The projective dimension of codimension two algebras presented by quadrics

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A B S T R A C T

Motivated by a question of Stillman, we find a sharp upper bound for the projective dimension of ideals of height two generated by quadrics. In a polynomial ring with arbitrary large number of variables, we prove that ideals generated by n quadrics define cyclic modules with projective dimension at most $2n - 2$. We refine this bound according to the multiplicity of the ideal. We ask whether tight upper bounds for the projective dimension of ideals generated by quadrics can be expressed only in terms of their height and number of minimal generators.

1. Introduction

This paper is concerned with bounding the projective dimension of ideals generated by homogeneous quadratic polynomials. Our main motivation arises from the following question posed by Stillman:

Question 1.1 (Stillman). (See [27, Problem 15.8].) *Is there a bound, independent of N, on the projective dimension of ideals in $R = K[x_1, \ldots, x_N]$ which are generated by n homogeneous polynomials of given degrees $d_1, \ldots, d_n$?*

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Recently, there has been a surge of interest in Stillman's question. We mention here some of the relevant works in order of their appearance: [13–15,25,4,1]. We also remark that this question has interesting connections to the study of Castelnuovo–Mumford regularity. As shown by Caviglia [7], Question 1.1 is equivalent to a similar problem in which projective dimension is replaced by regularity. For a survey of the developments regarding Stillman's question up to the moment when this paper was written see [26].

Although Stillman's question is open in general, it has been answered in the affirmative by Ananyan and Hochster [1] in the case of ideals generated by not necessarily homogeneous polynomials of degree at most two. These authors have announced upcoming improvements of their estimates as well as an extension of their result to ideals generated by cubics in [20]. However, the bounds on projective dimension produced by the methods in [1] are typically very large. Ananyan and Hochster find an upper bound with asymptotical order of growth $2n^{2n}$ for the projective dimension of an ideal generated by n polynomials of degree at most 2 [1, Section 6]. When the number of minimal generators for the ideal is small, the bounds obtained by Ananyan and Hochster are still large. For example, in the case of ideals generated by three polynomials of degree two, the concrete bound obtained by using the methods in [1] is 296 [26, Proposition 3.15], while the tight upper bound is 4 by a result of Eisenbud and Huneke (cf. [26, Theorem 3.1]).

Our paper stems from a desire to provide a *sharp* upper bound for the projective dimension of ideals generated by homogeneous quadratic polynomials. This is currently out of reach in complete generality. Our main result gives a complete answer in codimension two:

Theorem. *For any ideal I of height two generated by n homogeneous quadratic polynomials in a polynomial ring R, $\mathrm{pd}(R/I) \leqslant 2n - 2$. Moreover, this bound is tight.*

Our present work leads to the natural question of whether it is possible to find tight upper bounds on the projective dimension of ideals generated by quadratic polynomials of a fixed arbitrary height. In Section 6 we pose a more specific question on whether a bound on the projective dimension of ideals generated by quadratic polynomials can be given in terms of the minimal number of generators and the height of the ideal.

This paper is organized as follows: Section 2 covers the background needed for our developments, in Section 3 we give an outline of the proof of the main result and we prove this result in the simplest cases, Sections 4 and 5 develop the material needed to fill in the details of the proof of the main result in the most difficult case and Section 6 presents some open questions which stem from computations motivated by our work.

2. Background

2.1. Preliminaries

For the rest of this paper, R will denote a standard graded polynomial ring over a field K. We further let I denote an ideal of R generated by n homogeneous quadratic polynomials. We henceforth use the terminology *quadric* for homogeneous quadratic polynomial and *ideal of quadrics* for an ideal generated by quadrics.

For a finitely generated graded R-module M, there exists a unique, up to isomorphism, minimal graded free resolution $0 \longrightarrow F_p \xrightarrow{d_p} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$, that is, an exact sequence of finitely generated graded R-modules F_i with $M \cong \mathrm{Coker}(d_1)$. The resolution is called *minimal* if $\mathrm{Im}(d_i) \subseteq \mathfrak{m}F_{i-1}$ for $i \geqslant 1$. Here $\mathfrak{m}$ denotes the homogeneous maximal ideal of R. The length p of the minimal free resolution of M is called the *projective dimension* of M and is denote $\mathrm{pd}(M) = p$.

For the case of cyclic modules, computing $\mathrm{pd}(R/I)$ is equivalent to computing $\mathrm{pd}(I)$ as the two are related by $\mathrm{pd}(R/I) = \mathrm{pd}(I) + 1$. For uniformity and consistency with [1,4,14,15,25,26] we choose to bound the projective dimension of R/I throughout the paper. We often make use of the next two well-known remarks which relate the projective dimension of an ideal to the projective dimension of its colon ideals and allow for computations of such colon ideals, respectively.

Remark 2.1. Let I be an ideal in a polynomial ring R and let ℓ be any element of R. Then

(1) $\mathrm{pd}(R/I) \leqslant \max\{\mathrm{pd}(R/(I:(\ell))), \mathrm{pd}(R/(I+(\ell)))\}$,
(2) $\mathrm{pd}(R/(I:(\ell))) \leqslant \max\{\mathrm{pd}(R/I), \mathrm{pd}(R/(I+(\ell))) - 1\}$,
(3) $\mathrm{pd}(R/(I+(\ell))) \leqslant \max\{\mathrm{pd}(R/(I:(\ell))) + 1, \mathrm{pd}(R/I)\}$.

Remark 2.2. Let I be an ideal in a polynomial ring R and let x, f be elements of R. Then

$$\big(I + (xf)\big) : (x) = I : (x) + (f).$$

2.2. 1-generic matrices and heights of ideals of minors

The class of 1-generic matrices was introduced in Eisenbud's paper [8]. We explain in Section 4 how matrices of linear forms come up naturally when analyzing ideals generated by quadrics that are contained in a linear prime.

Let M be a matrix of linear forms over a polynomial ring R with coefficient field K. By a *generalized row* of M we mean a K-linear combination of the rows of M with not all coefficients 0. Similarly a *generalized column* is a non-zero K-linear combination of the columns of M. A *generalized entry* of M is simply a linear combination with non-zero coefficients of the entries of some generalized row. In the following we write $I_i(M)$ for the ideal of $i \times i$ minors of M.

Definition 2.3. We call M 1-*generic* if, after arbitrary K-linear row and column operations, M exhibits no generalized zero entries.

The following result was established by Eisenbud in [9] (see also Theorem 2.1 in [8] for a generalization allowing linear sections of small codimension of these determinantal varieties).

Theorem 2.4. *(See [9, Theorem 6.4].) If M is a 1-generic matrix of linear forms of size $p \times q$ ($p \leqslant q$) with entries in a polynomial ring R over an algebraically closed field K, then the ideal $I_p(M)$ generated by the maximal minors of M is prime of codimension $q - p + 1$. Its free resolution is given by an Eagon–Northcott complex and $R/I_p(M)$ is a Cohen–Macaulay domain.*

In addition to using the preceding theorem for 1-generic matrices, we shall also be concerned with ideals of minors of matrices which are far from being 1-generic; specifically we shall be interested in determining the height of ideals of minors of matrices whose rank is not maximal. In this situation, bounds on the height of ideals of minors for (not necessarily linear) matrices have been given by Eisenbud, Huneke and Ulrich [12] generalizing results of Eagon–Northcott, Bruns and Faltings.

Theorem 2.5. *(See [12, Theorem A].) Let R be a regular local ring, and let M be a matrix of size $p \times q$ with entries in R. Set $r = \mathrm{rank}(M)$ and consider an integer i such that $1 \leqslant i \leqslant r$ and $I_i(M) \neq R$. Then*

$$\mathrm{ht}\, I_i(M) \leqslant (r - i + 1)\big(\max\{p, q\} - i + 1\big) + i - 1.$$

Remark 2.6. The conclusion of the theorem above holds in the graded case as well. If R is a polynomial ring and M a matrix of size $p \times q$ and rank r whose entries are homogeneous forms, then

$$\mathrm{ht}\, I_i(M) \leqslant (r - i + 1)\big(\max\{p, q\} - i + 1\big) + i - 1.$$

This is easily deduced from Theorem 2.5 by localizing at the homogeneous maximal ideal.

