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# An equivalence of categories for graded modules over monomial algebras and path algebras of quivers

Cody Holdaway\*, S. Paul Smith

Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195, United States

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## ABSTRACT

Let  $A$  be a finitely generated connected graded  $k$ -algebra defined by a finite number of monomial relations, or, more generally, the path algebra of a finite quiver modulo a finite number of relations of the form “path = 0”. Then there is a finite directed graph,  $Q$ , the Ufnarovskii graph of  $A$ , for which there is an equivalence of categories  $\text{QGr } A \cong \text{QGr}(kQ)$ . Here  $\text{QGr } A$  is the quotient category  $\text{Gr } A / \text{Fdim}$  of graded  $A$ -modules modulo the subcategory consisting of those that are the sum of their finite dimensional submodules. The proof makes use of an algebra homomorphism  $A \rightarrow kQ$  that may be of independent interest.

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## 1. Introduction

1.1. Throughout  $k$  is a field.

Let  $A$  be an  $\mathbb{N}$ -graded  $k$ -algebra.

The category of  $\mathbb{Z}$ -graded right  $A$ -modules with degree-preserving homomorphisms is denoted by  $\text{Gr } A$  and  $\text{Fdim } A$  is its full subcategory consisting of modules that are the sum of their finite dimensional submodules. Since  $\text{Fdim } A$  is a Serre subcategory of  $\text{Gr } A$  (it is, in fact, a localizing subcategory) we may form the quotient category

\* Corresponding author.

E-mail addresses: [codyh3@math.washington.edu](mailto:codyh3@math.washington.edu) (C. Holdaway), [smith@math.washington.edu](mailto:smith@math.washington.edu) (S.P. Smith).

$$\text{QGr } A := \frac{\text{Gr } A}{\text{Fdim } A}.$$

We are interested in the structure of  $\text{QGr } A$  for monomial algebras.

1.2. A connected graded monomial algebra is a free algebra modulo an ideal generated by words in the letters generating the free algebra. More explicitly, if  $w_1, \dots, w_r$  are words in the letters  $x_1, \dots, x_g$ , then

$$A = \frac{k\langle x_1, \dots, x_g \rangle}{(w_1, \dots, w_r)} \quad (1.1)$$

is a finitely presented monomial algebra.

Our main result applies to a more general class of monomial algebras, namely those of the form  $kQ'/I$  where  $Q'$  is a finite quiver (Section 2.1) and  $I$  an ideal generated by a finite set of paths in  $Q'$ . Such algebras can be described without mentioning quivers: let  $K$  be a finite product of copies of  $k$ ,  $T_K V$  the tensor algebra of a  $K$ -bimodule  $V$  that has a finite  $k$ -basis  $a_1, \dots, a_g$ , and

$$A = \frac{T_K V}{(p_1, \dots, p_r)} \quad (1.2)$$

where each  $p_j$  is a word in the  $a_i$ 's.

### 1.3. The main result

**Theorem 1.1.** *Let  $A$  be a monomial algebra of the form (1.2). There is a quiver  $Q$  and an equivalence of categories*

$$\text{QGr } A \cong \text{QGr } kQ.$$

The structure and properties of  $\text{QGr } kQ$  are described in [5].

The proof of Theorem 1.1 uses result of Artin and Zhang, Proposition 2.1 below, in an essential way.

When  $A$  is of the form (1.1) we can take  $Q$  to be its Ufnarovskii graph (Section 3) and there is then a homomorphism  $f: A \rightarrow kQ$  such that the functor  $-\otimes_A kQ$  induces the equivalence in Theorem 1.1. This is proved in Section 4.1; see Theorem 4.2 for a precise statement.

In Section 4.2, Theorem 1.1 is proved for algebras of the form (1.2): if  $A$  is of the form (1.2) its subalgebra generated by  $k$  and  $A_1$  is of the form (1.1) and has finite codimension in  $A$  so, by Artin and Zhang's result and Theorem 1.1 for algebras of the form (1.1), Theorem 1.1 holds for algebras of the form (1.2).

### 1.4. Quadratic monomial algebras

If  $A$  is monomial algebra of the form (1.1) with  $\deg w_i = 2$  for all  $i$  we call  $A$  a *quadratic* monomial algebra. The proof of Theorem 1.1 for quadratic monomial algebras is much simpler than the general case. We give that proof in Section 6.1.

