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Isotropy over function fields of Pfister forms

James O'Shea^{a,b,*}

^a School of Mathematical Sciences, University College Dublin, Ireland

^b Zukunftscolleg, D216, Universität Konstanz, D-78457 Konstanz, Germany

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ABSTRACT

The question of which quadratic forms become isotropic when extended to the function field of a given form is studied. A formula for the minimum dimension of the minimal isotropic forms associated to such extensions is given, and some consequences thereof are outlined. Especial attention is devoted to function fields of Pfister forms. Here, the relationship between excellence concepts and the isotropy question is explored. Moreover, in the case where the ground field is formally real and has finite Hasse number, the isotropy question is answered for forms of sufficiently large dimension.

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1. Introduction

Certain field invariants in quadratic form theory (for example the u -invariant, the Hasse number, the Pythagoras number) are defined as the suprema of the dimensions of anisotropic quadratic forms of a given type. A fruitful method of establishing that such an invariant attains a particular value was introduced by Merkurjev (see [18]). It serves as one source of motivation for the following question.

Question 1.1. Given a quadratic form φ over a field F , which anisotropic quadratic forms over F become isotropic when extended to the function field of φ over F ?

While this question appears to be extremely difficult to resolve, some noteworthy progress has been made in this direction (see [17, Chapter X]). More is known regarding the following related question.

* Correspondence to: School of Mathematical Sciences, University College Dublin, Ireland.

E-mail address: james.oshea@ucd.ie.

Question 1.2. Given a quadratic form φ over a field F , which anisotropic forms over F become hyperbolic when extended to the function field of φ over F ?

The Cassels–Pfister Subform Theorem [17, Chapter X, Theorem 4.5] gives a partial answer to this question by providing necessary conditions in terms of subform containment. One would obtain a complete answer to Question 1.1, again in terms of subform containment, if one could classify the minimal forms with respect to subform containment that become isotropic over the function field of φ . Towards this end, we study the dimensions of such forms in the second section of this article.

In the case where φ is a Pfister form, a complete answer is known to Question 1.2 [17, Chapter X, Theorem 4.9]. Thus, it is justified to devote particular attention to Question 1.1 in the context of function fields of Pfister forms, particularly since the property of excellence can only arise for such function fields (see [16, Theorem 7.13]). Consequently, sections three, four and five of this paper are primarily concerned with addressing Question 1.1 for function fields of Pfister forms. The third section explores excellence concepts and their relation to Question 1.1. Building on this, the fourth section answers Question 1.1 for forms of certain dimensions, and provides bounds on the range of these dimensions. The final section tackles Question 1.1 for function fields of Pfister forms over formally real fields of finite Hasse number, and offers an answer for forms whose dimension is greater than the Hasse number.

Throughout, we highlight cases where our investigations allow for short or simple recoveries of established results.

Henceforth, we will let F denote a field of characteristic different from two and $n \in \mathbb{N}$. The term “form” will refer to a regular quadratic form. Every form over F can be diagonalised. Given $a_1, \dots, a_n \in F^\times$, one denotes by $\langle a_1, \dots, a_n \rangle$ the n -dimensional quadratic form $a_1X_1^2 + \dots + a_nX_n^2$. If φ and ψ are forms over F , we denote by $\varphi \perp \psi$ their orthogonal sum and by $\varphi \otimes \psi$ their tensor product. We will denote the orthogonal sum of n copies of φ by $n \times \varphi$. We use $a\varphi$ to denote $\langle a \rangle \otimes \varphi$ for $a \in F^\times$. We write $\varphi \simeq \psi$ to indicate that φ and ψ are isometric. Two forms φ and ψ over F are *similar* if $\varphi \simeq a\psi$ for some $a \in F^\times$. For φ a form over F and K/F a field extension, we write φ_K when we view φ as a form over K . A form over F is *isotropic* if it represents zero non-trivially, and *anisotropic* otherwise. Every form φ has a decomposition $\varphi \simeq \psi \perp i \times \langle 1, -1 \rangle$ where the anisotropic form ψ and the integer i are uniquely determined, with ψ being referred to as the *anisotropic part* of φ , denoted φ_{an} , and i being labelled the *Witt index* of φ , denoted $i_W(\varphi)$. A form φ is *hyperbolic* if its anisotropic part is trivial, whereby $i_W(\varphi) = \frac{1}{2} \dim \varphi$. A form τ is a *subform* of φ if $\varphi \simeq \tau \perp \gamma$ for some form γ , in which case we will write $\tau \subset \varphi$. The following basic fact (see [7, Lemma 3]) will be employed frequently.

Lemma 1.3. *If $\tau \subset \varphi$, then $i_W(\tau) \geq i_W(\varphi) - (\dim \varphi - \dim \tau)$. In particular, if $\tau \subset \varphi$ and $\dim \tau \geq \dim \varphi - i_W(\varphi) + 1$, then τ is isotropic.*

Given $n \in \mathbb{N}$, an n -fold Pfister form is a form isometric to $\langle 1, a_1 \rangle \otimes \dots \otimes \langle 1, a_n \rangle$ for some $a_1, \dots, a_n \in F^\times$. We let PF denote the class of Pfister forms over F , and $P_n F$ the class of n -fold Pfister forms over F . For $\pi \in PF$, a form τ over F is a *generalised Pfister neighbour* of π if there exists a form γ over F such that $\tau \subset \pi \otimes \gamma$ and $\dim \tau > \frac{1}{2} \dim(\pi \otimes \gamma)$. In particular, if $\dim \gamma = 1$ then τ is said to be a *Pfister neighbour* of π . Since isotropic Pfister forms are hyperbolic [17, Chapter X, Theorem 1.7], Lemma 1.3 demonstrates that the isotropy of a Pfister form implies the isotropy of its generalised Pfister neighbours.

For a form φ over F with $\dim \varphi = n \geq 2$ and $\varphi \not\simeq \langle 1, -1 \rangle$, the *function field* $F(\varphi)$ of φ is the quotient field of the integral domain $F[X_1, \dots, X_n]/(\varphi(X_1, \dots, X_n))$ (this is the function field of the affine quadric $\varphi(X) = 0$ over F). As per [17, Chapter X, Theorem 4.1], $F(\varphi)/F$ is a purely-transcendental extension if and only if φ is isotropic over F . To avoid case distinctions, we set $F(\varphi) = F$ if $\dim \varphi \leq 1$ or $\varphi \simeq \langle 1, -1 \rangle$. The positive integer $i_W(\varphi_{F(\varphi)})$ is called the *first Witt index* of φ , and is denoted by $i_1(\varphi)$. For all extensions K/F such that φ_K is isotropic, $i_1(\varphi) \leq i_W(\varphi_K)$ (see [15, Proposition 3.1 and Theorem 3.3]). For an anisotropic form φ over F , the *essential dimension* of φ , defined by Izhboldin in [12], is given by $\text{edim}(\varphi) = \dim \varphi - i_1(\varphi) + 1$. We will often invoke [7, Theorem 1]:

Theorem 1.4 (Hoffmann). *Let ψ be anisotropic over F . If $\dim \psi \leq 2^n < \dim \varphi$ for some $n \in \mathbb{N}$, then $\psi_{F(\varphi)}$ is anisotropic.*

Two anisotropic forms φ and ψ over F are *isotropy equivalent* if for every K/F we have that φ_K is isotropic if and only if ψ_K is isotropic.

