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Topological Jordan decompositions

Loren Spice [1]

The University of Michigan, Ann Arbor, MI 48109-1043, USA

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Abstract

The notion of a topological Jordan decomposition of a compact element of a reductive p-adic group has proven useful in many contexts. In this paper, we generalise it to groups defined over fairly general discretely valued fields and prove the usual existence and uniqueness properties, as well as an analogue of a fixed-point result of Prasad and Yu.

Keywords: Reductive group; Non-Archimedean field; Jordan decomposition

0. Introduction

In [15], Kazhdan defines the notions of $\mathfrak{f}$-semisimplicity and $\mathfrak{f}$-unipotence of an element of $\mathrm{GL}_n(F_0)$, where F is a discretely valued locally compact field with ring of integers F_0 and residue field $\mathfrak{f}$ (see the definition on p. 226 of [15]). An arbitrary element of $\mathrm{GL}_n(F_0)$ can be decomposed as a commuting product of an $\mathfrak{f}$-semisimple and an $\mathfrak{f}$-unipotent element (see Lemma 2 on p. 226 of [15]). Furthermore, stably conjugate $\mathfrak{f}$-semisimple elements are actually $\mathrm{GL}_n(F_0)$-conjugate (see Lemma 3 on p. 226 of [15], where the result is proven for rationally conjugate elements, and Lemma 13.1 of [13]).

Kazhdan uses this last result in his calculation of the ε-twisted orbital integral $I_\ell(f)$ (see Theorem 1 on p. 224 of [15], and the definition immediately preceding it). A detailed exposition appears in [14, Section 5]; see especially [14, Section 5.6]. An analogous result is used by Waldspurger in his computation of Shalika germs for $\mathrm{GL}(n)$ (see [26, Section 5]).

E-mail address: lspice@umich.edu.

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In [13], Hales defines absolute semisimplicity and topological unipotence (the analogues of $\mathfrak{f}$-semisimplicity and $\mathfrak{f}$-unipotence) for elements of unramified groups, and shows that every strongly compact element can be decomposed as a commuting product of an absolutely semisimple and a topologically unipotent element (the topological Jordan decomposition). He then defines transfer factors on unramified groups, and shows that the transfer factor at a strongly compact element may be expressed in terms of the transfer factor for the centraliser of its absolutely semisimple part, evaluated at its topologically unipotent part (see Theorem 10.18 and Lemma 13.2 of [13]). This suggests that the topological Jordan decomposition is important for the fundamental lemma. Indeed, in [12], Flicker uses a twisted analogue of the decomposition to prove a special case of the fundamental lemma (see the theorem on p. 509 of [12]).

The topological Jordan decomposition is also useful in character computations. Recall that the character of a Deligne–Lusztig representation of a finite group of Lie type is expressed by a reduction formula in terms of the (ordinary) Jordan decomposition (see Theorem 4.2 of [11]). The topological Jordan decomposition plays the same rôle for the characters of depth-zero supercuspidal representations of p-adic groups arising via compact induction from representations which are inflations of Deligne–Lusztig representations of reductive quotients (see Lemma 10.0.4 of [10]).

In [2, Section 5], as preparation for the (positive-depth) character computations of [3], Adler and the author define the notion of a *normal approximation* of an element of a reductive p-adic group, a refinement of the topological Jordan decomposition. However, for the results of that paper, one needs notions of absolute semisimplicity and topological unipotence that make sense over a discretely valued field F which is not necessarily locally compact.

In this paper, we offer two generalisations of these notions, the first an abstract one adapted to profinite groups, and the second adapted to the setting in which we are most interested, of reductive groups over discretely valued fields F as above. We prove the usual existence (Propositions 1.8 and 2.36) and uniqueness (Propositions 1.7 and 2.24) results for these decompositions. In the familiar case where F is locally compact, these definitions have significant overlap; see, for example, Lemmata 2.21, 2.28, and 2.30. Our main result, Theorem 2.38, is a strong existence result which is the analogue of item (7) in the list of properties of topological Jordan decompositions given in [13, Section 3]. An important ingredient in its proof is an analogue of a fixed-point result of Prasad and Yu (see Proposition 2.33).

1. Abstract groups

Fix a prime p, a Hausdorff topological group G, and a closed normal subgroup N. Note that G/N is also Hausdorff. For $g, \gamma \in G$, we define ${}^g\gamma := g\gamma g^{-1}$.

Definition 1.1. An element or subgroup of G is *compact modulo N* if its image in G/N belongs to a compact subgroup. If N is the trivial subgroup, we shall omit "modulo N."

Definition 1.2. The group G is *ind-locally-compact* (respectively, *ind-locally-profinite*; respectively, *ind-locally-pro-p*) if it is an inductive limit of a directed system of locally compact (respectively, locally profinite; respectively, locally pro-p) groups.

The main example we will have in mind of an ind-locally-compact group is the set of F-rational points of a linear algebraic F-group $\mathbf{G}$, where F is an algebraic extension of a locally compact field (see Remark 2.12). Another example is $\mathrm{GL}_\infty(F) := \varinjlim \mathrm{GL}_n(F)$, where F is a locally compact field; the limit is taken over positive integers n; and, for a given n, the map $\mathrm{GL}_n(F) \to \mathrm{GL}_{n+1}(F)$ comes from the natural embedding of GL_n into the Levi subgroup $\mathrm{GL}_n \times \mathrm{GL}_1$ of GL_{n+1}.

Definition 1.3. An element $\gamma \in G$ is *absolutely p-semisimple* if it has finite, coprime-to-p order. It is *topologically p-unipotent* if $\lim_{n\to\infty} \gamma^{p^n} = 1$. If the projection of γ to G/N is absolutely p-semisimple (respectively, topologically p-unipotent), then we will say that γ is *absolutely p-semisimple modulo N* (respectively, *topologically p-unipotent modulo N*).

Remark 1.4. Any power of an absolutely p-semisimple modulo N (respectively, topologically p-unipotent modulo N) element is absolutely p-semisimple modulo N (respectively, topologically p-unipotent modulo N). A p-power root of a topologically p-unipotent modulo N element is again topologically p-unipotent modulo N.

Remark 1.5. Clearly, an absolutely p-semisimple element is compact. Suppose that $\gamma \in G$ is topologically p-unipotent and G is ind-locally-compact. Then γ belongs to a locally compact subgroup of G, so, since $\gamma^{p^n} \to 1$, there is some $n \in \mathbb{Z}_{>0}$ such that γ^{p^n} belongs to a compact subgroup of G. Thus γ is compact.

Definition 1.6. A *topological p-Jordan decomposition modulo N* of an element $\gamma \in G$ is a pair of commuting elements $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ of G such that

- $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$,
- γ_{as} is absolutely p-semisimple modulo N, and
- γ_{tu} is topologically p-unipotent modulo N.

We will sometimes just say that $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is a topological p-Jordan decomposition modulo N. If N is the trivial subgroup, we will omit "modulo N."

In the statement of the following result, recall that p is *fixed*. It is certainly possible for an element to have distinct p- and ℓ-decompositions for ℓ a prime distinct from p (although it is an easy consequence of Remark 1.9 that, if G is ind-locally-pro-p, then this happens only for finite-order elements).

Proposition 1.7. *Suppose that $\gamma \in G$ has a topological p-Jordan decomposition $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$.*

(1) *If $\gamma = \gamma'_{\mathrm{as}}\gamma'_{\mathrm{tu}}$ is a topological p-Jordan decomposition, then $\gamma_{\mathrm{as}} = \gamma'_{\mathrm{as}}$ and $\gamma_{\mathrm{tu}} = \gamma'_{\mathrm{tu}}$.*
(2) *The closure of the group generated by γ contains γ_{as} and γ_{tu}.*
(3) *If G' is another Hausdorff topological group, and $f : G \to G'$ is a continuous homomorphism, then $f(\gamma) = f(\gamma_{\mathrm{as}})f(\gamma_{\mathrm{tu}})$ is a topological p-Jordan decomposition.*
(4) *For $g \in G$, we have that ${}^g\gamma = ({}^g\gamma_{\mathrm{as}})({}^g\gamma_{\mathrm{tu}})$ is a topological p-Jordan decomposition.*

A special case of the above was introduced in [13, Section 3] (especially items (3) and (4) of the list there), and Lemma 2 on [15, p. 226]. We omit the (straightforward) proof.

Proposition 1.8. *If G is ind-locally-profinite, then an element $\gamma \in G$ is topologically p-unipotent if and only if it belongs to a pro-p subgroup of G. If G is ind-locally-pro-p, then an element $\gamma \in G$ has a topological p-Jordan decomposition if and only if it is compact.*

Proof. Suppose that G is ind-locally-profinite. Note that G is ind-locally-compact and totally disconnected. Denote by K the closure of the subgroup of G generated by γ. Then K is also totally disconnected. By Proposition I.1.1.0 of [22], if K is compact, then it is profinite. In this case, since K is Abelian, by Proposition I.1.4.3 of [22] it has a unique Sylow pro-ℓ subgroup for each prime ℓ. Let K_p be its Sylow pro-p subgroup, and $K_{p'}$ the direct product of its Sylow pro-ℓ subgroups, taken over all $\ell \neq p$. Then K_p and $K_{p'}$ are profinite Abelian groups, of p-power and prime-to-p order, respectively. It is an easy consequence of Proposition I.1.4.4(b) of [22] that $K = K_p \times K_{p'}$. Write $\gamma = \gamma_p \gamma_{p'}$, with $\gamma_p \in K_p$ and $\gamma_{p'} \in K_{p'}$.

The 'if' direction of the first statement is obvious.

For the 'only if' direction of the first statement, suppose that γ is topologically p-unipotent. By Remark 1.5, K is compact. By the first paragraph of the proof, it is profinite. For any open subgroup U' of $K_{p'}$, we have that the map $g \mapsto g^p$ is an isomorphism on $K_{p'}/U'$. (Here, we have used commutativity of $K_{p'}$.) Since $\gamma_{p'}^{p^n} \in U'$ for some $n \in \mathbb{Z}_{>0}$, we have that $\gamma_{p'} \in U'$. Since U' was arbitrary and $K_{p'}$ is Hausdorff, we have that $\gamma_{p'} = 1$; i.e., $\gamma = \gamma_p \in K_p$. Since K_p is closed, in fact $K = K_p$; i.e., K is a pro-p group.

Now suppose that G is ind-locally-pro-p. The 'only if' direction of the second statement follows from Remark 1.5.

For the 'if' direction of the second statement, suppose that γ (equivalently, K) is compact. By the first paragraph of the proof, K is profinite. Then the intersection of $K_{p'}$ with an open ind-pro-p subgroup of H is an open ind-pro-p subgroup of $K_{p'}$. However, $K_{p'}$ contains no non-trivial pro-p subgroup, so this intersection is the trivial subgroup of $K_{p'}$. Thus $K_{p'}$ is discrete, so finite. Put $\gamma_{\mathrm{as}} = \gamma_{p'}$ and $\gamma_{\mathrm{tu}} = \gamma_p$. By the first statement of the lemma, γ_{tu} is topologically p-unipotent. Since $K_{p'}$ is finite, γ_{as} is absolutely p-semisimple. □

Remark 1.9. We isolate from the preceding proof a more refined, but technical, version of Proposition 1.8. Suppose that G is ind-locally-profinite, γ is a compact element of G, and K is the closure in G of the group generated by γ. Then K is profinite, and we may write $K = K_p \times K_{p'}$, where K_p and $K_{p'}$ are profinite groups of p-power and prime-to-p order, respectively. Write $\gamma = \gamma_p \gamma_{p'}$, with $\gamma_p \in K_p$ and $\gamma_{p'} \in K_{p'}$. Then

- γ is topologically p-unipotent if and only if $\gamma_{p'} = 1$, in which case $K_{p'} = \{1\}$.
- γ has a topological p-Jordan decomposition if and only if $\gamma_{p'}$ has finite order, in which case $K_{p'}$ is finite and we may take the topologically p-semisimple and topologically p-unipotent parts of γ to be $\gamma_{p'}$ and γ_p, respectively.
- If G is ind-locally-pro-p, then $K_{p'}$ is finite, so γ has a topological p-Jordan decomposition.

2. Algebraic groups

2.1. Unipotent elements

The adjective "unipotent" has sometimes carried several meanings (see [1, Section 3.7.1]). We begin by defining our notion of unipotence, then give a general result, essentially due to Kempf, relating the different meanings.

Definition 2.1. If F is a field, $\mathbf{G}$ is a linear algebraic F-group, and $\gamma \in \mathbf{G}(F)$, then γ is *unipotent* if there is an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n$, for some $n \in \mathbb{Z}_{\geqslant 0}$, such that the image of γ is an upper triangular matrix, with 1s on the diagonal.

Lemma 2.2. *If F is a field, $\mathbf{G}$ is a connected reductive F-group, and $\gamma \in \mathbf{G}(F)$ is unipotent, then there are a finite separable extension E/F and a unipotent radical $\mathbf{U}$ of a parabolic E-subgroup of $\mathbf{G}$ such that $\gamma \in \mathbf{U}(E)$. If E is equipped with a topology making it a non-discrete Hausdorff topological field, then there is a one-parameter subgroup λ of $\mathbf{G}$, defined over E, such that $\lim_{t\to 0} {}^{\lambda(t)}\gamma = 1$ in the E-analytic topology on $\mathbf{G}(E)$.*

Since $\mathbf{G}$ embeds as a closed subset of the affine space $\mathbb{A}^N$ for some $N \in \mathbb{Z}_{\geqslant 0}$, we may regard $\mathbf{G}(E)$ as a subset of E^N. By definition, the E-analytic topology on $\mathbf{G}(E)$ is just the subspace topology. This topology is finer than the Zariski topology on $\mathbf{G}(E)$. By [9, Appendix B] (especially Theorem B.1), it is independent of the choice of embedding.