Let  $A$  be an arbitrary finitely presented connected graded monomial algebra. By Backelin and Fröberg [2], the Veronese subalgebra  $A^{(n)} \subset A$  is quadratic for  $n \gg 0$ ; by Verevkin [9],  $\text{QGr } A \cong \text{QGr } A^{(n)}$ , so Theorem 1.1 holds for  $A$  if it holds for  $A^{(n)}$ . However, if Theorem 1.1 is proved for  $A$  by first proving it for  $A^{(n)}$  the quiver  $Q$  is the Ufnarovskii graph for  $A^{(n)}$  which is more complicated than that for  $A$  (see Section 6.3 for an example illustrating this).

That is why we prove Theorem 1.1 directly in Section 4.1, i.e., without passing to a Veronese subalgebra.

## 2. Preliminaries

### 2.1. Notation

The letter  $Q$  will always denote a directed graph, or quiver, with a finite number of vertices and arrows—loops and multiple arrows between vertices are allowed.

We write  $kQ$  for the path algebra of  $Q$ . The finite paths in  $Q$ , including the trivial paths at each vertex, form a basis for  $kQ$  and multiplication is given by concatenation of paths. If  $a$  is an arrow that ends where the arrow  $b$  begins we write

$$ab := \text{the path “}a \text{ followed by } b\text{”}.$$

We set  $ab = 0$  if  $b$  does not begin where  $a$  ends. Likewise, if a path  $p$  ends where a path  $q$  begins,  $pq$  denotes the path *first traverse  $p$  then  $q$* .

We make  $kQ$  an  $\mathbb{N}$ -graded algebra by declaring that a path is homogeneous of degree equal to its length.

2.2. Throughout, modules are *right* modules.

**Proposition 2.1.** (See [1, Prop. 2.5].) *Let  $\phi : A \rightarrow B$  be a homomorphism of graded  $k$ -algebras. If  $\ker \phi$  and  $\text{coker } \phi$  belong to  $\text{Fdim } A$ , then  $-\otimes_A B$  induces an equivalence of categories  $\text{QGr } A \rightarrow \text{QGr } B$ .*

**Lemma 2.2.** *Let  $A$  and  $B$  be  $\mathbb{N}$ -graded  $k$ -algebras generated by  $A_0 + A_1$  and  $B_0 + B_1$  respectively. Let  $\phi : A \rightarrow B$  be a homomorphism of graded  $k$ -algebras. If  $B_0\phi(A_m) \subset \phi(A_m)$  and  $B_1\phi(A_m) \subset \phi(A_{m+1})$  for some  $m \in \mathbb{N}$ , then  $\text{coker } \phi$  belongs to  $\text{Fdim } A$ .*

**Proof.** We can replace  $A$  by its image in  $B$  so we will do that; i.e., without loss of generality,  $A$  is a graded subalgebra of  $B$  and  $\phi$  is the inclusion map.

If  $n \geq 2$  and  $B_{n-1}A_m \subset A_{m+n-1}$ , then

$$B_n A_m = B_1 B_{n-1} A_m \subset B_1 A_{m+n-1} = B_1 A_m A_{n-1} \subset A_{m+1} A_{n-1} = A_{m+n}.$$

It follows that  $B_n A_m \subset A_{m+n}$  for all  $n \geq 0$ . Thus  $B/A$  is annihilated on the right by  $A_m$  and therefore belongs to  $\text{Fdim } A$ .  $\square$

## 3. The Ufnarovskii graph of a connected graded monomial algebra

Throughout this paper  $G$  is a fixed finite set of *letters* or *generators*,  $\langle G \rangle$  is the free monoid generated by  $G$ , and  $k\langle G \rangle$  is the free  $k$ -algebra generated by  $G$ . Elements of  $\langle G \rangle$  are called words. Throughout,  $F$  denotes a fixed *finite* set of words and

$$A := \frac{k\langle G \rangle}{(F)} \tag{3.1}$$

is the quotient by the ideal  $(F)$  generated by  $F$ . Such  $A$  is called a monomial algebra.

There is no loss of generality in assuming that  $G \cap F = \emptyset$ . We will make that assumption.

We make  $A$  a graded algebra by placing  $G$  in degree one. Thus  $A_1 = kG$ .

### 3.1. Words

The words in  $F$  are said to be forbidden. A word is illegal if it belongs to  $(F)$  and legal otherwise. The set of legal words is denoted by  $L$ , and  $L_r := L \cap G^r$  is the set of legal words of length  $r$ . The image of  $L_r$  in  $A$  is a basis for  $A_r$ ; see, for example, [3, Lem. 2.2].