Lemma 1.5. *Let φ , ψ and γ be anisotropic forms over F . Then*

- (a) φ and ψ are isotropy equivalent if and only if $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic.
- (b) If φ and ψ are isotropy equivalent, then $i_W(\gamma_{F(\varphi)}) = i_W(\gamma_{F(\psi)})$.

Proof. (a) The left-to-right implication is clear. Suppose that φ_K is isotropic for some K/F . Since $\psi_{F(\varphi)}$ is isotropic, we have that $\psi_{K(\varphi)}$ is isotropic. As φ_K is isotropic, $K(\varphi)/K$ is purely transcendental, whereby we can conclude that ψ_K is isotropic. The converse follows by symmetry.

(b) Invoking (a), we have that $F(\varphi, \psi)$ is a purely-transcendental extension of $F(\varphi)$ and of $F(\psi)$. Thus $i_W(\gamma_{F(\varphi)}) = i_W(\gamma_{F(\varphi, \psi)}) = i_W(\gamma_{F(\psi)})$. \square

Lemma 1.6. *If φ is an anisotropic form with $\dim \varphi = 2^n + k$ where $0 < k \leq 2^n$, then $i_1(\varphi) \leq k$ and $\text{edim}(\varphi) \geq 2^n + 1$. In particular, $i_1(\varphi) = k$ if and only if $\text{edim}(\varphi) = 2^n + 1$.*

Proof. Let τ be a 2^n -dimensional F -subform of φ . Theorem 1.4 implies that $\tau_{F(\varphi)}$ is anisotropic. If $i_1(\varphi) > k$, then $\dim \tau = 2^n \geq (2^n + k) - i_1(\varphi) + 1$, whereby Lemma 1.3 implies that $\tau_{F(\varphi)}$ is isotropic, a contradiction. \square

We say that a form φ as above has *maximal splitting* if $i_1(\varphi) = k$. Forms of dimension $2^n + 1$ clearly have maximal splitting.

Lemma 1.7. *Let ψ be an anisotropic Pfister neighbour of an n -fold Pfister form π . Then $\text{edim}(\psi) = 2^{n-1} + 1$. Thus, a Pfister neighbour has maximal splitting.*

Proof. Let $\dim \psi = 2^{n-1} + k$ where $0 < k \leq 2^{n-1}$. Combining Lemma 1.3 with Theorem 1.4, we have that $i_W(\psi_{F(\pi)}) = k$. Lemma 1.5(a) implies that ψ and π are isotropy equivalent. Hence $i_W(\psi_{F(\pi)}) = i_1(\psi)$ by Lemma 1.5(b), whereby $\text{edim}(\psi) = 2^{n-1} + 1$. \square

Given an extension K/F , a form ψ over F is *minimal K -isotropic* if ψ is anisotropic, ψ_K is isotropic and, for every proper F -subform φ of ψ , the form φ_K is anisotropic. Invoking Lemma 1.3, one can make the following observation.

Lemma 1.8. *Every minimal K -isotropic form ψ over F satisfies $i_W(\psi_K) = 1$.*

For further details regarding the above, we refer the reader to [17, Chapter X].

2. Minimal isotropy and essential dimension

Since every form over F that becomes isotropic over K contains a minimal K -isotropic form over F , a determination of the minimal K -isotropic forms over F would provide an answer to the question of which anisotropic forms over F become isotropic when extended to K . Towards this end, we introduce the following set:

$$\mathcal{M}(K/F) = \{\dim \psi \mid \psi \text{ is a minimal } K\text{-isotropic form over } F\}.$$

The invariants $\min \mathcal{M}(K/F)$ and $\sup \mathcal{M}(K/F)$ were introduced in [5], wherein they are denoted by $t_{\min}(K/F)$ and $t_{\max}(K/F)$, and have since been studied in the case where $K = F(\varphi)$ for φ a form

over F . In particular, it was shown in [10, Section 4] that $\sup \mathcal{M}(F(\varphi)/F)$ can be infinite when $\dim \varphi = 3$.

Throughout this section, we will regularly employ the isotropy criteria provided by [14, Theorem 4.1]:

Theorem 2.1 (Karpenko, Merkurjev). *Let φ and ψ be anisotropic forms over F .*

- (a) *If $\psi_{F(\varphi)}$ is isotropic, then $\text{edim}(\psi) \geq \text{edim}(\varphi)$.*
- (b) *If $\text{edim}(\psi) = \text{edim}(\varphi)$, then $\psi_{F(\varphi)}$ is isotropic if and only if $\varphi_{F(\psi)}$ is isotropic.*

Corollary 2.2. *If φ and ψ are anisotropic isotropy-equivalent forms over F , then $\text{edim}(\varphi) = \text{edim}(\psi)$.*

Proof. This follows immediately from combining Theorem 2.1(a) with Lemma 1.5. \square

Corollary 2.3. *Let φ and ψ be anisotropic forms over F . If $\psi_{F(\varphi)}$ is isotropic, then $\dim \psi \geq \text{edim}(\varphi)$.*

Proof. This follows immediately from Theorem 2.1(a). \square

Lemma 2.4. *Let φ and ψ be anisotropic forms over F .*

- (a) *Then $\min \mathcal{M}(F(\varphi)/F) \leq \text{edim}(\varphi)$.*
- (b) *If $\psi_{F(\varphi)}$ is isotropic, then $\text{edim}(\psi) \geq \dim \psi - i_W(\psi_{F(\varphi)}) + 1 \geq \min \mathcal{M}(F(\varphi)/F)$.*

Proof. (a) Lemma 1.3 implies that every F -subform of φ of dimension $\text{edim}(\varphi)$ is isotropic over $F(\varphi)$, whereby the result follows.

(b) Lemma 1.3 implies that every F -subform of ψ of dimension $\dim \psi - i_W(\psi_{F(\varphi)}) + 1$ is isotropic over $F(\varphi)$, whereby $\min \mathcal{M}(F(\varphi)/F) \leq \dim \psi - i_W(\psi_{F(\varphi)}) + 1$. Since $\psi_{F(\varphi)}$ is isotropic, we have that $i_1(\psi) \leq i_W(\psi_{F(\varphi)})$ and the result follows. \square

Remark 2.5. Theorem 2.1(a) was established through the usage of advanced algebro-geometric machinery. Given its importance as an isotropy criterion over function fields of quadratic forms, it would be desirable to obtain a proof of this result solely by means of classical quadratic form theory. To this end, the following argument demonstrates that it suffices to find such a proof of Corollary 2.3:

Let φ and ψ be anisotropic forms over F . Assuming Corollary 2.3, we have that $\min \mathcal{M}(F(\varphi)/F) \geq \text{edim}(\varphi)$. Hence, if $\psi_{F(\varphi)}$ is isotropic, Lemma 2.4(b) implies that $\text{edim}(\psi) \geq \min \mathcal{M}(F(\varphi)/F)$, whereby Theorem 2.1(a) follows.

Our next result relates the essential dimension of φ to the problem of determining the minimal $F(\varphi)$ -isotropic forms over F .