Proof. By Lemma 3.1 and Theorem 3.4 of [16], there is a one-parameter subgroup λ of $\mathbf{G}$ such that $\lim_{t\to 0} {}^{\lambda(t)}\gamma = 1$ in the Zariski topology. (Indeed, in the notation of Theorem 3.4(c) of [16], any $\lambda \in \Delta_{S=\{1\},x=\gamma}$ will do.) There is a finite separable extension E/F such that $\mathbf{G}$ is E-split and λ is defined over E. Then $\gamma \in \mathbf{U}(\lambda)(E)$, in the notation of [16, p. 305]. By Theorem 13.4.2(i) and Lemma 15.1.2(ii) of [23], $\mathbf{U}(\lambda)$ is the unipotent radical of a parabolic E-subgroup of $\mathbf{G}$.

Now let $\mathbf{T}$ be an E-split maximal torus containing the image of λ, and $\Phi(\mathbf{G},\mathbf{T})$ the root system of $\mathbf{T}$ in $\mathbf{G}$. Put $\Phi_\lambda^+ = \{\alpha \in \Phi(\mathbf{G},\mathbf{T}) \mid \langle\alpha,\lambda\rangle > 0\}$. By Proposition 14.4(2)(a) of [4], $\mathbf{U}(\lambda)$ is (as a variety) the Cartesian product of the root subgroups $\mathbf{U}_\alpha$ of $\mathbf{G}$ associated to roots $\alpha \in \Phi_\lambda^+$. By Theorem 18.7 of [4], there are E-isomorphisms $\mathsf{e}_\alpha : \mathrm{Add} \to \mathbf{U}_\alpha$ for $\alpha \in \Phi_\lambda^+$ such that ${}^\tau\mathsf{e}_\alpha(s) = \mathsf{e}_\alpha(\alpha(\tau)s)$ for $s \in E$ and $\tau \in \mathbf{T}(E)$. There are elements $s_\alpha \in E$ (for $\alpha \in \Phi_\lambda^+$) such that $\gamma = \prod_{\alpha\in\Phi_\lambda^+} \mathsf{e}_\alpha(s_\alpha)$. Now suppose that E is equipped with a topology making it a non-discrete Hausdorff topological field. By Theorem B.1 of [9], the maps $\mathsf{e}_\alpha : E \to \mathbf{U}_\alpha(E) \subseteq \mathbf{G}(E)$ are continuous for the E-analytic topology on $\mathbf{G}(E)$, so

$$\lim_{t\to 0} {}^{\lambda(t)}\gamma = \prod_{\alpha\in\Phi_\lambda^+} \lim_{t\to 0} {}^{\lambda(t)}\mathsf{e}_\alpha(s_\alpha) = \prod_{\alpha\in\Phi_\lambda^+} \mathsf{e}_\alpha\Bigl(\lim_{t\to 0} t^{\langle\alpha,\lambda\rangle} s_\alpha\Bigr) = \prod_{\alpha\in\Phi_\lambda^+} \mathsf{e}_\alpha(0) = 1$$

in the E-analytic topology on $\mathbf{G}(E)$, as desired. □

2.2. Algebraic groups: basic definitions and notation

Let

- F be a field, with non-trivial discrete valuation ord, that is an algebraic extension of a complete field with perfect residue field,
- $\overline{F}$ an algebraic closure of F,
- F^{un}/F the maximal unramified subextension of $\overline{F}/F$,
- F^{tame}/F the maximal tame subextension of $\overline{F}/F$,
- F_0 the ring of integers of F,
- F_{0+} the maximal ideal of F_0,
- $F_0^\times = F_0 \smallsetminus F_{0+}$,

- $F_{0+}^\times = 1 + F_{0+}$,
- $\mathfrak{f}$ the residue field F_0/F_{0+},
- $p = \operatorname{char} \mathfrak{f}$,
- $(F^\times)^{p^\infty} = \bigcap_{n=0}^\infty (F^\times)^{p^n}$,
- $\mathbf{G}$ a connected reductive F-group,
- $\mathbf{N}$ a closed normal F-subgroup of $\mathbf{G}$, and
- $\widetilde{\mathbf{G}}$ the quotient $\mathbf{G}/\mathbf{N}$.

(Many of our results apply to any Henselian field with perfect residue field; but we restrict our attention slightly so that we do not have to re-prove the results of [2, Section 2] in this generality.) We denote by $Z(\mathbf{G})$ the centre of $\mathbf{G}$; by $\mathbf{X}^*(\mathbf{G})$ and $\mathbf{X}_*(\mathbf{G})$ the characters and cocharacters, respectively, of $\mathbf{G}$; and by $\mathbf{X}^*_F(\mathbf{G})$ and $\mathbf{X}^F_*(\mathbf{G})$ those characters and cocharacters, respectively, defined over F. If necessary, we will write $\mathfrak{f}_F$ in place of $\mathfrak{f}$ to indicate the dependence on the field F.

We will assume without further mention that any algebraic extension of F is contained in $\overline{F}$. If E/F is such an extension, then we denote again by ord the unique extension of ord to a (not necessarily discrete) valuation on E; and by E_0, etc., the analogues for E of F_0, etc., above.

We will write $G = \mathbf{G}(F)$ and $N = \mathbf{N}(F)$, and similarly for other F-groups.

Definition 2.3. If $\gamma \in G$ is semisimple, then the *character values* of γ (in $\mathbf{G}$) are the elements of the set $\{\chi(\gamma) \mid \chi \in \mathbf{X}^*(\mathbf{T})\}$, where $\mathbf{T}$ is any maximal torus in $\mathbf{G}$ containing γ.

Definition 2.4. An element $\gamma \in G$ is *F-tame* if there exists an F^{tame}-split torus (equivalently, by Lemmata 3.2 and A.2 of [2], an F^{tame}-split F-torus) $\mathbf{S}$ in $\mathbf{G}$ such that $\gamma \in \mathbf{S}(F^{\text{tame}})$.

Definition 2.5. Let $\mathcal{B}(\mathbf{G}, F)$ be the (enlarged) Bruhat–Tits building of $\mathbf{G}$ over F and, for $x \in \mathcal{B}(\mathbf{G}, F)$, let G_x and G_x^+ be the parahoric subgroup associated to x and its pro-unipotent radical, respectively. (In general, the parahoric subgroup may be strictly smaller than the stabiliser of x (but see Lemma 2.32). In the language of Proposition 4.6.28(i) of [7], it is the *fixateur connexe* of the facet containing x.) Let G_x be the (not necessarily connected) $\mathfrak{f}_F$-group such that $\mathsf{G}_x(\mathfrak{f}_{\widetilde{F}}) = \operatorname{stab}_{\mathbf{G}(\widetilde{F})}(x)/\mathbf{G}(\widetilde{F})_x^+$ for all unramified extensions $\widetilde{F}/F$. If necessary, we will write G_x^F in place of G_x to indicate the dependence on the field F. Put $G_0 = \bigcup_{x \in \mathcal{B}(\mathbf{G}, F)} G_x$ and $G_{0+} = \bigcup_{x \in \mathcal{B}(\mathbf{G}, F)} G_x^+$.

Remark 2.6. We have $\mathsf{G}_x^\circ(\mathfrak{f}_{\widetilde{F}}) = \mathbf{G}(\widetilde{F})_x/\mathbf{G}(\widetilde{F})_x^+$ for all unramified extensions $\widetilde{F}/F$.

Definition 2.7. An element or subgroup of G is *bounded* if its orbits in $\mathcal{B}(\mathbf{G}, F)$ are bounded (in the sense of metric spaces). An element or subgroup of G is *bounded modulo* $\mathbf{N}$ if its image in $\widetilde{G}$ is bounded. If $\mathbf{G} = \mathbf{T}$ is a torus, then denote by T_{b} the maximal bounded subgroup of T.

Remark 2.8. If $\widetilde{\mathbf{G}}$ is semisimple, then the building $\mathcal{B}(\widetilde{\mathbf{G}}, F)$ is canonical. In general, we "canonify" it as in [25, Sections 1.2 and 2.1]. Since we will be concerned almost exclusively with the case $\mathbf{N} = Z(\mathbf{G})^\circ$, this "canonification" will not usually be necessary.

Remark 2.9. Consider a bounded element or subgroup of G and a non-empty, closed, convex, G-stable subset $\mathcal{S}$ of $\mathcal{B}(\mathbf{G}, F)$. By Proposition 3.2.4 of [6], the element or subgroup fixes a point

$\bar{x}$ of the image of $\mathcal{S}$ in the reduced building $\mathcal{B}^{\text{red}}(\mathbf{G}, F)$; hence, by boundedness, actually fixes any lift to $\mathcal{S}$ of $\bar{x}$. On the other hand, since G acts on $\mathcal{B}(\mathbf{G}, F)$ by isometries, an element or subgroup of G which fixes a point of $\mathcal{B}(\mathbf{G}, F)$ is bounded.

Remark 2.10. If F is locally compact, then a subgroup of G is bounded modulo $\mathbf{N}$ if and only if its closure is compact modulo N. If F is an algebraic extension of a locally compact field, then an element of G is bounded modulo $\mathbf{N}$ if and only if it belongs to a compact modulo N subgroup of G. Indeed, the 'if' direction is obvious. For the 'only if' direction, suppose that $\gamma \in G$ is bounded modulo $\mathbf{N}$. Then, by Lemma 2.2 of [2], there is some locally compact subfield F' of F such that $\mathbf{G}$, $\mathbf{N}$, and γ are all defined over F'. Thus γ is contained in a compact modulo $\mathbf{N}(F')$ subgroup of $\mathbf{G}(F')$, hence *a fortiori* a compact modulo N subgroup of G.

Remark 2.11. If $\mathbf{G} = \mathbf{T}$ is a torus, then $T_0 = T_x$ and $T_{0+} = T_x^+$ for any $x \in \mathcal{B}(\mathbf{T}, F)$. Concretely, T_{b} is the group of elements of T whose character values lie in $E_0^\times$ (by Lemme 4.2.19 of [7]) and T_x^+ is the group of elements of T_x whose character values lie in $E_{0+}^\times$, where E/F is the splitting field of $\mathbf{T}$. If $\mathbf{T}$ is F-split, then $T_x = T_{\mathrm{b}}$; so T_x^+ is the group of elements of T whose character values lie in $F_{0+}^\times$.

Remark 2.12. If F is an algebraic extension of a locally compact field, then $p > 0$ and, by Lemma 2.2 of [2], we have $G = \varinjlim \mathbf{G}(F')$, the limit taken over all locally compact subfields F' of F over which $\mathbf{G}$ is defined. For such a subfield, $\mathbf{G}(F')_x^+$ is an open pro-p subgroup of $\mathbf{G}(F')$ (for any $x \in \mathcal{B}(\mathbf{G}, F')$). Thus, G is ind-locally-pro-p.

Remark 2.13. Suppose that $\gamma \in G$ is unipotent. By Lemma 2.2, there are a finite separable extension E/F (which we may take, by passing to a further finite separable extension if necessary, to be a splitting field for $\mathbf{G}$) and a one-parameter subgroup $\lambda \in \mathbf{X}_*^E(\mathbf{G})$ such that $\lim_{t \to 0} {}^{\lambda(t)}\gamma = 1$. Since $\mathbf{G}(E)_{0+}$ is a neighbourhood of 1, there is an element $t \in E^\times$ such that ${}^{\lambda(t)}\gamma \in \mathbf{G}(E)_{0+}$. Then $\gamma \in {}^{\lambda(t)^{-1}}\mathbf{G}(E)_{0+} = \mathbf{G}(E)_{0+}$, so γ is topologically F-unipotent in the sense of Definition 2.15.

2.3. *A lifting of* $\bar{\mathfrak{f}}^\times$

In this section, we will define a $\mathrm{Gal}(F^{\mathrm{un}}/F)$-stable subgroup $\mathcal{F}(F)$ of $(F^{\mathrm{un}})_0^\times$ such that the map $\mathcal{F}(F) \to \bar{\mathfrak{f}}^\times$ is an isomorphism. Let F' be a complete subfield of F such that F/F' is unramified. (Such a subfield exists, by Lemma 2.2 of [2].)

If $p > 0$, then put $\mathcal{F}(F) := \bigcup_{L/F' \text{ finite unramified}} (L^\times)^{p^\infty}$. We have that $\mathcal{F}(F) \cap (F^{\mathrm{un}})_{0+}^\times = \{1\}$, so that the map $\mathcal{F}(F) \to \bar{\mathfrak{f}}^\times$ is injective. By Lemma 7 of [8], since $\mathfrak{f}$ is perfect, $(L^\times)^{p^\infty}$ maps onto $\mathfrak{f}_L^\times$ for every finite unramified extension L/F'; so the map $\mathcal{F}(F) \to \bar{\mathfrak{f}}^\times$ is also surjective, hence again an isomorphism. Note that the p^nth power map on $\mathcal{F}(F)$ is also an isomorphism for all $n \in \mathbb{Z}_{\geqslant 0}$. Note that, if E/F' is an arbitrary finite extension with maximal unramified subextension L/F', then $(E^\times)^{p^\infty}$ contains $(L^\times)^{p^\infty}$, and both map isomorphically onto $\mathfrak{f}_E = \mathfrak{f}_L$; so, in fact, $(E^\times)^{p^\infty} = (L^\times)^{p^\infty}$, and we could take the union defining $\mathcal{F}(F)$ over *all* finite extensions E/F'.

