



The strengthened Alperin–McKay conjecture for p -solvable groups

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ABSTRACT

The Alperin–McKay conjecture is a well-known conjecture. It is known to be true for p -solvable groups by work of Dade and Okuyama–Wajima. Recently, this conjecture has been strengthened by work of Isaacs–Navarro, Navarro and Turull. This refinement involves the degrees modulo p of the characters involved, the field of values over the p -adic numbers of the relevant characters, and their p -local Schur indices. In this paper, we prove that this strengthened version of the conjecture is true for all p -solvable groups.

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1. Introduction

The McKay conjecture, originally proposed in [4], is an important conjecture in the area of representation theory of finite groups. In its current standard form, it claims that given any finite group G and any prime p , if P is any Sylow p -subgroup of G , the number of irreducible characters of p' degree of G is equal to the number of irreducible characters of p' degree of $N_G(P)$, the normalizer of P . This conjecture was refined and generalized by Alperin [1]. The Alperin–McKay conjecture claims that if G is any finite group, and p is any prime, and B is any p -block of G with defect group D , and b is the block of $N_G(D)$ which corresponds to B under the Brauer correspondence, then

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|.$$

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As is standard, we denote by $\text{Irr}(B)$ the set of ordinary irreducible characters of a block B , and by $\text{Irr}_0(B)$ the set of ordinary irreducible characters of B of height zero. The Alperin–McKay conjecture implies the McKay conjecture by taking all blocks of full defect.

More recently, a number of strengthenings of these conjectures have been proposed by Isaacs and Navarro [3], by Navarro [6] and by Turull [8]. This work led to the following strengthened version of the McKay conjecture. In order to state it properly, we need to introduce some notation. Let p be any prime number. If n is any positive integer, we denote by n_p the p -part of n , and by $n_{p'}$ the p' -part of n , so n_p is a power of p , $n_{p'}$ is prime to p , and $n = n_p n_{p'}$. If G is a group and D is a p -subgroup of G , we denote by $\text{Irr}(G, D)$ the set of ordinary irreducible characters of G which belong to some p -block with defect group D . We denote by $\text{Irr}_0(G, D)$ the set of all elements of $\text{Irr}(G, D)$ which have height zero. If $\chi \in \text{Irr}(G)$, we denote by $\text{codeg}(\chi)$ its *codegree*, i.e., we set $\text{codeg}(\chi) = |G|/\chi(1)$. Furthermore, we denote by $[\chi]_p$ the element of the Brauer group over the local field of values $\text{Br}(\mathbf{Q}_p(\chi))$ corresponding to χ , so, in particular, the p -local Schur index $m_p(\chi)$ is the order of $[\chi]_p$ as an element of the Brauer group $\text{Br}(\mathbf{Q}_p(\chi))$. Of course, here and throughout, \mathbf{Q}_p denotes the field of p -adic numbers, and we denote by $\overline{\mathbf{Q}_p}$ an algebraic closure of \mathbf{Q}_p .

Conjecture (Strengthened Alperin–McKay). *Let p be a prime number, let G be any finite group, and let D be a p -subgroup of G . Then there exists a bijection*

$$f : \text{Irr}_0(G, D) \rightarrow \text{Irr}_0(N_G(D), D)$$

satisfying all of the following conditions:

- (1) If $\chi \in \text{Irr}_0(B)$, and B is a block of G of defect D , then $f(\chi) \in \text{Irr}_0(b)$, where b is the Brauer correspondent block of $N_G(D)$ for the block B .
- (2) If $\chi \in \text{Irr}_0(G, D)$, then $\text{codeg}(f(\chi))_p^2 \equiv \text{codeg}(\chi)_p^2 \pmod{p}$.
- (3) f commutes with the action of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, so, in particular, $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(f(\chi))$ for every $\chi \in \text{Irr}_0(G, D)$.
- (4) For every $\chi \in \text{Irr}_0(G, D)$, we have $[f(\chi)]_p = [\chi]_p$, so, in particular, $m_p(f(\chi)) = m_p(\chi)$, and $f(\chi)$ and χ have the same p -local Schur index.