2.3. Residual intersections

The second result that we use extensively appeared in the context of residual intersections, a notion introduced by Artin and Nagata [3]. An ideal J is called an *s-residual intersection* of an ideal I if there exists an s-generated ideal $A \subseteq I$ such that $J = A : I$ and $\mathrm{ht}(J) \geqslant s \geqslant \mathrm{ht}(I)$. If, in addition, $\mathrm{ht}(I + J) > \mathrm{ht}(J)$, then J is said to be an *s-geometric residual intersection* of I. The notion of residual intersection generalizes that of liaison, for which $s = \mathrm{ht}(I)$.

One of the most common settings in which residual intersections arise naturally is the following: take A to be an ideal generated by at most s elements, assume A is not unmixed, and let I be the intersection of its primary components of height at most s (in any of its irredundant primary decompositions). If $I \neq A$, then $J = A : I$ is a geometric residual intersection of I, in particular J is actually the intersection of all the primary components of A of height at least $s + 1$ and $A = I \cap J$.

Definition 2.7. Let I be an ideal in a polynomial ring R and fix $\underline{f} = f_1, \ldots, f_n$ to be any minimal set of generators of I. We define I to be strongly Cohen–Macaulay if the homology modules $H_i(\underline{f}, R)$ of the Koszul complex associated to $\underline{f}$ are either zero or Cohen–Macaulay modules for all i.

Although this definition seems to depend on the chosen generating set $\underline{f}$ of I, one can check that this is actually a property of the ideal I (see for instance [21, Remark 1.2]).

Definition 2.8. (See [3].) We say I satisfies condition G_s if $\mu(I_{\mathfrak{p}}) \leqslant \dim R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$, with $I \subset \mathfrak{p}$ and $\dim R_{\mathfrak{p}} \leqslant s - 1$.

In the course of our proofs, we shall exploit the following result which is a translation of [21, Theorem 3.1] into purely algebraic language.

Theorem 2.9. *(See [21, Theorem 3.1].) Let R be a Cohen–Macaulay local ring. Let I be an ideal of R such that I is strongly Cohen–Macaulay and satisfies G_s. Let $J = A : I$ be an s-geometric residual intersection of I. Then:*

(1) *R/J is Cohen–Macaulay and $\mathrm{ht} \, J = s$,*
(2) *$A = I \cap J$,*
(3) *$I + J$ is strongly Cohen–Macaulay, and*
(4) *$\mathrm{depth} \, R/A \geqslant \dim R - s$.*

The interested reader may find generalizations of this result by Herzog–Vasconcelos–Villareal and Huneke–Ulrich in [19] and [23], respectively. However, for the purpose of this paper, we only need the following special case of Theorem 2.9, where I is taken to be a homogeneous complete intersection ideal.

Corollary 2.10. *Let R be a polynomial ring. Let C be a complete intersection homogeneous ideal in R. Let $A = (a_1, \ldots, a_s)$ denote an ideal contained in C. Set $J = A : C$ and assume $\mathrm{ht} \, J \geqslant s$ and $\mathrm{ht}(C + J) \geqslant s + 1$. Then $\mathrm{ht}(J) = s$ and $\mathrm{pd}(R/A) \leqslant s$.*

Proof. The problem can be reduced to the local case by localizing with respect to the homogeneous maximal ideal $\mathfrak{m}$ of R. Clearly $C_{\mathfrak{m}}$ is strongly Cohen–Macaulay and G_s, hence we can apply Theorem 2.9 to obtain $\mathrm{ht} \, J_{\mathfrak{m}} = s$ and $\mathrm{depth} \, R_{\mathfrak{m}}/A_{\mathfrak{m}} \geqslant \dim R_{\mathfrak{m}} - s$. This latter inequality combined with the Auslander–Buchsbaum formula implies $\mathrm{pd}(R_{\mathfrak{m}}/A_{\mathfrak{m}}) \leqslant s$. The conclusion now follows from the equalities $\mathrm{ht}(J) = \mathrm{ht}(J_{\mathfrak{m}})$ and $\mathrm{pd}_R(R/A) = \mathrm{pd}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/A_{\mathfrak{m}})$. □

2.4. Hilbert–Samuel multiplicity

In our quest to analyze the projective dimension of ideals generated by quadrics, we shall resort to a case analysis based on the minimal primes associated to our ideals. To facilitate the understanding of

what minimal primes may occur, we use the notion of Hilbert–Samuel multiplicity and a result known as the associativity formula. In the following, we denote by $e(M)$ the Hilbert–Samuel multiplicity of a module M and by $\lambda(M)$ the length of an Artinian module M.

Theorem 2.11 (*Associativity Formula*). (*See [24, Theorem 14.7].*) *Let I be an ideal of R. Then*

$$e(R/I) = \sum_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}(\mathfrak{p})=\mathrm{ht}(I)}} e(R/\mathfrak{p})\lambda(R_\mathfrak{p}/J_\mathfrak{p}).$$

Let I^{un} denote the unmixed part of I, defined as the intersection of the primary components of I with height equal to $\mathrm{ht}(I)$. It is easy to see from the formula above that $e(R/I) = e(R/I^{un})$.

We recall a classical lower bound for the multiplicity of non-degenerate prime ideals. A homogeneous ideal is called degenerate if it has a linear form as a minimal generator.

Proposition 2.12. (*See [10, Proposition 0], [11, Corollary 18.12].*) *Let $\mathfrak{p}$ be a homogeneous prime ideal in a polynomial ring defined over an algebraically closed field. If $\mathfrak{p}$ is non-degenerate, then $e(R/\mathfrak{p}) \geqslant \mathrm{ht}(\mathfrak{p}) + 1$.*

In view of this bound, an ideal $\mathfrak{p}$ is said to have *minimal multiplicity* if $e(R/\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) + 1$. A variety defined by a prime ideal of minimal multiplicity is called a variety of minimal multiplicity. These varieties are well known and have been classified. Proofs of this classification in characteristic zero can be found in [5,17,29] and a characteristic-independent proof can be found in [10]. The case of interest to us, that of varieties of codimension 2 and minimal multiplicity, has been first considered in [2,28].

Theorem 2.13 (*Del Pezzo–Bertini*). (*See [10, Theorem 1].*) *A non-degenerate irreducible variety of codimension at least two and minimal multiplicity defined over an algebraically closed field is of one of the following types* (1) *a rational normal scroll,* (2) *the second Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ or* (3) *a cone over one of the previous two varieties.*

3. Main results

3.1. The method of proof

The main result of this paper provides a tight upper bound on the projective dimension of ideals I of height two generated by quadrics. The first step in our proof is to classify the minimal primes of such ideals. In order to accomplish this task, we bound the multiplicity of height two ideals generated by quadrics then we use the associativity formula to find the possible multiplicities of the individual minimal primes. As a consequence of this analysis, we conclude that a height two ideal of quadrics I is contained in at least one prime $\mathfrak{p}$ of one of the following types: (1) a prime of multiplicity one and height two, i.e. $\mathfrak{p} = (x, y)$, with x, y independent linear forms, (2) a prime of multiplicity two and height two, i.e. $\mathfrak{p} = (x, q)$, with x a linear form and q an irreducible quadric or (3) a prime of multiplicity three and height two, i.e. the defining ideal of one of the varieties of minimal multiplicity classified in Theorem 2.13 (we shall also see that degenerate ideals of multiplicity three and height two cannot occur).

We prove bounds on the projective dimension of ideals of quadrics contained in primes of each of the types described above separately. The first case is the most intricate and is dealt with in Section 5, while the other two cases are covered in this section. We combine these bounds in Theorem 3.5, which conveys the overall estimate.