The definition of $\mathcal{F}(F)$ is slightly more complicated if $p = \mathrm{char}\, F$. Let $\mathfrak{f}'_F$ be a subfield of F'_0 satisfying the following property.

($\mathbf{CF}_F$) The restriction to $\mathfrak{f}'_F$ of the natural map $F'_0 \to \mathfrak{f}_F$ is an isomorphism onto $\mathfrak{f}_F$.

By Theorem 9 of [8], $\mathfrak{f}'_F$ exists.

If L/F is a finite unramified extension, say of degree n, then $\mathfrak{f}_L/\mathfrak{f}_F$ is separable, so there exists a primitive element $\bar{\theta}$ for $\mathfrak{f}_L/\mathfrak{f}_F$, say with minimal polynomial $\overline{m}(x)$ over $\mathfrak{f}_F$. Since $\mathfrak{f}_L/\mathfrak{f}_F$ is separable, so is $\overline{m}(x)$. Let $m(x)$ be the unique preimage in $\mathfrak{f}'_F[x]$ of $\overline{m}(x)$, and θ the unique root of $m(x)$ lifting $\bar{\theta}$. Note that $F'[\theta]$ is a finite unramified extension of F', hence complete.

Suppose that there exists a subfield $\mathfrak{f}'_L$ of L_0, containing $\mathfrak{f}'_F$, with property ($\mathbf{CF}_L$). Then $[\mathfrak{f}'_L : \mathfrak{f}'_F] = n$, and there is a lift θ' in $\mathfrak{f}'_L$ of $\bar{\theta}$, say with minimal polynomial $m'(x)$ over $\mathfrak{f}'_F$. Then $\deg m'(x) \leqslant [\mathfrak{f}'_L : \mathfrak{f}'_F] = n$ and $\bar{\theta}$ is a root of the image in $\mathfrak{f}_F[x]$ of $m'(x)$, so $m'(x)$ is the preimage in $\mathfrak{f}'_F[x]$ of $\overline{m}(x)$; that is, $m'(x) = m(x)$. Thus $\theta' = \theta$, so $\mathfrak{f}'_L = \mathfrak{f}'_F[\theta]$. Note that $\mathfrak{f}'_L$ lies in the complete field $F'[\theta]$.

Since $\mathfrak{f}'_F[\theta]$ clearly has property ($\mathbf{CF}_L$), we have shown that it is the unique subfield of L_0 containing $\mathfrak{f}'_F$ with this property. In particular, if L/F is Galois (which is not automatic, since we have not assumed that $\mathfrak{f}_F$ is finite), then $\mathfrak{f}'_L \subseteq L$ is $\mathrm{Gal}(L/F)$-, hence $\mathrm{Gal}(F^{\mathrm{un}}/F)$-, stable. Put $\mathcal{F}(F) := \bigcup_{L/F \text{ finite unramified}} (\mathfrak{f}'_L)^\times$. By Theorem 10(b) of [8], if $p > 0$ (in addition to $p = \operatorname{char} F$), then this definition coincides with the one given above.

It is clear that $\mathcal{F}(F) \cup \{0\}$ is a $\mathrm{Gal}(F^{\mathrm{un}}/F)$-stable field satisfying ($\mathbf{CF}_{F^{\mathrm{un}}}$) that contains $\mathfrak{f}'_F$ and is contained in $(F^{\mathrm{un}})_0^\times$.

Remark 2.14. It is easy to verify that the group $\mathcal{F}(F)$ does not depend on the choice of F'. However, for $p = \operatorname{char} F$, it *does* depend on the choice of $\mathfrak{f}'_F$ (and, of course, of $\overline{F}$). Since $\mathfrak{f}'_F$ may fail to be unique (see Theorem 10(a) of [8]), so may $\mathcal{F}(F)$; but this ambiguity seems unavoidable.

Note that, regardless of the values of p and $\operatorname{char} F$, we have $\mathcal{F}(F) = \mathcal{F}(E)$ for any discretely valued algebraic extension E/F.

2.4. Absolute semisimplicity and topological unipotence: definitions and basic results

Definition 2.15. An element $\gamma \in G$ is *topologically F-unipotent* (in G) if it belongs to $\mathbf{G}(E)_{0+}$ for some finite extension E/F. It is *absolutely F-semisimple* (in G) if it is semisimple and its character values belong to $\mathcal{F}(F)$. If the image of γ in $\widetilde{G}$ is absolutely F-semisimple (respectively, topologically F-unipotent), then we will say that γ is *absolutely F-semisimple modulo* $\mathbf{N}$ (respectively, *topologically F-unipotent modulo* $\mathbf{N}$).

Note that an absolutely F-semisimple element need not belong to G_0, and a topologically F-unipotent element need not belong to G_{0+} (but see Proposition 2.43). We will show later (see Corollary 2.37) that an absolutely F-semisimple element must be F-tame.

Remark 2.16. Any power of an absolutely F-semisimple modulo $\mathbf{N}$ (respectively, topologically F-unipotent modulo $\mathbf{N}$) element is again absolutely F-semisimple modulo $\mathbf{N}$ (respectively, topologically F-unipotent modulo $\mathbf{N}$).

Remark 2.17. It is clear that a topologically F-unipotent element of G is bounded. The character values of an absolutely F-semisimple element lie in $\mathcal{F}(F) \subseteq (F^{\mathrm{un}})_0^\times$; so, by Remark 2.11, γ is bounded. It is an easy consequence that an element which is absolutely F-semisimple or topologically F-unipotent modulo $\mathbf{N}$ is bounded modulo $\mathbf{N}$.

Lemma 2.18. *If $\gamma, \delta \in G$ commute, γ is bounded, and δ lies in G_{0+}, then there is a point $x \in \mathcal{B}(\mathbf{G}, F)$ such that $\gamma \cdot x = x$ and $\delta \in G_x^+$.*

Proof. Let ε be a positive real number such that $\mathcal{S} := \{x \in \mathcal{B}(\mathbf{G}, F) \mid \delta \in G_{x,\varepsilon}\}$ is non-empty. (Here, $G_{x,\varepsilon}$ is a Moy–Prasad filtration subgroup. See [19, Section 2.6] and [20, Section 3.2].) Certainly, $\mathcal{S}$ is also closed and convex. If $x \in \mathcal{S}$, then $\delta = {}^{\gamma}\delta \in G_{\gamma \cdot x,\varepsilon}$, so $\gamma \cdot x \in \mathcal{S}$. That is, $\mathcal{S}$ is γ-stable. By Remark 2.9, γ fixes a point of $\mathcal{S}$. $\square$

Lemma 2.19. *Suppose that $\gamma \in G$ has (ordinary) Jordan decomposition $\gamma = \gamma_{\mathrm{ss}}\gamma_{\mathrm{un}}$, and that $\gamma_{\mathrm{ss}}, \gamma_{\mathrm{un}} \in G$. Then γ is topologically F-unipotent modulo* $\mathbf{N}$ *if and only if γ_{ss} is.*

Proof. Clearly, it suffices to prove this in case $\mathbf{N}$ is the trivial subgroup.

Suppose that $\delta, \delta_+ \in G$ commute, δ is topologically F-unipotent, and δ_+ is unipotent. There is a finite extension E/F such that $\delta \in \mathbf{G}(E)_{0+}$; say $z \in \mathcal{B}(\mathbf{G}, E)$ is such that $\delta \in \mathbf{G}(E)_z^+$. By Remark 2.13, there is a finite separable extension K/E such that $\delta_+ \in \mathbf{G}(K)_{0+}$. By Lemma 2.5 of [2], $\delta \in \mathbf{G}(K)_z^+$. In particular, δ is bounded, so, by Lemma 2.18, there is a point $y \in \mathcal{B}(\mathbf{G}, K)$ such that $\delta \cdot y = y$ and $\delta_+ \in \mathbf{G}(E)_y^+$. By Lemma 2.9 of [2], $\delta \in \mathbf{G}(E)_y$. By Lemma 2.8 of [2], for $x \in (y, z)$ sufficiently close to y, we have that $\delta \in \mathbf{G}(E)_x^+$. If, in addition, x is so close to y that it is contained in a facet whose closure contains y, then $\mathbf{G}(E)_y^+ \subseteq \mathbf{G}(E)_x^+$, so $\delta\delta_+ \in \mathbf{G}(E)_x^+$. That is, $\delta\delta_+$ is topologically F-unipotent.

If we take $\delta = \gamma_{\mathrm{ss}}$ and $\delta_+ = \gamma_{\mathrm{un}}$, then we see that the topological F-unipotence of γ_{ss} implies that of γ. If we take $\delta = \gamma$ and $\delta_+ = \gamma_{\mathrm{un}}^{-1}$, then we see that the topological F-unipotence of γ implies that of γ_{ss}. $\square$

Remark 2.20. Suppose that $\gamma \in G$ has (ordinary) Jordan decomposition $\gamma = \gamma_{\mathrm{ss}}\gamma_{\mathrm{un}}$. Put $\widetilde{\mathbf{H}} = C_{\mathbf{G}}(\gamma_{\mathrm{ss}})$ and $\mathbf{H} = \widetilde{\mathbf{H}}^\circ$.

(1) If char $F = 0$, then $\gamma_{\mathrm{ss}} \in G$. By Propositions 1.2(a), 9.1(1), and 13.19 of [4], $\mathbf{H}$ is a connected reductive F-group. Certainly, $\gamma_{\mathrm{ss}} \in H$ and $\gamma_{\mathrm{un}} \in \widetilde{H}$. Since the image of γ_{un} in the component group $(\widetilde{\mathbf{H}}/\mathbf{H})(F)$ is unipotent and has finite order, it is trivial. That is, $\gamma_{\mathrm{un}} \in H$, so $\gamma \in H$.
(2) If char $F > 0$, then, by [4, Section 4.1(a)], there is some $a \in \mathbb{Z}_{\geqslant 0}$ such that $\gamma_{\mathrm{un}}^{p^a} = 1$. Then $\gamma_{\mathrm{ss}}^{p^a} = \gamma^{p^a} \in G$. Since an easy GL_n calculation shows that $\widetilde{\mathbf{H}} = C_{\mathbf{G}}(\gamma_{\mathrm{ss}}^{p^a})$, hence that $\mathbf{H} = C_{\mathbf{G}}(\gamma_{\mathrm{ss}}^{p^a})^\circ$, we have again that $\mathbf{H}$ is a connected reductive F-group. We have $\gamma^{p^a} = \gamma_{\mathrm{ss}}^{p^a} \in \mathbf{H}(\overline{F}) \cap G = H$.

Lemma 2.21. *Suppose that $p > 0$. Then an element of G is topologically F-unipotent modulo* $\mathbf{N}$ *if and only if it is topologically p-unipotent modulo N.*

Proof. Recall that, for every finite extension E/F, Moy and Prasad have defined (in [19, Section 2.6] and [20, Section 3.2]), for each $x \in \mathcal{B}(\mathbf{G}, E)$, an exhaustive filtration $(\mathbf{G}(E)_{x,r})_{r \in \mathbb{R}_{\geqslant 0}}$ of $\mathbf{G}(E)$ by subgroups such that $\mathbf{G}(E)_{x,0} = \mathbf{G}(E)_x$ and $\mathbf{G}(E)_{x,\varepsilon} = \mathbf{G}(E)_x^+$ for sufficiently small positive ε. Since $\mathbf{G}(E)_{x,r}/\mathbf{G}(E)_{x,r+}$ is a p-group for $(x, r) \in \mathcal{B}(\mathbf{G}, E) \times \mathbb{R}_{>0}$, we have that a topologically F-unipotent modulo $\mathbf{N}$ element is topologically p-unipotent modulo N.

If $\gamma \in G$ is topologically p-unipotent modulo N, then its image in $\widetilde{G}$ is topologically p-unipotent. Thus it suffices to prove that, if γ is topologically p-unipotent, then it is topologically F-unipotent. Let γ_{ss} and γ_{un} be the semisimple and unipotent parts, respectively, of the (ordinary) Jordan decomposition of γ; and put $\mathbf{H} = C_{\mathbf{G}}(\gamma_{\mathrm{ss}})^\circ$.

If char $F = 0$, then, by Remark 2.20, we have $\gamma, \gamma_{\mathrm{un}} \in H$. By Remark 2.13, there is a finite separable extension E/F such that $\mathbf{H}$ is E-split and $\gamma_{\mathrm{un}} \in \mathbf{H}(E)_{0+}$; say $x \in \mathcal{B}(\mathbf{H}, E)$ is such that $\gamma_{\mathrm{un}} \in \mathbf{H}(E)_x^+$.

If char $F > 0$, then, again by Remark 2.20, there is $a \in \mathbb{Z}_{\geqslant 0}$ such that $\gamma^{p^a} \in H$ and $\gamma_{\mathrm{un}}^{p^a} = 1$. In this case, let E/F be a finite separable extension such that $\mathbf{H}$ is E-split, and x any point of $\mathcal{B}(\mathbf{H}, E)$.