Theorem 2.6. *Let φ be an anisotropic form over F . Then $\min \mathcal{M}(F(\varphi)/F) = \text{edim}(\varphi)$.*

Proof. Corollary 2.3 implies that $\min \mathcal{M}(F(\varphi)/F) \geq \text{edim}(\varphi)$. Equality follows from Lemma 2.4(a). \square

[1, Example 1.5] demonstrates that there exist anisotropic 5-dimensional isotropy-equivalent forms which are non-similar. In particular, since an anisotropic 5-dimensional form φ over F trivially has maximal splitting, and therefore satisfies $\min \mathcal{M}(F(\varphi)/F) = 5$ by Theorem 2.6, this shows that minimal $F(\varphi)$ -isotropic forms of minimum dimension need not be similar to subforms of φ .

Corollary 2.7. *Let ψ and φ be anisotropic isotropy-equivalent forms over F . Every F -subform of ψ of dimension $\dim \psi - i_W(\psi_{F(\varphi)}) + 1$ is a minimal $F(\varphi)$ -isotropic form of minimum dimension.*

Proof. Lemma 1.3 implies that such subforms of ψ are isotropic over $F(\varphi)$. Since ψ and φ are isotropy-equivalent forms, invoking Lemma 1.5(b) and Corollary 2.2, we have that $\dim \psi - i_W(\psi_{F(\varphi)}) + 1 = \text{edim}(\psi) = \text{edim}(\varphi)$. Hence, Theorem 2.6 implies that $\dim \psi - i_W(\psi_{F(\varphi)}) + 1 = \min \mathcal{M}(F(\varphi)/F)$. \square

Corollary 2.8. *An anisotropic form φ over F satisfies $i_1(\varphi) = 1$ if and only if φ is a minimal $F(\varphi)$ -isotropic form.*

Proof. Lemma 1.8 gives the right-to-left implication. The converse follows from Theorem 2.1(a), Theorem 2.6 or Corollary 2.7. \square

Corollary 2.7 does not hold for arbitrary pairs of anisotropic forms. As per [5, Section 3.3], there exists an example of a field F and anisotropic forms γ and π over F , where $\dim \gamma = 6$ and $\pi \in P_2 F$, such that $i_W(\gamma_{F(\pi)}) = 1$ but γ is not a minimal $F(\pi)$ -isotropic form (indeed, it is shown that γ contains two non-similar 5-dimensional minimal $F(\pi)$ -isotropic forms over F). In Section 5 we will provide a complementary example, Example 5.8, which demonstrates that for φ and γ anisotropic forms over a field F such that $i_W(\gamma_{F(\varphi)}) = n$, the form γ need not contain any minimal $F(\varphi)$ -isotropic forms over F of dimension $\dim \gamma - n + 1$.

Theorem 2.6 allows us to describe those forms which have maximal splitting.

Corollary 2.9. *An anisotropic form ψ over F has maximal splitting if and only if there exists an anisotropic form φ over F with $\dim \varphi - i_W(\varphi_{F(\psi)}) = 2^{n-1}$, where n is such that $2^{n-1} < \dim \varphi$ and $2^{n-1} < \dim \psi \leq 2^n$.*

Proof. Letting $\varphi = \psi$ gives the left-to-right implication. Conversely, if φ is such that $\dim \varphi - i_W(\varphi_{F(\psi)}) = 2^{n-1}$, then Lemma 1.3 implies that every F -subform of φ of dimension $2^{n-1} + 1$ is isotropic over $F(\psi)$. Hence, $\min \mathcal{M}(F(\psi)/F) = 2^{n-1} + 1$ by Theorem 1.4, whereby Theorem 2.6 and Lemma 1.6 imply that ψ has maximal splitting. \square

Returning to Theorem 2.1 itself, our next two results highlight the extreme cases, where equality of the respective essential dimensions is forced.

Corollary 2.10. *Let φ and ψ be anisotropic forms over F such that $2^{n-1} < \dim \psi \leq 2^n$ and $2^{n-1} < \dim \varphi \leq 2^n$. Assume that ψ has maximal splitting. Then $\psi_{F(\varphi)}$ is isotropic if and only if φ and ψ are isotropy equivalent.*

Proof. The right-to-left implication follows from Lemma 1.5(a). Conversely, by Lemma 1.6, we have that $\text{edim}(\varphi) \geq 2^{n-1} + 1 = \text{edim}(\psi)$. Invoking Theorem 2.1(a), we have that $\text{edim}(\varphi) = \text{edim}(\psi)$, whereby Theorem 2.1(b) and Lemma 1.5(a) imply that φ and ψ are isotropy equivalent. \square

Corollary 2.11. *Let φ and ψ be anisotropic forms over F such that $\dim \varphi = \dim \psi$ and $i_1(\varphi) = 1$. The following are equivalent:*

- (a) $\psi_{F(\varphi)}$ is isotropic,
- (b) φ and ψ are isotropy equivalent,
- (c) ψ is a minimal $F(\varphi)$ -isotropic form over F of minimum dimension.

Proof. Assuming (a), the hypotheses and Theorem 2.1(a) give $\dim \psi \geq \text{edim}(\psi) \geq \text{edim}(\varphi) = \dim \varphi = \dim \psi$. Thus $\text{edim}(\psi) = \text{edim}(\varphi)$, whereby Theorem 2.1(b) implies (b). Since $\dim \psi = \dim \varphi = \text{edim}(\varphi)$, the hypothesis of (b) and Theorem 2.6 imply that ψ is a minimal $F(\varphi)$ -isotropic form over F of minimum dimension, establishing (c). Finally, (c) clearly implies (a). \square

We next record some characterisations of anisotropic Pfister neighbours. The equivalence of (a) and (b) in the following result is due to Hoffmann [7, Proposition 2].

Proposition 2.12. *Let π be an anisotropic n -fold Pfister form over F and γ an anisotropic form over F . The following are equivalent:*

- (a) γ is a Pfister neighbour of π ,
- (b) γ and π are isotropy equivalent,
- (c) γ has maximal splitting, $\dim \gamma \leq \dim \pi$ and $\gamma_{F(\pi)}$ is isotropic.

Proof. As per [7, Proposition 2], statements (a) and (b) are equivalent. Moreover, invoking Lemma 1.7, Pfister neighbours have maximal splitting, whereby (a) clearly implies (c).

Assuming (c), since $\gamma_{F(\pi)}$ is isotropic, Theorem 1.4 implies that $\dim \gamma > 2^{n-1}$, whereby $\dim \gamma = 2^{n-1} + k$ for some $0 < k \leq 2^{n-1}$. As γ has maximal splitting, we have that $\text{edim}(\gamma) = 2^{n-1} + 1 = \text{edim}(\pi)$, whereby Theorem 2.1(b) and Lemma 1.5(a) imply that (b) holds. \square

Corollary 2.13. *Let π be an anisotropic n -fold Pfister form over F and γ an anisotropic form over F such that $\dim \gamma = 2^{n-1} + 1$. If $\gamma_{F(\pi)}$ is isotropic, then γ is a Pfister neighbour of π .*

Proof. This follows directly from Proposition 2.12, since $\text{edim}(\gamma) = 2^{n-1} + 1$. \square

As per Lemma 1.7, Pfister neighbours have maximal splitting. In general, forms with maximal splitting need not be Pfister neighbours (see [7, Example 2] for some non-trivial examples). However, this correspondence does hold for forms of certain dimension (see [13] for more details), and is invoked in the following proof of [7, Theorem 3].