3.2. Minimal associated primes and consequences on projective dimension

We begin by characterizing the minimal primes associated to height two ideals of quadrics and classifying them, over an algebraically closed field, according to their multiplicity.

Lemma 3.1. *Let I be an ideal of R minimally generated by $n \geqslant 3$ quadrics and with $\mathrm{ht}(I) = 2$, then $e(R/I) \leqslant 3$.*

Proof. Let f, g be a regular sequence of quadratic forms contained in I, thus $e(R/(f,g)) = 4$. Passing to the artinian reduction of R/I by going modulo a sequence of linear forms $\underline{\ell}$, yields $e(R/I) \leqslant \lambda(R/(\underline{\ell}, I)) \leqslant \lambda(R/(\underline{\ell}, f, g)) = e(R/(f,g)) = 4$. We have the series of containments $(f,g) \subset I \subset I^{un}$. Note that (f,g) and I^{un} are unmixed ideals of height two. If $e(R/I) = e(R/I^{un}) = e(R/(f,g)) = 4$, then $(f,g) = I^{un}$ by [14, Lemma 8]. But this would force $(f,g) = I$, contradicting that I has at least three minimal generators. Thus we must have $e(R/I) \leqslant 3$. □

Combining this lemma with the associativity formula one obtains:

Corollary 3.2. *Let I be an ideal of R minimally generated by $n \geqslant 3$ quadrics and with $\mathrm{ht}(I) = 2$. Let $\mathfrak{p}$ be a minimal prime of I with $\mathrm{ht}\,\mathfrak{p} = 2$. Then $e(R/\mathfrak{p}) \leqslant 3$.*

To make use of this result, one needs to understand prime ideals of height two and multiplicity 1, 2 and 3. According to Theorem 2.13, any prime of height two and multiplicity 1 or 2 is degenerate (contains a linear form). Going modulo this linear form, the image is a principal ideal. In particular, a prime of height two and multiplicity one is minimally generated by two independent linear forms, and a prime of height two and multiplicity two is minimally generated by a linear form and a quadric. This gives the first two types of primes mentioned in the paragraph at the beginning of this section. Similarly, it is easy to see that the degenerate primes of height two and multiplicity three are minimally generated by a linear form and a cubic, while the non-degenerate ones are listed in Theorem 2.13.

We begin our case analysis by considering ideals generated by quadrics contained in a prime ideal of multiplicity three and height two.

Proposition 3.3. *Let R be a polynomial ring over an algebraically closed field. Any height two ideal I generated by quadrics contained in a prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$ and $e(R/\mathfrak{p}) = 3$ has $\mathrm{pd}(R/I) = 2$. In particular, for any such ideal I, R/I is Cohen–Macaulay.*

Proof. We start by considering the degenerate case $\mathfrak{p} = (x, c)$, with x a linear form, c an irreducible cubic polynomial and $\mathrm{ht}(x, c) = 2$. Since $I \subseteq \mathfrak{p}$ is generated by quadrics, all minimal generators of I must be multiples of x. However, in that case the height of I is one, a contradiction.

Next assume $\mathfrak{p}$ is the non-degenerate defining ideal of one of the varieties listed in Theorem 2.13. The ideal of the second Veronese of $\mathbb{P}^2$ has height three and must be excluded. Note that each of the defining ideals of rational normal scrolls of height two is minimally generated by 3 quadrics. Since the minimal generators of I are linear combinations of these quadrics, we obtain, depending on the number of minimal generators of I, that either I is a complete intersection or $I = \mathfrak{p}$. In the first case $\mathrm{pd}(R/I) = 2$ and in the second case $\mathrm{pd}(R/I) = 2$ as well, as $R/\mathfrak{p}$ is Cohen–Macaulay by the Hilbert–Burch theorem applied to each of the primes defining rational normal scrolls. □

We continue with the case of ideals generated by quadrics contained in a prime ideal of multiplicity two and height two.

Proposition 3.4. *Let I be an ideal generated by n quadrics which is contained in a prime ideal of height two and multiplicity two. Then $\mathrm{pd}(R/I) \leqslant n$.*

Proof. Let $\mathfrak{p}$ be the minimal prime mentioned in the hypothesis. It is necessarily of the form $\mathfrak{p} = (x, q)$, with x a linear form and q a quadric. We may assume that:

$$I = (q + x\ell_1, x\ell_2, \ldots, x\ell_n)$$

where ℓ_i are linear forms such that $\{\ell_2, \ldots, \ell_n\}$ form a linearly independent set. By first using Remark 2.2 and subsequently using that x is regular on $R/(q)$ and hence also on $R/(q + x\ell_1)$, we compute

$$I : (x) = (q + x\ell_1) : (x) + (\ell_2, \ldots, \ell_n) = (q + x\ell_1) + (\ell_2, \ldots, \ell_n).$$

If $q + x\ell_1 \in (\ell_2, \ldots, \ell_n)$, then $I : (x) = (\ell_2, \ldots, \ell_n)$ and $\mathrm{pd}(R/I : (x)) = n - 1$. Otherwise, $q + x\ell_1$ is a regular element on $R/(\ell_2, \ldots, \ell_n)$ and thus $\mathrm{pd}(R/(q + x\ell_1, \ell_2, \ldots, \ell_n)) = n$. Hence in any case we obtain $\mathrm{pd}(R/(I : (x))) \leqslant n$. Now since $I + (x) = (q, x)$, we obtain $\mathrm{pd}(R/(I + (x))) = 2$ and Remark 2.1 yields $\mathrm{pd}(R/I) \leqslant n$. □

The last and most involved case is that of ideals generated by quadrics contained in a prime ideal of multiplicity one. We defer the detailed analysis of this case to Section 5.

3.3. Proof of the main theorem

The following theorem, which is our main result, combines the bounds obtained throughout this paper.

Theorem 3.5. *For any ideal I of height two generated by n quadrics in a polynomial ring R, one has $\mathrm{pd}(R/I) \leqslant 2n - 2$.*

Proof. For the purpose of computing projective dimension we may assume the ground field is algebraically closed by tensoring R with an appropriate extension of the original field. The case when $n = 2$ yields R/I is a complete intersection, thus in this case $\mathrm{pd}(R/I) = 2$ agrees with the conclusion of the theorem. Henceforth we assume $n \geqslant 3$.

By Corollary 3.2, any ideal of height two generated by quadrics falls under the hypotheses of at least one of Theorem 5.7, Proposition 3.4 or Proposition 3.3. These three results insure that $\mathrm{pd}(R/I) \leqslant 2n - 2$, $\mathrm{pd}(R/I) \leqslant n$ or $\mathrm{pd}(R/I) = 2$ respectively. Taking the maximum of these bounds yields $\mathrm{pd}(R/I) \leqslant 2n - 2$ overall. □

We complement the statement of our principal result by the important observation that the bound in Theorem 3.5 is tight as we demonstrate in Example 3.6. The example below is a particular case of the main result in [25], specifically the ideal discussed below appears under the name $I_{2,n-2,2}$ in [25]. Note that the multiplicity of the family of examples given below is one as soon as the minimal number of generators is at least four and it is two for the example minimally generated by three quadrics.

Example 3.6. Let $n \geqslant 2$ and $R = K[x, y, a_{1,1}, \ldots, a_{2,n-2}]$. Consider the n-generated ideal of quadrics $I = (x^2, y^2, a_{1,1}x + a_{2,1}y, \ldots, a_{1,n-2}x + a_{2,n-2}y)$. Then $\mathrm{pd}(R/I) = 2n - 2$.