In either case, $\gamma^{p^n} \in \gamma_{\mathrm{ss}}^{p^n} \mathbf{H}(E)_x^+$ for all sufficiently large integers n. Since also $\gamma^{p^n} \in \mathbf{H}(E)_x^+$ for all sufficiently large n (by topological p-unipotence), we have that $\gamma_{\mathrm{ss}}^{p^n} \in \mathbf{H}(E)_x^+$ for some $n \in \mathbb{Z}_{\geqslant 0}$. Let $\mathbf{T}$ be an E-split maximal torus in $\mathbf{H}$ (hence in $\mathbf{G}$) such that x belongs to the apartment of $\mathbf{T}$ in $\mathcal{B}(\mathbf{H}, E)$. By Lemma 2.6 of [2], we have that $\gamma_{\mathrm{ss}}^{p^n} \in \mathbf{T}(E)_{0+}$. Let K/E be a finite extension such that $\gamma_{\mathrm{ss}} \in \mathbf{G}(K)$. By Remark 2.11, the character values of $\gamma_{\mathrm{ss}}^{p^n}$ lie in $E_{0+}^\times \subseteq K_{0+}^\times$, so the character values of γ_{ss} lie in $K_{0+}^\times$, so $\gamma_{\mathrm{ss}} \in \mathbf{T}(K)_{0+}$. By another application of Lemma 2.6 of [2], $\gamma_{\mathrm{ss}} \in \mathbf{G}(K)_{0+}$. Thus γ_{ss} is topologically K-unipotent. By Lemma 2.19, γ is topologically K-unipotent. It is then clear from the definition (see Definition 2.15) that it is topologically F-unipotent. □

Remark 2.22. Let E/F be a discretely valued algebraic extension. Since $\mathcal{F}(F) = \mathcal{F}(E)$, an element of G is absolutely F-semisimple modulo $\mathbf{N}$ if and only if it is absolutely E-semisimple modulo $\mathbf{N}$. If $p = 0$, then Lemma 2.7 of [2] shows that an element of G is topologically F-unipotent modulo $\mathbf{N}$ if and only if it is topologically E-unipotent modulo $\mathbf{N}$. If $p > 0$, then, by Lemma 2.21, the topological F-unipotence and topological E-unipotence of an element of G are both equivalent to its topological p-unipotence, hence to one another. Thus an element of G is topologically F-unipotent modulo $\mathbf{N}$ if and only if it is topologically E-unipotent modulo $\mathbf{N}$.

Our definition of topological F-Jordan decompositions is almost the analogue one would expect of the definition of a topological p-Jordan decomposition (see Definition 1.6), except for one somewhat surprising condition about tori. Proposition 2.42 will show that this condition can be omitted.

Definition 2.23. A *topological F-Jordan decomposition modulo* $\mathbf{N}$ of an element $\gamma \in G$ is a pair of commuting elements $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ of G such that

- the images of γ_{ss} and γ_{as} in $\widetilde{\mathbf{G}}(\overline{F})$ belong to a common $\overline{F}$-torus there,
- $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$,
- γ_{as} is absolutely F-semisimple modulo $\mathbf{N}$, and
- γ_{tu} is topologically F-unipotent modulo $\mathbf{N}$.

We will sometimes just say that $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is a topological F-Jordan decomposition modulo $\mathbf{N}$. If $\mathbf{N}$ is the trivial subgroup, then we will omit "modulo $\mathbf{N}$."

2.5. Uniqueness of topological Jordan decompositions

Proposition 2.24. *An element of G has at most one topological F-Jordan decomposition.*

Proof. Suppose that $\gamma_{\rm as}\gamma_{\rm tu} = \gamma = \gamma'_{\rm as}\gamma'_{\rm tu}$ are two topological F-Jordan decompositions of an element $\gamma \in G$. By Remark 2.22, they remain topological F-Jordan decompositions if we replace F by a finite extension, so we do so whenever necessary.

Since $\gamma_{\rm as}$ and $\gamma'_{\rm as}$ are semisimple and commute with $\gamma_{\rm tu}$ and $\gamma'_{\rm tu}$, respectively, we have that $(\gamma_{\rm tu})_{\rm un} = \gamma_{\rm un} = (\gamma'_{\rm tu})_{\rm un}$. Upon replacing F by a finite extension, we may, and hence do, assume that $\gamma_{\rm ss}$ (hence also $(\gamma_{\rm tu})_{\rm ss}$ and $(\gamma'_{\rm tu})_{\rm ss}$) lie in G. By Lemma 2.19, $(\gamma_{\rm tu})_{\rm ss}$ and $(\gamma'_{\rm tu})_{\rm ss}$ are topologically F-unipotent. Thus $\gamma_{\rm as}(\gamma_{\rm tu})_{\rm ss} = \gamma_{\rm ss} = \gamma'_{\rm as}(\gamma'_{\rm tu})_{\rm ss}$ are two topological F-Jordan decompositions of $\gamma_{\rm ss}$, so we may, and hence do, assume that γ is semisimple.

We show that γ, $\gamma_{\rm as}$, and $\gamma'_{\rm as}$ (hence also $\gamma_{\rm tu}$ and $\gamma'_{\rm tu}$) lie in a common torus. Let $\mathbf{T}$ and $\mathbf{T}'$ be maximal $\overline{F}$-tori in $\mathbf{G}$ such that $\gamma, \gamma_{\rm as} \in \mathbf{T}(\overline{F})$ and $\gamma, \gamma'_{\rm as} \in \mathbf{T}'(\overline{F})$ (hence $\gamma'_{\rm tu} \in \mathbf{T}'(\overline{F})$). Upon replacing F by a finite extension, we may, and hence do, assume that $\mathbf{T}'$ is F-split and $\gamma'_{\rm tu} \in G_{0+}$. By Lemma 2.6 of [2], we have $\gamma'_{\rm tu} \in T'_{0+}$, so that, by Remark 2.11, the character values of $\gamma'_{\rm tu}$ lie in $F^\times_{0+}$. If α is a root of $\mathbf{T}'$ in $C_{\mathbf{G}}(\gamma)^\circ$, then $\alpha(\gamma) = 1$, so $\alpha(\gamma'_{\rm as}) = \alpha(\gamma'_{\rm tu})^{-1} \in \mathcal{F}(F) \cap F^\times_{0+} = \{1\}$. That is, $\gamma'_{\rm as}$ and $\gamma'_{\rm tu}$ are central in $C_{\mathbf{G}}(\gamma)^\circ$, hence belong to $\mathbf{T}$.

Now the character values of $\gamma_{\rm as}$ and of $\gamma'_{\rm as}$, hence of $\gamma'^{-1}_{\rm as}\gamma_{\rm as}$, lie in $\mathcal{F}(F)$; and those of $\gamma_{\rm tu}$ and of $\gamma'_{\rm tu}$, hence of $\gamma'_{\rm tu}\gamma_{\rm tu}^{-1}$, lie in $F^\times_{0+}$; so, since $\mathcal{F}(F) \cap F^\times_{0+} = \{1\}$, we have that $\gamma'^{-1}_{\rm as}\gamma_{\rm as} = \gamma'_{\rm tu}\gamma_{\rm tu}^{-1}$ equals 1. □

It is an easy observation that, if $\mathbf{G}'$ is a connected reductive F-group and $f : \mathbf{G} \to \mathbf{G}'$ is an F-morphism, then $f(\gamma)$ is absolutely F-semisimple as long as γ is. Although we do not do so here, one can formulate a condition on f such that $f(\gamma)$ is topologically F-unipotent as long as γ is. We record three consequences.

Lemma 2.25. *If $\gamma = \gamma_{\rm as}\gamma_{\rm tu}$ is a topological F-Jordan decomposition, then $\gamma_{\rm as}, \gamma_{\rm tu} \in Z(C_G(\gamma))$.*

Proof. Fix $g \in C_G(\gamma)$. Then $\gamma = {}^g\gamma = ({}^g\gamma_{\rm as})({}^g\gamma_{\rm tu})$ is a topological F-Jordan decomposition. By Proposition 2.24, ${}^g\gamma_{\rm as} = \gamma_{\rm as}$ and ${}^g\gamma_{\rm tu} = \gamma_{\rm tu}$. □

Lemma 2.26. *Suppose that*

- *$\gamma \in G$,*
- *E/F is a discretely valued separable extension, and*
- *$\gamma = \gamma_{\rm as}\gamma_{\rm tu}$ is a topological E-Jordan decomposition.*

Then $\gamma_{\rm as}, \gamma_{\rm tu} \in G$, and $\gamma = \gamma_{\rm as}\gamma_{\rm tu}$ is a topological F-Jordan decomposition.

Proof. By Remark 2.22, $\gamma = \gamma_{\rm as}\gamma_{\rm tu}$ is a topological $\widetilde{E}$-Jordan decomposition, where $\widetilde{E}/F$ is the Galois closure of E/F. Then $\gamma = \sigma(\gamma_{\rm as})\sigma(\gamma_{\rm tu})$ is also a topological $\widetilde{E}$-Jordan decomposition for $\sigma \in \mathrm{Gal}(\widetilde{E}/F)$. By Proposition 2.24, $\gamma_{\rm as}, \gamma_{\rm tu} \in \mathbf{G}(\widetilde{E})^{\mathrm{Gal}(\widetilde{E}/F)} = G$. The last statement follows from another application of Remark 2.22. □

Corollary 2.27. *With the notation and hypotheses of Lemma* 2.26*, suppose that $g \in \mathbf{G}(E)$ is such that ${}^g\gamma \in G$. Then ${}^g\gamma_{\rm as}, {}^g\gamma_{\rm tu} \in G$, and ${}^g\gamma = ({}^g\gamma_{\rm as})({}^g\gamma_{\rm tu})$ is a topological F-Jordan decomposition.*

Proof. It is clear that ${}^g\gamma = ({}^g\gamma_{\rm as})({}^g\gamma_{\rm tu})$ is a topological E-Jordan decomposition. Now the result is an immediate consequence of Lemma 2.26. □

2.6. Relationship between algebraic and abstract groups

We now relate the abstract setting of Section 1 to our present setting. We have already seen that topological F- and p-unipotence are equivalent when $p > 0$ (see Lemma 2.21). We prove below the analogous results for absolute F-semisimplicity and topological F-Jordan decompositions; but note that the formulations are slightly more complicated.

Lemma 2.28. *Suppose that $p > 0$ and $\gamma \in G$. If γ is absolutely p-semisimple modulo N, then it is absolutely F-semisimple modulo* $\mathbf{N}$*. If γ is absolutely F-semisimple modulo* $\mathbf{N}$*, then it is absolutely p-semisimple modulo N if and only if some finite power of it is topologically p-unipotent modulo N.*

Proof. Suppose that γ is absolutely p-semisimple modulo N. Certainly, some finite power of it is topologically p-unipotent modulo N (in fact, lies in N). Further, its image $\overline{\gamma}$ in $\widetilde{G}$ has finite, prime-to-p order, say M, hence is semisimple. (Indeed, the unipotent part $(\overline{\gamma})_{\text{un}}$ of $\overline{\gamma}$ also has finite, prime-to-p order. If $\operatorname{char} F = 0$ (respectively, $\operatorname{char} F > 0$), then any non-trivial unipotent element has infinite (respectively, p-power) order; so $(\overline{\gamma})_{\text{un}} = 1$.) By Lemma 2.2 of [2], there is a complete subfield F' of F such that

- F/F' is unramified,
- $\widetilde{\mathbf{G}}$ is defined over F', and
- $\gamma \in \widetilde{\mathbf{G}}(F')$.

Let

- $\widetilde{\mathbf{T}}$ be a maximal F'-torus in $\widetilde{\mathbf{G}}$ such that $\overline{\gamma} \in \widetilde{\mathbf{T}}(F')$,
- E/F' the splitting field of $\widetilde{\mathbf{T}}$, and
- a an integer such that $ap \equiv 1 \pmod{M}$.

Then, for $\chi \in \mathbf{X}^*(\mathbf{T})$, we have $\chi(\overline{\gamma}) = \chi(\overline{\gamma}^{a^m})^{p^m} \in (E^\times)^{p^m}$ for all $m \in \mathbb{Z}_{\geqslant 0}$, so $\chi(\overline{\gamma}) \in (E^\times)^{p^\infty} \subseteq \mathcal{F}(F)$. That is, $\overline{\gamma}$ is absolutely F-semisimple, so γ is absolutely F-semisimple modulo $\mathbf{N}$.

Suppose that γ is absolutely F-semisimple modulo $\mathbf{N}$, and $M \in \mathbb{Z}_{>0}$ is such that γ^M is topologically p-unipotent modulo N. Write $M = p^m M'$, with $m \in \mathbb{Z}_{\geqslant 0}$ and $M' \in \mathbb{Z}_{>0}$ such that M' is coprime to p. Then $\gamma^{M'}$ is also topologically p-unipotent modulo N, hence (by Lemma 2.21) topologically F-unipotent modulo $\mathbf{N}$. On the other hand, by Remark 2.16, $\gamma^{M'}$ is absolutely F-semisimple modulo $\mathbf{N}$. By Proposition 2.24, the image of $\gamma^{M'}$ in $\widetilde{G}$ is trivial, so $\gamma^{M'} \in N$; that is, γ is absolutely p-semisimple modulo N. □

Corollary 2.29. *If F is an algebraic extension of a locally compact field, then an element of G is absolutely F-semisimple modulo* $\mathbf{N}$ *if and only if it is absolutely p-semisimple modulo N.*

Proof. The 'if' direction is clear. For the 'only if' direction, suppose that $\gamma \in G$ is absolutely F-semisimple modulo $\mathbf{N}$. By Remark 2.17, γ is bounded modulo $\mathbf{N}$; so, by Remark 2.9, it fixes some point $x \in \mathcal{B}(\widetilde{\mathbf{G}}, F)$. By Lemma 2.2 of [2], there is a locally compact subfield F' of F such that

- $\mathbf{G}$ and $\mathbf{N}$ are defined over F',
- $\gamma \in \mathbf{G}(F')$, and
- $x \in \mathcal{B}(\widetilde{\mathbf{G}}, F')$.