Theorem 2.14 (Hoffmann). *Let ψ be an anisotropic form over F with $\dim \psi = 2^{n-1} + 1$ for some $n \geq 4$. Let γ be an anisotropic form over F with $2^n - 3 \leq \dim \gamma$. Then $\psi_{F(\gamma)}$ is isotropic if and only if there exists an n -fold Pfister form π such that ψ and γ are Pfister neighbours of π .*

Proof. The right-to-left implication is clear. Conversely, since $\dim \psi = 2^{n-1} + 1$, we have that $\text{edim}(\psi) = 2^{n-1} + 1$. If $\psi_{F(\gamma)}$ is isotropic, then $\dim \gamma \leq 2^n$ by Theorem 1.4, whereby ψ and γ are isotropy equivalent by Corollary 2.10. Then $\text{edim}(\gamma) = \text{edim}(\psi)$ by Corollary 2.2, whereby γ has maximal splitting. As $2^n - 3 \leq \dim \gamma \leq 2^n$, [15, Theorem 5.8] and [16, Corollary 8.2] imply that γ is a Pfister neighbour of some $\pi \in P_n F$. Hence, Lemma 1.5(a) implies that γ and π are isotropy equivalent, whereby ψ and π are isotropy equivalent. Thus, Proposition 2.12 implies that ψ is a Pfister neighbour of π . \square

3. Excellence

A field extension K/F is said to be *excellent* if, for every form ϑ over F , the anisotropic part of ϑ_K is defined over F , that is, $(\vartheta_K)_{\text{an}} \simeq \gamma_K$ for some form γ over F . For $m \in \mathbb{N}$, we say that K/F is *m -excellent* if, for every form ϑ over F with $\dim \vartheta \leq m$, there exists a form γ over F such that $(\vartheta_K)_{\text{an}} \simeq \gamma_K$.

Combining [16, Theorem 7.13] and [7, Proposition 3], the following is known:

Proposition 3.1 (Knebusch (\Rightarrow) and Hoffmann (\Leftarrow)). *Let φ be an anisotropic form over F . Then $(\varphi_{F(\varphi)})_{\text{an}}$ is defined over F if and only if φ is a Pfister neighbour.*

Thus, the only anisotropic quadratic forms whose function fields can be excellent are Pfister neighbours. Indeed, for $\pi \in P_n F$ anisotropic, the extension $F(\pi)/F$ is excellent when $n \leq 2$, and is not excellent in general when $n \geq 3$ (see [2, Chapter IV, Section 29]).

As a result of Proposition 3.1, if φ is not a Pfister neighbour, then $F(\varphi)/F$ is not $(\dim \varphi)$ -excellent. However, one may justifiably examine m -excellence for arbitrary function fields $F(\varphi)/F$ when m is less than $\dim \varphi$, and the opening comments of this section address this topic.

Proposition 3.2. *If φ is an anisotropic form over F with $\dim \varphi > 2^n$, then $F(\varphi)/F$ is 2^n -excellent.*

Proof. This follows directly from Theorem 1.4, since an anisotropic form ψ over F with $\dim \psi \leq 2^n$ is such that $\psi_{F(\varphi)}$ is anisotropic. \square

Proposition 3.3. *Let φ and ψ be anisotropic isotropy-equivalent forms over F and γ an anisotropic form over F . Then $(\gamma_{F(\varphi)})_{\text{an}}$ is defined over F if and only if $(\gamma_{F(\psi)})_{\text{an}}$ is defined over F .*

Proof. Invoking Lemma 1.5(a), we have that $F(\varphi, \psi)$ is a purely-transcendental extension of $F(\varphi)$ and of $F(\psi)$. Assume that $(\gamma_{F(\varphi)})_{\text{an}}$ is defined over F , with $(\gamma_{F(\varphi)})_{\text{an}} \simeq \delta_{F(\varphi)}$ for some form δ over F . Thus $\gamma \perp -\delta$ is hyperbolic over $F(\varphi, \psi)$, whereby it is hyperbolic over $F(\psi)$. Hence $(\gamma_{F(\psi)})_{\text{an}} \simeq (\delta_{F(\psi)})_{\text{an}}$. Moreover, since $\delta_{F(\varphi, \psi)}$ is anisotropic, we have that $(\gamma_{F(\psi)})_{\text{an}} \simeq \delta_{F(\psi)}$. The converse follows by symmetry. \square

Our next observation is that Proposition 3.2 cannot be improved in general.

Proposition 3.4. *Let ψ and φ be anisotropic forms over F such that $2^n + 1 = \dim \psi \leq \dim \varphi$ for some $n \in \mathbb{N}$ and $\psi_{F(\varphi)}$ is isotropic. Then $(\psi_{F(\varphi)})_{\text{an}}$ is defined over F if and only if ψ is a Pfister neighbour.*

Proof. Since $\psi_{F(\varphi)}$ is isotropic, Theorem 2.1(a) implies that $\text{edim}(\psi) \geq \text{edim}(\varphi)$. Invoking Lemma 1.6, we have that $\text{edim}(\varphi) \geq 2^n + 1 = \dim \psi \geq \text{edim}(\psi)$. Thus $\text{edim}(\varphi) = \text{edim}(\psi)$, and hence Theorem 2.1(b) and Lemma 1.5(a) imply that φ and ψ are isotropy equivalent. Thus, $(\psi_{F(\varphi)})_{\text{an}}$ is defined over F if and only if $(\psi_{F(\psi)})_{\text{an}}$ is defined over F by Proposition 3.3, which occurs if and only if ψ is a Pfister neighbour by Proposition 3.1. \square

For the rest of this section, we will consider how the aforementioned excellence concepts relate to $F(\pi)/F$ when π is an anisotropic Pfister form. We begin by examining which forms φ over F are such that $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F . Pfister neighbours of π have this property. Indeed, if $a \in F^\times$ and α, μ are forms over F such that $\alpha \perp \mu \simeq a\pi$ with $\dim \alpha > \dim \mu$, then Theorem 1.4 implies that $\mu_{F(\pi)}$ is anisotropic, whereby it follows that $(\alpha_{F(\pi)})_{\text{an}} \simeq -\mu_{F(\pi)}$. We next show that certain forms containing Pfister neighbours of π also possess this property.

Proposition 3.5. *Suppose that an anisotropic form φ over F contains a Pfister neighbour τ of an n -fold Pfister form π over F .*

- (a) *If $i_W(\varphi_{F(\pi)}) = 1$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F .*
- (b) *If $\dim \varphi - \dim \tau \leq \dim \tau - 2^{n-1} + 1$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F . In particular, if $\dim \varphi \leq 2^{n-1} + 3$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F .*

Proof. Let $\varphi \simeq \varphi' \perp \tau$ for some F -form φ' , where τ is a Pfister neighbour of π such that $\tau \perp \gamma \simeq a\pi$ for some F -form γ and $a \in F^\times$. Hence, we have that $\varphi_{F(\pi)} \simeq \varphi'_{F(\pi)} \perp i_W(\tau_{F(\pi)}) \times \langle 1, -1 \rangle \perp -\gamma_{F(\pi)}$ and $(\varphi_{F(\pi)})_{\text{an}} \simeq ((\varphi' \perp -\gamma)_{F(\pi)})_{\text{an}}$.