Proof. By the graded Auslander–Buchsbaum formula, showing that $\mathrm{pd}(R/I) = 2n - 2$ is equivalent to showing depth $R/I = 0$ or, equivalently, that the maximal ideal of R is associated to R/I. It is not hard to see that xy is a non-zero socle element in R/I, which finishes the proof. □

3.4. A more detailed analysis

Even though our bound is tight, it turns out that it can be refined in an important number of cases. The purpose of the additional analysis we perform in the rest of this section is to point out in which contexts such refinements are possible. Following Engheta [13], we introduce the following notation to keep track of the possibilities for the associated primes of minimal height of an ideal J.

Table 1
Refining the bound on projective dimension according to multiplicity.

$\langle \underline{e}; \underline{\lambda} \rangle$	Bound on pd(R/I)	Reference
$\langle 1; 1 \rangle$	$2n - 2$	contained in a linear prime (see Theorem 5.7)
$\langle 2; 1 \rangle$	n	contained in a multiplicity two prime (see Proposition 3.4)
$\langle 1; 2 \rangle$	$n + 2$	multiple structure on a linear prime (see Proposition 5.8)
$\langle 1, 1; 1, 1 \rangle$	$n + 1$	contained in 2 linear primes (see Proposition 5.9)
$\langle 3; 1 \rangle$	2	prime of minimal multiplicity (see Proposition 3.3)
$\langle 1; 3 \rangle$	$n + 2$	multiple structure on a linear prime (see Proposition 5.8)
$\langle 1, 2; 1, 1 \rangle$	n	contained in a multiplicity two prime (see Proposition 3.4)
$\langle 1, 1; 1, 2 \rangle$	$n + 1$	contained in 2 linear primes (see Proposition 5.9)
$\langle 1, 1, 1; 1, 1, 1 \rangle$	$n + 1$	contained in 2 linear primes (see Proposition 5.9)

Definition 3.7. We say I is of type $\langle e = e_1, e_2, \ldots, e_m; \ \lambda = \lambda_1, \lambda_2, \ldots, \lambda_m \rangle$ if I has m associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of minimal height with $e(R/\mathfrak{p}_i) = e_i$ and with $\lambda(R_{\mathfrak{p}_i}/I_{\mathfrak{p}_i}) = \lambda_i$, for $i = 1, \ldots, m$.

Remark 3.8. If I is of type $\langle e = e_1, e_2, \ldots, e_m; \ \lambda = \lambda_1, \lambda_2, \ldots, \lambda_m \rangle$, then $e(R/I) = \sum_{i=1}^{m} e_i \lambda_i$.

We classify the possible types of height two ideals generated by quadrics as follows:

Proposition 3.9. Let I be an ideal of R minimally generated by $n \geqslant 3$ quadrics and with $\mathrm{ht}(I) = 2$, then the minimal associated primes of I fall into one of the following categories (1) $\langle 1; 1 \rangle$, (2) $\langle 2; 1 \rangle$, (3) $\langle 1; 2 \rangle$, (4) $\langle 1, 1; 1, 1 \rangle$, (5) $\langle 3; 1 \rangle$, (6) $\langle 1; 3 \rangle$, (7) $\langle 1, 2; 1, 1 \rangle$, (8) $\langle 1, 1; 1, 2 \rangle$, (9) $\langle 1, 1, 1; 1, 1, 1 \rangle$.

Proof. The result of combining Theorem 2.13 and the associativity formula of Theorem 2.11 is the inequality

$$ e(R/I) = \sum_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}(\mathfrak{p}) = 2}} e(R/\mathfrak{p}) \lambda(R_{\mathfrak{p}}/I_{\mathfrak{p}}) \leqslant 3. $$

The possible types listed above are all the possibilities of partitioning the integers 1, 2 and 3 according to the associativity formula. □

A finer case analysis than the one performed in the proof of Theorem 3.5 is undertaken in Table 1, which summarizes the bounds we are able to obtain on the projective dimension of ideals of quadrics of each of the types described above. We base our refined analysis on the observation that better bounds can be established for ideals contained in multiple structures with linear support (Proposition 5.8) or in at least two distinct linear primes (Proposition 5.9). For each bound, we reference the relevant results in the paper for the reader's convenience. Note that by taking the maximum among the bounds listed in Table 1, we recover our general result, Theorem 3.5.

The case breakdown in Table 1 identifies that, for ideals of height two minimally generated by at least four quadrics, there is only one type that can achieve the maximum projective dimension, namely ideals of multiplicity one. For ideals minimally generated by three quadrics, the situation is quite different: the sharp bound of 4 is never attained by an ideal of multiplicity one (see [22] for a proof). The bound of $n + 2$ on the projective dimension of ideals of height two generated by quadrics and having multiplicity strictly larger than one may be of separate interest.

4. Matrices of linear forms and ideals of quadrics contained in linear primes

It is easy to see [18, Exercise 7.6] that any prime ideal of multiplicity one of a polynomial ring over an algebraically closed field is generated by linear forms. We shall call such a prime ideal generated by linear forms a *linear prime*.

We give a matrix-theoretic approach to ideals generated by quadrics contained in a linear prime by viewing them as subideals of the determinantal ideal associated to certain matrices of linear forms. While our ideals are not the full determinantal ideal, we often use residual intersection techniques to recover determinantal ideals as residuals of our ideals or of subideals thereof (see Proposition 5.1 for a typical example). However, this technique requires 1-genericity assumptions on the matrix of linear forms, therefore we begin by studying linear algebraic properties of matrices associated to ideals of quadrics.

For the rest of the section, we consider the case of an ideal I generated by n quadrics and such that $I \subset (x, y)$, where x and y are linearly independent linear forms.

Definition 4.1. Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \end{pmatrix}.$$

Let I be an ideal generated by the n quadrics which are entries of the $1 \times n$ matrix $(-y \ \ x)A$, where x, y are linear forms. In this context we say that I is *represented by coefficients* by the matrix A.

Definition 4.2. Let

$$M = \begin{pmatrix} x & a_{11} & \ldots & a_{1n} \\ y & a_{21} & \ldots & a_{2n} \end{pmatrix}$$

be the matrix obtained by prepending the column vector $(x \ \ y)^T$ to the matrix A described in Definition 4.1. Let I be the ideal generated by the subset of the 2 by 2 minors of this matrix which involve the first column of M. In this context we say that I is *represented by minors* by the matrix M.

Remark 4.3. The same ideal I is obtained from the two matrix representations described in Definitions 4.1 and 4.2 respectively.

The example below shows that the same ideal I can be represented by several distinct matrices both by coefficients and also by minors.

Example 4.4. Both matrices A and A' displayed below represent by coefficients the ideal

$$I = (x^2, xy, ax + by, cx + dy) \subset K[x, y, a, b, c, d].$$

Furthermore, both matrices M and M' represent the same ideal I by minors:

$$A = \begin{pmatrix} 0 & 0 & -b & -d \\ x & y & a & c \end{pmatrix}, \qquad A' = \begin{pmatrix} 0 & -x & -b & -d \\ x & 0 & a & c \end{pmatrix},$$

$$M = \begin{pmatrix} x & 0 & 0 & -b & -d \\ y & x & y & a & c \end{pmatrix}, \qquad M' = \begin{pmatrix} x & 0 & -x & -b & -d \\ y & x & 0 & a & c \end{pmatrix}.$$

Note that M' in the example above is obtained from M by subtracting the first column from the third column. We use elementary operations of this type frequently in this section. We define a *sequence of invertible elementary operations* applied to a matrix to mean multiplication on either side by invertible matrices with scalar entries. Note that this is closely related to the notions of generalized rows and columns defined in the introduction. We say that two matrices are *equivalent* if one is obtained from the other by performing a sequence of invertible elementary operations.

Remark 4.5. The ideals of quadrics represented by coefficients by a pair of equivalent matrices are equal. To have that two ideals of quadrics represented by minors by a pair of equivalent matrices are equal, we must further require that any elementary column operations involved preserves the first column. We refer to matrix operations with this property by the name of *ideal-preserving operations*.

A fundamental idea of our analysis is to systematically exploit the pattern of generalized zero entries of the matrix representing an ideal of quadrics by minors. We now explain a procedure to reduce matrices of linear forms using ideal-preserving operations to a short list of canonical forms, which shall be analyzed in detail in Section 5.