Then the image of γ in $\widetilde{\mathbf{G}}(F')$ lies in $\operatorname{stab}_{\widetilde{\mathbf{G}}(F')}(x)$. By Remarks 2.9 and 2.10, $\operatorname{stab}_{\widetilde{\mathbf{G}}(F')}(x)$ is bounded, hence compact; so its open subgroup $\widetilde{\mathbf{G}}(F')_x^+$ has finite index. That is, some (finite) power of γ is topologically p-unipotent modulo N. Now the result follows from Lemma 2.28. □

Lemma 2.30. *Suppose $p > 0$ and $\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}} \in G$. If $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ is a topological p-Jordan decomposition modulo N, then it is a topological F-Jordan decomposition modulo $\mathbf{N}$. If it is a topological F-Jordan decomposition modulo $\mathbf{N}$, then it is a topological p-Jordan decomposition modulo N if and only if γ_{as} is absolutely p-semisimple.*

Proof. It is clear from Lemma 2.21 that, if $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ is a topological F-Jordan decomposition modulo $\mathbf{N}$, then it is a topological p-Jordan decomposition modulo N if and only if γ_{as} is absolutely p-semisimple.

Suppose that $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ is a topological p-Jordan decomposition modulo N. Put $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$. By Lemmata 2.21 and 2.28, γ_{as} is absolutely F-semisimple modulo $\mathbf{N}$ and γ_{tu} is topologically F-unipotent $\mathbf{N}$. Let E/F be a finite extension such that γ_{ss} (hence $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$) belongs to $\mathbf{G}(E)$. By Lemmata 2.19 and 2.21, $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$ is topologically p-unipotent modulo $\mathbf{N}(E)$. Since γ_{as} commutes with γ_{tu}, it commutes also with $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$, so $\gamma_{\mathrm{ss}} = \gamma_{\mathrm{as}}(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$ is a topological p-Jordan decomposition modulo $\mathbf{N}(E)$. By Proposition 1.7(2), the image of γ_{as} in $\widetilde{\mathbf{G}}(E)$ belongs to any maximal $\overline{F}$-torus containing the image there of γ_{ss}. Thus $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is a topological F-Jordan decomposition modulo $\mathbf{N}$. □

Corollary 2.31. *If F is an algebraic extension of a locally compact field, then a topological F-Jordan decomposition modulo $\mathbf{N}$ is a topological p-Jordan decomposition modulo N, and conversely.*

2.7. Stabilisers and parahorics

It is a minor inconvenience in our arguments that the stabiliser of a point in $\mathcal{B}(\mathbf{G}, F)$ may be strictly larger than the associated parahoric subgroup. The next result shows that, under some circumstances, we may bring an element of the stabiliser into the parahoric by passing to a tame extension.

Lemma 2.32. *Suppose that*

- *$x \in \mathcal{B}(\mathbf{G}, F)$,*
- *$g \in \operatorname{stab}_G(x)$, and*
- *$g^n \in G_x$ for some $n \in \mathbb{Z}_{>0}$ indivisible by p.*

Then there exists a finite tame extension L/F such that $g \in \mathbf{G}(L)_x$.

Proof. Upon replacing F by the splitting field of a maximal F^{tame}-split torus, we may, and hence do, assume that $\mathbf{G}$ is F-quasisplit. Let $\mathbf{S}$ be a maximal F-split (hence maximal F^{tame}-

split) torus in $\mathbf{G}$ such that x belongs to the apartment $\mathcal{A}(\mathbf{S}, F)$ of $\mathbf{S}$, and $\mathbf{T}$ the maximal F-torus in $\mathbf{G}$ containing $\mathbf{S}$. If a is a root of $\mathbf{S}$ in $\mathbf{G}$, then let α be a root of $\mathbf{T}$ in $\mathbf{G}$ restricting to α, and write F_a for the fixed field in F^{sep} of $\text{stab}_{\text{Gal}(F^{\text{sep}}/F)}(\alpha)$ (the field denoted by L_a in [7, Sections 4.1.8 and 4.1.14]). Up to F-isomorphism, this field does not depend on the choice of α. By replacing F by a further finite tame extension if necessary, we may, and hence do, assume that F contains the nth roots of unity in F^{sep}, and all the extensions F_a/F are totally wildly ramified. Let L'/F be a totally (tamely) ramified extension of degree n. Note that $\mathbf{S}$ is still a maximal L'-split torus. Fix a root a of $\mathbf{S}$ in $\mathbf{G}$. With the obvious notation, L'_a is the fixed field in F^{sep} of $\text{stab}_{\text{Gal}(F^{\text{sep}}/L')}(\alpha) = \text{Gal}(F^{\text{sep}}/L') \cap \text{stab}_{\text{Gal}(F^{\text{sep}}/F)}(\alpha)$—that is, $L'_a = L'F_a$. Since F_a/F is totally wildly ramified and L'/F is tamely ramified, it follows that L'_a/F_a is a totally ramified extension of degree n.

Choose a chamber C in $\mathcal{A}(\mathbf{S}, F)$ containing x in its closure, and a special vertex o in the closure of C. By regarding o as an origin, we may, and hence do, identify $\mathcal{A}(\mathbf{S}, F)$ with $\mathbf{X}_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, hence the affine F-roots on $\mathcal{A}(\mathbf{S}, F)$ (in the sense of [19, Section 2.5], not [2, Section 2.2]) with certain functions on $\mathbf{X}_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ of the form $y \mapsto \langle a, y\rangle + r$ with a a root of $\mathbf{S}$ in $\mathbf{G}$ and $r \in \mathbb{R}$. (Here, $\langle\cdot,\cdot\rangle$ is the usual pairing between $\mathbf{X}^*(\mathbf{S})$ and $\mathbf{X}_*(\mathbf{S})$.) Specifically, r must belong to the set denoted by Γ'_a in [6, Section 6.2.2]. By [7, Section 4.2.21] (adapted to our choice of origin, which is different from the one in [7, Section 4.2.2]), we have $\Gamma'_a = \text{ord}(F_a^\times)$. Denote by ${}_F\mathcal{H}$ the collection of zero-sets of affine F-roots.

We have that ${}_F W_{\text{aff}} := N_G(T)/T_{\text{b}}$, viewed as a group of affine transformations of $\mathcal{A}(\mathbf{S}, F)$, is isomorphic to the semi-direct product ${}_F\Lambda \rtimes {}_F W$, where ${}_F\Lambda = T/T_{\text{b}}$ is a lattice of translations, and ${}_F W \cong N_G(T)/T$ is the finite group generated by the reflections through the hyperplanes in ${}_F\mathcal{H}$ passing through o. Let ${}_F W'$ be the (normal) subgroup of ${}_F W_{\text{aff}}$ generated by reflections through the hyperplanes in ${}_F\mathcal{H}$. Then ${}_F W' \cap {}_F\Lambda$ is generated by translations by elements of the form $\gamma' a^\vee$, where a is a root of $\mathbf{S}$ in $\mathbf{G}$, $a^\vee$ is the associated coroot, and $\gamma' \in \text{ord}(F_a^\times)$. We will denote by a left subscript L' the analogues over L' of the objects defined over F above. Then the fact that $\text{ord}({L'_a}^\times) = \frac{1}{n}\text{ord}(F_a^\times)$ and the obvious analogue for L' of our discussion above for F show that

$(*)$ if $\tau \in {}_F\Lambda$ satisfies $\tau^n \in {}_F W'$, then $\tau \in {}_{L'} W'$.

Let Ω be the image of $\{x\}$ in the reduced building $\mathcal{B}^{\text{red}}(\mathbf{G}, F)$, and $f = f'_\Omega$ the optimisation of the function f_Ω of [7, Section 4.6.26]. Then, by Proposition 4.6.28(i) and Définition 5.2.6 of [7], the group of integer points of the scheme $\mathfrak{G}_f^0$ of [7, Section 4.6.2] is the parahoric G_x. By Corollaire 4.6.12 of [7], there exists, for each root $a \in \Phi_f$, an affine transformation $w_a \in W_f$ such that the linear part of w_a is reflection in the zero-set of a; and W_f is generated by the elements w_a. Here, Φ_f is the set of gradients of affine F-roots vanishing at x, and W_f is as in [7, 4.6.3(6)]. Fix $a \in \Phi_f$, and let ψ be the affine F-root with gradient a that vanishes at x. Since w_a fixes x, it must actually be reflection in the zero-set of ψ. That is, W_f is generated by the reflections through hyperplanes in ${}_F\mathcal{H}$ passing through x. By Proposition V.3.2 of [5], it is actually the stabiliser of x in ${}_F W'$. Since $N_G(T) \cap G_x = N_f^0$, in the notation of [5, 4.6.3(5)], and since W_f is the image in ${}_F W_{\text{aff}}$ of N_f^0, we have that

$(**_F)$ the image of $N_G(T) \cap G_x$ in ${}_F W_{\text{aff}}$ is the stabiliser in ${}_F W'$ of x.

Of course, there is an analogous statement, which we will denote by $(**_{L'})$, when F is replaced by L'.

By Proposition 4.6.28 of [7], $N_G(T) \cap gG_x \neq \emptyset$. We may, and hence do, replace g by an element of this intersection. Then write $w(g)$ for the image of g in ${}_F W_{\mathrm{aff}}$, and let $\tau \in {}_F\Lambda$ be such that $w(g) \in \tau \cdot {}_F W'$. Then $w(g)^n \in \tau^n \cdot {}_F W'$. Since $g^n \in N_G(T) \cap G_x$, we have by $(**_F)$ that $w(g)^n \in {}_F W'$. Thus, $\tau^n \in {}_F W'$. By $(*)$, we have that $\tau \in {}_{L'} W'$, so

$(***)$ $w(g) \in \tau \cdot {}_F W' \subseteq {}_{L'} W'$.

Since $w(g)$ stabilises x, we have by $(**_{L'})$ and $(***)$ that it belongs to the image of $N_{\mathbf{G}}(\mathbf{T})(L') \cap \mathbf{G}(L')_x$ in ${}_{L'} W_{\mathrm{aff}}$. That is, $g \in (N_{\mathbf{G}}(\mathbf{T})(L') \cap \mathbf{G}(L')_x)\mathbf{T}(L')_{\mathrm{b}}$. In particular, $\mathbf{G}(L')_x \cdot g$ contains an element of $\mathbf{T}(L')$, say t. Then $t^n \in \mathbf{G}(L')_x$. By Lemma 2.6 of [2], we have that $t^n \in \mathbf{T}(L')_0$. Now we imitate the proof of Lemma 2.4 of [2] to show that there is a finite tame extension L/L' such that $t \in \mathbf{T}(L)_0$. Denote by M a totally ramified extension of L'^{un} of degree n. Then, in the notation of [17, Section 7.3] (except that our M and L' are Kottwitz's L' and L, respectively; so β is the inclusion of $\mathbf{T}(L')$ in $\mathbf{T}(M)$), we have by (7.3.2) of [7] that

$$\alpha\bigl(w_{\mathbf{T}(M)}\bigl(\beta(t)\bigr)\bigr) = \alpha\bigl(N\bigl(w_{\mathbf{T}(L')}(t)\bigr)\bigr) = n w_{\mathbf{T}(L')}(t) = w_{\mathbf{T}(L')}\bigl(t^n\bigr).$$

By Lemma 2.3 of [21], $\mathbf{T}(L')_0 = \ker w_{\mathbf{T}(L')}$ and $\mathbf{T}(M)_0 = \ker w_{\mathbf{T}(M)}$. In particular, $\alpha(w_{\mathbf{T}(M)}(\beta(t))) = 0$, so, since α is an injection, $t = \beta(t) \in \mathbf{T}(M)_0$. Now let L/L' be any finite subextension of M/L' such that M/L is unramified. Then $t \in \mathbf{T}(M)_0^{\mathrm{Gal}(M/L)} = \mathbf{T}(L)_0$.

By Lemma 2.6 of [2], we have that $t \in \mathbf{G}(L)_x$, so $g \in \mathbf{G}(L)_x \cdot t = \mathbf{G}(L)_x$. □

2.8. Existence of topological Jordan decompositions

The following two results show that the answers are "yes" to the analogues of the questions posed in [18, Sections 5.7 and 5.10], where semisimplicity and unipotence are replaced by absolute F-semisimplicity and topological F-unipotence. We must impose at first a somewhat artificial tameness hypothesis, but Corollary 2.37 below will show that it can be omitted.

