over hyperfields. Let H be a hyperfield. Let $\mathcal{M}$ be a set of matroids over H which is closed under taking direct sums and minors. Let $\mathcal{M}_{\text{iso}}$ be the set of isomorphism classes of elements in $\mathcal{M}$, where the isomorphism class is defined by Definition 4.1. Then, $\mathcal{M}_{iso}$ has a canonical monoid structure as follows:

$$\cdot\colon \mathcal{M}_{\text{iso}} \times \mathcal{M}_{\text{iso}} \to \mathcal{M}_{\text{iso}}, \quad ([M_1],[M_2]) \mapsto [M_1 \oplus M_2]. \tag{13}$$

Note that (13) is well-defined by Proposition 4.4 and the isomorphism class of the *empty matroid* $[\emptyset]$ is the identity element. Let k be a field. Then we have the monoid algebra $k[\mathcal{M}_{\text{iso}}]$ over k with the unit map $\eta\colon k \to k[\mathcal{M}_{\text{iso}}]$ sending 1 to $[\emptyset]$ and the multiplication

$$\mu\colon k[\mathcal{M}_{\text{iso}}] \otimes_k k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}], \text{ generated by } [M_1] \otimes [M_2] \mapsto [M_1 \oplus M_2].$$

Proposition 5.1. *Let k be a field and H be a hyperfield. Let $(\mathcal{M}_{\text{iso}}, \cdot)$ be the monoid and $k[\mathcal{M}_{\text{iso}}]$ be the monoid algebra over k as above. Then $\mathscr{H} := k[\mathcal{M}_{\text{iso}}]$ is a bialgebra with the following maps.*

- *(Comultiplication)*

$$\Delta\colon \mathscr{H} \longrightarrow \mathscr{H} \otimes_k \mathscr{H}, \quad [M] \mapsto \sum_{A \subseteq E} [M\,|_A] \otimes_k [M/A]. \tag{14}$$

- *(Counit)*

$$\varepsilon\colon \mathscr{H} \longrightarrow k, \quad [M] \mapsto \begin{cases} 1 & \text{if } E_M = \emptyset \\ 0 & \text{if } E_M \neq \emptyset. \end{cases} \tag{15}$$

Furthermore, $\mathscr{H}$ is graded and connected and hence has a unique Hopf algebra structure.

Proof. There is a canonical grading on $\mathscr{H}$ via the cardinality of the underlying set of each element $[M]$ and this is clearly compatible with the bialgebra structure of $\mathscr{H}$. In this case, $[\emptyset]$ has degree 0 and hence $\mathscr{H}$ is connected. The last assertion follows from the result of [16]. □

Example 5.2. Let H be an arbitrary hyperfield and let k be a field. Consider the uniform rank-1 matroid on two elements, denoted U_2^1. Let $\mathcal{M}_{\text{iso}}$ be the smallest set of matroid isomorphism classes containing $[U_2^1]$ and closed under taking direct sums and minors; then $\mathcal{M}_{\text{iso}}$ is a monoid with multiplication given by direct sum. Observe that $\mathcal{M}_{\text{iso}}$ is the free monoid generated by two elements, namely $[U_1^1]$ and $[U_2^1]$.

Let $\mathcal{H}$ be the matroid-minor Hopf algebra associated to $\mathcal{M}_{\text{iso}}$. As an algebra, $\mathcal{H}$ is isomorphic to the polynomial algebra $k[X,Y]$, where $X = [U_1^1]$ and $Y = U_2^1$. Note that

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \Delta(Y) = 1 \otimes Y + 2(X \otimes 1) + Y \otimes 1.$$

Let $\mathcal{M}_{\text{iso}}^H$ be the set of isomorphism classes of those H-matroids with pushforward in $\mathcal{M}_{\text{iso}}$, and let M_1, M_2 be H-matroids. If M_1 and M_2 both pushforward to U_1^1, then $M_1 = M_2$ as U_1^1 has a unique basis. Suppose M_1 has pushforward U_2^1 and let $\varphi\colon \{a,b\} \to H$ be a Grassmann-Plücker function for M_1; as the pushforward of M_1 is U_2^1, we have $\varphi(a) \neq 0_H \neq \varphi(b)$. Thus φ is uniquely determined by the ratio $q \coloneqq \varphi(b)\varphi(a)^{-1} \in H^\times$. If M_1 and M_2 both pushforward to U_2^1, then M_1 and M_2 are isomorphic if and only if $q_2 = q_1$ as the only nontrivial automorphism of U_2^1 is a transposition, and this isomorphism lifts to an isomorphism of $M_1 \to M_2$ if and only if $\varphi_1 = \alpha\varphi_2$ for some $\alpha \in H^\times$.