Lemma 4.6. *Let M be a $2 \times (n+1)$ matrix of linear forms representing a height two ideal by minors and having at least one generalized zero. Then there is a sequence of ideal-preserving operations producing an equivalent matrix M' of the form*

$$M' = \begin{pmatrix} x' & 0 & a_{12} & \ldots & a_{1n} \\ y' & \ell & a_{22} & \ldots & a_{2n} \end{pmatrix},$$

where all the entries of M' are either linear forms or zero and $\mathrm{ht}(x', y') = 2$.

Proof. Recall that a generalized zero of M is a linear combination of the entries of a generalized column of M. Since M represents a height two ideal, the entries of the first column of M are linearly independent, thus this generalized column cannot be equal to the first column of M. Therefore at least one of the last n columns of M is involved in the linear combination giving the previously mentioned generalized column. The matrix operation replacing this column of M by the generalized column is an ideal-preserving invertible column operation. Next an appropriate row operation (which is in turn ideal-preserving) can be performed to yield a zero entry in the upper entry of this column, followed by a permutation of the columns which places the zero entry in the position indicated in M'. □

Proposition 4.7. *Let M be a $2 \times (n+1)$ matrix of linear forms with $n \geqslant 2$ representing a height two ideal of quadrics by minors. Then M is equivalent via a sequence of ideal-preserving elementary operations to a matrix M' of one of the following types, where the entries of all matrices below are either zero or linear forms and $\mathrm{ht}(x, y) = 2$:*

(1) M' is 1-generic;
(2) $M' = \begin{pmatrix} x & 0 & a_{12} \ldots a_{1n} \\ y & a_{21} & a_{22} \ldots a_{2n} \end{pmatrix}$, where $D = \begin{pmatrix} x & a_{12} \ldots a_{1n} \\ y & a_{22} \ldots a_{2n} \end{pmatrix}$ is 1-generic;
(3) $M' = \begin{pmatrix} x & 0 & 0 & a_{13} \ldots a_{1n} \\ y & a_{21} & a_{22} & a_{23} \ldots a_{2n} \end{pmatrix}$, with no additional restrictions;
(4) $M' = \begin{pmatrix} x & 0 & a_{12} & a_{13} \ldots a_{1n} \\ y & a_{21} & 0 & a_{23} \ldots a_{2n} \end{pmatrix}$, where $D = \begin{pmatrix} x & a_{13} \ldots a_{1n} \\ y & a_{23} \ldots a_{2n} \end{pmatrix}$ is 1-generic;
(5) $M' = \begin{pmatrix} x & 0 & a_{12} & a_{13} & a_{14} \ldots a_{1n} \\ y & a_{21} & 0 & \lambda a_{13} & a_{24} \ldots a_{2n} \end{pmatrix}$, where λ is a scalar and there are no additional restrictions.

Proof. We argue based on the number of generalized zeros of M. If M has no generalized zeros, then we set $M' = M$ which is a matrix of the first type listed. If M has at least one generalized zero, then we use Lemma 4.6 to produce by ideal-preserving operations a matrix $M' = \begin{pmatrix} x' & 0 & a_{12} \ldots a_{1n} \\ y' & a_{21} & a_{22} \ldots a_{2n} \end{pmatrix}$. We set $D = \begin{pmatrix} x' & a_{12} \ldots a_{1n} \\ y' & a_{22} \ldots a_{2n} \end{pmatrix}$ and analyze the pattern of generalized zeros of the submatrix D. If D has no generalized zeros then M' is of the second type listed in the statement. Next we consider the case when D has at least one generalized zero. Using Lemma 4.6, D is equivalent to a matrix of the type $D' = \begin{pmatrix} x'' & 0 & a'_{13} \ldots a'_{1n} \\ y'' & a'_{22} & a'_{23} \ldots a'_{2n} \end{pmatrix}$ via ideal-preserving elementary row operations. Applying the same operations to the matrix M' yields a result of the form

$$M'' = \begin{pmatrix} x'' & \alpha a_{21} & 0 & a'_{13} & \ldots & a'_{1n} \\ y'' & \beta a_{21} & a'_{22} & a'_{23} & \ldots & a'_{2n} \end{pmatrix}.$$

Now either $\alpha = 0$, which gives a matrix of the third type on our list, or $\alpha \neq 0$ and then subtracting a multiple of the first row from the second yields an equivalent matrix

$$M'' = \begin{pmatrix} x'' & \alpha a_{21} & 0 & a'_{13} & \dots & a'_{1n} \\ y''' & 0 & a''_{22} & a''_{23} & \dots & a''_{2n} \end{pmatrix}.$$

At this stage, either the submatrix $\begin{pmatrix} x'' & a'_{13} & \dots & a'_{1n} \\ y'' & a''_{23} & \dots & a''_{2n} \end{pmatrix}$ has no generalized zeros, which gives the fourth form in the statement of this proposition, or we may use Lemma 4.6 to find ideal-preserving operations which replace a'_{13} by zero. Acting by these matrix operations on M'' yields

$$M''' = \begin{pmatrix} x''' & \alpha' a_{21} & \gamma' a_{22} & 0 & a'''_{14} & \dots & a'''_{1n} \\ y''' & \beta' a_{21} & \delta' a_{22} & a'''_{23} & a'''_{14} & \dots & a'''_{2n} \end{pmatrix}.$$

If α' is zero, then the matrix is of the third type on our list up to a permutation of columns. Otherwise, by subtracting a β'/α' multiple of the first row from the last and permuting the third and fourth columns, we obtain a matrix of the last type on our list. □

Finally, we give a result concerning colon ideals. We strengthen the containments presented below to equalities in the case when A is a 1-generic matrix in Proposition 5.1.

Lemma 4.8. *Let I be an ideal of quadrics I contained in a height two linear prime (x, y) and let A and M be matrices representing I by coefficients and minors respectively. Then*

(1) $I_2(A) \subseteq I : (x, y)$,
(2) $I_2(M) \subseteq I : (x, y)$.

Proof. This first assertion is a generalization of the familiar Cramer's rule. The second assertion puts together the first containment and the trivial containment $I \subseteq I : (x, y)$, using that $I_2(M) = I_2(A) + I$. □

5. Bounding the projective dimension of ideals of quadrics contained in a linear prime of height two

We henceforth start an analysis of ideals I generated by $n \geqslant 2$ quadrics which are contained in a linear prime ideal of height two by considering ideals represented by each of the matrices M listed in Proposition 4.7. Recall that we work with ideals of quadrics in a polynomial ring R over a field K. For the rest of this section we assume that K is algebraically closed so that we can apply Theorem 2.4 on 1-generic matrices. This assumption is easily met by tensoring with an extension of the original field without changing the projective dimension of I. Moreover, we always assume that the polynomial ring R is generated by a basis of the span of the entries of M. Since any polynomial ring containing I is free over this R, the projective dimension of I is unaffected by this choice of ambient ring. It is easy to see that the dimension of R is bounded above by the number of entries of M, namely $2n + 2$, thus $\mathrm{pd}(R/I) \leqslant 2n + 2$ follows immediately by Hilbert's Syzygy Theorem. In this section we work towards lowering this upper bound to a *sharp* estimate given by $\mathrm{pd}(R/I) \leqslant 2n - 2$, which we prove in Theorem 5.7.

We begin with the case when the ideal is represented by a 1-generic matrix (equivalently a matrix with no generalized zeros).

Proposition 5.1. *Let I be an ideal minimally generated by n quadrics such that I is contained in a prime ideal (x, y) with x, y independent linear forms. Let M be any matrix representing I by minors and assume that M is 1-generic. Then*

(1) $\mathrm{pd}(R/I) \leqslant n$ and
(2) $I_2(M) = I : (x, y) = I : (x) = I : (y)$.