Proposition 2.33. *If γ is absolutely semisimple and F-tame, then $\mathcal{B}(C_{\mathbf{G}}(\gamma), F) = \{x \in \mathcal{B}(\mathbf{G}, F) \mid \gamma \cdot x = x\}$.*

Proof. Denote the right-hand set above by $\mathcal{B}(\gamma)$. By Proposition 3.4 of [2], we have that $C_{\mathbf{G}}(\gamma)$ is a compatibly filtered F-subgroup of $\mathbf{G}$, in the sense of Definition 3.3 of [2]. In particular, $\mathcal{B}(C_{\mathbf{G}}(\gamma), E)$ may be regarded non-canonically as a subset of $\mathcal{B}(\mathbf{G}, E)$ for all discretely valued tame extensions E/F, so that the statement makes sense. Since $\mathcal{B}(C_{\mathbf{G}}(\gamma), F) = \mathcal{B}(C_{\mathbf{G}}(\gamma), E)^{\mathrm{Gal}(E/F)}$ for any discretely valued, tame, Galois extension E/F, we may, and hence do, assume that F is strictly Henselian (hence that $\mathbf{G}$ is F-quasisplit) and that γ belongs to a maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$. Since γ is bounded (by Remark 2.17) and $S_{\mathrm{b}} = S_0$ (by Remark 2.11), we have $\gamma \in S_0 \subseteq G_0$ and $\mathcal{B}(C_{\mathbf{G}}(\gamma), F) \subseteq \mathcal{B}(\gamma)$.

Suppose that $x \in \mathcal{B}(C_{\mathbf{G}}(\gamma), F)$, and $y \in \mathcal{B}(\gamma)$ lies in a facet of $\mathcal{B}(\mathbf{G}, F)$ whose closure contains x. Denote by $g \mapsto \bar{g}$ the reduction map $G_x \to \mathsf{G}_x^\circ(\mathfrak{f})$.

We have that $G_x^+ \subseteq G_y^+ \subseteq G_y \subseteq G_x$, and the images in $\mathsf{G}_x^\circ(\mathfrak{f})$ of G_y and G_y^+ are the groups of $\mathfrak{f}$-points of a parabolic $\mathfrak{f}$-subgroup P_y and of its unipotent radical U_y, respectively. Let T be the $\mathfrak{f}$-split maximal torus in G_x° such that the image of S_0 in $\mathsf{G}_x^\circ(\mathfrak{f})$ is $\mathsf{T}(\mathfrak{f})$. By Lemma 2.9 of [2], we have that $\gamma \in G_y \subseteq G_x$, so $\bar{\gamma} \in \mathsf{T}(\mathfrak{f}) \cap \mathsf{P}_y(\mathfrak{f})$ is semisimple. Thus it lies in a maximal $\mathfrak{f}$-torus T' of P_y.

We claim that there is an F-split torus $\mathbf{S}'$ in $C_{\mathbf{G}}(\gamma)^\circ$ such that x lies in the apartment of $\mathbf{S}'$ and the image of S'_0 in $\mathsf{G}^\circ_x(\mathfrak{f})$ is $\mathsf{T}'(\mathfrak{f})$. Indeed, since $\mathbf{S}$ is F-split, there is an isomorphism $i : \mathbf{X}^*(\mathbf{S}) \to \mathbf{X}^*(\mathsf{T})$ such that, for all $\chi \in \mathbf{X}^*(\mathbf{S})$, the image in $\mathfrak{f}^\times$ of $\chi(\gamma) \in F_0^\times$ is $i(\chi)(\bar\gamma)$. By Proposition 3.5.4 of [24], $C_{\mathsf{G}_x}(\bar\gamma)^\circ(\mathfrak{f})$ is generated by $\mathsf{T}(\mathfrak{f})$ and the $\mathfrak{f}$-points of those root subgroups corresponding to roots of T in G°_x that vanish at $\bar\gamma$. Let $\bar\alpha$ be such a root. By Corollaire 4.6.12(i) of [7] (applied to the function $f = f'_\Omega$ occurring in the proof of Lemma 2.32), $\alpha := i^{-1}(\bar\alpha)$ is a root of $\mathbf{S}$ in $\mathbf{G}$. Let $U \subseteq U_\alpha \cap G_x$ be the affine root subgroup of G that maps onto the $\mathfrak{f}$-points of the root subgroup $\mathsf{U}_{\bar\alpha}$ of G°_x. Since the image of $\alpha(\gamma)$ in $\mathfrak{f}^\times$ is $\bar\alpha(\bar\gamma) = 1$, we have that $\alpha(\gamma) \in \mathcal{F}(F) \cap F_{0+}^\times = \{1\}$. By Proposition 3.5.4 of [24], α is a root of $\mathbf{S}$ in $C_{\mathbf{G}}(\gamma)^\circ$, so $U \subseteq C_G(\gamma)^\circ \cap G_x$. That is, the image in $\mathsf{G}^\circ_x(\mathfrak{f})$ of $C_G(\gamma)^\circ \cap G_x$ includes $C_{\mathsf{G}_x}(\bar\gamma)^\circ(\mathfrak{f})$. (Although we do not need to do so here, one can show that the image is precisely $C_{\mathsf{G}^\circ_x}(\bar\gamma)(\mathfrak{f})$.) Since T and T' are maximal $\mathfrak{f}$-tori in $C_{\mathsf{G}_x}(\bar\gamma)^\circ$ and $\mathfrak{f}$ is algebraically closed, there is an element $\bar c \in C_{\mathsf{G}_x}(\bar\gamma)^\circ(\mathfrak{f})$ such that $\mathsf{T}' = {}^{\bar c}\mathsf{T}$. Let $c \in C_G(\gamma)^\circ \cap G_x$ be an element whose image in $\mathsf{G}^\circ_x(\mathfrak{f})$ is $\bar c$. Then $\mathbf{S}' := {}^c\mathbf{S}$ certainly contains x in its apartment, and has the property that the image of S'_0 in $\mathsf{G}^\circ_x(\mathfrak{f})$ is $\mathsf{T}'(\mathfrak{f})$.

Note that we may, and hence do, also regard T' as a torus in $\mathsf{P}_y/\mathsf{U}_y = \mathsf{G}^\circ_y$. By Proposition 5.1.10 of [7], there is an F-split torus $\mathbf{S}''$ in $\mathbf{G}$ such that the apartment of $\mathbf{S}''$ contains y and the image of S''_0 in $\mathsf{G}^\circ_y(\mathfrak{f})$ is $\mathsf{T}'(\mathfrak{f})$. Since y lies in a facet whose closure contains x, the apartment of $\mathbf{S}''$ also contains x. By Proposition 4.6.28(iii) of [7], there is an element $k \in G_x$ such that $\mathbf{S}'' = {}^k\mathbf{S}'$. Since S''_0 and S'_0 have the same image, namely $\mathsf{T}'(\mathfrak{f})$, in $\mathsf{G}^\circ_x(\mathfrak{f})$, we have that $\bar k \in N_{\mathsf{G}^\circ_x}(\mathsf{T}')(\mathfrak{f})$. As in the proof of Lemma 2.32, one sees from Corollaire 4.6.12(ii) of [7] that $\bar k$ lies in the image in $\mathsf{G}^\circ_x(\mathfrak{f})$ of $N_G(S') \cap G_x$. Thus, there are $k_+ \in G_x^+ \subseteq G_y^+$ and $n \in N_G(S') \cap G_x$ such that $k = k_+ n$. Then $y = k_+^{-1} y$ belongs to the apartment of ${}^{k_+^{-1}}\mathbf{S}'' = {}^n\mathbf{S}' = \mathbf{S}'$, hence is contained in $\mathcal{B}(C_{\mathbf{G}}(\gamma), F)$.

We have shown that $\mathcal{B}(C_{\mathbf{G}}(\gamma), F)$ is open in $\mathcal{B}(\gamma)$. Since it is a union of apartments, it is also closed there. Since $\mathcal{B}(\gamma)$ is connected (even convex), and since $\mathcal{B}(C_{\mathbf{G}}(\gamma), F)$ is non-empty, we have the desired equality. □

Lemma 2.34. *Suppose that*

- $x \in \mathcal{B}(\mathbf{G}, F)$,
- $\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}} \in \operatorname{stab}_G(x)$ *are absolutely F-semisimple and topologically F-unipotent, respectively, and*
- γ_{as} *is F-tame.*

Then the images of γ_{as} and γ_{tu} in $\mathsf{G}_x(\mathfrak{f})$ are semisimple and unipotent, respectively.

Proof. We first show that the image of γ_{tu} is unipotent. If $p > 0$, then we have by Lemma 2.21 that γ_{tu} is topologically p-unipotent, hence that the image of γ_{tu} in $\mathsf{G}_x(\mathfrak{f})$ has p-power order. By [4, Section 4.1(a)], it is unipotent.

If $p = 0$, then let E/F be a finite extension such that $\gamma_{\mathrm{tu}} \in \mathbf{G}(E)_{0+}$. Since E/F is tame, we have by Lemma 2.7 of [2] that $\gamma_{\mathrm{tu}} \in G_{0+}$. Choose $z \in \mathcal{B}(\mathbf{G}, F)$ such that $\gamma_{\mathrm{tu}} \in G_z^+$. By Lemma 2.9 of [2], $\gamma_{\mathrm{tu}} \in G_x$. By Lemma 2.8 of [2], there is a point $y \in (x, z)$ such that $\gamma_{\mathrm{tu}} \in G_y^+$ and y belongs to a facet of $\mathcal{B}(\mathbf{G}, F)$ whose closure contains x. Then the image of G_y^+ in G_x is the group of $\mathfrak{f}$-points of the unipotent radical of a parabolic $\mathfrak{f}$-subgroup of G°_x. In particular, the image of γ_{tu} is unipotent.

Now we show that the image of γ_{as} is semisimple. Let L/F^{un} be a finite tame extension such that γ_{as} belongs to an L-split torus. By Lemma 2.5 of [2], $\mathbf{G}(L)_x^+ \cap \mathbf{G}(F^{un}) = \mathbf{G}(F^{un})_x^+$, so

$$\mathsf{G}_x(\bar{\mathfrak{f}}) = \mathrm{stab}_{\mathbf{G}(F^{un})}(x)/\mathbf{G}\big(F^{un}\big)_x^+ \subseteq \mathrm{stab}_{\mathbf{G}(L)}(x)/\mathbf{G}(L)_x^+ = \mathsf{G}_x^L(\bar{\mathfrak{f}});$$

that is, G_x is an $\bar{\mathfrak{f}}$-subgroup of G_x^L. Thus we may, and hence do, assume, upon replacing F by L, that γ_{as} belongs to an F-split torus. By Proposition 2.33, we have that there is a maximal F-split torus $\mathbf{S}$ whose apartment contains x such that $\gamma_{as} \in S$. Since γ_{as} is bounded (by Remark 2.17) and $S_b = S_0$ (by Remark 2.11), we have $\gamma_{as} \in S_0$. Then the image of γ_{as} in $\mathsf{G}_x(\mathfrak{f})$ belongs to the image of S_0 there, which is the group of $\mathfrak{f}$-rational points of an $\mathfrak{f}$-torus. □

Lemma 2.35. *If $\gamma = \gamma_{as}\gamma_{tu}$ is a topological F-Jordan decomposition, then a point x of $\mathcal{B}(\mathbf{G}, F)$ is fixed by γ if and only if it is fixed by γ_{as} and γ_{tu}.*

Proof. The 'if' direction is obvious, so we need only prove the 'only if' direction. By Remark 2.22, it suffices to prove this result over any finite extension of F; so we may, and hence do, assume that γ_{as} belongs to an F-split maximal torus in $\mathbf{G}$.

Denote by $\mathcal{B}(\gamma)$ the fixed points of γ, and similarly for γ_{as} and γ_{tu}. Suppose that $x \in \mathcal{B}(\gamma_{as}) \cap \mathcal{B}(\gamma_{tu}) \subseteq \mathcal{B}(\gamma)$, and $y \in \mathcal{B}(\gamma)$ belongs to a facet whose closure contains x. Denote by $g \mapsto \bar{g}$ the reduction map $G_x \to \mathsf{G}_x^\circ(\mathfrak{f})$.

The image of G_y in $\mathsf{G}_x^\circ(\mathfrak{f})$ is the group of $\mathfrak{f}$-points of a parabolic $\mathfrak{f}$-subgroup P_y of G_x°. Since γ normalises G_y, $\bar{\gamma}$ normalises $\mathsf{P}_y(\mathfrak{f})$; so, by Theorem 11.16 of [4], $\bar{\gamma} \in \mathsf{P}_y(\mathfrak{f})$. Then also $(\bar{\gamma})_{ss} \in \mathsf{P}_y(\mathfrak{f})$ and $(\bar{\gamma})_{un} \in \mathsf{P}_y(\mathfrak{f})$. By Lemma 2.34, $(\bar{\gamma})_{ss} = \overline{\gamma_{as}}$ and $(\bar{\gamma})_{un} = \overline{\gamma_{tu}}$. Since the preimage of $\mathsf{P}_y(\mathfrak{f})$ in G_x is G_y, we have that γ_{as} and γ_{tu} lie in G_y. In particular, $y \in \mathcal{B}(\gamma_{as}) \cap \mathcal{B}(\gamma_{tu})$. That is, $\mathcal{B}(\gamma_{as}) \cap \mathcal{B}(\gamma_{tu})$ is open in $\mathcal{B}(\gamma)$. Since it is also closed, and since $\mathcal{B}(\gamma)$ is connected (even convex), we have equality, as desired. □

Now we are in a position to prove an existence result for topological F-Jordan decompositions analogous to Proposition 1.8. A more refined version of this result appears as Theorem 2.38 below.

Proposition 2.36. *An element $\gamma \in G$ has a topological F-Jordan decomposition $\gamma = \gamma_{as}\gamma_{tu}$ if and only if it is bounded. In this case, γ_{as} is F-tame.*

Proof. Suppose that $\gamma = \gamma_{as}\gamma_{tu}$ is a topological F-Jordan decomposition. By Remark 2.17, γ_{as} and γ_{tu} are bounded. By Lemma 2.18, there is a point $x \in \mathcal{B}(\mathbf{G}, F)$ fixed by both, so $\gamma \cdot x = x$. By Remark 2.9, γ is bounded.