The simple existence of the bijection f satisfying condition (1) is the Alperin–McKay conjecture. It is open in general for finite groups. Dade [2] and Okuyama and Wajima [7] proved the Alperin–McKay conjecture for all p -solvable groups. The existence of a bijection f satisfying (1) and (2) was already known for all p -solvable groups [3]. The existence of f satisfying (2), (3), (4) in the case when D is a Sylow p -subgroup of G and G is p -solvable was proved in [9]. In the present paper, we prove the full strengthened conjecture for all p -solvable groups G and all p -subgroups D of G . In our proof, we use the character correspondence whose existence is proved in [9]. Further properties of this character correspondence than those originally described are needed for the proof here. These are proved as follows. We use the identity of elements of the Brauer–Clifford group proved in [9, Theorem 6.5] together with the results in [10] to conclude the existence of a suitable endoisomorphism. By [11], this endoisomorphism implies the existence of an excellent character correspondence, and the properties we need then follow easily. The properties we need about the correspondence relate mainly to characters, and they could alternatively have been proved by analyzing further the character correspondences as described in [9], but we find it is more efficient to simply use the more detailed results described in [11].

Note that we systematically write all functions on the left, and compose them from right to left. This allows, in particular, to compose characters with elements of Galois groups. We also use left exponential notation (i.e. ga for the action of a group element g on an algebra element a).

2. Some preliminary results

Let p be a prime. We fix F to be a field of characteristic zero, and let \bar{F} to be an algebraic closure of F . We will eventually take F to be the field \mathbf{Q}_p of p -adic numbers. We take the characters of finite groups to be functions from the group to \bar{F} . Let $\chi \in \text{Irr}(G)$ be an irreducible character of some finite group G . We denote, as is usual, by $F(\chi)$ the field extension of F by the image of the function χ . Suppose N is a normal subgroup of G , and $\theta \in \text{Irr}(N)$, then the *inertia group* of θ in G is

$$I_G(\theta) = \{g \in G: {}^g\theta = \theta\}.$$

We define the *extended inertia group* of θ with respect to F to be

$$\tilde{I}_G(\theta, F) = \{g \in G: \text{there exists } \sigma \in \text{Gal}(\bar{F}/F) \text{ such that } {}^g\theta = \sigma\theta\}.$$

Of course, both $I_G(\theta)$ and $\tilde{I}_G(\theta, F)$ are subgroups of G , and they satisfy $N \subseteq I_G(\theta) \subseteq \tilde{I}_G(\theta, F)$. By $\text{Irr}(G|\theta)$ we mean the set of all elements of $\text{Irr}(G)$ whose restriction to N involves θ as a summand. By $\text{Irr}(G|\theta, F)$ we mean the set of all elements of $\text{Irr}(G)$ whose restriction to N involves θ or some conjugate of θ under the action of $\text{Gal}(\bar{F}/F)$ as a summand. In general, for each ordinary irreducible character χ , we denote by $[\chi]_F$ the element of the Brauer group $\text{Br}(F(\chi))$ corresponding to χ .

The following proposition is very close to some well-known results, which we state here in a form convenient for our purposes. We include a short proof for the reader's convenience.

Proposition 2.1. *Let G be a finite group and let N be a normal subgroup of G . Suppose $\theta \in \text{Irr}(N)$. Set $T = \tilde{I}_G(\theta, F)$. Then ordinary induction of characters provides a bijection $\text{Ind}_T^G: \text{Irr}(T|\theta, F) \rightarrow \text{Irr}(G|\theta, F)$. Furthermore, it satisfies all of the following.*

- (1) *If $\chi \in \text{Irr}(T|\theta, F)$, then $\text{codeg}(\text{Ind}_T^G(\chi)) = \text{codeg}(\chi)$.*
- (2) *Ind_T^G commutes with the action of $\text{Gal}(\bar{F}/F)$, so, in particular, $F(\chi) = F(\text{Ind}_T^G(\chi))$ for every $\chi \in \text{Irr}(T|\theta, F)$.*
- (3) *For every $\chi \in \text{Irr}(T|\theta, F)$, we have $[\text{Ind}_T^G(\chi)]_F = [\chi]_F$.*

