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On a subfactor generalization of Wall's conjecture

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ABSTRACT

In this paper we discuss a conjecture on intermediate subfactors which is a generalization of Wall's conjecture from the theory of finite groups. We explore special cases of this conjecture and present supporting evidence. In particular we prove special cases of this conjecture related to some finite dimensional Kac algebras of Izumi–Kosaki type which include relative version of Wall's conjecture for solvable groups.

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1. Introduction

Let M be a factor represented on a Hilbert space and N a subfactor of M which is irreducible, i.e., $N' \cap M = \mathbb{C}$. Let K be an intermediate von Neumann subalgebra for the inclusion $N \subset M$. Note that $K' \cap K \subset N' \cap M = \mathbb{C}$, K is automatically a factor. Hence the set of all intermediate subfactors for $N \subset M$ forms a lattice under two natural operations $\wedge$ and $\vee$ defined by:

$$K_1 \wedge K_2 = K_1 \cap K_2, \qquad K_1 \vee K_2 = (K_1 \cup K_2)''.$$

The commutant map $K \to K'$ maps an intermediate subfactor $N \subset K \subset M$ to $M' \subset K' \subset N'$. This map exchanges the two natural operations defined above.

Let $M \subset M_1$ be the Jones basic construction of $N \subset M$. Then $M \subset M_1$ is canonically isomorphic to $M' \subset N'$, and the lattice of intermediate subfactors for $N \subset M$ is related to the lattice of intermediate subfactors for $M \subset M_1$ by the commutant map defined as above.

Let G_1 be a group and G_2 be a subgroup of G_1. An interval sublattice $[G_1/G_2]$ is the lattice formed by all intermediate subgroups K, $G_2 \subseteq K \subseteq G_1$.

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By cross product construction and Galois correspondence, every interval sublattice of finite groups can be realized as intermediate subfactor lattice of finite index. Hence the study of intermediate subfactor lattice of finite index is a natural generalization of the study of interval sublattice of finite groups. The study of intermediate subfactors has been very active in recent years (cf. [4,10,18,17,16,20,28,26] for only a partial list). By a result of S. Popa (cf. [25]), if a subfactor $N \subset M$ is irreducible and has finite index, then the set of intermediate subfactors between N and M is finite. This result was also independently proved by Y. Watatani (cf. [28]). In [28], Y. Watatani investigated the question of which finite lattices can be realized as intermediate subfactor lattices. Related questions were further studied by P. Grossman and V.F.R. Jones in [10] under certain conditions. As emphasized in [10], even for a lattice consisting of six elements with shape a hexagon, it is not clear if it can be realized as intermediate subfactor lattice with finite index. This question has been solved recently by M. Aschbacher in [1] among other things. In [1], M. Aschbacher constructed a finite group G_1 with a subgroup G_2 such that the interval sublattice $[G_1/G_2]$ is a hexagon. The lattices that appear in [10,28,1] can all be realized as interval sublattice of finite groups. There are a number of old problems about interval sublattice of finite groups. It is therefore a natural programme to investigate if these old problems have any generalizations to subfactor setting. The hope is that maybe subfactor theory can provide new perspective on these old problems.

In [30] we consider the problem whether the very simple lattice M_n consisting of a largest, a smallest and n pairwise incomparable elements can be realized as subfactor lattice. We showed in [30] all M_{2n} are realized as the lattice of intermediate subfactors of a pair of hyperfinite type III_1 factors with finite depth. Since it is conjectured that infinitely many M_{2n} cannot be realized as interval sublattices of finite groups (cf. [3] and [24]), our result shows that if one is looking for obstructions for realizing finite lattice as lattice of intermediate subfactors with finite index, then the obstruction is very different from what one may find in finite group theory.

In 1961 G.E. Wall conjectured that the number of maximal subgroups of a finite group G is less than $|G|$, the order of G (cf. [27]). In the same paper he proved his conjecture when G is solvable. See [19] for more recent result on Wall's conjecture.

Wall's conjecture can be naturally generalized to a conjecture about maximal elements in the lattice of intermediate subfactors. What we mean by maximal elements are those subfactors $K \neq M$, N with the property that if K_1 is an intermediate subfactor and $K \subset K_1$, then $K_1 = M$ or K. Minimal elements are defined similarly where N is not considered as a minimal element. When M is the cross product of N by a finite group G, the maximal elements correspond to maximal subgroups of G, and the order of G is the dimension of second higher relative commutant. Hence a natural generalization of Wall's conjecture as proposed in [29] is the following:

Conjecture 1.1. *Let $N \subset M$ be an irreducible subfactor with finite index. Then the number of maximal intermediate subfactors is less than dimension of $N' \cap M_1$ (the dimension of second higher relative commutant of $N \subset M$).*

We note that since maximal intermediate subfactors in $N \subset M$ correspond to minimal intermediate subfactors in $M \subset M_1$, and the dimension of second higher relative commutant remains the same, the conjecture is equivalent to a similar conjecture as above with maximal replaced by minimal.

In [29], Conjecture 1.1 is verified for subfactors coming from certain conformal field theories. These are subfactors not related to groups in general. In this paper we consider those subfactors which are more closely related to groups and more generally Hopf algebras.

If we take N and M to be cross products of a factor P by H and G with H a subgroup of G, then the minimal version of Conjecture 1.1 in this case states that the number of minimal subgroups of G which strictly contain H is less than the number of double cosets of H in G. This follows from simple counting argument. The nontrivial case is the maximal version of the above conjecture. In this case it gives a generalization of Wall's conjecture which we call relative version of Wall's conjecture. The relative version of Wall's conjecture states that the number of maximal subgroups of G strictly containing a subgroup H is less than the number of double cosets of H in G. As a simple example when this can be proved, consider $G = H \times H$, and

$D \cong H$ is a diagonal subgroup of G. Then the set of maximal subgroups of G containing D is in one-to-one correspondence with the set of maximal normal subgroups of H, and it is easy to check that the set of maximal normal subgroups has cardinality less than the number of irreducible representations of H. On the other hand the number of double cosets of H in G is the same as the number of conjugacy classes of H, and this is the same as the number of irreducible representations of H. So we have proved the relative version of Wall's conjecture in this case.