(a) If $i_W(\varphi_{F(\pi)}) = 1$, then $\varphi'_{F(\pi)} \perp -\gamma_{F(\pi)}$ is anisotropic. Thus, we have that $(\varphi_{F(\pi)})_{\text{an}} \simeq (\varphi' \perp -\gamma)_{F(\pi)}$ in this case.

(b) If $\dim \varphi' = \dim \varphi - \dim \tau \leq \dim \tau - 2^{n-1} + 1$, then $\dim(\varphi' \perp -\gamma) \leq 2^{n-1} + 1$. If $\dim(\varphi' \perp -\gamma)_{\text{an}} \leq 2^{n-1}$, then $(\varphi' \perp -\gamma)_{\text{an}}$ remains anisotropic over $F(\pi)$ by Theorem 1.4, whereby $(\varphi_{F(\pi)})_{\text{an}} \simeq ((\varphi' \perp -\gamma)_{\text{an}})_{F(\pi)}$. So we may assume that $\dim(\varphi' \perp -\gamma) = 2^{n-1} + 1$ and that $\varphi' \perp -\gamma$ is anisotropic. If $(\varphi' \perp -\gamma)_{F(\pi)}$ is isotropic, then $\varphi' \perp -\gamma$ is a Pfister neighbour of π by Corollary 2.13. Hence $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F in this case too. \square

In the case where π is a 3-fold Pfister form, [6, Corollary 4.2] states that $F(\pi)/F$ is 6-excellent. Owing to this fact, we can obtain a slight improvement of Proposition 3.5(b) in this case.

Proposition 3.6. Suppose that an anisotropic form φ over F contains a Pfister neighbour τ of a 3-fold Pfister form π over F . If $\dim \varphi - \dim \tau \leq \dim \tau - 2$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F . In particular, if $\dim \varphi \leq 8$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F .

Proof. As above, letting $\varphi \simeq \varphi' \perp \tau$, where τ is a Pfister neighbour of π such that $\tau \perp \gamma \simeq a\pi$, we have that $(\varphi_{F(\pi)})_{\text{an}} \simeq ((\varphi' \perp -\gamma)_{F(\pi)})_{\text{an}}$. If $\dim \varphi' = \dim \varphi - \dim \tau \leq \dim \tau - 2$, then $\dim(\varphi' \perp -\gamma) \leq 6$. Thus, [6, Corollary 4.2] implies that $((\varphi' \perp -\gamma)_{F(\pi)})_{\text{an}}$ is defined over F . \square

As a converse to the above results, we note the following:

Proposition 3.7. Let φ be an anisotropic form over F with $\dim \varphi \leq 2^n$ and π an anisotropic n -fold Pfister form over F . If $\varphi_{F(\pi)}$ is isotropic with $(\varphi_{F(\pi)})_{\text{an}}$ defined over F , then φ contains a Pfister neighbour of π .

Proof. Let $(\varphi_{F(\pi)})_{\text{an}} \simeq \delta_{F(\pi)}$ for some form δ over F . As $(\varphi \perp -\delta)_{F(\pi)}$ is hyperbolic, [17, Chapter X, Theorem 4.11] implies that $(\varphi \perp -\delta)_{\text{an}} \simeq \pi \otimes \vartheta$ for some form ϑ over F . Since $\dim(\varphi \perp -\delta) < 2^{n+1}$, we have that $(\varphi \perp -\delta)_{\text{an}} \simeq \varphi' \perp -\delta' \simeq a\pi$ for some $a \in F^\times$ and F -forms $\varphi' \subset \varphi$ and $\delta' \subset \delta$. Since $\dim \varphi > \dim \delta$, we have that $\dim \varphi' > \dim \delta'$, whereby φ' is a Pfister neighbour of π . \square

For certain generalised Pfister neighbours α of π , we can prove that $(\alpha_{F(\pi)})_{\text{an}}$ is defined over F .

Proposition 3.8. Let α be an anisotropic generalised Pfister neighbour of an n -fold Pfister form π , with μ and ϑ forms over F such that $\alpha \perp \mu \simeq \pi \otimes \vartheta$ with $\dim \alpha > \dim \mu$. If $i_W(\alpha_{F(\pi)}) = 1$, then $(\alpha_{F(\pi)})_{\text{an}} \simeq -\mu_{F(\pi)}$.

Proof. Since $\alpha \perp \mu \simeq \pi \otimes \vartheta$, we have that $(\alpha_{F(\pi)})_{\text{an}} \simeq -(\mu_{F(\pi)})_{\text{an}}$. Moreover, as $\dim \alpha > \dim \mu$, we have that $\dim \alpha - 2 \geq \dim \mu$. Thus, since $i_W(\alpha_{F(\pi)}) = 1$, we have that $\dim(\alpha_{F(\pi)})_{\text{an}} \geq \dim \mu$, whereby $(\alpha_{F(\pi)})_{\text{an}} \simeq -\mu_{F(\pi)}$. \square

We note that the above result does not follow from Proposition 3.5(a), as generalised Pfister neighbours need not contain Pfister neighbours:

Example 3.9. As per [10, Section 4], for a certain field F and a particular $\pi \in P_2 F$, there exists a minimal $F(\pi)$ -isotropic form ψ_m of dimension $2m + 1$ for every $m \in \mathbb{N}$. Since $F(\pi)/F$ is an excellent extension (see [2, Section 29]), [5, Lemma 3.1.2] implies that ψ_m is a generalised Pfister neighbour of π for every $m \in \mathbb{N}$. By minimality, ψ_m does not contain a Pfister neighbour of π when $m \geq 2$.

The “only if” part of the following result is due to Hoffmann [5, Lemma 3.1.2].

Proposition 3.10. Let π be an anisotropic Pfister form over F and ψ a minimal $F(\pi)$ -isotropic form over F . Then $(\psi_{F(\pi)})_{\text{an}}$ is defined over F if and only if ψ is a generalised Pfister neighbour of π .

Proof. If $(\psi_{F(\pi)})_{\text{an}} \simeq \delta_{F(\pi)}$ for some form δ over F , then $\psi \perp -\delta$ becomes hyperbolic over $F(\pi)$. Moreover, $\psi \perp -\delta$ is anisotropic over F , as otherwise we would have that $\psi \simeq \langle d \rangle \perp \psi'$ and $\delta \simeq \langle d \rangle \perp \delta'$ for some $d \in F^\times$ and forms ψ' and δ' over F , whereby $\psi' \perp -\delta'$ would also be hyperbolic over $F(\pi)$. However, since $\dim \psi > \dim \delta$, this cannot occur, as otherwise $\psi'_{F(\pi)}$ would be isotropic, contradicting the minimality of ψ . Thus, [17, Chapter X, Theorem 4.11] implies that $\psi \perp -\delta \simeq \pi \otimes \vartheta$ for some form ϑ over F , whereby ψ is a generalised Pfister neighbour of π .

Conversely, since ψ is a minimal $F(\pi)$ -isotropic form, we have that $i_W(\psi_{F(\pi)}) = 1$ by Lemma 1.8. Hence, if ψ is a generalised Pfister neighbour of π , Proposition 3.8 implies that $(\psi_{F(\pi)})_{\text{an}}$ is defined over F . \square

We conclude this section with some characterisations of excellence and m -excellence for function fields of Pfister forms.