Throughout we use the notation

$$\begin{aligned}\ell + 1 &:= \text{the longest length of a forbidden word} \\ &= \max\{\ell + 1 \mid F \cap G^{\ell+1} \neq \emptyset\}, \quad \text{and} \\ L_{\leq r} &:= \{\text{legal words of length } \leq r\}.\end{aligned}$$

### 3.2. Notation

The letters  $s, t, u, v, w$ , will always denote words.

If  $u$  and  $w$  are words we write

$$u \triangleleft w$$

if  $w = uv$  for some word  $v$ .

The symbols  $x, y$ , and  $x_i$ , will always denote elements of  $G$ . The notation  $x_i \triangleleft w$  therefore means that  $x_i$  is the first letter of  $w$ .

### 3.3. The Ufnarovskii graph

The Ufnarovskii graph of  $A$  is the directed graph  $Q$ , or  $Q(A)$  if we need to specify  $A$ , defined as follows (see [3, Sect. 12.2], [7,8]).

The set of vertices of  $Q$  is

$$Q_0 = L_\ell.$$

The set of arrows of  $Q$  is in bijection with the set  $L_{\ell+1}$  as follows,

$$Q_1 = \{a_w \mid w \in L_{\ell+1}\}.$$

If  $w \in L_{\ell+1}$ , then there are unique  $s, t \in Q_0$  and unique  $x, y \in G$  such that  $w = sy = xt \in L$  and we declare that the arrow  $a_w$  corresponding to  $w$  goes from  $s$  to  $t$ .

Given  $s, t \in Q_0$ , there is at most one arrow from  $s$  to  $t$ .

Suppose  $n > 0$ . If  $x_1 \dots x_{n+\ell}$  is a legal word of length  $n + \ell$  there is a length- $n$  path

$$x_1 \dots x_\ell \longrightarrow x_2 \dots x_{\ell+1} \longrightarrow \dots \longrightarrow x_{n+1} \dots x_{n+\ell} \quad (3.2)$$

in  $Q$ . This provides a bijection between legal words of length  $n + \ell$  and paths of length  $n$  (see the proof of [7, Thm. 3] and the remark at [3, p. 157]).

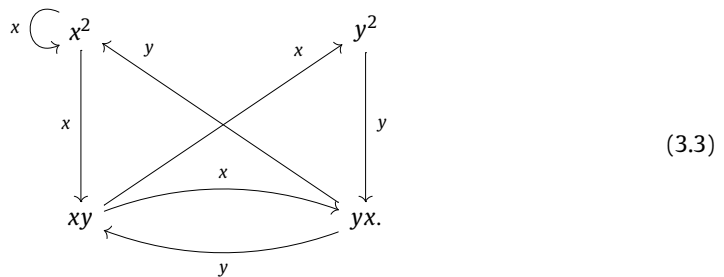
### 3.4. Labeling arrows and paths

We write  $a_w$  for the arrow corresponding to  $w \in L_{\ell+1}$ . The path in (3.2) is therefore  $a_{x_1 \dots x_{\ell+1}} a_{x_2 \dots x_{\ell+2}} \dots a_{x_n \dots x_{n+\ell}}$ .

Suppose there is an arrow  $u \rightarrow v$ . Then  $uy = xv$  for unique  $x$  and  $y$  in  $G$ , and we attach the label  $x$  to the arrow  $u \rightarrow v$ . We denote this by  $u \xrightarrow{x} v$ . The following facts will be used often:

- The label attached to the arrow  $a_w$  is the first letter of  $w$ .
- The existence of an arrow  $u \xrightarrow{x} v$  implies that  $x \triangleleft u$  and  $u \triangleleft xv$ .

We extend the labeling to paths: the label attached to a concatenation of arrows is the concatenation of the labels attached to the arrows in the path—for example, the label attached to the path in (3.2) is  $x_1 \dots x_n$ . In general, there will be different paths with the same label: for example, the labels on the Ufnarovskii graph for  $A = k\langle x, y \rangle / (y^3)$  are



The Ufnarovskii graph for  $k\langle x, y, z \rangle / (z^2, zy)$  appears in Section 5. The following observation is surely known to the experts.

**Lemma 3.1.** Suppose there is a path with the label  $x_1 \dots x_r$ , say

$$v_0 \xrightarrow{x_1} v_1 \xrightarrow{x_2} \dots \xrightarrow{x_r} v_r. \quad (3.4)$$

Let  $v_r = x_{r+1} \dots x_{r+\ell}$ .

- (1)  $v_{i-1} = x_i \dots x_{i+\ell-1}$  for all  $i = 1, \dots, r+1$ .
- (2)  $x_1 \dots x_r v_r$  is a legal word.
- (3)  $x_1 \dots x_r \notin (F)$ .