Proof. Let O be the orbit of θ under the action of $\text{Gal}(\bar{F}/F)$ on $\text{Irr}(N)$. Since the actions of G and of $\text{Gal}(\bar{F}/F)$ on $\text{Irr}(N)$ commute, and the orbit of θ under the action of T is contained in O , we have that T acts on O . Let O_1, \dots, O_r be the orbits of T on O , with $\theta \in O_1$, and pick representatives $\theta_1, \dots, \theta_r$ with $\theta_1 = \theta$ and $\theta_i \in O_i$ for $i = 1, \dots, r$. Set $I = I_G(\theta)$ to be the inertia group of θ . Then, I is the inertia group of every element of O . Let $\psi \in O$, and let H be any subgroup of G which contains T . Of course, $T = \tilde{I}_G(\psi, F)$. Furthermore, if $\chi \in \text{Irr}(H|\psi)$, then the elements of O which are summands of the restriction of χ to N are exactly those in the orbit of ψ under the action of T . It follows that $\text{Irr}(H|\theta, F)$ is the disjoint union of $\text{Irr}(H|\theta_i)$ for $i = 1, \dots, r$. By Clifford's Theorem, Ind_I^H provides a bijection $\text{Irr}(I|\theta_i) \rightarrow \text{Irr}(H|\theta_i)$. Using this in the cases $H = T$ and $H = G$, we get that Ind_T^G provides a bijection $\text{Irr}(T|\theta_i) \rightarrow \text{Irr}(G|\theta_i)$. Putting these bijections together, we get a bijection

$$\text{Ind}_T^G: \text{Irr}(T|\theta, F) \rightarrow \text{Irr}(G|\theta, F).$$

It is clear that this bijection satisfies (1) and (2).

Let $\chi \in \text{Irr}(T|\theta, F)$. Set $\chi_0 = \text{Ind}_T^G(\chi)$. Then $\chi_0 \in \text{Irr}(G|\theta, F)$, and by (2) we have that $F(\chi) = F(\chi_0)$. We set $K = F(\chi) = F(\chi_0)$. Let M be a non-zero KG -module affording the character $n\chi_0$, for some positive integer n . Then $\text{End}_{KG}(M)$ is a central simple algebra over K of dimension n^2 , and its class in the Brauer group $\text{Br}(K)$ is $[\chi_0]_F$. Let N be the χ -homogeneous component of $\text{Res}_T^G(M)$. By Frobenius reciprocity, N affords the character $n\chi$. Hence, $\text{End}_{KG}(N)$ is a central simple algebra over K of dimension n^2 , and its class in the Brauer group $\text{Br}(K)$ is $[\chi]_F$. If $\phi \in \text{End}_{KG}(M)$, then ϕ commutes in particular with the action of T , so that $\phi(N) \subseteq N$. Hence, restriction of maps provides

$$\text{Res}_N^M : \text{End}_{KG}(M) \rightarrow \text{End}_{KT}(N),$$

a homomorphism of algebras over K which sends the identity to the identity. Since both algebras have the same dimension and the first one is simple, it follows that $\text{End}_{KG}(M)$ and $\text{End}_{KT}(N)$ are isomorphic as algebras over K . This implies that $[\text{Ind}_T^G(\chi)]_F = [\chi]_F$. Hence (3) holds, and this completes the proof of the proposition. \square

Our next theorem is very close to a special case of a well-known result of Fong and Reynolds. For convenience, we regard blocks as sets of ordinary irreducible characters, so that if B is a block, we have $B = \text{Irr}(B)$.

Theorem 2.2. *Assume the hypotheses of Proposition 2.1. Assume furthermore that N be a p' -subgroup of G . Then, the bijection*

$$\text{Ind}_T^G : \text{Irr}(T|\theta, F) \rightarrow \text{Irr}(G|\theta, F)$$

of Proposition 2.1 satisfies, in addition, all the following.