Theorem 3.11. *For an arbitrary extension of fields K/F and $m \in \mathbb{N}$, the following statements are equivalent.*

- (a) K/F is m -excellent,
- (b) $(\psi_K)_{\text{an}}$ is defined over F for every minimal K -isotropic form ψ over F of dimension m or less.

If π is an anisotropic Pfister form over F and $K = F(\pi)$, then these statements are equivalent to the following statement.

- (c) Every minimal $F(\pi)$ -isotropic form over F of dimension m or less is a generalised Pfister neighbour of π .

Proof. Proposition 3.10 establishes the equivalence of (b) and (c) in the case where $K = F(\pi)$ for π an anisotropic Pfister form over F . Clearly, (a) implies (b). To conclude, we will show that (b) implies (a).

Let $\varphi_{F(\pi)}$ be isotropic, where $\dim \varphi \leq m$. Then $\varphi \simeq \psi \perp \gamma$ for ψ some minimal $F(\pi)$ -isotropic form over F and γ some form over F . Since $(\psi_{F(\pi)})_{\text{an}} \simeq \delta_{F(\pi)}$ for some form δ over F , Lemma 1.8 implies that $\psi_{F(\pi)} \simeq \langle 1, -1 \rangle_{F(\pi)} \perp \delta_{F(\pi)}$. Hence $\varphi_{F(\pi)} \simeq \langle 1, -1 \rangle_{F(\pi)} \perp \delta_{F(\pi)} \perp \gamma_{F(\pi)}$. Now consider $\varphi_1 := \delta \perp \gamma$. If φ_1 is anisotropic over $F(\pi)$, then $(\varphi_{F(\pi)})_{\text{an}} \simeq (\varphi_1)_{F(\pi)}$ and we are done. Otherwise, φ_1 is isotropic over $F(\pi)$ and then we finish the proof by induction on $\dim \varphi$. \square

Corollary 3.12. *For an arbitrary extension of fields K/F and $m \in \mathbb{N}$, the following statements are equivalent.*

- (a) K/F is excellent,
- (b) $(\psi_K)_{\text{an}}$ is defined over F for every minimal K -isotropic form ψ over F .

If π is an anisotropic Pfister form over F and $K = F(\pi)$, then these statements are equivalent to the following statement.

- (c) Every minimal $F(\pi)$ -isotropic form over F is a generalised Pfister neighbour of π .

In the case where $K = F(\pi)$ for π an anisotropic Pfister form over F , the equivalence of (a) and (c) in the above was proved by Hoffmann [5, Theorem 3.1.3]. This equivalence provides an answer to Question 1.1 for function fields of Pfister forms that are excellent extensions.

4. Bounds on the dimensions where excellence holds

For $\pi \in P_n F$ anisotropic, the extension $F(\pi)/F$ is known to be excellent when $n \leq 2$. This result, an easy exercise in the case where $n = 1$, was established by Arason [4, Appendix II] for $n = 2$. Izhboldin [11, Proposition 1.2] proved that for every $n \geq 3$ and any anisotropic $\pi \in P_n F$, there exists a field K/F such that $K(\pi)/K$ is not excellent. In particular, there exists a form φ over K with $\dim \varphi = \dim \pi$ such that $(\varphi_{K(\pi)})_{\text{an}}$ is not defined over K (see [11, Lemma 2.4]).

As remarked above, Corollary 3.12 provides an answer to Question 1.1 for function fields of Pfister forms that are excellent extensions. In light of Izhboldin's results, one cannot use Corollary 3.12 to obtain information regarding isotropy over $F(\pi)$ when $n \geq 3$ without first placing restrictions on either F or $\pi \in P_n F$. However, Theorem 3.11 provides isotropy criteria over all function fields of Pfister forms. Let $\Phi : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ be given by

$$\Phi(n) = \sup\{m \in \mathbb{N} \mid F(\pi)/F \text{ is } m\text{-excellent for every field } F \text{ and } \pi \in P_n F\}.$$

Reinterpreting the opening comments of this section, we note that $\Phi(n)$ is infinite for $n \leq 2$ and finite for $n \geq 3$. Indeed, $\Phi(n) < 2^n$ for $n \geq 3$ by [11, Lemma 2.4].

Proposition 4.1. *Let $n \geq 3$ and let π be an n -fold Pfister form over F . An anisotropic form φ over F with $\dim \varphi \leq \Phi(n)$ is isotropic over $F(\pi)$ if and only if it contains a Pfister neighbour of π .*

Proof. The right-to-left implication is clear. Conversely, φ contains a minimal $F(\pi)$ -isotropic form ψ over F . Since $F(\pi)/F$ is $\Phi(n)$ -excellent and $\dim \psi \leq \Phi(n)$, Theorem 3.11 implies that ψ is a generalised Pfister neighbour of π . Since $\Phi(n) < \dim \pi$, it follows that ψ is a Pfister neighbour of π . \square

Question 4.2. For each $n \geq 3$, what is the value of $\Phi(n)$?

Proposition 4.3. $\Phi(n) \geq 2^{n-1} + 1$ for every $n \in \mathbb{N}$.

Proof. Let $\pi \in P_n F$ and ψ be anisotropic forms over F such that $\dim \psi \leq 2^{n-1} + 1$ and $\psi_{F(\pi)}$ is isotropic. Theorem 1.4 implies that $\dim \psi = 2^{n-1} + 1$. Hence, Corollary 2.13 implies that ψ is a Pfister neighbour of π , whereby $(\psi_{F(\pi)})_{\text{an}}$ is defined over F . Hence, $F(\pi)/F$ is $(2^{n-1} + 1)$ -excellent. \square

For $n > 3$, the lower bounds on the values of $\Phi(n)$ given by the above result are the best currently known. For $n = 3$ however, Hoffmann [6, Corollary 4.2] obtained a sharper bound, showing that $\Phi(3) \geq 6$.

Corollary 4.4. Let π be an anisotropic 3-fold Pfister form over F and ψ a 6-dimensional anisotropic form over F . Then $\psi_{F(\pi)}$ is isotropic if and only if ψ contains a Pfister neighbour of π . In particular, there are no 6-dimensional minimal $F(\pi)$ -isotropic forms over F .

Proof. Since $\Phi(3) \geq 6$, Proposition 4.1 gives the equivalence. Consequently, every 6-dimensional anisotropic form over F that becomes isotropic over $F(\pi)$ necessarily contains a 5-dimensional Pfister neighbour of π over F , and hence cannot be a minimal $F(\pi)$ -isotropic form. \square

In order to establish upper bounds on the values of $\Phi(n)$ when $n \geq 3$, we will require the following result:

Proposition 4.5. Let π be an anisotropic n -fold Pfister form over F and ψ a minimal $F(\pi)$ -isotropic form over F . If $(\psi_{F(\pi)})_{\text{an}}$ is defined over F , then $\dim \psi = m2^{n-1} + 1$ for some $m \in \mathbb{N}$.

Proof. By Proposition 3.10, ψ is a generalised Pfister neighbour of π , whereby there exists a form γ over F such that $\psi \subset \pi \otimes \gamma$ and $\dim \psi > \frac{1}{2} \dim(\pi \otimes \gamma)$. Letting $\dim \gamma = m$ for some $m \in \mathbb{N}$, the minimality of ψ implies the result. \square

Corollary 4.6 (Hoffmann). Let π be an anisotropic 2-fold Pfister form over F . Every minimal $F(\pi)$ -isotropic form over F has odd dimension.