**Proof.** The hypothesis implies  $v_{i-1} \triangleleft x_i v_i$  and  $x_i \triangleleft v_{i-1}$  for all  $i = 1, \dots, r$ . An induction argument, or simply noticing the pattern in the equalities

$$\begin{aligned} v_r &= x_{r+1} \dots x_{r+\ell}, \\ v_{r-1} &= x_r \dots x_{r+\ell-1}, \\ v_{r-2} &= x_{r-1} \dots x_{r+\ell-2}, \quad \text{etc.} \end{aligned}$$

proves (1).

(2) To prove  $x_1 \dots x_r v_r$  is legal it suffices to show its subwords of length  $\ell + 1$  are legal. Such a subword is of the form  $x_i \dots x_{i+\ell-1} x_{i+\ell}$  for some  $i$  in the range  $1 \leq i \leq r$ ; this subword is equal to  $v_{i-1} x_{i+\ell} = x_i v_i$  and is legal because there is an arrow  $v_{i-1} \rightarrow v_i$ .

(3) Since a subword of a legal word is legal, (3) follows from (2).  $\square$

**Lemma 3.2.** *If  $x_1 \dots x_r$  is an illegal word, then there are no paths labeled  $x_1 \dots x_r$ .*

$$Q = a_{yy} \circlearrowleft y \xrightarrow{a_{yx}} x$$
$$y \circlearrowleft y \xrightarrow{y} x. \quad (3.5)$$
$$f(x) = \sum_{\substack{w \in Q_1 \\ x \triangleleft w}} a_w \quad \text{if } xL_\ell \cap L_{\ell+1} \neq \emptyset.$$

Write  $v_r = x_{r+1} \dots x_{r+\ell}$ . Since  $x_i v_i$  is legal for all  $i = 1, \dots, r$  and  $x_i v_i = x_i x_{i+1} \dots x_{i+\ell-1}$  all subwords of  $x_1 \dots x_r v_r$  of length  $\ell + 1$  are legal. It follows that  $x_1 \dots x_r v_r$  is legal.

( $\Leftarrow$ ) Suppose  $x_1 \dots x_r L_\ell \cap L \neq \emptyset$ . Let  $v_r = x_{r+1} \dots x_{r+\ell}$  be a vertex such that  $x_1 \dots x_r v_r$  is legal. For  $i = 1, \dots, r$ , define

$$v_{i-1} := x_i \dots x_{i+\ell-1}.$$

This is a legal word, of length  $\ell$ , because it is a subword of the legal word  $x_1 \dots x_r v_r$ . Since  $v_{i-1} \triangleleft x_i v_i$  there is an arrow  $v_{i-1} \xrightarrow{x_i} v_i$ . Concatenating these arrows produces a path labeled  $x_1 \dots x_r$ .  $\square$

**Lemma 3.5.** *Let  $x_1 \dots x_r$  be a legal word of length  $r \geq \ell$ . There is a path labeled  $x_1 \dots x_r$  if and only if there is a path labeled  $x_{r-\ell+1} \dots x_r$ .*

**Proof.** The lemma is true for  $r = \ell$  so suppose  $r > \ell$ .

( $\Rightarrow$ ) This is obvious.

( $\Leftarrow$ ) Suppose there is a path

$$v_{r-\ell} \xrightarrow{x_{r-\ell+1}} v_{r-\ell+1} \longrightarrow \dots \longrightarrow v_{r-1} \xrightarrow{x_r} v_r.$$

Write  $v_r = x_{r+1} \dots x_{r+\ell}$ .

By Lemma 3.4,  $x_1 \dots x_r$  is legal if  $x_1 \dots x_r v_r$  is. The word  $x_1 \dots x_r v_r$  is legal if all its subwords of length  $\ell + 1$  are legal. The proof of Lemma 3.4 showed that  $x_{r-\ell+1} \dots x_r v_r$  is legal. All subwords of  $x_{r-\ell+1} \dots x_r x_{r+1} \dots x_{r+\ell}$  are therefore legal so it only remains to show that  $x_i \dots x_{i+\ell}$  is legal for all  $i \leq r - \ell$ . If  $i \leq r - \ell$ , then  $x_i \dots x_{i+\ell}$  is a subword of  $x_1 \dots x_r$  and therefore legal.  $\square$

### 3.6. The kernel of $f$

The homomorphism  $f$  need not be injective: for example, by looking at the labels on the quiver (3.5) above one sees that  $f(x) = 0$  when  $A = k\langle x, y \rangle / (xy, x^2)$ .