- (1) $\text{Irr}(T|\theta, F)$ and $\text{Irr}(G|\theta, F)$ are unions of blocks of their respective groups.
- (2) Ind_T^G sends each block b contained in $\text{Irr}(T|\theta, F)$ to a block B of $\text{Irr}(G|\theta, F)$, this block is $B = b^G$, and this provides a bijection from the set of all the blocks of T contained in $\text{Irr}(T|\theta, F)$ to the set of all the blocks of G contained in $\text{Irr}(G|\theta, F)$.
- (3) Let b be a block of T contained in $\text{Irr}(T|\theta, F)$, and let $B = b^G$. Then any defect group of b is a defect group of B .
- (4) Let $\chi \in \text{Irr}(T|\theta, F)$ be of height h in the block b . Then $\text{Ind}_T^G(\chi)$ has the same height h in $B = b^G$.

Proof. Since N is a p' -group, each p -block of N consists of a single irreducible character. Hence, (1) follows, for example, from [5, Theorem 9.2]. The Theorem of Fong and Reynolds, see for example [5, Theorem 9.14], provides results similar to (2), (3), and (4), for induction from the inertia group to an arbitrary subgroup containing the inertia group. Using this and the information in the proof of Proposition 2.1, one can deduce the rest of the stated properties. This completes the proof of the theorem. \square

Our next result is a generalization of a well-known result of Fong.

Theorem 2.3. *Let G be a finite p -solvable group and let $N = \text{O}_{p'}(G)$ be the largest normal p' -subgroup of G . Suppose that $\theta \in \text{Irr}(N)$, and that the orbit of θ under the action of G on $\text{Irr}(N)$ is contained in the orbit of θ under the action of $\text{Gal}(\bar{F}/F)$ on $\text{Irr}(N)$. Then $\text{Irr}(G|\theta)$ is a single block B of G . Furthermore, the defect groups of B are the Sylow p -subgroup of the inertia subgroup $I = I_G(\theta)$, and*

$$\text{Irr}_0(B) = \{\text{Ind}_I^G(\psi) : \psi \in \text{Irr}(I|\theta) \text{ and } p \nmid \psi(1)\}.$$

Proof. Let O be the orbit of θ under the action of G . Set $K = F(\theta)$. Then, for all $\psi \in O$, we have $F(\psi) = K$. Now, $\text{Gal}(K/F)$ acts on the set of all $\psi \in \text{Irr}(G)$ such that $F(\psi) = K$, a set which contains O . By our assumption, for each $g \in G$, there exists a unique $\sigma = \phi(g) \in \text{Gal}(K/F)$ such that ${}^g\theta = \sigma\theta$. The map $\phi : G \rightarrow \text{Gal}(K/F)$ is a group homomorphism. Set I to be the kernel of ϕ . Then $I = I_G(\theta)$ is the inertia group of θ . Since $I \trianglelefteq G$ and $N \subseteq I$, it follows that $N = \text{O}_{p'}(I)$. Now by Fong's Theorem, see for example [5, Theorem 10.20], it follows that $b = \text{Irr}(I|\theta)$ is a p -block of I , and the defect groups of this block are exactly the Sylow p -subgroups of I . Furthermore, we have

$$\text{Irr}_0(b) = \{\psi \in \text{Irr}(I|\theta) : p \nmid \psi(1)\}.$$

Now the block stabilizer in G of b is I . Hence, by the Fong–Reynolds Theorem, see for example [5, Theorem 9.14], we have that there is exactly one block B of G covering b , and this block consists of all the characters of b induced up to G , and that corresponding characters have the same height. Furthermore, the theorem also tells us that every defect group of b is a defect group of B . Hence, we have $B = \text{Irr}(G|\theta)$, and the theorem follows. \square

As is well known, there is a strong relationship between the Glauberman correspondence and the Brauer correspondence of blocks. We will use the following.