Proof. As shown in Lemma 4.8, one has the containment $I_2(M) \subset I : (x, y)$. To prove that the other containment holds, we consider the possibilities for $\mathrm{ht}(I : (x, y))$. First note that by Theorem 2.4, $\mathrm{ht}(I_2(M)) = n$ and since $I_2(M) \subseteq I : (x, y)$, the inequality $\mathrm{ht}(I : (x, y)) \geqslant n$ follows. Our first aim is to prove that in fact $\mathrm{ht}(I : (x, y)) = n$. We assume by way of contradiction that $\mathrm{ht}(I : (x, y)) \geqslant n+1$. Then, trivially, $\mathrm{ht}((I : (x, y)) + (x, y)) \geqslant n+1$ and the hypotheses of Corollary 2.10 are verified for $A = I$ and $C = (x, y)$. This yields $\mathrm{ht}(I : (x, y)) = n$, a contradiction.

We may now conclude that $\mathrm{ht}(I : (x, y)) = n$ and we turn to computing $I : (x, y)$. Since $I_2(M)$ is by Theorem 2.4 a prime ideal minimally generated by quadrics, neither of the linear forms x and y is contained in $I_2(M)$. Thus $\mathrm{ht}(I_2(M) + (x, y)) > \mathrm{ht}(I_2(M))$, or equivalently $\mathrm{ht}(I_2(M) + (x, y)) \geqslant n+1$. Now the hypotheses of Corollary 2.10 are verified for $A = I$ and $C = (x, y)$ and an application of this corollary yields the inequality $\mathrm{pd}(R/I) \leqslant n$, completing the proof of part (1) of our proposition.

To prove the formula for the colon ideals in part (2), note that the ideals on both sides of the containment $I_2(M) \subseteq I : (x, y)$ have the same height n and the ideal on the left is prime. This yields that in fact equality $I_2(M) = I : (x, y)$ must hold. A similar argument applies to show that the equalities $I : (x) = I : (y) = I_2(M)$ hold.　□

We continue our analysis of ideals I generated by quadrics which are contained in a linear prime ideal of height two by considering ideals represented by matrices of the second type listed in Proposition 4.7.

Proposition 5.2. *Assume that I is an ideal minimally generated by n quadrics which is represented via minors by a matrix of the form $M = \begin{pmatrix} x & 0 & a_{12} & a_{13} & \dots & a_{1n} \\ y & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \end{pmatrix}$, where $D = \begin{pmatrix} x & a_{12} & \dots & a_{1n} \\ y & a_{22} & \dots & a_{2n} \end{pmatrix}$ is 1-generic. Then $\mathrm{pd}(R/I) \leqslant n$.*

Proof. Note that we have $I = (a_{21}x) + I'$, where D represents I' by minors. Furthermore, the following holds by an application of Proposition 5.1:

$$I : (x) = (a_{21}) + I' : (x) = (a_{21}) + I_2(D).$$

By Theorem 2.4, $I_2(D)$ is a prime ideal such that $R/I_2(D)$ is Cohen–Macaulay of codimension $n-1$. Since the linear form a_{21} is not an element of the prime ideal $I_2(D)$, it is regular on the module $R/I_2(D)$. Together with $\mathrm{pd}(R/I_2(D)) = n-1$, this implies that $\mathrm{pd}(R/(I_2(D) + (a_{21}))) = n$. Since $I + (x) = (y)(a_{12}, a_{13}, \dots, a_{1n}) + (x)$, it is easy to see that $\mathrm{pd}(R/(I + (x))) \leqslant n$. Since $\mathrm{pd}(R/(I : (x))) = \mathrm{pd}(R/(I_2(D) + (a_{21}))) = n$ and $\mathrm{pd}(R/(I + (x))) \leqslant n$, we may now conclude by Remark 2.1 that $\mathrm{pd}(R/I) \leqslant n$.　□

Next we analyze the third situation from Proposition 4.7.

Proposition 5.3. *Assume that I is an ideal generated by $n \geqslant 3$ quadrics represented by minors by a matrix M which has at least two generalized zeros in the same row, i.e. is of the form*

$$M = \begin{pmatrix} x & 0 & 0 & a_{13} & \dots & a_{1n} \\ y & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \end{pmatrix},$$

where the a_{ij} are either zero or linear forms. Then $\mathrm{pd}(R/I) \leqslant 2n - 2$.

Proof. We denote by D the matrix

$$D = \begin{pmatrix} x & a_{13} & \dots & a_{1n} \\ y & a_{23} & \dots & a_{2n} \end{pmatrix}.$$

It is easy to see that $I = (xa_{21}, xa_{22}) + I'$, where D represents I' by minors. We conclude by Remark 2.2 and Proposition 5.1 that

$$I : (x) = (a_{21}, a_{22}) + I_2(D).$$

We now set $\overline{R} = R/(a_{21}, a_{22})$ and we let $\overline{D}$ denote the $2 \times (n-1)$ matrix with entries in $\overline{R}$ obtained by reducing the entries of D modulo the linear forms a_{21} and a_{22}. If $\dim \overline{R} \leqslant 2n - 4$, we must have $\dim R \leqslant 2n - 2$ and the desired bound $\mathrm{pd}(R/I) \leqslant 2n - 2$ follows by the Hilbert Syzygy Theorem.

We henceforth assume $\dim \overline{R} \geqslant 2n - 3$ (i.e. either $\dim \overline{R} = 2n - 3$ or $\dim \overline{R} = 2n - 2$). If $\overline{D}$ is 1-generic, then $\mathrm{pd}_{\overline{R}}(\overline{R}/I_2(\overline{D})) = n - 2$ by Theorem 2.4. Otherwise, $\overline{D}$ has at least one generalized zero, thus, after a linear change of coordinates in $\overline{R}$, it is equivalent to a matrix $\overline{D}'$ with $2n - 3$ generic entries (each is a distinct variable) and one zero. By a result of Boocher [6, Theorem 4.1], the minimal free resolution of $\overline{R}/I_2(\overline{D}')$ is obtained by appropriately pruning the Eagon–Northcott complex, hence $\mathrm{pd}_{\overline{R}}(\overline{R}/I_2(\overline{D})) = \mathrm{pd}_{\overline{R}}(\overline{R}/I_2(\overline{D}')) \leqslant n - 2$.

To conclude, note that since $I_2(\overline{D})$ and (a_{21}, a_{22}) use disjoint sets of variables, we have

$$\mathrm{pd}\big(R/(I : (x))\big) = \mathrm{pd}_R\big(R/\big((a_{21}, a_{22}) + I_2(D)\big)\big) = \mathrm{pd}_{\overline{R}}\big(\overline{R}/I_2(\overline{D})\big) + 2 \leqslant n.$$

It is easily seen that $\mathrm{pd}(R/(I + (x))) = \mathrm{pd}(R/(y(a_{23}, \dots, a_{2n}) + (x))) \leqslant n - 1$ and Remark 2.1 now yields $\mathrm{pd}(R/I) \leqslant n$. $\quad\square$

Before we can analyze the next case described by Proposition 4.7, an additional lemma is needed.

Lemma 5.4. *Let I be an ideal generated by quadratic polynomials, contained in a linear prime ideal (x, y) and represented by minors by a 1-generic matrix D. Let a be a linear form that is not in the linear span of the entries of D. Then*

$$(ay) + I = \big((y) + I\big) \cap \big((a) + I_2(D)\big).$$

Proof. The containment $(ay) + I \subseteq ((y) + I) \cap ((a) + I_2(D))$ is obvious. As for the opposite containment, let $f \in (y + I) \cap ((a) + I_2(D))$. Since $f \in ((y) + I)$, we have $f = yh + g$, with $g \in I$ and $h \in R$. As $g \in I \subset (a) + I_2(D)$, the assertion $f \in (a) + I_2(D)$ is equivalent to $yh \in (a) + I_2(D)$.