**Lemma 3.6.** *Let  $w_1, \dots, w_n$  be pairwise distinct legal words. If  $f(w_i) \neq 0$  for all  $i$ , then  $\{f(w_1), \dots, f(w_n)\}$  is linearly independent.*

**Proof.** Since  $f$  preserves degree we can assume that  $w_1, \dots, w_n$  have the same length, say  $r$ . By definition,  $f(w_i)$  is the sum of the paths labeled  $w_i$ ; hence if  $i \neq j$  no path that appears in  $f(w_i)$  appears in  $f(w_j)$ . But the paths of length  $r$  are linearly independent elements of  $kQ$  so  $\{f(w_1), \dots, f(w_n)\}$  is linearly independent.

**Theorem 3.7.** *The kernel of the homomorphism  $f : k\langle G \rangle \rightarrow kQ$  is equal to  $(F) + I$  where  $I$  is the left ideal generated by the set*

$$S := \{x_1 \dots x_s \in G^S \mid s \leq \ell \text{ and there is no path labeled } x_1 \dots x_s\}.$$

**Proof.** By Proposition 3.3,  $\ker f$  contains the ideal  $(F)$ . Since  $f(x_1 \dots x_r)$  is the sum of all the paths labeled  $x_1 \dots x_r$ ,  $S \subset \ker f$ . Hence  $(F) + I \subset \ker f$ .

Since  $(F)$  is spanned by words, Lemma 3.6 implies  $\ker f$  is spanned by  $(F)$  and various legal words. Suppose  $x_1 \dots x_r$  is a legal word such that  $f(x_1 \dots x_r) = 0$ . This implies there is no path labeled  $x_1 \dots x_r$  so, if  $r \leq \ell$ ,  $x_1 \dots x_r$  is in  $S$  and therefore in  $I$ . On the other hand, if  $r \geq \ell + 1$ , Lemma 3.5 implies  $x_{r-\ell+1} \dots x_r$  is in  $S$ , whence  $x_1 \dots x_r \in I$ .  $\square$

Information about the cokernel of  $f$  is given in Proposition 4.1.

#### 4. The proof of Theorem 1.1

##### 4.1. The proof of Theorem 1.1 when $A$ is as in (1.1)

Let  $A$  be as in (1.1) and adopt the notation in (3.1). We will prove Theorem 1.1 by applying Proposition 2.1 to the induced homomorphism  $\bar{f} : A \rightarrow kQ$ . Before doing that we must check that the hypotheses of Proposition 2.1 hold: we must show that the kernel and cokernel of  $\bar{f}$  belong to  $\text{Fdim } A$ .

**Proposition 4.1.** *Let  $\bar{f} : A \rightarrow kQ$  be the homomorphism induced by  $f$ . Then  $\ker \bar{f}$  and  $\text{coker } \bar{f}$  belong to  $\text{Fdim } A$ .*

**Proof.** Let  $I$  and  $S$  be as in Theorem 3.7 and write  $\bar{I}$  and  $\bar{S}$  for their images in  $A$ . Thus,  $\bar{I} = \ker \bar{f}$  and  $\ker \bar{f}$  is generated as a left ideal by  $\bar{S}$ .

Given the description of  $\ker f$  in Theorem 3.7, it suffices to show that  $\bar{I}A_\ell = 0$ .

Let  $x_1 \dots x_s \in S$ . By Lemma 3.4,  $x_1 \dots x_r L_\ell \cap L = \emptyset$ ; in other words,  $x_1 \dots x_r L_\ell \subset (F)$ . Taking the image of this equality in  $A$  we conclude that  $\bar{S}A_\ell = 0$ . It follows that  $\bar{I}A_\ell = 0$ . Thus  $\ker \bar{f}$  belongs to  $\text{Fdim } A$ .

By Lemma 2.2, to show  $\text{coker } \bar{f}$  belongs to  $\text{Fdim } A$  it suffices to show that

$$(kQ_0)\bar{f}(A_\ell) \subset \bar{f}(A_\ell) \quad \text{and} \quad (kQ_1)\bar{f}(A_\ell) \subset \bar{f}(A_{\ell+1}).$$

To do this it suffices to show that  $Q_0 f(L_\ell) \subset f(L_\ell)$  and  $Q_1 f(L_\ell) \subset f(L_{\ell+1})$ .

Let  $x_1 \dots x_\ell \in L_\ell$ . By Lemma 3.1(1), every path labeled  $x_1 \dots x_\ell$  begins at the vertex  $v_0 = x_1 \dots x_\ell$ .