The above argument yields every H-matroid M with pushforward U_2^1 is determined up to isomorphism by the $H^\times$-ratio q_M. Thus the monoid $\mathcal{M}_{\text{iso}}^H$ is the free monoid on generators $\{S\} \cup \{T_q \mid q \in H^\times\}$, where S is the isomorphism class of the H-matroid with pushforward U_1^1, and T_q is the isomorphism class of the H-matroid with pushforward U_2^1 and having ratio q. Hence the Hopf algebra $k[\mathcal{M}_{\text{iso}}^H]$ is $k[S, T_q]_{q \in H^\times}$ as a k-algebra. We have a map

$$\pi\colon k[\mathcal{M}_{\text{iso}}^H] = k[S, T_q]_{q \in H} \longrightarrow k[\mathcal{M}_{\text{iso}}] = k[X,Y], \quad S \mapsto X \text{ and } T_q \mapsto Y,$$

which is trivially a surjection. Moreover, $\ker(\pi)$ is generated by elements of the form $T_q - T_r$ for $q, r \in H^\times$.

6. Future directions

6.1. Relation to matroids over partial hyperfields

We review Baker and Bowler's more generalized framework, namely matroids over partial hyperfields [5] and briefly explain how our work generalizes to this setting.

Definition 6.1. [5, §1] A *tract* is an abelian group G together with a designated subset N_G of the group semiring $\mathbb{N}[G]$ such that

(1) $0_{\mathbb{N}[G]} \in N_G$ and $1_G \notin N_G$.
(2) $\exists\,!\ \varepsilon \in G$ such that $1 + \varepsilon \in N_G$.
(3) $G \cdot N_G = N_G$.

In this generalization, ε plays the role of -1 and N_G encodes "non-trivial dependence" relations; in the case of fuzzy rings, one has a designated subset K_0 of "zeros", however, by using ε one can always change K_0 to N_G as above.

For a hyperfield $(H, \boxplus, \odot)$, one can canonically associate a tract (G, N_G) by setting $G = H^\times$ and letting $f = \sum a_i g_i \in \mathbb{N}[G]$ be in N_G if and only if

$$0_H \in \boxplus(a_i \odot g_i) \quad \text{(as elements of } H). \tag{16}$$

Recall that partial fields are introduced by C. Semple and G. Whittle in [15] to study realizability of matroids. A partial field $(G \cup \{0_R\}, R)$ consists of a commutative ring R and a multiplicative subgroup G of $R^\times$ such that $-1 \in G$ and G generates R. Inspired by this definition (along with hyperfields), Baker and Bowler define the following.

Definition 6.2. [5, §1] A *partial hyperfield* is a hyperdomain R (a hyperring without zero divisors) together with a designated subgroup G of $R^\times$.

One can naturally associate a tract to a partial hyperfield (G, R) in a manner similar to the previous association of a tract to a hyperfield by stating that $\sum a_i g_i \in \mathbb{N}[G]$ if and only if (16) holds.

With tracts (or partial hyperfields), Baker and Bowler generalize their previous work on matroids over hyperfields. Their main idea is that in their proofs for matroids over hyperfields, one only needs the three conditions of tracts given in Definition 6.1. Although we focus on matroids over hyperfields, our results readily generalize to matroids over partial hyperfields.

6.2. Tutte polynomials of Hopf algebras and universal Tutte characters

The *Tutte polynomial* is one of the most interesting invariants of graphs and matroids. In [10], T. Krajewsky, I. Moffatt, and A. Tanasa introduced Tutte polynomials associated to Hopf algebras. More recently, C. Dupont, A. Fink, and L. Moci introduced *universal Tutte characters* generalizing [10]. In fact, both [7] and [10] consider the case when one has combinatorial objects which have notions of "deletion" and "contraction" (e.g. graphs and matroids). In the context of our work, the following is straightforward.

Proposition 6.3. *Let H be a hyperfield. A set $\mathcal{M}_{\text{iso}}$ of isomorphism classes of matroids over H, which is stable under taking direct sums and minors, satisfies the axioms of a* minor system *in [10].*

Proof. This directly follows from Corollary 3.10. □

The notion of *minors system* is used in [7] to define universal Tutte characters. The following is an easy consequence of §3.

Proposition 6.4. *Let H be a hyperfield and $\mathbf{Mat}_H$ be the set species such that $\mathbf{Mat}_H(E)$ is the set of matroids over H with an underlying set E. Then $\mathbf{Mat}_H$ is a connected multiplicative minors system as in [7].*

Proof. Let $\mathbf{S} \coloneqq \mathbf{Mat}_H$. As the empty matroid over H is the only object of $\mathbf{S}[\varnothing]$, $\mathbf{S}$ is connected. The multiplicative structure of $\mathbf{S}$ comes from direct sums, and Axioms (M1)–(M3), (M4')–(M8') are easy to check. $\square$

The construction in [10, §2] can be applied to define the Tutte polynomial for $k[\mathcal{M}_{\mathrm{iso}}]$ by our above observations. One can also associate universal Tutte characters in our setting.

Remark 6.5. Rank-type axioms are not currently known for matroids over hyperfields, and the traditional Tutte polynomials are defined in terms of this feature. This construction of Tutte polynomials circumvents the issue, replacing rank axioms by access to the Hopf algebra.

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