In Section 2 we will prove this relative version of Wall's conjecture for G solvable. We will present two proofs. The first proof is motivated by an idea of V.F.R. Jones which is to seek linear independent vectors associated with minimal subfactors in the space of second higher relative commutant. This proof is indirect but we hope that the idea will prove to be useful for more general case. We formulate a conjecture for general subfactors (cf. Conjecture 2.1) which is stronger than Conjecture 1.1, and for solvable groups this conjecture is proved in [29]. Here we modify the proof in [29] to prove a linear independence result (cf. Theorem 2.7), and this result implies the relative Wall conjecture for solvable groups. The second proof is a more direct proof using properties of maximal subgroups of solvable groups.

The cross product by finite group subfactor is a special case of depth 2 subfactor. If we take $N \subset M$ to be depth 2, by [5,22] such a subfactor comes from cross product by a finite dimensional $*$-Hopf algebra or Kac algebra $\mathcal{A}$. By [16] or [22] the intermediate subfactors are in one-to-one correspondence to the set of left (or right) coideals of $\mathcal{A}$. Then Conjecture 1.1 states that the number of maximal (resp. minimal) right coideals of $\mathcal{A}$ is less than the dimension of $\mathcal{A}$. In Section 3 we will prove this for the case of Kac algebras $\mathcal{A}$ of Izumi–Kosaki type with solvable groups as considered in [14]. We also prove Conjecture 1.1 for the intermediate subfactors of Izumi–Kosaki type with solvable groups as considered in [14] which are not necessarily of depth 2 (cf. Theorem 3.13). Theorem 3.13 generalizes Theorem 2.9. It is interesting to note that the same type of first cohomology problem encountered in Remark 2.8 also appears here but in a different way and solvability is once again used to ensure that the first cohomology group is trivial (cf. Lemmas 3.6 and 3.7).

We note that recently lattices of intermediate or other types of Kac algebras have been obtained in [6]. Our conjecture can be verified in the examples of [6] where complete lattice of intermediate subfactors is determined. The maximal (or minimal) coideals are very few compared with the dimension of the Kac algebra in these examples of [6].

In Section 4 we first present a lemma which bounds the number of maximal subgroups of a group $X \times Y$ which does not contain either X nor Y. This lemma gives a proof of Wall's conjecture for $X \times Y$ assuming that Wall's conjecture is true for X and Y. We then propose a natural conjecture about tensor products of subfactors.

At the end of this introduction let us consider a fusion algebra version of Conjecture 1.1. Let $\rho_i \in \mathrm{End}(M)$, $i = 1, \ldots, n$ be a finite system of irreducible sectors of a properly infinite factor M which is closed under fusion. Consider the Longo–Rehren subfactor associated with such a system (cf. [21]). By [13], the intermediate subfactors are in one-to-one correspondence with the fusion subalgebras which are generated by a subset of simple objects ρ_i, and Conjecture 1.1 states that the number of such maximal fusion subalgebras is bounded by n which is the number of simple objects. This motivates us to make the following conjecture:

Conjecture 1.2. *Let $\mathcal{F}$ be a finite dimensional semisimple fusion algebra with n simple objects. Then the number of maximal fusion subalgebras which are generated by a subset of the simple objects of $\mathcal{F}$ is less than n.*

If we take $\mathcal{F}$ to be the group algebra of G, then Conjecture 1.2 is equivalent to Wall's conjecture.

If we take $\mathcal{F}$ to be the fusion algebra of representations of a finite group G, then the maximal fusion subalgebras are in one-to-one correspondence to minimal normal subgroups of G, and the number of such subgroups is less than the number of conjugacy classes of G, which is the same as the number of simple objects of $\mathcal{F}$. This is a special case of a more general result of D. Nikshych and V. Ostrik, who prove that Conjecture 1.2 is true for commutative $\mathcal{F}$ [23].

2. Relative version of Wall's conjecture for solvable groups

In this section we will prove Theorem 2.9, which confirms the relative version of Wall's conjecture for solvable groups. We will give two proofs of this result. The first proof is motivated by the following conjecture, formulated as Conjecture A.1 in [29], which can be stated for general subfactors:

Conjecture 2.1. *Let $N \subset M$ be an irreducible subfactor with finite Jones index, and let P_i, $1 \leqslant i \leqslant n$ be the set of minimal intermediate subfactors. Denote by $e_i \in N' \cap M_1$, $1 \leqslant i \leqslant n$ the Jones projections e_i from M onto P_i and e_N the Jones projections e_N from M onto N. Then there are vectors ξ_i, $\xi \in N' \cap M_1$ such that $e_i\xi_i = \xi_i$, $1 \leqslant i \leqslant n$, $e_N\xi = \xi$, and ξ_i, $1 \leqslant i \leqslant n$, ξ are linearly independent.*

Remark 2.2. We note that unlike Conjecture 1.1, the conjecture above makes use of the algebra structure of $N' \cap M_1$ and therefore does not immediately imply the dual version or if one replaces minimal by maximal.

By definition Conjecture 2.1 implies Conjecture 1.1. In the case of subfactors from groups, it is easy to check that Conjecture 2.1 is equivalent to:

Conjecture 2.3. *Let K_i, $1 \leqslant i \leqslant n$ be a set of maximal subgroups of G. Then there are vectors $\xi_i \in l(G)$, $1 \leqslant i \leqslant n$ such that $e_G\xi_i = 0$, ξ_i are K_i invariant and linearly independent.*

This conjecture is proved in [29] when G is solvable. It turns out a modification of the proof presented in [29] gives a proof of a stronger statement. Let us make the following stronger conjecture. First we need to introduce some notation. If H is a subgroup of G, let $\ell(H) = \ell(G, H)$ be the permutation module $\mathbb{C}_H^G$. Let $\ell_0(H)$ denote the hyperplane of weight zero vectors in $\ell(H)$ (i.e. the complement to the one-dimensional G-fixed space on $\ell(H)$).