Let  $e$  be a trivial path and  $p$  a path labeled  $x_1 \dots x_\ell$ ; since  $p$  begins at  $v_0$ ,  $ep = p$  if  $e$  is the trivial path at  $v_0$ , and  $ep = 0$  if  $e$  is some other trivial path. Hence  $ef(x_1 \dots x_\ell)$  is either 0 or  $f(x_1 \dots x_\ell)$ . It follows that  $Q_0 f(x_1 \dots x_\ell) = \{f(x_1 \dots x_\ell)\}$  and  $Q_0 f(L_\ell) = f(L_\ell)$ .

Let  $a$  be an arrow and  $p$  a path labeled  $x_1 \dots x_\ell$ . If  $a$  does not end at  $v_0$ , then  $ap = 0$  because  $p$  begins at  $v_0$ ; thus, if  $a$  does not end at  $v_0$ , then  $af(x_1 \dots x_\ell) = 0$ .

We now assume  $a$  ends at  $v_0$ ; i.e.,  $v_{-1} \xrightarrow{a} v_0$  and the arrow  $a$  is labeled by the first letter of  $v_{-1}$ , say  $x_0$ . The path  $ap$  is therefore labeled  $x_0 x_1 \dots x_\ell$ . Since  $v_0 \triangleleft x_0 v_1$ ,  $a$  is the only arrow labeled  $x_0$  that ends at  $v_0$ . Therefore

$$\begin{aligned} af(x_1 \dots x_\ell) &= f(x_0)f(x_1 \dots x_\ell) \\ &= f(x_0 x_1 \dots x_\ell). \end{aligned}$$

In particular,  $af(x_1 \dots x_\ell) \in f(L_{\ell+1})$ .

This completes the proof that  $Q_1 f(L_\ell) \subset f(L_{\ell+1})$  and, as explained before, this implies  $\text{coker } \bar{f}$  belongs to  $\text{Fdim } A$ .  $\square$

**Theorem 4.2.** *Let  $A$  be a connected graded monomial algebra as in (1.1) and/or (3.1). Let  $Q$  be its Ufnarowski graph and view  $kQ$  as a left  $A$ -module through the homomorphism  $\bar{f} : A \rightarrow kQ$ . Then  $-\otimes_A kQ$  induces an equivalence of categories  $\text{QGr } A \cong \text{QGr } kQ$ .*

**Proof.** This follows from Propositions 2.1 and 4.1.  $\square$

##### 4.2. The proof of Theorem 1.1 when $A$ is as in (1.2)

Let  $Q'$  be a finite quiver and  $A = kQ'/I$  the quotient of its path algebra by an ideal generated by a finite number of paths. (Thus  $A$  is a more general kind of monomial algebra.) The subalgebra

$$A' = k \oplus A_1 \oplus A_2 \oplus \dots$$



is of finite codimension in  $A$  so  $A/A' \in \text{Fdim } A'$ . Proposition 2.1 therefore implies that  $-\otimes_{A'} A$  induces an equivalence of categories

$$\text{QGr } A' \equiv \text{QGr } A. \quad (4.1)$$

Since  $A'$  is a monomial algebra of the form (1.1), Theorem 4.2 gives an equivalence

$$\text{QGr } A' \equiv \text{QGr } kQ \quad (4.2)$$

where  $Q$  is the Ufnarovskii graph of  $A'$ . By (4.1) and (4.2),

$$\text{QGr } A \equiv \text{QGr } kQ.$$

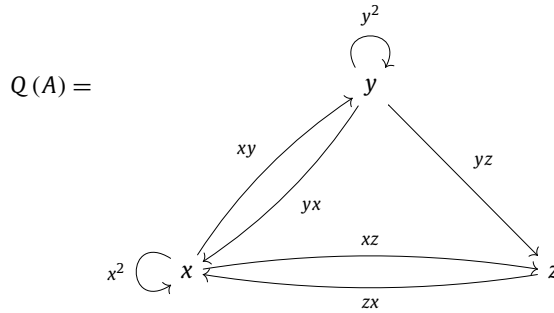
This completes the proof of Theorem 1.1 for  $kQ'/I$ .