Theorem 2.4. *Let G be a finite group, let p be a prime, let B be a p -block of G , and let D be a defect group of B . Let b be the block of $N_G(D)$ Brauer-corresponding to B . Let N be a normal p' -subgroup of G , so, of course, $C_N(D) \trianglelefteq N_G(D)$, let $\theta \in \text{Irr}(N)$ be D -invariant, and let $\mu \in \text{Irr}(C_N(D))$ be the Glauberman correspondent to θ under the action of D . Then $B \subseteq \text{Irr}(G|\theta)$ if and only if $b \subseteq \text{Irr}(N_G(D)|\mu)$.*

Proof. We let R be the ring of all algebraic integers in \mathbf{C} , the field of complex numbers, and we choose M to be a maximal ideal of R containing pR . We set $K = R/M$. Then K is a field of characteristic p . We fix an identification of the algebraic integers in \bar{F} with R . So the values of every irreducible character on any of its group elements, and the value of any central character on any conjugacy class sum is in R . Let $\text{Br}_D : Z(KG) \rightarrow Z(KN_G(D))$ be the usual Brauer map. It is a linear map defined as follows. If C is any conjugacy class of G , and \hat{C} is the corresponding conjugacy class sum, then

$$\text{Br}_D(\hat{C}) = \sum_{x \in C \cap C_G(D)} x.$$

Let $\lambda_b : Z(KN_G(D)) \rightarrow K$ and $\lambda_B : Z(KG) \rightarrow K$ be the algebra homomorphisms corresponding respectively to the blocks b and B . Then

$$\lambda_B = \lambda_b \text{Br}_D,$$

see for example [5] for details. Let

$$e_\theta = \frac{\theta(1)}{|N|} \sum_{n \in N} \theta(n^{-1})n$$

be the central idempotent corresponding to θ , and let

$$e_\mu = \frac{\mu(1)}{|C_N(D)|} \sum_{c \in C_N(D)} \mu(c^{-1})c$$

be the central idempotent corresponding to μ . Since N is a p' -group, we may project e_θ into an element $\bar{e}_\theta \in Z(KN)$, and we may project e_μ into an element $\bar{e}_\mu \in Z(KC_N(D))$. Since D acts on N , we have $|N| \equiv |C_N(D)| \pmod{p}$. By well-known properties of the Glauberman correspondence, there is some sign $\epsilon \in \{1, -1\}$ such that $\mu(c) \equiv \epsilon\theta(c) \pmod{M}$ for all $c \in C_N(D)$. It follows that $\bar{e}_\mu = \text{Br}_D(\bar{e}_\theta)$. Suppose first that $B \subseteq \text{Irr}(G|\theta)$. Then, we have $\lambda_B(\bar{e}_\theta) = 1$, and this implies that $\lambda_b \text{Br}_D(\bar{e}_\theta) = 1$, which implies that $\lambda_b(\bar{e}_\mu) = 1$. Hence, $b \subseteq \text{Irr}(N_G(D)|\mu)$, in this case, as desired. Conversely, suppose $b \subseteq \text{Irr}(N_G(D)|\mu)$. Then, $1 = \lambda_b(\bar{e}_\mu) = \lambda_b \text{Br}_D(\bar{e}_\theta) = \lambda_B(\bar{e}_\theta)$, and it follows that $B \subseteq \text{Irr}(G|\theta)$. Hence, the theorem holds. \square

3. Character correspondences

Our proof ultimately depends on properties of some character correspondences. We prove these in this section. By results in [9] we know there is an equality of certain elements of the Brauer–Clifford group. This implies, by results in [11], the existence of certain character correspondences with excellent properties, and we take these as our starting point here. We refer the reader to [9–11] for unexplained definitions and notation.