Conjecture 2.4. *Let K_i, $1 \leqslant i \leqslant n$ be a set of maximal subgroups of G. Set $H = \bigcap K_i$. Then there are vectors $\xi_i \in \ell_0(H)$, $1 \leqslant i \leqslant n$ that are K_i-invariant and linearly independent. In particular, this implies that $n \leqslant \dim \ell_0(H)^H < |H/G\backslash H|$.*

We will prove Conjecture 2.4 for solvable groups by modifying the arguments of [29]. We begin with some preparations that hold for all finite groups.

Lemma 2.5. *Suppose that $K_1, \ldots, K_n$ are conjugate maximal subgroups of the finite group G. Then Conjecture 2.4 holds for $\{K_1, \ldots, K_n\}$.*

Proof. Set $K = K_1$. If $n = 1$, the result is obvious (in particular, if K is normal in G). So assume that $n > 1$. Let $H = \bigcap K_i$. Of course, $\ell_0(K)$ is a submodule of $\ell_0(H)$. Let $K_1, \ldots, K_m$, $m \geqslant n$ be the set of all conjugates of K. Since K is not normal in G, K is self normalizing whence if we choose a permutation basis $\{v_i \mid 1 \leqslant i \leqslant m\}$ for $\ell(K)$, then the stabilizers of the v_i are precisely the K_i. If $m > n$, then the vectors $v_i - v_0 \in \ell_0(K)$, $1 \leqslant i \leqslant n$ are clearly linearly independent (here $v_0 = \sum v_i$ is fixed by G). So it suffices to assume that $m = n$ and so in particular, H is normal in G. So we may assume that $H = 1$. Let V be a nontrivial irreducible submodule of $\ell_0(K)$. Then K does not act trivially on V. Note that, by Frobenius reciprocity, the multiplicity of V in $\ell_0(K)$ is precisely $\dim V^K < \dim V$. Of course $\dim V$ is the multiplicity of V in $\ell_0(H)$. Thus, $\ell_0(K) \oplus V$ is a submodule of $\ell_0(H)$. Now choose vectors $v_i - v_0$, $1 \leqslant i < m$ as above and w_m any fixed vector of K_m in V. These are obviously linearly independent. $\square$

Next we prove a reduction theorem for Conjecture 2.4. Note that the reduction depends on the existence of the vectors and not just on cardinality.

Lemma 2.6. *Let $\mathcal{S}$ be a family of finite simple groups. Let $\mathcal{F}(\mathcal{S})$ denote the family of all finite groups with all composition factors in $\mathcal{S}$. Let $K_1, \dots, K_n$ be maximal subgroups of the finite group G in $\mathcal{F}(\mathcal{S})$ and assume that Conjecture 2.4 fails with $n|G|$ minimal. Then each K_i has trivial core in G. In particular, G is a primitive permutation group.*

Proof. Suppose that N is a nontrivial normal subgroup of G contained in K_1. Set $H = \bigcap K_i$. If each K_i contains N, then $\ell_0(H)$ is a G/N-module and so $G/N, K_1/N, \dots, K_n/N$ would give a counterexample to the conjecture.

Reorder the K_i so that $N \leqslant K_i$ if and only if $i \leqslant s < n$. Note that $NK_j = G$ for $j > s$. By the minimality of $|G|n$, we can choose $v_1, \dots, v_n \in \ell_0(H)$ with $K_j v_j = v_j$ for all j such that $\{v_1, \dots, v_s\}$ and $\{v_{s+1}, \dots, v_n\}$ are linearly independent. It thus suffices to show that spans of $v_1, \dots, v_s$ and $v_{s+1}, \dots, v_n$ have trivial intersection. Suppose that u is in this intersection.

Since $e_N e_{K_j} = e_G$ for $j > s$ (since $G = NK_j$), it follows that $0 = e_G v_j = e_N e_{K_j} v_j = e_N v_j$ for $j > s$. Thus, $e_N u = 0$. Since N fixes v_i, $i \leqslant s$, it follows that $e_N u = u$. Thus, $u = 0$ and the result follows. □

Theorem 2.7. *Conjecture 2.4 is true for G solvable.*

Proof. Consider a counterexample with $|G|n$ minimal. By Lemma 2.6, none of the K_i contain a normal subgroup. It follows that G is a solvable primitive permutation group, whence $G = AK$ where A is elementary abelian and K acts irreducibly on A. In particular, any maximal subgroup of G either contains A or is a complement to A. Since the core of each K_i is trivial, $G = AK_i$ for each i. Since G is solvable, $H^1(K, A) = 0$, whence all of the K_i are conjugate. Now apply Lemma 2.5 to complete the proof. □

Remark 2.8. As we have seen, a minimal counterexample to Conjecture 2.4 would be a primitive permutation group, and the set of maximal subgroups must all have trivial core. Such groups are classified by Aschbacher–O'Nan–Scott theorem (cf. §4 of [8]). The first case is when G is the semidirect product of an elementary abelian group V by K_1, and the action of K_1 on V is irreducible. When G is not solvable, maximal subgroups K of G with trivial core are not conjugates of K_1, and our proof as above does not work. Such maximal subgroups are related to the first cohomology of K_1 with coefficients in V, and Conjecture 2.4 implies that the order of this cohomology is less than $|K_1|$ (cf. Question 12.2 of [12]). Unfortunately even though it is believed that the order of this cohomology is small (cf. [11]), the bound $|K_1|$ has not been achieved yet.

We give a second proof of Conjecture 2.4 for solvable groups which is not inductive.