## 5. An example

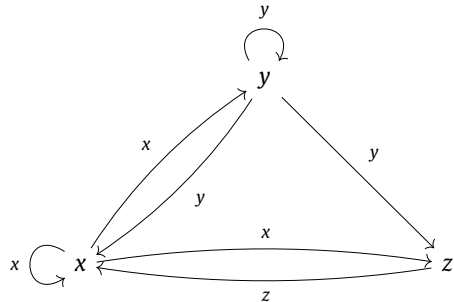
Let  $A = k\langle x, y, z \rangle / (z^2, zy)$ . Since  $\ell = 1$ ,  $Q_0 = \{x, y, z\}$ . The arrows for  $Q(A)$  correspond to the legal words of length two, namely

$$\{x^2, xy, xz, y^2, yx, yz, z^2, zx, zy\} - \{z^2, zy\}.$$

The Ufnarovskii graph of  $A$  is therefore



(the arrows are denoted by  $w$  rather than  $a_w$ ) with labels



Thus, the homomorphism  $f$  is

$$f(x) = a_{x^2} + a_{xy} + a_{xz},$$

$$f(y) = a_{y^2} + a_{yx} + a_{yz},$$

$$f(z) = a_{zx}.$$

## 6. Connected graded quadratic monomial algebras

Section 6.1 contains a short proof of Theorem 4.2 for connected graded monomial algebras with quadratic relations. Section 6.2 shows that Theorem 4.2 for an arbitrary finitely presented connected graded monomial algebra  $A$  can be deduced from the quadratic case.

6.1. Let  $A$  be a quadratic monomial algebra and  $Q$  its Ufnarovskii graph.

The defining relations for  $A$  have length 2 so  $\ell = 1$ . The set of vertices for  $Q$  is therefore in bijection with  $G$ . There is an arrow  $a_{xy}$  from vertex  $x$  to vertex  $y$  if and only if  $xy \notin F$  and that arrow is labeled  $x$  if it exists. It follows that the map  $f : k\langle G \rangle \rightarrow kQ$  defined in Section 3 can be defined as follows:

$$f(x) = \text{the sum of all arrows that start at } x.$$

Thus, if  $r \geq 2$ , then

$$f(x_1 \dots x_r) = \begin{cases} pf(x_r) & \text{where } p \text{ is the unique path labeled} \\ & x_1 \dots x_{r-1} \text{ that ends at vertex } x_r; \\ 0 & \text{if there is no such } p. \end{cases}$$

In particular, if  $xy \in F$ , there is no arrow from  $x$  to  $y$  so  $f(xy) = 0$ . Thus  $f(F) = 0$  and there is an induced map  $\tilde{f} : A \rightarrow kQ$ .

The lemmas in Section 3 are either trivial or unnecessary in the quadratic case. The proof that  $\ker \tilde{f}$  belongs to  $\text{Fdim } A$  is also much simpler.

6.2. Let  $n$  be a positive integer. The  $n$ th Veronese subalgebra of a  $\mathbb{Z}$ -graded algebra  $B$  is

$$B^{(n)} := \bigoplus_{i \in \mathbb{Z}} B_{in}.$$

**Theorem 6.1** (Backelin–Fröberg). (See [2, Prop. 3].) If  $A$  is a connected graded  $k$ -algebra with defining relations of degree  $\leq d + 1$ , then  $A^{(n)}$  is a quadratic algebra for all  $n \geq d$ .

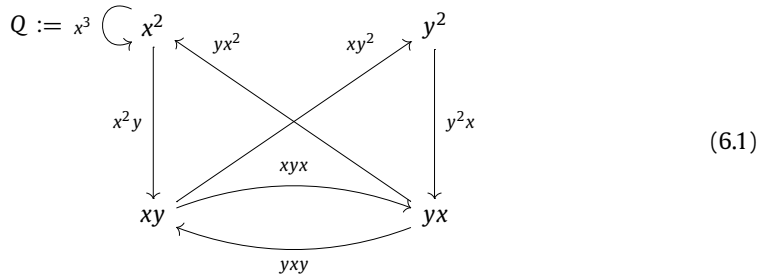
**Theorem 6.2** (Verevkin). (See [9, Thm. 4.4].) Let  $A$  be a connected graded algebra generated by  $A_1$ . Then  $\text{QGr } A \equiv \text{QGr } A^{(n)}$  for all positive integers  $n$ .

**Proposition 6.3.** If Theorem 1.1 holds for connected graded quadratic monomial algebras it holds for all connected graded monomial algebras.