Theorem 3.1. *Let p be a prime number, let G be any finite group, let Q be a p -subgroup of G , let N be a normal p' -subgroup of G , and assume that $N_G(Q)N = G$. Set $\bar{G} = G/N$, and let $\pi_1 : G \rightarrow \bar{G}$ and $\pi_2 : N_G(Q) \rightarrow \bar{G}$ be the respective (surjective) natural projection homomorphisms. Let U be a normal subgroup of G such that $U \subseteq Q$. Let $\theta \in \text{Irr}(N)$ be Q -invariant, and let $\mu \in \text{Irr}(C_N(Q))$ be the Glauberman correspondent to θ under the action of Q . Then there exist a θ -quasi-homogeneous $\mathbf{Q}_p G$ -module M_1 with kernel containing U , a μ -quasi-homogeneous $\mathbf{Q}_p N_G(Q)$ -module M_2 with kernel containing U , and an endoisomorphism*

$$\epsilon : \text{End}_{\mathbf{Q}_p N}(M_1) \rightarrow \text{End}_{\mathbf{Q}_p C_N(Q)}(M_2)$$

with respect to π_1 and π_2 . Furthermore the character correspondence κ_ϵ induced by the endoisomorphism ϵ satisfies the following properties, where we set $q = \mu(1)/\theta(1)$.

(1) By restriction κ_ϵ yields a bijection

$$\text{Res}(\kappa_\epsilon) : \text{Irr}(G|\theta, \mathbf{Q}_p) \rightarrow \text{Irr}(N_G(Q)|\mu, \mathbf{Q}_p).$$

(2) $\kappa_\epsilon(\theta) = \mu$.

(3) Let $\eta \in \text{Irr}(U)$ be Q -invariant. Then $\eta \otimes \theta \in \text{Irr}(UN)$, $\eta \otimes \mu \in \text{Irr}(UC_N(Q))$, and

$$\kappa_\epsilon(\eta \otimes \theta) = \eta \otimes \mu.$$

(4) q is a quotient of integers prime to p , $q^2 \equiv 1 \pmod{p}$, and if $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p)$, then $f(\chi)(1) = q\chi(1)$ so that the degree of corresponding character is equal to q times the degree of the original one.

(5) κ_ϵ commutes with the action of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, so, in particular, $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(\kappa_\epsilon(\chi))$ for every $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p)$.

(6) For every $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p)$, we have $[\kappa_\epsilon(\chi)]_p = [\chi]_p$, so, in particular, $m_p(\kappa_\epsilon(\chi)) = m_p(\chi)$, and $\kappa_\epsilon(\chi)$ and χ have the same p -local Schur index.

Proof. Note that q is a quotient of degrees of irreducible characters of p' -groups, so q is a quotient of integers prime to p . That $q^2 \equiv 1 \pmod{p}$ is a well-known property of the Glauberman correspondence. Since U is a normal p -subgroup of G , it centralizes N . Set $G' = G/U$. Let $\pi : G \rightarrow G'$ be the natural projection homomorphism. Let $N' = \pi(N)$ and $Q' = \pi(Q)$. Then Q' acts on N' and $C_{N'}(Q') = \pi(C_N(Q))$. Let θ' be the character of N' corresponding to θ (so that $\theta'\pi = \theta$ as functions of N), and let μ' be the character of $C_{N'}(Q')$ corresponding to μ (so that $\mu'\pi = \mu$ as functions of $C_N(Q)$). By the properties of the Glauberman correspondence, we know that μ' is the Glauberman correspondent of θ' under the action of Q' . Notice that we also have that $\pi(N_G(Q)) = N_{G'}(Q')$ and $N_{G'}(Q')N' = G'$. Let $\bar{G}' = G'/N'$, and let $\pi'_1 : G' \rightarrow \bar{G}'$ and $\pi'_2 : N_{G'}(Q') \rightarrow \bar{G}'$ be the respective (surjective) natural projection homomorphisms. Then, in the notation of [9], there is a unique isomorphism

$$\alpha : Z(\theta', \pi'_1, \mathbf{Q}_p) \rightarrow Z(\mu', \pi'_2, \mathbf{Q}_p)$$

which sends the central character associated with θ' to the central character associated with μ' by [9, Proposition 6.4]. Furthermore, by [9, Theorem 6.5], we have

$$\bar{\alpha}([\theta', \pi'_1, \mathbf{Q}_p]) = [\mu', \pi'_2, \mathbf{Q}_p].$$