By Theorem 2.4, $I_2(D)$ is a prime ideal. Since a is a linear form which is not in the span of the entries of D, the ideal $(a) + I_2(D)$ is in turn prime. Now $yh \in (a) + I_2(D)$ implies $y \in (a) + I_2(D)$ or $h \in (a) + I_2(D)$. Since the only linear forms in $(a) + I_2(D)$ are scalar multiples of a, which is not a multiple of y by hypothesis, the first alternative is impossible. It remains that $h \in (a) + I_2(D)$, thus finishing the proof. $\quad\square$

We continue our analysis with the fourth among the cases listed in Proposition 4.7.

Proposition 5.5. *Let I be an ideal generated by $n \geqslant 3$ quadrics and represented by minors by a matrix M of the form*

$$M = \begin{pmatrix} x & 0 & a_{12} & a_{13} & \dots & a_{1n} \\ y & a_{21} & 0 & a_{23} & \dots & a_{2n} \end{pmatrix},$$

where the submatrix $D = \begin{pmatrix} x & a_{13} & \dots & a_{1n} \\ y & a_{23} & \dots & a_{2n} \end{pmatrix}$ is 1-generic. Then $\mathrm{pd}(R/I) \leqslant 2n - 2$.

Proof. Clearly the ideal I has the form $I = (a_{12}y, a_{21}x) + I'$, where I' is the ideal represented by minors by the 1-generic matrix D. Should both a_{21} and a_{12} be in the span of the entries of D, then the conclusion would easily follow from Hilbert's Syzygy Theorem. Therefore, without loss of

generality, we may assume that a_{12} is not in the span of the entries of D. Furthermore, if a_{21} is in the span of $\{y, a_{23}, \ldots, a_{2n}\}$, then the second column of M can be replaced by ideal-preserving matrix operations by a column having a zero entry in the bottom, and then the ideal I can be represented by minors by a matrix that has already been analyzed in Proposition 5.3. Henceforth, we assume that a_{21} is not in the span of $\{y, a_{23}, \ldots, a_{2n}\}$.

It is easy to see that $\mathrm{pd}(R/(I + (x))) = \mathrm{pd}(R/(x, ya_{12}, ya_{13}, \ldots, ya_{1n})) \leqslant n$. We now proceed to compute $I : (x)$ using Remark 2.2 and Lemma 5.4:

$$I : (x) = (a_{21}) + \big(a_{12}y + I'\big) : (x) = (a_{21}) + \big[\big((y) + I'\big) \cap \big((a_{12}) + I_2(D)\big)\big] : (x)$$

$$= (a_{21}) + \big[(y, a_{23}x, \ldots, a_{2n}x) : (x)\big] \cap \big[\big((a_{12}) + I_2(D)\big) : (x)\big]$$

$$= (a_{21}) + (y, a_{23}, \ldots, a_{2n}) \cap \big((a_{12}) + I_2(D)\big).$$

The last equality follows because the ideal $(a_{12}) + I_2(D)$ is a prime ideal not containing any linear form in the span of x. We now estimate the projective dimension of each term appearing above. First, by Theorem 2.4 we have $\mathrm{pd}(R/I_2(D)) = n - 2$ and since a_{12} is a regular form on $I_2(D)$, we obtain that $\mathrm{pd}(R/((a_{12}) + I_2(D))) = n - 1$. Furthermore, we have that $\mathrm{pd}(R/(y, a_{23}, \ldots, a_{2n})) = n - 1$ and

$$\mathrm{pd}\big(R/\big((y, a_{23}, \ldots, a_{2n}) + \big((a_{12}) + I_2(D)\big)\big)\big) = \mathrm{pd}\big(R/(y, a_{23}, \ldots, a_{2n}, a_{12})\big) \leqslant n.$$

Using a standard short exact sequence and the three previous inequalities we deduce:

$$\mathrm{pd}\big(R/\big((y, a_{23}, \ldots, a_{2n}) \cap \big((a_{12}) + I_2(D)\big)\big)\big) \leqslant n - 1.$$

Since $(y, a_{23}, \ldots, a_{2n}) \cap ((a_{12}) + I_2(D))$ is the intersection of two prime ideals and since $a_{21} \notin (y, a_{23}, \ldots, a_{2n})$, it is easily seen that a_{21} is a non-zero divisor on $R/((y, a_{23}, \ldots, a_{2n}) \cap ((a_{12}) + I_2(D)))$, if and only if a_{21} is not in the span of a_{12}. Thus, if a_{21} is not in the span of a_{12}, then we deduce from the previous inequality that

$$\mathrm{pd}\big(R/\big((a_{21}) + (y, a_{23}, \ldots, a_{2n}) \cap \big((a_{12}) + I_2(D)\big)\big)\big) \leqslant n.$$

Since $\mathrm{pd}(R/(I + (x))) = \mathrm{pd}(R/(x, ya_{12}, ya_{13}, \ldots, ya_{1n})) \leqslant n$, it follows that $\mathrm{pd}(R/I) \leqslant n$.

The remaining case is when a_{12} is a scalar multiple of a_{21}. Recall that a_{12} is not in the span of the entries of D, which allows us to compute $I : (a_{12}) = (x, y) + I' : (a_{12}) = (x, y) + I' = (x, y)$ and $I + (a_{12}) = I' + (a_{12})$. By Proposition 5.1, we have that $\mathrm{pd}(R/I') \leqslant n + 1$, since D is 1-generic. Since a_{12} is a non-zero divisor on R/I', it follows that $\mathrm{pd}(R/(a_{12} + I')) \leqslant n + 2$; therefore $\mathrm{pd}(R/I) \leqslant n + 1$ in this case. $\quad\square$

Finally, we consider the last case described in Proposition 4.7. To handle this case, we need to assume an inductive bound on the projective dimension of ideals of quadrics with fewer generators. We shall use Proposition 5.6 below in an inductive argument to compute the overall bound given in Theorem 5.7.

Proposition 5.6. *Let I be an ideal generated by $n \geqslant 3$ quadrics and represented by minors by a matrix M of the form*

$$M = \begin{pmatrix} x & 0 & a_{12} & a_{13} & a_{14} & \ldots & a_{1n} \\ y & a_{21} & 0 & \lambda a_{13} & a_{24} & \ldots & a_{2n} \end{pmatrix},$$

where the a_{ij} are either zero or linear forms. Assume additionally that any ideal J of height two generated by $n - 1$ quadrics in a polynomial ring S is known to have $\mathrm{pd}(S/J) \leqslant 2n - 4$. Then $\mathrm{pd}(R/I) \leqslant 2n - 2$.

Proof. Note that, if the non-zero entries of M form a linearly dependent set, then the conclusion follows immediately by the Hilbert Syzygy Theorem. We henceforth assume that the $2n-1$ non-zero entries of M form a linearly independent set. It can be read off M that the ideal I has the form

$$I = \left(a_{12}y, a_{21}x, a_{13}(\lambda x - y)\right) + I',$$

where I' is an ideal generated by quadrics represented by minors by the matrix given by the first column and last $n-2$ columns of M. Next we compute

$$I : (a_{12}) = (y) + \left(\left(a_{21}x, a_{13}(\lambda x - y)\right) + I'\right) : (a_{12}) = (y) + \left(a_{21}x, a_{13}(\lambda x - y)\right) + I',$$

where the first equality follows from Remark 2.2 and the second from knowing that the ideal $(a_{21}x, a_{13}(\lambda x - y)) + I'$ is written in terms of a set of variables disjoint from a_{12}. Since the previous computation gives $I : (a_{12}) = (y) + (a_{21}, a_{13}, a_{24}, \ldots, a_{2n})x$, we may conclude that $\mathrm{pd}(R/(I : a_{12})) = n$.