Proof. Since $F(\pi)/F$ is excellent, Proposition 4.5 implies that every minimal $F(\pi)$ -isotropic form over F is of dimension $2m + 1$ for some $m \in \mathbb{N}$. \square

Corollary 4.7. Let π be an anisotropic n -fold Pfister form over F and ψ a minimal $F(\pi)$ -isotropic form over F . If $2^{n-1} + 2 \leq \dim \psi \leq 2^n$, then $(\psi_{F(\pi)})_{\text{an}}$ is not defined over F .

Proof. This is an immediate corollary of Proposition 4.5. \square

In [11, Lemma 2.4], for π an anisotropic n -fold Pfister form over a certain field F , Izhboldin established the existence of 2^n -dimensional minimal $F(\pi)$ -isotropic forms ψ over F for all $n \geq 3$. Additionally, he proved that $(\psi_{F(\pi)})_{\text{an}}$ is not defined over F , a result we can recover directly by invoking Corollary 4.7. These examples belong to the class of twisted Pfister forms, $P_{n,m}F$, which Hoffmann studied in [8]. For $1 \leq m < n$, a form φ over F is contained in $P_{n,m}F$ if $\dim \varphi = 2^n$ and $\varphi \simeq (\pi_1 \perp -\pi_2)_{\text{an}}$, where π_1 and π_2 are respectively n -fold and m -fold anisotropic Pfister forms

over F . For all $n \geq 3$ and for all m satisfying $1 \leq m \leq n - 2$, Hoffmann provided examples in [8, Section 8] of fields F , forms $\varphi \in P_{n,m}F$ and anisotropic n -fold Pfister forms π over F such that $(\varphi_{F(\pi)})_{\text{an}}$ is not defined over F . The following result concerns the minimal $F(\pi)$ -isotropic forms over F contained within these examples.

Proposition 4.8. *For $n \geq 3$, let a field F , forms $\varphi \in P_{n,m}F$ and an anisotropic n -fold Pfister form π over F be as in [8, Example 8.1] or [8, Example 8.3]. Let ψ be a minimal $F(\pi)$ -isotropic form over F such that $\psi \subset \varphi$. Then $(\psi_{F(\pi)})_{\text{an}}$ is not defined over F .*

Proof. We note that the forms $\varphi \in P_{n,m}F$, defined for all m such that $1 \leq m \leq n - 2$, satisfy the criteria of [8, Proposition 7.6]. As a consequence, the minimal $F(\pi)$ -isotropic forms $\psi \subset \varphi$ satisfy $2^{n-1} + 2 \leq \dim \psi \leq 2^n - 2^{m-1} + 1$. Thus, Corollary 4.7 implies that $(\psi_{F(\pi)})_{\text{an}}$ is not defined over F . \square

Corollary 4.9. $\Phi(n) \leq 2^n - 2^{n-3}$ for every $n \geq 3$.

Proof. For certain fields F and certain anisotropic n -fold Pfister forms π over F , Proposition 4.8 implies the existence of minimal $F(\pi)$ -isotropic forms ψ over F such that $(\psi_{F(\pi)})_{\text{an}}$ is not defined over F . As per the proof of Proposition 4.8, these forms ψ satisfy $2^{n-1} + 2 \leq \dim \psi \leq 2^n - 2^{m-1} + 1$, where $1 \leq m \leq n - 2$. The result follows by taking $m = n - 2$. \square

For $n \geq 3$, the upper bounds on the values of $\Phi(n)$ given by the above result are the best currently known. For $n = 3$, the upper bound coincides with that derivable from Izhboldin's result [11, Lemma 2.4]. In particular, we note that $6 \leq \Phi(3) \leq 7$.

5. Formally real fields of finite Hasse number

We will let X_F denote the space of orderings of F , with $r_P^+(\varphi)$ and $r_P^-(\varphi)$ respectively denoting the number of positive and negative coefficients in a diagonalisation of φ with respect to $P \in X_F$. A field F is *formally real* if -1 is not a sum of squares in F , a condition which is equivalent to $X_F \neq \emptyset$ (see [17, Chapter VIII, Theorem 1.10]). It is known that $P \in X_F$ extends to at least one ordering of $F(\varphi)$ if and only if φ is *indefinite at P* , that is, $r_P^+(\varphi) > 0$ and $r_P^-(\varphi) > 0$ (see [3, Theorem 3.5]). Thus, every ordering on F extends to an ordering of $F(\varphi)$ if and only if φ is *totally indefinite*, that is, indefinite at every $P \in X_F$. The *Hasse number* of F is defined to be

$$\tilde{u}(F) := \sup\{\dim \varphi \mid \varphi \text{ is anisotropic and totally indefinite over } F\}.$$

In this section, we study a special case of Question 1.1 for function fields of Pfister forms, namely that where F is formally real and $\tilde{u}(F)$ is finite. The next result provides an answer to this question for those forms over F of dimension greater than $\tilde{u}(F)$, by offering a classification of the minimal $F(\pi)$ -isotropic forms over F contained therein, where π is a Pfister form over F .

Theorem 5.1. *Let π be an anisotropic n -fold Pfister form over F , and φ an anisotropic form over F such that $\dim \varphi > \tilde{u}(F)$. Then $\varphi_{F(\pi)}$ is isotropic if and only if φ contains a Pfister neighbour of π .*

Proof. The right-to-left implication is clear. Conversely, suppose that $\varphi_{F(\pi)}$ is isotropic. Since $\tilde{u}(F) < \dim \varphi$ and φ is anisotropic over F , we can conclude that F is formally real (as otherwise φ would trivially be totally indefinite and hence isotropic over F) and that there exists $Q \in X_F$ such that φ is definite at Q . Since $\varphi_{F(\pi)}$ is isotropic, Theorem 1.4 implies that $\dim \varphi \geq 2^{n-1} + 1$. If $P \in X_F$ is such that π is definite with respect to P , then [5, Lemma 4.4.3] implies that the statement $0 < r_P^+(\varphi), r_P^-(\varphi) \leq 2^{n-1}$ does not hold, whereby we can conclude that either $r_P^+(\varphi) > 2^{n-1}$ or $r_P^-(\varphi) > 2^{n-1}$. Since isotropic forms must necessarily be totally indefinite, if $R \in X_F$ extends to an ordering of $F(\pi)$ (that is, π is indefinite with respect to R), then φ must be indefinite with respect to R . Hence, invoking [5, Lemma 4.4.5], one concludes that φ contains a Pfister neighbour of π . \square

We offer the following improvement of [5, Theorem 4.4.6] as a corollary of Theorem 5.1:

Theorem 5.2. For $\tilde{u}(F) \leq 2^{n-1} + 1$ and π an anisotropic n -fold Pfister form over F , the minimal $F(\pi)$ -isotropic forms over F are exactly the Pfister neighbours of π of dimension $2^{n-1} + 1$. In particular, $F(\pi)/F$ is excellent.