Now suppose that γ is bounded. By Remark 2.22 and Lemma 2.26, we may, and hence do, replace F by discretely valued tame extensions as necessary. In particular, we will assume throughout that $F = F^{un}$. Put $\mathbf{H} = C_{\mathbf{G}}(\gamma_{ss})^\circ$.

If $p > 0$, then, as in Remark 2.20, let $a \in \mathbb{Z}_{\geqslant 0}$ be so large that γ^{p^a} and $\gamma_{un}^{p^a}$ belong to H. Let E/F be a finite separable extension such that $\mathbf{H}$ is E-split. By Proposition 3.4 of [2], $\mathbf{H}$ is a compatibly filtered E-subgroup of $\mathbf{G}$, in the sense of Definition 3.3 of [2]. In particular, the building of $\mathcal{B}(\mathbf{H}, E)$ may be embedded isometrically and γ^{p^a}-equivariantly into $\mathcal{B}(\mathbf{G}, E)$. Thus the orbits of γ^{p^a} in $\mathcal{B}(\mathbf{H}, E)$, hence in $\mathcal{B}(\mathbf{H}, F)$, are bounded; that is, γ^{p^a} is bounded (in H). By Remark 2.9, there is a point $x \in \mathcal{B}(\mathbf{H}, F)$ fixed by γ^{p^a}. Denote by $h \mapsto \bar{h}$ the reduction map $\operatorname{stab}_H(x) \to \mathsf{H}_x(\mathfrak{f})$. Let $b \in \mathbb{Z}_{\geqslant 0}$ be so large that the order of $\gamma^{p^{a+b}} \in \operatorname{stab}_H(x)$ modulo H_x is indivisible by p. By Lemma 2.32, we may, and hence do, assume, upon replacing F by a finite tame extension, that $\gamma^{p^{a+b}} \in H_x$. By Remark 2.13 and Lemma 2.21, $\gamma_{\mathrm{un}}^{p^{a+b}}$ is topologically p-unipotent. Let $c \in \mathbb{Z}_{\geqslant 0}$ be so large that $\gamma_{\mathrm{un}}^{p^{a+b+c}} \in H_x^+$, and put $n = a + b + c$. Since $\gamma_{\mathrm{ss}}^{p^n} \in Z(H)$ and $\mathfrak{f}$ is algebraically closed, we have that $\overline{\gamma_{\mathrm{ss}}^{p^n}} \in Z(\mathsf{H}_x^\circ(\mathfrak{f})) = Z(\mathsf{H}_x^\circ)(\mathfrak{f})$. Let T be a maximal $\mathfrak{f}$-torus (necessarily $\mathfrak{f}$-split) in H_x°, so that $\overline{\gamma_{\mathrm{ss}}^{p^n}} \in \mathsf{T}(\mathfrak{f})$. By Proposition 5.1.10 of [7], there exists a maximal F-split torus $\mathbf{S}$ in $\mathbf{H}$ such that x is contained in the apartment of $\mathbf{S}$, and the image of S_0 in $\mathsf{H}_x^\circ(\mathfrak{f})$ is $\mathsf{T}(\mathfrak{f})$. Let δ be a preimage in S_0 of $\overline{\gamma_{\mathrm{ss}}^{p^n}}$, so that $\delta^{-1}\gamma_{\mathrm{ss}}^{p^n} \in H_x^+$. For $\chi \in \mathbf{X}^*(\mathbf{S})$, let s'_χ be the unique element of $\mathcal{F}(F)$ such that $\chi(\delta) \equiv s'_\chi \pmod{F_{0+}^\times}$, and s_χ the unique element of $\mathcal{F}(F)$ such that $s_\chi^{p^n} = s'_\chi$. Finally, let γ_{as} be the unique element of S such that $\chi(\gamma_{\mathrm{as}}) = s_\chi$ for all $\chi \in \mathbf{X}^*(\mathbf{S})$. Clearly, γ_{as} is absolutely F-semisimple and F-tame (even F-split, in the obvious language). Put $(\gamma_{\mathrm{ss}})_{\mathrm{tu}} := \gamma_{\mathrm{as}}^{-1}\gamma_{\mathrm{ss}}$. By Remark 2.11, $\gamma_{\mathrm{as}}^{-p^n}\delta \in S_{0+} \subseteq H_x^+$. Thus $(\gamma_{\mathrm{ss}})_{\mathrm{tu}}^{p^n} = \gamma_{\mathrm{as}}^{-p^n}\gamma_{\mathrm{ss}}^{p^n} \in H_x^+$, so $(\gamma_{\mathrm{ss}})_{\mathrm{tu}}$ is topologically p-unipotent. By Lemma 2.21, $(\gamma_{\mathrm{ss}})_{\mathrm{tu}}$ is topologically K-unipotent (where K/F is a finite extension such that $\gamma_{\mathrm{ss}} \in \mathbf{H}(K)$). Thus, $\gamma_{\mathrm{ss}} = \gamma_{\mathrm{as}}(\gamma_{\mathrm{ss}})_{\mathrm{tu}}$ is a topological K-Jordan decomposition. By Lemma 2.25, γ_{as} and $(\gamma_{\mathrm{ss}})_{\mathrm{tu}}$ commute with $C_{\mathbf{G}(K)}(\gamma_{\mathrm{ss}})$; in particular, with γ_{un}. Put $\gamma_{\mathrm{tu}} := \gamma_{\mathrm{as}}^{-1}\gamma = (\gamma_{\mathrm{ss}})_{\mathrm{tu}}\gamma_{\mathrm{un}} \in G$. Since $\gamma_{\mathrm{tu}}^{p^n} = (\gamma_{\mathrm{ss}})_{\mathrm{tu}}^{p^n}\gamma_{\mathrm{un}}^{p^n} \in H_x^+$, we have that γ_{tu} is topologically p-unipotent, hence, by another application of Lemma 2.21, topologically F-unipotent. Thus, $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is the desired topological F-Jordan decomposition.

If $p = 0$, then $\gamma_{\mathrm{un}} \in H$. By Remark 2.13, we may, and hence do, assume, upon replacing F by a finite (necessarily tame) extension, that $\gamma_{\mathrm{un}} \in H_{0+}$ and $\mathbf{H}$ is F-split. By Lemma 2.18, there is a point $x \in \mathcal{B}(\mathbf{H}, F)$ such that $\gamma \cdot x = x$ and $\gamma_{\mathrm{un}} \in H_x^+$. Let $\mathbf{T}$ be an F-split maximal torus in $\mathbf{H}$ whose apartment contains x. Then $\gamma_{\mathrm{ss}} \in T$ fixes x, hence is bounded. By Remark 2.11, the character values of γ_{ss} lie in $F_0^\times$. For $\chi \in \mathbf{X}^*(\mathbf{T})$, let s_χ be the unique element of $\mathcal{F}(F)$ such that $\chi(\gamma_{\mathrm{ss}}) \equiv s_\chi \pmod{F_{0+}^\times}$. In particular, $s_\alpha = 1$ for all roots α of $\mathbf{T}$ in $\mathbf{H}$. Let γ_{as} be the unique element of T such that $\chi(\gamma_{\mathrm{as}}) = s_\chi$ for all $\chi \in \mathbf{X}^*(\mathbf{T})$. In particular, $\alpha(\gamma_{\mathrm{as}}) = 1$ for all roots α of $\mathbf{T}$ in $\mathbf{H}$, so $\gamma_{\mathrm{as}} \in Z(H)$. Clearly, γ_{as} is F-tame and absolutely F-semisimple, and belongs to an $\overline{F}$-torus containing γ_{ss}. Moreover, by Remark 2.11, $\gamma_{\mathrm{as}}^{-1}\gamma_{\mathrm{ss}} \in T_0^+ \subseteq H_x^+$. Thus $\gamma_{\mathrm{tu}} := \gamma_{\mathrm{as}}^{-1}\gamma = (\gamma_{\mathrm{as}}^{-1}\gamma_{\mathrm{ss}})\gamma_{\mathrm{un}} \in H_x^+$. By Proposition 3.4 of [2], $\mathbf{H}$ is a compatibly filtered F-subgroup of $\mathbf{G}$, in the sense of Definition 3.3 of [2]. In particular, we may regard x (non-canonically) as a point of $\mathcal{B}(\mathbf{G}, F)$. Then $H_x^+ \subseteq G_x^+$, so γ_{tu} is topologically F-unipotent (in G). Clearly, $\gamma_{\mathrm{tu}} \in H$ commutes with $\gamma_{\mathrm{as}} \in Z(H)$. Thus, $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is the desired topological F-Jordan decomposition. □

Now we show that we can drop the tameness hypotheses of Proposition 2.33 and Lemma 2.34.

Corollary 2.37. *If $\gamma \in G$ is absolutely F-semisimple, then it is F-tame.*

Proof. By Remark 2.17, γ is bounded. By Proposition 2.36, there is a topological F-Jordan decomposition $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ with γ_{as} F-tame. By Proposition 2.24, $\gamma = \gamma_{\mathrm{as}}$. □

The following rather technical result, which is now an immediate consequence of Lemmata 2.34 and 2.35 and Proposition 2.36, is really the heart of the paper. It should be viewed as a quite precise existence result about topological F-Jordan decompositions.

Theorem 2.38. *If $x \in \mathcal{B}(\mathbf{G}, F)$ and $\gamma \in \operatorname{stab}_G(x)$, then there is a topological F-Jordan decomposition $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ such that γ_{as} and γ_{tu} project to the semisimple and unipotent parts, respectively, of the image of γ in $\mathsf{G}_x(\mathfrak{f})$.*

Remark 2.39. If F is an algebraic extension of a locally compact field, then the proof of Theorem 2.38 can be considerably simplified. Indeed, in this case G is ind-locally-pro-p, by Remark 2.12; so, by Propositions 1.7(2) and 1.8, an element $\gamma \in \operatorname{stab}_G(x)$ has a topological p-, hence F-, Jordan decomposition $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ with $\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}} \in \operatorname{stab}_G(x)$. Now Lemma 2.34 shows that the images of γ_{as} and γ_{tu} in $\mathsf{G}_x(\mathfrak{f})$ are as desired.

Corollary 2.40. *For $x \in \mathcal{B}(\mathbf{G}, F)$, any normal subgroup of G_x consisting entirely of topologically F-unipotent elements lies in G_x^+.*

Proof. Suppose that $H \subseteq G_x$ is normal and consists entirely of topologically F-unipotent elements. By Lemma 2.34, the image of H in $\mathsf{G}_x^\circ(\mathfrak{f})$ consists entirely of unipotent elements. Denote by H its Zariski closure in G_x°. Then H° is a connected, normal, unipotent subgroup of the reductive group G_x°, hence trivial. By Lemma 22.1 of [4], H is central in G_x°, hence consists entirely of semisimple elements. Since we have already observed that it consists entirely of unipotent elements, H is trivial. $\square$

We already have an existence result (Proposition 2.36) for topological F-Jordan decompositions modulo the trivial group. The next result handles such decompositions modulo any group $\mathbf{N}$, for some fields F.

Proposition 2.41. *Suppose that F is an algebraic extension of a locally compact field. Then the following statements about an element $\gamma \in G$ are equivalent.*

(1) *γ has a topological p-Jordan decomposition modulo N.*
(2) *γ has a topological F-Jordan decomposition modulo* $\mathbf{N}$.
(3) *γ is bounded modulo* $\mathbf{N}$.

Proof. By Corollary 2.29 and Lemma 2.30, a topological p-Jordan decomposition modulo N is a topological F-Jordan decomposition modulo $\mathbf{N}$, and conversely. Thus the equivalence (1) $\Leftrightarrow$ (2) is clear.

Denote by $g \mapsto \bar{g}$ the natural map $\mathbf{G} \to \widetilde{\mathbf{G}}$. By Remark 2.12, $\widetilde{G}$ is ind-locally-pro-p, so we have by Proposition 1.8 that $\bar{\gamma}$ has a topological p-Jordan decomposition if and only if $\bar{\gamma}$ is compact; that is, if and only if γ is compact modulo N (equivalently, by Remark 2.10, bounded modulo $\mathbf{N}$). Thus, to prove the equivalence (1) $\Leftrightarrow$ (3), it suffices to prove that γ has a topological p-Jordan decomposition modulo N if and only if $\bar{\gamma}$ has a topological p-Jordan decomposition.

The ‘only if’ direction is easy. For the ‘if’ direction, suppose that $\bar{\gamma}$ has a topological p-Jordan decomposition $\bar{\gamma} = (\bar{\gamma})_{\mathrm{as}}(\bar{\gamma})_{\mathrm{tu}}$. Denote by H the closure of the group generated by γ. By Remark 3.1 of [2], the image in $\widetilde{G}$ of H is closed. By Proposition 1.7(2), $(\bar{\gamma})_{\mathrm{as}}$ belongs to this image. Let γ_{as} be a preimage of $(\bar{\gamma})_{\mathrm{as}}$ in H. Then $\gamma_{\mathrm{tu}} := \gamma_{\mathrm{as}}^{-1}\gamma$ is a preimage of $(\bar{\gamma})_{\mathrm{tu}}$, and clearly γ_{as} and γ_{tu} commute. Thus, $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ is a topological p-Jordan decomposition modulo N. $\square$

We close by showing that the “common torus” condition of Definition 2.23 can be omitted.