Next we compute $I + (a_{12})$. Since a_{12} only appears in one minimal generator of I, one has $I + (a_{12}) = J + (a_{12})$, where J in the ideal minimally generated by the $n-1$ quadratic generators of I not involving a_{12}. Consider the polynomial ring $S = K[x, y, a_{21}, a_{13}, a_{ij}]$ (for $1 \leqslant i \leqslant 2$ and $4 \leqslant j \leqslant n$) and view J as an ideal of S. The hypothesis on ideals generated by $n-1$ quadrics yields $\mathrm{pd}(S/J) \leqslant 2n-4$. Since a_{12} is a variable not appearing in S, we have $\mathrm{pd}_R(R/((a_{12}) + J)) = \mathrm{pd}_S(S/J) + 1 \leqslant 2n-3$. Finally, we use Remark 2.1 to estimate $\mathrm{pd}(R/I) \leqslant \max\{n, 2n-3\} \leqslant 2n-2$ for all $n \geqslant 3$. □

To complete the analysis, we combine Propositions 5.1, 5.2, 5.3 and 5.5 to obtain:

Theorem 5.7. *Assume that I is a height two ideal minimally generated by n quadrics which has a linear minimal associated prime. Then $\mathrm{pd}(R/I) \leqslant 2n - 2$.*

Proof. We proceed by induction on n, noting that the assumption on the height of I forces $n \geqslant 2$. The base case, $n = 2$ is easily seen to be a complete intersection thus having projective dimension two.

Let $n \geqslant 3$. Using the inductive hypothesis, we may assume that any ideal J generated by $n-1$ quadrics, which has a minimal associated prime ideal generated by two linear forms has $\mathrm{pd}(S/J) \leqslant 2n-4$, where S is an ambient polynomial ring for J. Now we consider an ideal I generated by n quadrics having a linear minimal prime ideal of height two.

We finish the proof by analyzing the various possibilities for matrices representing I by minors listed in Proposition 4.7:

(1) if M has no generalized zeros, we apply Propositions 5.1 to obtain $\mathrm{pd}(R/I) \leqslant n+1$;
(2) if M is of one of the types stated in Propositions 5.2, 5.3 or 5.5, then we obtain $\mathrm{pd}(R/I) \leqslant 2n-2$;
(3) if M is of the type stated in Proposition 5.6 we rely on the inductive hypothesis to obtain again $\mathrm{pd}(R/I) \leqslant 2n-2$. □

5.1. Better bounds

In Theorem 5.7 we have successfully bounded the projective dimension of ideals generated by $n \geqslant 2$ quadrics under the assumption that the ideal is contained in a linear prime ideal of height two. We now improve on this bound under the additional condition that the multiplicity of the ideal at this linear prime is at least two. We recall that, by Lemma 3.1, the local multiplicity of an ideal of height two generated by quadrics at such a prime is at most three.

Proposition 5.8. *Let R be a polynomial ring; let I be an ideal generated by $n \geqslant 2$ quadrics, contained in a height two linear prime ideal (x, y) and assume $e(R_{(x,y)}/I_{(x,y)}) \geqslant 2$. Then $\mathrm{pd}(R/I) \leqslant n+2$.*

Proof. Let M be any matrix representing I by minors and let A be the submatrix representing I by coefficients. Since $e(R_{(x,y)}/I_{(x,y)}) \geqslant 2$, we must have that $I_2(M)R_{(x,y)} \neq R_{(x,y)}$; otherwise, the containment $(x,y)R_{(x,y)} \subseteq IR_{(x,y)}$ would hold by Lemma 4.8, leading to $e(R_{(x,y)}/I_{(x,y)}) = 1$. The fact that $I_2(M)R_{(x,y)} \neq R_{(x,y)}$ is equivalent to $I_2(M) \subseteq (x,y)$, which in particular implies that the containment $I_2(A) \subseteq (x,y)$ holds. Now consider the matrix $\overline{A}$ obtained by taking the image of A under the canonical projection to the polynomial factor ring $\overline{R} = R/(x,y)$. Since $I_2(A) \subseteq (x,y)$, it follows that $I_2(\overline{A}) = (0)$ in $\overline{R}$. An application of Theorem 2.5 for $\overline{A}$, with $r = 1$ and $i = 1$, shows that the entries of $\overline{A}$ can be written in terms of at most n independent linear forms. The entries of the matrix A and hence also the ideal I can then be written in terms of at most $n + 2$ independent linear forms. An application of the Hilbert Syzygy Theorem finishes the proof. $\square$

Proposition 3.4 on ideals contained in a prime of multiplicity two can be imitated in the context of bounding the projective dimension of ideals contained in linear primes as well, with the additional hypothesis that there be at least two distinct such linear minimal primes.

Proposition 5.9. *Let I be an ideal generated by n quadratic polynomials which is contained in two distinct linear prime ideals of height 2. Then* $\mathrm{pd}(R/I) \leqslant n$.

Proof. Let the two linear primes be (x,y) and (z,w). If $\mathrm{ht}(x,y,z,w) = 4$, then the intersection of these two ideals is generated by quadrics and since the minimal generators of I must be linear combinations of these quadrics, the variables x, y, z, w are the only ones required to express I, hence $\mathrm{pd}(R/I) \leqslant 4$. Otherwise if $\mathrm{ht}(x,y,z,w) < 4$, we may assume without loss of generality that $x = w$. Now $I \subset (x,y) \cap (x,z) = (x,yz)$. Setting $q = yz$, one may apply the technique in the proof of Proposition 3.4 to see that $\mathrm{pd}(R/I) \leqslant n$. $\square$

6. Final questions

Throughout this paper two key invariants were used to obtain our upper bound on $\mathrm{pd}(R/I)$: the number of minimal generators of I and the additional assumption that $\mathrm{ht}\,I = 2$. In this section, we formulate two questions aimed towards establishing whether these invariants suffice in general for bounding the projective dimension of ideals generated by quadrics. Note that the main result of [1] shows the number of minimal generators alone suffices to give a bound on the projective dimension of ideals of quadrics, if one does not insist on an optimal bound. The main result of this paper answers both question below in the affirmative for the case $h = 2$. Together with a result of Eisenbud and Huneke (cf. [26, Theorem 3.1]) for the case $n = 3$ and easy arguments in the remaining cases, this verifies that the two questions have an affirmative answer whenever $n \leqslant 3$ or $h \leqslant 2$.

Question 6.1. *Let R be a polynomial ring and let I be an ideal of R generated by n quadrics and having $\mathrm{ht}\,I = h$. Is there a sharp upper bound for $\mathrm{pd}(R/I)$ expressed only in terms of n and h?*

Based on computational evidence and on the bound given by our main result in the case of height two ideals of quadrics, we refine the previous question by suggesting a specific candidate for an upper bound on the projective dimension of ideals of quadrics.

Question 6.2. *Let R be a polynomial ring and let I be an ideal of R generated by n quadrics and having $\mathrm{ht}\,I = h$. Is it true that* $\mathrm{pd}(R/I) \leqslant h(n - h + 1)$?

We note the existence of a family of ideals of quadrics which achieves the equality in the bound proposed in Question 6.2, namely the ideals $I_{m,n,2}$ described in [25]. These ideals have $m + n$ quadratic generators, $\mathrm{ht}(I_{m,n,2}) = m$ and $\mathrm{pd}(I_{m,n,2}) = m(n + 1)$. Therefore, if the answer to Question 6.1 is affirmative, any such bound must be at least as large as the bound we suggest in Question 6.2. However, as soon as the generators are allowed to have arbitrary degrees, one cannot expect any upper bound on the projective dimension to have asymptotic growth bounded by the expression in Question 6.2

or even by the product of height and minimal number of generators. To see this, consider again, this time for $d \geqslant 3$, the family of ideals $I_{m,n,d}$ described in [25]. These ideals have ht $I_{m,n,d} = m$ and are minimally generated by $m+n$ homogeneous polynomials of degree d, while their projective dimension is given by $\mathrm{pd}(R/I_{m,n,d}) = m + n\frac{(m+d-2)!}{(m-1)!(d-1)!}$.

Acknowledgments

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