Proof. All anisotropic F -forms which become isotropic over $F(\pi)$ are necessarily of dimension at least $2^{n-1} + 1$ by Theorem 1.4. Since $\tilde{u}(F) \leq 2^{n-1} + 1$, Theorem 5.1 implies that all the minimal $F(\pi)$ -isotropic forms over F are of dimension $2^{n-1} + 1$. Moreover, all such forms are Pfister neighbours of π by Corollary 2.13. Hence, $F(\pi)/F$ is excellent by Corollary 3.12. \square

We will proceed to list some further corollaries of Theorem 5.1, beginning by making explicit the consequence thereof employed in the above proof. We note that the following result is contained in [9, Theorem 5.3], where an analogous statement is presented for iterated function fields of Pfister forms, and thus, Corollary 5.3 may be viewed as a short recovery of [9, Theorem 5.3] for function fields of a single Pfister form:

Corollary 5.3 (Hoffmann). $2^{n-1} + 1 \leq \sup \mathcal{M}(F(\pi)/F) \leq \max\{2^{n-1} + 1, \tilde{u}(F)\}$, for π an anisotropic n -fold Pfister form over F .

Proof. Theorem 1.4 gives the lower bound. Letting ψ be an anisotropic form over F such that $\psi_{F(\pi)}$ is isotropic, if $\dim \psi > \tilde{u}(F)$ then Theorem 5.1 implies that ψ contains a Pfister neighbour of π of dimension $2^{n-1} + 1$. Hence if ψ is a minimal $F(\pi)$ -isotropic form, $\dim \psi \leq \max\{2^{n-1} + 1, \tilde{u}(F)\}$. \square

As a corollary of the above, we can give a short proof of [10, Proposition 2.6], a result concerning function fields of conics (or equivalently, function fields of 2-fold Pfister forms):

Corollary 5.4 (Hoffmann, Van Geel). Let F be formally real with $\tilde{u}(F) \leq 2n$ for $n \in \mathbb{N}$, and ρ an anisotropic conic over F . Then $\sup \mathcal{M}(F(\rho)/F) \leq \max\{3, 2n - 1\}$.

Proof. Since $\dim \rho = 3$, $\sup \mathcal{M}(F(\rho)/F) = \sup \mathcal{M}(F(\pi)/F)$ for some $\pi \in P_2 F$. Hence, $\sup \mathcal{M}(F(\rho)/F) \leq \max\{3, \tilde{u}(F)\}$ by Corollary 5.3. The statement follows, since Corollary 4.6 implies that $F(\rho)$ -minimal forms over F are of odd dimension. \square

Proposition 5.5. Let φ and a Pfister form π be anisotropic forms over F . If φ is such that $\dim(\varphi_{F(\pi)})_{\text{an}} \geq \tilde{u}(F) - 1$, then $(\varphi_{F(\pi)})_{\text{an}}$ is defined over F .

Proof. Without loss of generality, we may assume that $\varphi_{F(\pi)}$ is isotropic, whereby $\dim \varphi \geq \tilde{u}(F) + 1$. Theorem 5.1 implies that $\varphi \simeq \varphi' \perp \tau$, where τ is a Pfister neighbour of π with $\tau \perp \gamma \simeq a\pi$ for some $a \in F^\times$ and form γ over F . Hence $\varphi_{F(\pi)} \simeq \varphi'_{F(\pi)} \perp i_W(\tau_{F(\pi)}) \times \langle 1, -1 \rangle \perp -\gamma_{F(\pi)}$. Thus $(\varphi_{F(\pi)})_{\text{an}} \simeq ((\varphi' \perp -\gamma)_{F(\pi)})_{\text{an}}$. Let $\varphi_1 := (\varphi' \perp -\gamma)_{\text{an}}$. If φ_1 is anisotropic over $F(\pi)$, we have that $(\varphi_{F(\pi)})_{\text{an}} \simeq (\varphi_1)_{F(\pi)}$ and we are done. Hence, we may assume that φ_1 is isotropic over $F(\pi)$. Since $(\varphi_{F(\pi)})_{\text{an}} \simeq ((\varphi_1)_{F(\pi)})_{\text{an}}$, we have that $\dim((\varphi_1)_{F(\pi)})_{\text{an}} \geq \tilde{u}(F) - 1$, whereby $\dim \varphi_1 \geq \tilde{u}(F) + 1$. Applying Theorem 5.1 to φ_1 and iterating our argument, we will obtain that $(\varphi_{F(\pi)})_{\text{an}} \simeq (\varphi_n)_{F(\pi)}$ for some $n \in \mathbb{N}$. \square

Proposition 5.6. Let π be an anisotropic n -fold Pfister form over F . Then $F(\pi)/F$ is excellent if and only if $F(\pi)/F$ is $\tilde{u}(F)$ -excellent.

Proof. The left-to-right implication is clear, as is the right-to-left one in the case where $\tilde{u}(F) = \infty$, so we will assume that $F(\pi)/F$ is $\tilde{u}(F)$ -excellent where $\tilde{u}(F) < \infty$. If $\tilde{u}(F) \leq 2^{n-1} + 1$, Corollary 5.2 gives the result. Hence, we may assume that $\tilde{u}(F) > 2^{n-1} + 1$, whereby Corollary 5.3 implies that

there are no minimal $F(\pi)$ -isotropic forms over F of dimension greater than $\tilde{u}(F)$. Since $F(\pi)/F$ is $\tilde{u}(F)$ -excellent, every minimal $F(\pi)$ -isotropic form ψ over F is such that $(\psi_{F(\pi)})_{\text{an}}$ is defined over F , whereby Corollary 3.12 implies that $F(\pi)/F$ is excellent. \square

Suppose that π is an anisotropic n -fold Pfister form. Then $F(\pi)/F$ is 6-excellent for $n \geq 4$ by Proposition 4.3. Moreover, $F(\pi)/F$ is 6-excellent for $n = 3$ by Hoffmann's result [6, Corollary 4.2] that $\Phi(3) \geq 6$. Hence, $F(\pi)/F$ is 6-excellent for all Pfister forms π over F , allowing us to invoke Proposition 5.6 to recover the following component of [9, Corollary 4.8]:

Corollary 5.7 (Hoffmann). *Let F be a field such that $\tilde{u}(F) \leq 6$. Then $F(\pi)/F$ is excellent for every anisotropic Pfister form π over F .*

Proof. As above, $F(\pi)/F$ is 6-excellent for every $\pi \in PF$. Since $\tilde{u}(F) \leq 6$, $F(\pi)/F$ is $\tilde{u}(F)$ -excellent for every $\pi \in PF$, whereby Proposition 5.6 establishes the result. \square

We conclude by invoking Theorem 5.1 to establish the example referred to previously in Section 2.

Example 5.8. Let F be a formally real field with $\tilde{u}(F) = 4$. Let $n \geq 1$ and let γ be an anisotropic form over F of dimension $5 + n$ that becomes isotropic over $F(\pi)$, where $\pi \in P_2 F$. Since [17, Chapter X, Theorem 4.9] implies that γ cannot become hyperbolic over $F(\pi)$ in the case where $n = 1$, we have that $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1 > 4$ for all n . Since every F -subform of γ of dimension $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1$ is isotropic over $F(\pi)$ by Lemma 1.3, Theorem 5.1 implies that these F -subforms contain Pfister neighbours of π . Thus γ contains no minimal $F(\pi)$ -isotropic forms over F of dimension $\dim \gamma - i_W(\gamma_{F(\pi)}) + 1$.

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