Let G be a solvable group. Let $H \leqslant G$ and let $K_1, \dots, K_r$ denote a maximal collection of maximal subgroups of G containing H which are not conjugate. Let K_{ij}, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$ denote the set of all maximal subgroups of G containing H where K_{ij} is conjugate to K_i.

It is easy to see that $G = K_i K_j$ for $i \neq j$ (cf. [2]). Thus, $\mathrm{Hom}(\ell_0(K_i), \ell_0(K_j)) = 0$ if $i \neq j$. If K_i is normal in G, set $V_i = 0$. If not, let V_i be a nontrivial irreducible submodule of $\ell_0(K_i)$ such that $\ell_0(K_i) \oplus V_i$ embeds in $\ell_0(H)$ (as in the proof of Lemma 2.5). Thus $X := \bigoplus_i (W_i \oplus V_i)$ embeds in $\ell_0(H)$ and as above, we can choose v_{ij} in $W_i \oplus V_i$ linearly independent with K_{ij} the stabilizer of v_{ij}.

Of course, this gives:

Theorem 2.9. *Let G be a finite solvable group. Let H be a subgroup of G. Then the number of maximal subgroups of G which contain H is less than $|H/G\backslash H|$.*

3. Kac algebras of Izumi–Kosaki type for solvable groups

In this section we will prove Conjecture 1.1 for Kac algebras of Izumi–Kosaki type for solvable groups. These Kac algebras are introduced in [14] and in more details in [15] by considering compositions of group type subfactors, and they are special cases of bicrossed products from factorisable groups in the theory of Hopf algebras (cf. [7,15]). Let us first recall some definitions from [14] to set up our notations. The reader is referred to [15] for more details. Let $G = N \rtimes H$ be a semidirect product of two finite groups N, H. For $n \in N$, $h \in H$, we define $n^h := h^{-1}nh$. Denote by $L(N)$ the set of complex-valued functions on N. For $f \in L(N)$, $f^h(n) := f(h^{-1}nh)$, $h \in H$.

Definition 3.1. Denote by $\eta_h(n_1, n_2)$, $\xi_n(h_1, h_2)$ $U(1)$-valued cocycles as defined in §2 of [14] which verify the following cocycle conditions:

$$\eta_h(n_1, n_2)\eta_h(n_1n_2, n_3) = \eta_h(n_1, n_2n_3)\eta_h(n_2, n_3),$$

$$\xi_n(h_1h_2, h_3)\xi_n(h_1, h_2) = \xi_n(h_1, h_2h_3)\xi_n(h_2, h_3).$$

Moreover, these cocycles verify the following Pentagon equation:

$$\frac{\eta_{h_1}(n_1, n_2)\eta_{h_2}(n_1^{h_1}, n_2^{h_2})}{\eta_{h_1}(n_1, n_2)} = \frac{\xi_{n_1n_2}(h_1, h_2)}{\xi_{n_1}(h_1, h_2)\xi_{n_2}(h_1, h_2)}$$

and normalizations:

$$\eta_h(e, n_2) = \eta_h(n_1, e) = \xi_n(e, h_2) = \xi_n(h_1, e) = \eta_e(n_1, n_2) = 1.$$

For subfactor motivations for introducing these cocycles, we refer the reader to §2 of [14].

Definition 3.2. Kac algebras of Izumi–Kosaki type are defined as Hopf algebras $\mathcal{A} = L(N) \rtimes_\xi H$ whose Hopf algebra structures are given in [14] as follows:

(1) Algebra products:

$$\bigl(f_1(n), h_1\bigr)\bigl(f_2(n), h_2\bigr) = \bigl(f_1(n) f_2^{h_1}(n)\xi_n(h_1, h_2), h_1h_2\bigr)$$

where $f_2^{h_1}(n) := f_2(h_1^{-1}nh_1)$;

(2) Coproducts:

$$\Delta(n, h) = \sum_{n_2} \eta_h\bigl(nn_2^{-1}, n_2\bigr)\bigl(nn_2^{-1}, h\bigr) \otimes (n_2, h);$$

(3) $*$ structure:

$$(f, h)^* = \bigl(\overline{f\xi\bigl(h, h^{-1}\bigr)}^{h^{-1}}, h^{-1}\bigr).$$

The following two operators on $L(N)$ will play an important role:

Definition 3.3.

$$(L_{n,\eta_h} f)(m) := f(nm)\eta_h(n, m), \qquad (R_{n,\eta_h} f)(m) := f(mn)\eta_h(n, m), \quad \forall n, m \in N,\ h \in H.$$

The following lemma summarizes the properties of these operators which follow from definitions:

Lemma 3.4.

$$L_{n_1,\eta_h}L_{n_2,\eta_h}=L_{n_2n_1,\eta_h}\eta_h(n_2,n_1),\qquad R_{n_1,\eta_h}R_{n_2,\eta_h}=R_{n_1n_2,\eta_h}\eta_h(n_1,n_2),$$
$$L_{n_1,\eta_h}R_{n_2,\eta_h}=R_{n_2,\eta_h}L_{n_1,\eta_h}.$$

The subfactor associated with $\mathcal{A}$ is of the form $\mathcal{L}^{\mathcal{A}}\subset\mathcal{L}$ where $\mathcal{L}^{\mathcal{A}}$ is the fixed point subfactor of a factor $\mathcal{L}$ under the action of $\mathcal{A}$ as defined in §4 of [16]. By [16], any intermediate subfactor of $\mathcal{L}^{\mathcal{A}}\subset\mathcal{L}$ is of the form $\mathcal{L}^{\mathcal{B}}\subset L$, where $\mathcal{B}$ is a right coideal of $\mathcal{A}$, i.e., an $*$ subalgebra of $\mathcal{A}$ which verifies that $\Delta(\mathcal{B})\subset\mathcal{B}\otimes\mathcal{A}$. The following theorem gives a characterization of coideals of $\mathcal{A}$:

Theorem 3.5. *Let $\mathcal{B}$ be a right (resp. left) coideal of $\mathcal{A}$. Then there are subgroups $H_1\leqslant H$, $N_1\leqslant N$ and $U(1)$-valued function $\lambda: N_1\times H_1\to U(1)$ such that:*

$\forall h\in H_1$, $hN_1h^{-1}=N_1$;

$$\lambda(n_1,h)\lambda(n_2,h)=\lambda(n_1n_2,h)\eta_h(n_1,n_2),$$
$$\lambda(n,h_1)\lambda\big(h_1^{-1}nh_1,h\big)\xi_n(h_1,h_2)=\lambda(n,h_1h_2)\tag{1}$$

and $\mathcal{B}=\bigoplus_{h\in H_1}(C(h),h)$ where each $C(h)\subset L(N)$ consists of functions $f\in L(N)$ such that $L_{n,\eta_h}f=\lambda(n,h)f$ (resp. $R_{n,\eta_h}f=\lambda(n,h)f$) $\forall n\in N_1$, $h\in H_1$.

Conversely, any triple (N_1,H_1,λ) which verify the above conditions uniquely determine a coideal of $\mathcal{A}$.

Proof. We will prove the theorem for the case when $\mathcal{B}$ is a right coideal of $\mathcal{A}$. The remaining case is similar. We write the elements of $\mathcal{B}$ as $\sum_h(f_h,h)$ where $f_h\in L(N)$. We have

$$\Delta\sum_h(f_h,h)=\sum_{n_2,h}(R_{n_2,\eta_h}f_h,h)\otimes(n_2,h).$$

Since $\mathcal{B}$ is a right coideal, it follows that for each fixed (n_2,h), $(R_{n_2,\eta_h}f_h,h)\in\mathcal{B}$. So we have $\mathcal{B}=\bigoplus_h(C(h),h)$ with $C(h)$ a subspace of $L(N)$ which is mapped by $R_{n,h}$ to itself. Since $\mathcal{B}$ is also an algebra, we have

$$C(h_1)C(h_2)^{h_1}\xi(h_1,h_2)\subset C(h_1h_2).\tag{2}$$

In particular $C(e)$ is a subalgebra of $L(N)$ which affords a right representation of N. It follows that there is a subgroup $N_1\leqslant N$ such that $C(e)$ is the space of N_1-left invariant functions on N. Let $N=\bigcup_i N_1b_i$, $1\leqslant i\leqslant k$ with $k=|N|/|N_1|$ be the left coset decompositions of N. Then δ_{Nb_i} is a basis of $C(e)$.

Since $\mathcal{B}$ is an $*$ algebra, it follows that if $(f_h,h)\in\mathcal{B}$, then

$$(f_h,h)^*\in\big(C\big(h^{-1}\big),h^{-1}\big),\qquad(f_h,h)(g_h,h)^*=(f_h\bar{g}_h,e)\in\big(C(e),e\big).$$

Let $H_1:=\{h\in H\mid C(h)\neq 0\}$. It follows easily from the above that H_1 is a subgroup of H. By Eq. (2) $C(e)C(h)\subset C(h)$, so it follows that $C(h)=\bigoplus_i C(h)\delta_{N_1b_i}$. Assume that $h\in H_1$ so $C(h)\neq 0$. Since $R_{b_i^1,\eta_h}$ maps $C(h)$ to itself, we can assume that $C(h)\delta_{N_1}\neq 0$. Let $f_i\neq 0\in C(h)\delta_N$, $i=1,2$ then $f_1\bar{f}_2\in C(e)\delta_{N_1}=\mathbb{C}\delta_{N_1}$. We conclude that $C(h)\delta_{N_1}$ is one-dimensional, and by using operator $R_{b_i^{-1},\eta_h}$, we conclude that $C(h)\delta_{N_1b_i}$ is one-dimensional. Let $f_h\neq 0\in C(h)\delta_{N_1}$, then since R_{n_1,η_h} maps $C(h)\delta_{N_1}$ to itself, there is a function $\lambda: N_1\times H_1\to U(1)$ such that

$$R_{n_1,\eta_h}f_h=\lambda(n_1,h)f_h.$$

We can assume that $f_h(e) = 1$. Then we have

$$(R_{n_1,\eta_h} f_h)(e) = \lambda(n_1, h) = f_h(n_1).$$

Eq. (1) follows from Lemma 3.4 and Eq. (2). Let us show that $C(h)$, $h \in H_1$ is the subspace of $L(N)$ which verifies

$$L_{n_1,\eta_h} f = \lambda(n_1, h) f.$$

We note that by Definition 3.3 $f_h(n_1) = \lambda(n_1, h)$ verifies

$$(L_{n_1,\eta_h} f_h)(m) = \lambda(n_1 m, h)\eta_h(n_1, m) = \lambda(n, h)\lambda(m, h) = \lambda(n, h) f_h(m)$$

where in the last equation we have used Eq. (1). Since $C(h)$ is the linear span of $R_{n,\eta_h} f_h$, by Lemma 3.4 we have proved that for any $f \in C(h)$, $L_{n_1,\eta_h} f = \lambda(n_1, h) f$. On the other hand by counting dimensions we conclude that $C(h)$, $h \in H_1$ is the subspace of $L(N)$ which verifies