**Proof.** Let  $A$  be a monomial algebra and give  $\ell$ ,  $F$  and  $G$  the meanings they have in Section 3.

By Theorem 6.1,  $A^{(\ell)}$  is a quadratic algebra. Because  $A$  is a monomial algebra so is  $A^{(\ell)}$ . By Theorem 6.2,  $\text{QGr } A \equiv \text{QGr } A^{(\ell)}$ . Hence if Theorem 1.1 holds for  $A^{(\ell)}$ , then  $\text{QGr } A \equiv \text{QGr } kQ'$  where  $Q'$  is the Ufnarovskii graph for  $A^{(\ell)}$ .  $\square$

6.3. The Ufnarovskii graph for  $A^{(\ell)}$  is more complicated than that for  $A$ . For example, the Ufnarovskii graph for  $A = k\langle x, y \rangle / (y^3)$  is



where the arrows are denoted by  $w$  rather than  $a_w$ . The homomorphism  $\bar{f} : A \rightarrow kQ$  is given by

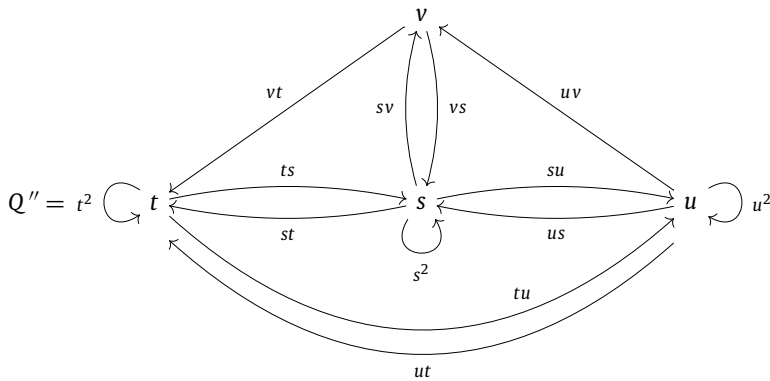
$$\bar{f}(x) = a_{x^3} + a_{x^2y} + a_{xyx} + a_{xy^2},$$

$$\bar{f}(y) = a_{yx^2} + a_{yxy} + a_{y^2x}.$$

The 2-Veronese subalgebra of  $A$  is generated by  $s = x^2$ ,  $t = xy$ ,  $u = yx$ , and  $v = y^2$ . We have

$$A^{(2)} \cong \frac{k\langle s, t, u, v \rangle}{(vu, tv, v^2)}$$

so its Ufnarovskii graph is



The homomorphism  $f : k\langle s, t, u, v \rangle / (vu, tv, v^2) \rightarrow kQ''$  is given by

$$\bar{f}(s) = a_{s^2} + a_{st} + a_{su} + a_{sv},$$

$$\bar{f}(t) = a_{t^2} + a_{ts} + a_{tu},$$

$$\bar{f}(u) = a_{u^2} + a_{us} + a_{ut} + a_{uv},$$

$$\bar{f}(v) = a_{vs} + a_{vt}.$$

## 7. A remark

The results in [5] and [6] show that many different  $Q$  give rise to the equivalent categories  $\text{QGr}kQ$ . Thus, given a finitely presented connected graded monomial algebra  $A$ , the Ufnarovskii graph is not the only  $Q$  for which  $\text{QGr} A$  is equivalent to  $\text{QGr}kQ$ .

Consider, in particular,

$$A = \frac{k\langle x, y \rangle}{(y^3)}.$$

The Ufnarovskii graphs for  $A$  and  $A^{(2)}$  appear in Section 6.3. Since  $A^{(\ell)}$  is quadratic for all  $\ell \geq 2$ ,  $\text{QGr}kQ(A) \equiv \text{QGr}kQ(A^{(\ell)})$  for all  $\ell \geq 2$ .

Furthermore, by [4],  $\text{QGr} A$  is also equivalent to  $\text{QGr}kQ'$  where

$$Q' = \begin{array}{c} \text{0} \xrightarrow{\quad} \text{1} \xrightarrow{\quad} \text{2} \\ \text{0} \xleftarrow{\quad} \text{1} \xleftarrow{\quad} \text{2} \end{array} \quad (7.1)$$

There is a direct proof of the equivalence  $\text{QGr}kQ(A) \equiv \text{QGr}kQ'$ .

**Theorem 7.1.** (See [6].) Let  $L$  and  $R$  be  $\mathbb{N}$ -valued matrices such that  $LR$  and  $RL$  make sense. Let  $Q^{LR}$  be the quiver with incidence matrix  $LR$  and  $Q^{RL}$  the quiver with incidence matrix  $RL$ . There is an equivalence of categories

$$\text{QGr}kQ^{LR} \equiv \text{QGr}kQ^{RL}.$$

The equivalence  $\text{QGr}kQ(A) \equiv \text{QGr}kQ'$  follows from Theorem 7.1 because  $Q(A) = Q^{LR}$  and  $Q' = Q^{RL}$  where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

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