Proposition 2.42. *Suppose that $\gamma \in G$ and $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ is a pair of commuting elements of G such that*

- $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$,
- γ_{as} *is absolutely F-semisimple modulo* $\mathbf{N}$*, and*
- γ_{tu} *is topologically F-unipotent modulo* $\mathbf{N}$.

Then $(\gamma_{\mathrm{as}}, \gamma_{\mathrm{tu}})$ is a topological F-Jordan decomposition modulo $\mathbf{N}$.

Proof. It remains only to show that the images of γ_{ss} and γ_{as} in $\widetilde{\mathbf{G}}(\overline{F})$ belong to a common $\overline{F}$-torus there. We will show the equivalent statement that the images of γ_{as} and $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$ belong to a common $\overline{F}$-torus. Clearly, it suffices to assume that $\mathbf{N}$ is the trivial subgroup, so we do so. By Remark 2.22, the hypotheses remain valid if we replace F by a finite extension, so we may, and hence do, make such replacements as necessary.

Upon replacing F by a finite extension, we may, and hence do, assume that γ_{ss} (hence $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$) lies in G and γ_{as} is F-split. By Lemma 2.19, $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$ is topologically F-unipotent. Thus, upon replacing F by a finite extension, we may, and hence do, assume that $(\gamma_{\mathrm{tu}})_{\mathrm{ss}} \in G_{0+}$. By Remark 2.17 and Lemma 2.18, there is an element $x \in \mathcal{B}(\mathbf{G}, F)$ such that $\gamma_{\mathrm{as}} \cdot x = x$ and $\gamma_{\mathrm{tu}} \in G_x^+$. By Proposition 3.4 of [2], $\mathbf{H} := C_{\mathbf{G}}(\gamma_{\mathrm{as}})$ is a compatibly filtered F-subgroup of $\mathbf{G}$, in the sense of Definition 3.3 of [2]. Thus, $\mathcal{B}(\mathbf{H}, F)$ may be regarded (non-canonically) as a subset of $\mathcal{B}(\mathbf{G}, F)$ in such a way that $G_z^+ \cap H = H_z^+$ for $z \in \mathcal{B}(\mathbf{H}, F)$. By Proposition 2.33, we have that $x \in \mathcal{B}(\mathbf{H}, F)$, so $(\gamma_{\mathrm{tu}})_{\mathrm{ss}} \in G_x^+ \cap H = H_x^+ \subseteq H^\circ$. In particular, $(\gamma_{\mathrm{tu}})_{\mathrm{ss}}$ belongs to some maximal F-torus $\mathbf{T}$ in $\mathbf{H}$. Since γ_{as} is central in $\mathbf{H}$, it also belongs to $\mathbf{T}$. □

2.9. Topological unipotence and tameness

We have already seen that an absolutely F-semisimple element is F-tame (see Corollary 2.37). Of course, a topologically F-unipotent element need not be F-tame (or even semisimple). In the next result, we see that, for F-tame topologically F-unipotent elements, it is not necessary to introduce the finite extension E/F of Definition 2.15.

Proposition 2.43. *The topologically F-unipotent part of a bounded and F-tame element belongs to G_{0+}.*

Proof. Let γ be a bounded and F-tame element. By Proposition 2.36, it has a topological F-Jordan decomposition $\gamma = \gamma_{\mathrm{as}}\gamma_{\mathrm{tu}}$ with γ_{as} F-tame. By Lemma 2.7 of [2], we may, and hence do, replace F by a finite tame extension so that $\mathbf{G}$ is F-quasisplit, and γ and γ_{as} belongs to F-split tori.

If $p = 0$, then let E/F be a finite extension such that $\gamma_{\mathrm{tu}} \in \mathbf{G}(E)_{0+}$. By another application of Lemma 2.7 of [2], $\gamma_{\mathrm{tu}} \in G_{0+}$.

If $p > 0$, then let

- $\mathbf{S}$ be a maximal F-split torus containing γ,
- $\mathbf{T}$ the maximal torus containing $\mathbf{S}$, and
- E/F the splitting field of $\mathbf{T}$.

By Lemma 2.25, $\gamma_{\mathrm{as}} \in Z(C_G(\gamma)) \subseteq T$. Thus, $\mathbf{S}$ commutes with γ_{as}, hence is a maximal F-split torus in $C_{\mathbf{G}}(\gamma_{\mathrm{as}})^\circ$. By Lemma A.2 of [2], we have $\gamma_{\mathrm{as}} \in S$. Therefore $\gamma_{\mathrm{tu}} \in S$ also. Since γ_{tu} is bounded (by Remark 2.17) and $S_{\mathrm{b}} = S_0$ (by Remark 2.11), we have $\gamma_{\mathrm{tu}} \in S_0$. By Lemma 2.21, γ_{tu} is topologically p-unipotent, hence has p-power order modulo S_{0+}. On the other hand, $\mathsf{S}(\mathfrak{f}) = S_0/S_{0+}$ is the group of $\mathfrak{f}$-rational points of an $\mathfrak{f}$-split torus, hence contains no non-trivial elements of p-power order. That is, $\gamma_{\mathrm{tu}} \in S_{0+}$. By Lemma 2.4 of [21], we have $S_0 \subseteq T_0$. By Remark 2.11, we have $\mathbf{S}(E)_{0+} \subseteq \mathbf{T}(E)_{0+}$. By Lemmata 2.4 and 2.6 of [2],

$$S_{0+} = S_0 \cap \mathbf{S}(E)_{0+} \subseteq T_0 \cap \mathbf{T}(E)_{0+} = T_{0+} \subseteq G_{0+}.$$

In particular, $\gamma_{\mathrm{tu}} \in G_{0+}$, as desired. $\square$

2.10. *Lifting*

In this subsection, put $\widetilde{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})^\circ$. (This is consistent with the notation in the earlier part of the paper, as long as we take $\mathbf{N} = Z(\mathbf{G})^\circ$.) Denote by $g \mapsto \bar{g}$ the natural map $\mathbf{G} \to \widetilde{\mathbf{G}}$.

We show that elements of $\widetilde{G}$ which are absolutely F-semisimple or topologically F-unipotent can be lifted, upon passing to suitable finite extensions E/F, to elements of $\mathbf{G}(E)$ which are absolutely E-semisimple or topologically E-unipotent, respectively.

Proposition 2.44. *If $\gamma \in G$ is absolutely F-semisimple modulo $Z(\mathbf{G})^\circ$, then there is a finite separable extension E/F such that $\gamma Z(\mathbf{G})^\circ(E)$ contains an absolutely E-semisimple element.*

Proof. Since $\bar{\gamma}$ is semisimple, the unipotent part of γ lies in $Z(\mathbf{G})^\circ(\overline{F})$, hence is trivial. That is, γ is semisimple. Let

- $\mathbf{T}$ be an F-torus such that $\gamma \in T$,
- E/F the splitting field of $\mathbf{T}$, and
- e_γ the homomorphism $\mathbf{X}^*(\mathbf{T}) \to \operatorname{ord}(E^\times)$ sending $\chi \in \mathbf{X}^*(\mathbf{T})$ to $\operatorname{ord}(\chi(\gamma))$.

By Remark 2.17, γ is bounded modulo $Z(\mathbf{G})^\circ$, so e_γ is trivial on $\widetilde{\mathbf{Y}}^* := \mathbf{X}^*(\mathbf{T}/(Z(\mathbf{G})^\circ \cap \mathbf{T}))$. Since $\operatorname{ord}(E^\times)$ is torsion-free and $\mathbf{Y}^* := \mathbf{X}^*(\mathbf{T}/(Z(\mathbf{G}) \cap \mathbf{T})^\circ)$ has finite index in $\widetilde{\mathbf{Y}}^*$, also e_γ is trivial on $\mathbf{Y}^*$, hence induces a homomorphism λ from $\mathbf{X}^*(\mathbf{T})/\mathbf{Y}^* \cong \mathbf{X}^*((Z(\mathbf{G}) \cap \mathbf{T})^\circ)$ to $\operatorname{ord}(E^\times)$. By choosing a uniformiser for E, hence an isomorphism $\operatorname{ord}(E^\times) \cong \mathbb{Z}$, we may, and hence do, regard λ as an element of $\mathbf{X}_*((Z(\mathbf{G}) \cap \mathbf{T})^\circ)$. Denote by $z \in Z(\mathbf{G})^\circ$ the value of λ at the chosen uniformiser, so that $\operatorname{ord}(\chi(z)) = e_\gamma(\chi) = \operatorname{ord}(\chi(\gamma))$ for all $\chi \in \mathbf{X}^*(\mathbf{T})$. Then $\delta := \gamma z^{-1} \in \mathbf{T}(E)$ is bounded.

By Proposition 2.36, there exists a topological E-Jordan decomposition $\delta = \delta_{\mathrm{as}}\delta_{\mathrm{tu}}$. Notice that $\overline{\delta_{\mathrm{as}}}$ and $\overline{\delta_{\mathrm{tu}}}$ belong to a common $\overline{F}$-torus (namely, the image in $\widetilde{\mathbf{G}}$ of any $\overline{F}$-torus in $\mathbf{G}$ containing both δ_{as} and δ). Clearly, $\overline{\delta_{\mathrm{as}}}$ is absolutely E-semisimple. As in the proof of Proposition 2.24, the character values of δ_{tu} lie in $K_{0+}^\times$ for some finite extension K/E. The character values of $\overline{\delta_{\mathrm{tu}}}$, being a subset of those of δ_{tu}, thus also belong to $K_{0+}^\times$. By replacing K by a further finite (separable) extension if necessary, we may, and hence do, assume that $\mathbf{T}/Z(\mathbf{G})^\circ$ is K-split, so that Remark 2.11 gives $\overline{\delta_{\mathrm{tu}}} \in (\mathbf{T}/Z(\mathbf{G})^\circ)(K)_{0+}$. By Lemma 2.8 of [2], we have that $\overline{\delta_{\mathrm{tu}}} \in \widetilde{\mathbf{G}}(K)_{0+}$, so $\overline{\delta_{\mathrm{tu}}}$ is topologically E-unipotent. That is, $\bar{\gamma} = \bar{\delta} = \overline{\delta_{\mathrm{as}}} \cdot \overline{\delta_{\mathrm{tu}}}$ is a topological E-Jordan decomposition of $\bar{\gamma}$. By Proposition 2.24, $\overline{\delta_{\mathrm{tu}}} = \bar{1}$, so $\delta_{\mathrm{tu}} \in Z(\mathbf{G})^\circ(E)$. Thus $\gamma Z(\mathbf{G})^\circ(E)$ contains an absolutely E-semisimple element, namely $\delta_{\mathrm{as}} = \gamma z^{-1}\delta_{\mathrm{tu}}^{-1}$, as desired. $\square$

Remark 2.45. The field E/F occurring in Proposition 2.44 may be taken to be the splitting field for any F-torus containing γ. In particular, if γ is F-tame, then E/F may be chosen to be tame. (Notice that Corollary 2.37 only guarantees that $\overline{\gamma}$, not γ itself, is F-tame.) We do not know an equally satisfactory answer to when the field extension E/F in the next proposition may be taken to be tame.

Proposition 2.46. *If $\gamma \in G$ is topologically F-unipotent modulo $Z(\mathbf{G})^\circ$, then there is a finite extension E/F such that $\gamma Z(\mathbf{G})^\circ(E)$ contains a topologically E-unipotent element.*

Proof. By Remark 2.22, we may, and hence do, replace F by a finite extension so that $\mathbf{G}$ is F-split and $\overline{\gamma} \in \widetilde{G}_{0+}$; say $x \in \mathcal{B}(\mathbf{G}, F)$ is such that $\gamma \in \widetilde{G}_{\bar{x}}^+$ (where $\bar{x}$ is the image of x in $\mathcal{B}^{\mathrm{red}}(\mathbf{G}, F) = \mathcal{B}(\widetilde{\mathbf{G}}, F)$), and $\mathbf{T}$ is an F-split maximal torus in $\mathbf{G}$ whose apartment contains x.

It suffices to show that the image of G_x^+ under the natural map $G \to \widetilde{G}$ includes $\widetilde{G}_{\bar{x}}^+$. By Remark 2.1 of [2], since the affine root subgroups of $\widetilde{\mathbf{G}}$ are naturally isomorphic to those of $\mathbf{G}$, it suffices to show that the image of T_{0+} includes $\widetilde{T}_{0+}$ (where $\widetilde{\mathbf{T}} := \mathbf{T}/Z(\mathbf{G})^\circ$). The following square commutes:

$$\begin{array}{ccc} T_{0+} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}^*(\mathbf{T}), F_{0+}^\times) \\ \big\downarrow & & \big\downarrow \\ \widetilde{T}_{0+} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}^*(\widetilde{\mathbf{T}}), F_{0+}^\times) \end{array}$$

(where the vertical maps are the obvious ones, the top horizontal map takes $t \in T_{0+}$ to the "evaluation at t" homomorphism, and the bottom horizontal map is the analogous map for $\widetilde{\mathbf{T}}$). By Remark 2.11, the top and bottom horizontal arrows are isomorphisms. The cokernel of the right-hand vertical map is $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbf{X}^*(Z(\mathbf{G})^\circ), F_{0+}^\times)$, which is trivial since $\mathbf{X}^*(Z(\mathbf{G})^\circ)$ is a free $\mathbb{Z}$-module. Thus the left-hand map is surjective, as desired. □

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