$$L_{n_1,\eta_h} f = \lambda(n_1, h) f.$$

Let us show that $\forall h \in H_1$, $hN_1h^{-1} = N_1$. By Eq. (2) we have $f_h(n_1)\delta^h_{N_1} \subset C(h)\delta_{N_1}$ which is one-dimensional, and it follows that $\delta^h_{N_1} = \delta_{N_1}$ for all $h \in H_1$, i.e., $hN_1h^{-1} = N_1$. Conversely for any triple (N_1, H_1, λ) which verify the conditions in Theorem 3.5, we can simply define $\mathcal{B} := \bigoplus_{h_1 \in H_1}(C(h), h)$ where $C(h) \subset L(N)$ consists of functions $f \in L(N)$ such that $L_{n,\eta_h} f = \lambda(n, h) f$. We need to check that $\mathcal{B}$ is a right coideal. By inspection it is enough to check Eq. (2). By definition we need to check that if $f_{h_i} \in C(h_i)$, $i = 1, 2$, then $g(n) := f_{h_1}(n) f^{h_1}_{h_2}(n)\xi_n(h_1, h_2) \in C(h_1h_2)$. So we need to show that $(L_{n,\eta_{h_1h_2}} g)(m) = \lambda(n, h_1h_2) g(m)$, $\forall n \in N_1$, $m \in N$. Since $f_{h_i} \in C(h_i)$, $i = 1, 2$, we have

$$f_{h_i}(nm)\eta_{h_i}(n, m) = \lambda(n, h_1) f_{h_i}(m), \quad i = 1, 2.$$

By using the above equation and Eq. (1) it follows that $(L_{n,\eta_{h_1h_2}} g)(m) = \lambda(n, h_1h_2) g(m)$, $\forall n \in N_1$, $m \in N$ iff the following holds:

$$\frac{\eta_{h_1}(n, m)\eta_{h_2}(n^{h_1}, m^{h_2})}{\eta_{h_1h_2}(n, m)} = \frac{\xi_{nm}(h_1, h_2)}{\xi_n(h_1, h_2)\xi_m(h_1, h_2)}$$

which is the Pentagon equation in Definition 3.1. □

For a coideal $\mathcal{B}$ with (N_1, H_1, λ) as in Theorem 3.5, we shall refer to (N_1, H_1, λ) as *the triple* associated with $\mathcal{B}$. We note that by Theorem 3.5, such triple uniquely determine $\mathcal{B}$. Moreover, suppose that the triples associated with $\mathcal{B}_i$ are given by (N_i, H_i, λ_i), $i = 1, 2$. Then $\mathcal{B}_1 \subset \mathcal{B}_2$ iff $N_1 \supset N_2$, $H_1 \subset H_2$ and λ_1, λ_2 agree on $N_2 \times H_1$.

Lemma 3.6. *Let $\mathcal{B}$ be a right coideal of $\mathcal{A}$ as in Theorem* 3.5 *with triple (N_1, H_1, λ). The number of right coideals of $\mathcal{A}$ with the same (N_1, H_1) are given as follows: Let $\hat{N}_1$ be the set of homomorphisms from N_1 to $U(1)$ and form a group $\hat{N}_1 \rtimes H_1$. Then the right coideals of $\mathcal{A}$ with the same (N_1, H_1) are in one-to-one correspondence with the set of cocycles from H_1 to $\hat{N}_1$, i.e., maps $\mu : H_1 \to \hat{N}_1$ such that*

$$\mu(h_1)\mu(h_2)^{h_1} = \mu(h_1h_2).$$

Proof. Let $\mathcal{B}_1$ be a right coideal of $\mathcal{A}$ with triple (N_1, H_1, λ_1) as in Theorem 3.5. Let $\mu := \lambda_1/\lambda$. By Eq. (1) we conclude that μ is a cocycle from H_1 to $\hat{N}_1$. Conversely, if μ is a cocycle from H_1 to $\hat{N}_1$, then $\mathcal{B}_1$ associated with the triple $(N_1, H_1, \lambda\mu)$ is a right coideal of $\mathcal{A}$ by Theorem 3.5. $\square$

Lemma 3.7. *Let $H_1 \subset H$, $N_1 \neq \{e\} \subset N$ such that $hN_1h^{-1} = N_1$, $\forall h \in H_1$. Let $\hat{N}_1$ be the set of homomorphisms from N_1 to $U(1)$ and form a group $\hat{N}_1 \rtimes H_1$. Assume that H_1 acts irreducibly on $\hat{N}_1$ and H_1 is solvable. Then the number of cocycles from H_1 to $\hat{N}_1$, i.e., maps $\mu : H_1 \to \hat{N}_1$ such that*

$$\mu(h_1)\mu(h_2)^{h_1} = \mu(h_1h_2)$$

is less or equal to $(|N_1| - 1)|H_1|$.

Proof. If H_1 acts trivially on $\hat{N}_1$, since H_1 acts irreducibly on $\hat{N}_1$, it follows that $\hat{N}_1$ is an abelian group of prime order, and the number of cocycles from H_1 to $\hat{N}_1$ is bounded by $|H_1|$. If H_1 acts nontrivially on $\hat{N}_1$, then H_1 is a maximal subgroup of $\hat{N}_1 \rtimes H_1$ with trivial core, and since H_1 is solvable, all cocycles from H_1 to $\hat{N}_1$ are coboundaries by Theorem 16.1 of [9], and is bounded by $|\hat{N}_1|$. Since $|H_1| > 1$ in this case the lemma is proved. $\square$

Theorem 3.8. *Let $\mathcal{A} = L(N) \rtimes_\xi H$ be Kac algebras of Izumi–Kosaki type as in Definition* 3.2. *Assume that N, H are solvable groups. Then the number of maximal* (*resp. minimal*) *right coideals is less than the dimension of $\mathcal{A}$.*

Proof. Assume that $\mathcal{B}$ is a right coideal of $\mathcal{A}$ and let (N_1, H_1, λ) be the triple as in Theorem 3.5. Let us first prove the minimal case. Since $\mathcal{B} \supset (L(N/N_1), e)$, if $N_1 \neq N$, we must have $\mathcal{B} = (L(N/N_1), e)$, and N_1 must be maximal in N. The number of such N_1 is less than $|N|$ by Theorem 2.9.