By [11, Corollary 9.9], it follows that there exist modules M_1 and M_2 such that M_1 is a θ -quasihomogeneous $\mathbf{Q}_p G$ -module with kernel containing U , M_2 is a μ -quasihomogeneous $\mathbf{Q}_p N_G(Q)$ -module with kernel containing U , and an endomorphism ϵ with respect to π_1 and π_2 from M_1 to M_2 satisfying all the properties described in [11, Theorem 9.8] and [11, Corollary 9.9].

Let ζ be the sum of the $G \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ orbit of θ , and let ζ' be the sum of the $N_G(Q) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ orbit of μ . Then by restriction κ_ϵ yields a bijection

$$\text{Res}(\kappa_\epsilon) : \text{Irr}(G|\zeta) \rightarrow \text{Irr}(N_G(Q)|\zeta').$$

Moreover, it follows from Clifford theory that $\text{Irr}(G|\zeta) = \text{Irr}(G|\theta, \mathbf{Q}_p)$ and $\text{Irr}(N_G(Q)|\zeta') = \text{Irr}(N_G(Q)|\mu, \mathbf{Q}_p)$, so that (1) immediately follows. All the other listed properties follow immediately, except, possibly, (3). By [11, Theorem 9.8], κ_ϵ sends irreducible characters contained in a restriction of the character afforded by M_1 to irreducible characters contained in a restriction of the character afforded by M_2 . It then follows that $\kappa_\epsilon(1_U \otimes \theta) = 1_U \otimes \mu$ since U is in the kernel of both M_1 and M_2 , κ_ϵ commutes with restriction, and $\kappa_\epsilon(\theta) = \mu$. Let $\eta \in \text{Irr}(U)$ be Q -invariant. Since, again by [11, Theorem 9.8], κ_ϵ commutes with multiplication by characters of $\pi_1(U)$, we have

$$\kappa_\epsilon(\eta \otimes \theta) = \eta \otimes \mu,$$

as desired. This completes the proof of the theorem. \square

Theorem 3.2. *Let p be a prime number, let G be any finite group, let Q be a p -subgroup of G , let N be a normal p' -subgroup of G , and assume that $N_G(Q)N = G$. Let U be a normal subgroup of G and $\eta \in \text{Irr}(U)$ be such that $U \subseteq Q$, and η is Q -invariant. Let $\theta \in \text{Irr}(N)$ be Q -invariant, and let $\mu \in \text{Irr}(C_N(Q))$ be the Glauberman correspondent to θ under the action of Q . Then there exists a bijection*

$$f : \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(N_G(Q)|\mu, \mathbf{Q}_p) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p)$$

satisfying all of the following conditions, where we set $q = \mu(1)/\theta(1)$.

- (1) $f(\text{Irr}(G|\theta) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)) = \text{Irr}(N_G(Q)|\mu) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p)$.
- (2) q is a quotient of integers prime to p , $q^2 \equiv 1 \pmod{p}$, and if $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, then $f(\chi)(1) = q\chi(1)$ so that the degree of corresponding character is equal to q times the degree of the original one.
- (3) If $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, then $f(\chi)(1)_p = \chi(1)_p$ so that the p -parts of the degree of corresponding characters are equal.
- (4) If $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, then

$$\text{codeg}(f(\chi))_{p'}^2 \equiv \text{codeg}(\chi)_{p'}^2 \pmod{p}.$$

- (5) f commutes with the action of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, so, in particular, $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(f(\chi))$ for every $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$.
- (6) For every $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, we have $[f(\chi)]_p = [\chi]_p$, so, in particular, $m_p(f(\chi)) = m_p(\chi)$, and $f(\chi)$ and χ have the same p -local Schur index.