If $N_1 = N$, then $H_1 \neq e$. Let $\mathbb{Z}_p \leqslant H_1$ be any minimal subgroups of H_1, then the triple $(N, \mathbb{Z}_p, \lambda)$ will give rise to a right coideal of $\mathcal{A}$ by Theorem 3.5 which is contained in $\mathcal{B}$. It follows that $H_1 = \mathbb{Z}_p$. By Lemma 3.6, the number of such triple is bounded by $|\hat{N}| \leqslant |N|$. So minimal right coideal of $\mathcal{A}$ is bounded by the sum of number of maximal subgroups of N and the product of the number of minimal subgroups of H by $|N|$, and it follows that the number of minimal right coideals is less than the dimension of $\mathcal{A}$.

Now assume that $\mathcal{B}$ is maximal. If N_1 is trivial, then H_1 is maximal in H, and by Theorem 2.9 the number of maximal H_1 is less than $|H|$.

If N_1 is nontrivial, then $\bigoplus_{h \in H_1}(L(N), h) \supset \mathcal{B}$, and it follows that $H_1 = H$. We claim that N_1 is generated by $\mathrm{Ad}_H(x)$ for any nontrivial $x \in N_1$. In fact let $N_1' \subset N_1$ be a subgroup generated by $\mathrm{Ad}_H(x)$ for a nontrivial $x \in N_1$. Then the right coideal determined by the triple (N_1', H, λ) contains $\mathcal{B}$, and by maximality of $\mathcal{B}$ we have $N_1' = N_1$. It follows that H acts irreducibly on $N_1/[N_1, N_1]$, and therefore acts irreducibly on its dual $\hat{N}_1$. By Lemma 3.7 such $\mathcal{B}$ with fixed (N_1, H) is bounded by $(|N_1| - 1)|H|$. Note that different N_1's intersect only at identity. It follows that the number of maximal $\mathcal{B}$'s is bounded by

$$\big(|H| - 1\big) + |H|\big(|N| - 1\big) = |H||N| - 1. \qquad \square$$

We consider the intermediate subfactors of $\mathcal{L}^{\mathcal{B}} \subset L$ corresponding to $\mathcal{B}$ as in Theorem 3.5.

Lemma 3.9. *The dimension of second higher relative commutant associated with the subfactor $\mathcal{L}^{\mathcal{B}} \subset \mathcal{L}$ is given by*

$$\sum_{h \in H_1} \dim\big(C_R(h) \cap C_L(h)\big)$$

where

$$C_R(h) \cap C_L(h) := \{ f \in L(N),\ R_{n_1,\eta_h} f = \lambda(n_1, h) f = L_{n_1,\eta_h} f \}.$$

Proof. This follows from §3 of [22] and Theorem 3.5. □

Lemma 3.10. *Let $\mathcal{B}$ be as in Theorem 3.5 with triple (N_1, H_1, λ). Suppose that λ can be extended to $N_i \supset N_1$ such that the triple (N_i, H_1, λ), $i = 1, 2, \dots, n$ gives a right coideal of $\mathcal{A}$ via Theorem 3.5, and $N_i \cap N_j = N_1$, $\forall i \neq j$. Let k_i be the number of double cosets of N_1 in N_i. Then*

$$\dim\big(C_R(h) \cap C_L(h)\big) \geqslant 1 + (k_2 - 1) + \cdots + (k_n - 1).$$

Proof. On each double coset $N_1 b N_1$ of N_i, we can define a function such that its value on the double coset is simply the value of $\lambda(.,h)$ and zero elsewhere. It is easy to check that these functions belong to $C_R(h) \cap C_L(h)$, and they are linearly independent since they have different support, and the lemma follows. □

The following two lemmas are straightforward consequences of the definitions:

Lemma 3.11. *Let $N_2 \supset N_1$. Then the number of homomorphisms from N_2 to $U(1)$ which takes value 1 on N_1 is bounded by the number of double cosets of N_1 in N_2.*

Lemma 3.12. *Let $N_2 \subset N$ be a minimal extension of N_1 which is Ad_{H_1} invariant. Then the natural action of H_1 on $N_2/N_1[N_2, N_2]$ is irreducible. The dual of $N_2/N_1[N_2, N_2]$ is abelian group of homomorphisms from N_2 to $U(1)$ which takes value 1 on N_1.*

Theorem 3.13. *Let $\mathcal{B} \subset \mathcal{A}$ be a right coideal of $\mathcal{A}$ as in Theorem 3.5. Then both minimal version and maximal version of Conjecture 1.1 is true for subfactors $\mathcal{L}^{\mathcal{B}} \subset \mathcal{L}$ when H, N are solvable.*

Proof. Let $\mathcal{B}_i \subset \mathcal{B}$ and (N_i, H_i, λ_i) be the associated triple of $\mathcal{B}_i$ as in Theorem 3.5. We have $N_i \supset N_1$, $H_i \subset H_1$, and λ_i agrees with λ on $N_1 \times H_i$.

Let us first prove the maximal case. Assume that $\mathcal{B}_i \subset \mathcal{B}$, $i = 1, 2, 3, \dots, m$ is the list of maximal right coideals of $\mathcal{A}$ that is contained in $\mathcal{B}$. Let (N_j, H_j), $j = 1, 2, 3, \dots, n$ be the list of different pairs of groups that are associated with $\mathcal{B}_i$'s as in Theorem 3.5.

If $N_j = N_1$, then $H_j \subset H_1$ is maximal in H_1. The number of such maximal H_j is less than $|H_1|$ by Theorem 2.9. If $N_j \neq N_1$, since $\mathrm{Ad}_{H_j}(N_1) = N_1$, it follows that $H_j = H_1$, and N_j is a minimal extension of N_1 which is invariant under Ad_{H_1}. Let k_j be the number of double cosets of N_1 in N_j. Note that these N_j's only intersect at N_1. By Lemmas 3.12, 3.11 and 3.7 the number of such $\mathcal{B}_i$ with fixed N_j is bounded by $(k_j - 1)|H_1|$. So $m \leqslant |H_1| - 1 + [(k_2 - 1) + \cdots + (k_n - 1)]|H_1|$ and the theorem follows from Lemmas 3.9 and 3.10.

Now assume that $\mathcal{B}_i \subset \mathcal{B}$, $i = 1, 2, 3, \dots, m$ is the list of minimal right coideals of $\mathcal{A}$ that is contained in $\mathcal{B}$. Let (N_j, H_j), $j = 1, 2, 3, \dots, p$ be the list of different pairs of groups that are associated with $\mathcal{B}_i$'s as in Theorem 3.5. By considering $(L(N/N_j), e)$ it follows that if H_j is trivial, then $N_j \subset N$ is a maximal subgroup, and the number of such maximal subgroups is bounded by the double cosets of N_1 in N by Theorem 2.9. If H_j is nontrivial, then $N_j = N$, and it follows that H_j has to be a minimal nontrivial subgroup of H_1. The number of such subgroups of H_1 is bounded by $|H_1| - 1$ if H_1 is not an abelian group of prime order, and 1 if H_1 is an abelian group of prime order. For each fixed (N, H_j), the possible λ_j's which agrees with λ on $N_1 \times H_j$ is clearly bounded by the number of homomorphisms from N to $U(1)$ which vanishes on N_1, and by Lemma 3.11 this number is bounded by the number of double cosets of N_1 in N which is denoted by p_1. It follows that $p \leqslant p_1 - 1 + (|H_1| - 1)p_1$, and by Lemma 3.9 we are done. □

Remark 3.14. If we set H to be a trivial group in Theorem 3.13, then we recover Theorem 2.9.

Remark 3.15. For subfactor $\mathcal{L}^{\mathcal{A}} \subset \mathcal{L}^{\mathcal{B}}$, we can map its intermediate subfactors to certain right coideals of $\hat{\mathcal{A}}$ which is the dual of $\mathcal{A}$ (cf. §2 of [14]). Essentially similar argument as in the proof of Theorem 3.13 shows that subfactor $L^{\mathcal{A}} \subset L^{\mathcal{B}}$ verifies the maximal and minimal version of Conjecture 1.1 when H, N is solvable.

4. Tensor product conjecture

Lemma 4.1. *Let $G = X \times Y$ be finite groups with $|X| = x$ and $|Y| = y$. Then the number of maximal subgroups of G which contain neither X nor Y is at most $(x-1)(y-1)$ (with equality if and only if X and Y are elementary abelian 2-groups).*

Proof. Let M be a maximal subgroup of G containing neither X nor Y. Let $f : G \to G/K$ be the natural homomorphism where K is the core of M in G. Then $f(X)$ and $f(Y)$ are normal nontrivial subgroups which commute in the primitive group G/K and moreover, they generate G/K together. By the Aschbacher–O'Nan–Scott theorem (although this can be proved easily in this case) this implies that either $f(X) = f(Y)$ has prime order p for some prime p or $G/K = f(X) \times f(Y) = S \times S$ with S a nonabelian simple group.

Thus, passing to the quotient by the intersection of all the cores of such maximal subgroups, we may assume that X and Y are direct products of simple groups. Write $X = \prod_p X_p \times \prod_S X_S$ where X_p is the maximal elementary abelian p-quotient of X and X_S is the maximal quotient of X that is a direct product of nonabelian simple groups each isomorphic to S. Write Y in a similar manner. The previous paragraph shows that we may reduce to the case that either X and Y are each elementary abelian p-groups or are both direct products of a fixed nonabelian simple group S.

In the first case, it is trivial to see that the total number of maximal subgroups is $(xy-1)/(p-1)$ while the number of maximal subgroups containing either X or Y is $(x-1)/(p-1) + (y-1)/(p-1)$. Thus, the total number of maximal subgroups containing neither X nor Y is $(x-1)(y-1)/(p-1)$.

In the second case, write $X = S^a$ and $Y = S^b$. If M is a maximal subgroup not containing X or Y, then $M \cap X$ is normal in X with $X/(M \cap X) \cong S$. Thus M is a direct factor of X isomorphic to S^{a-1}. There are a such factors. Thus, the number of maximal subgroups of $X \times Y$ not containing X or Y is abc where c is the number of maximal subgroups of $S \times S$ not containing either factor. This is precisely $|\mathrm{Aut}(S)|$ (since any such maximal subgroup is $\{s, \sigma(s) \mid s \in S\}$ where $\sigma \in \mathrm{Aut}(S)$). Thus, in this case the number of maximal subgroups not containing either factor is $ab|\mathrm{Aut}(S)|$. To complete the proof, we only need to know that $|\mathrm{Aut}(S)| < (|S|-1)^2$.

This is well known (and in fact much better inequalities can be shown). All such existing proofs depend upon the classification of finite simple groups. We note that the inequality we need follows from the fact that every finite nonabelian simple group can be generated by two elements (note that if $s \in S$, then $|\{\sigma(s) \mid \sigma \in \mathrm{Aut}(S)\} < |S| - 3$ for there are at least 4 different orders of elements and so at the very least 4 different orbits on S – if s, t are generators, then any automorphism is determined by its images on s, t whence the inequality). □

The following corollary follows immediately:

Corollary 4.2. *Let $G = X \times Y$ be finite groups such that both X and Y verify Wall's conjecture, then G also verifies Wall's conjecture.*

Based on Lemma 4.1, we propose the following tensor product conjecture:

Conjecture 4.3. *Let $N_i \subset M_i$, $i = 1, 2$ be two irreducible subfactors with finite index. Then the number of minimal intermediate subfactors in $N_1 \otimes N_2 \subset M_1 \otimes M_2$ which is not of the form $N_1 \otimes P$, $P \times N_2$ is less or equal to $(n_1 - 1)(n_2 - 1)$ where n_i is the dimension of second higher relative commutant of $N_i \subset M_i$, $i = 1, 2$.*

This conjecture is nontrivial even for subfactors coming from groups, where we have seen in the proof of Lemma 4.1 that we have used classification of finite simple groups to bound the number of automorphisms of a simple group.

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