Proof. The hypotheses of Theorem 3.1 are satisfied, so we also assume the further notation and conclusion of Theorem 3.1. Since U centralizes N , we know that the orbit of η under $G \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ is the orbit of η under $N_G(Q) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. Now for $\chi \in \text{Irr}(G|\theta, \mathbf{Q}_p)$, we have $\chi \in \text{Irr}(G|\eta, \mathbf{Q}_p)$ if and only if there is some $N_G(Q) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ conjugate η' of η such that $\eta' \otimes \theta$ is contained in the restriction of χ to UH . Similarly, for $\psi \in \text{Irr}(N_G(Q)|\mu, \mathbf{Q}_p)$, we have $\psi \in \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p)$ if and only if there is some $N_G(Q) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ conjugate η' of η such that $\eta' \otimes \mu$ is contained in the

restriction of ψ to $UC_H(Q)$. Since κ_ϵ commutes with both conjugation and restriction, it follows that the restriction of $\text{Res}(\kappa_\epsilon)$ does yield a bijection

$$f : \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(N_G(Q)|\mu, \mathbf{Q}_p) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p).$$

Since κ_ϵ commutes with restriction and $\kappa_\epsilon(\theta) = \mu$, (1) follows. (2) follows directly from [Theorem 3.1](#). (3) follows immediately from (2). Since Q is a p -group, the order of N is congruent modulo p with the order of $C_N(Q)$, and this implies that the p' -part of the order of G is congruent modulo p with the p' -part of the order of $N_G(Q)$. Then (4) follows from this and (2). Furthermore, (5) and (6) follow from [Theorem 3.1](#). Hence, the theorem holds. \square

4. The main results

If G is a finite group, then $\text{Gal}(\bar{F}/F)$ acts on the set of p -blocks of G . Of course, all blocks in a single orbit have the same defect group. It is convenient for our purposes to consider the ordinary irreducible characters contained in an orbit of this action.

Definition 4.1. Let G be a finite group and let B be a p -block of G . We let

$$\text{Irr}(B, F) = \{\sigma\chi : \chi \in \text{Irr}(B) \text{ and } \sigma \in \text{Gal}(\bar{F}/F)\}.$$

Furthermore, we set

$$\text{Irr}_0(B, F) = \{\chi \in \text{Irr}(B, F) : \chi \text{ has height zero in its } p\text{-block}\}.$$

With this notation, we can state the main result of this paper.

Theorem 4.2. Let p be a prime number, let G be any finite p -solvable group, let B be a p -block of G , let D be a defect group of B , and let b be the block of $N_G(D)$ Brauer correspondent of B . Let U be a normal subgroup of G , and $\eta \in \text{Irr}(U)$ be such that $U \subseteq D$, and η is D -invariant. Then there exists a bijection

$$f : \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

satisfying all of the following conditions:

- (1) If $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, and $\chi \in \text{Irr}(B')$ where B' is a p -block of G , then $f(\chi) \in \text{Irr}_0(b')$, where b' is the Brauer correspondent block of $N_G(D)$ of the block B' .
- (2) If $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, then

$$\text{codeg}(f(\chi))_{p'}^2 \equiv \text{codeg}(\chi)_{p'}^2 \pmod{p}.$$

- (3) f commutes with the action of $\text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$, so, in particular, $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(f(\chi))$ for every $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$.
- (4) For every $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, we have $[f(\chi)]_p = [\chi]_p$, so, in particular, $m_p(f(\chi)) = m_p(\chi)$, and $f(\chi)$ and χ have the same p -local Schur index.

Proof. Assume the theorem is false. Among all counterexamples, choose those with $|G|$ as small as possible, and among these, choose one where $[G : U]$ is as small as possible. Set $N = \mathcal{O}_{p'}(G)$. Then there exists some $\theta \in \text{Irr}(N)$ such that $B \subseteq \text{Irr}(G|\theta)$ and θ is D -invariant. Let $\mu \in \text{Irr}(C_N(D))$ be the Glauberman correspondent to θ under the action of D . By [Theorem 2.4](#), we have $b \subseteq \text{Irr}(N_G(D)|\mu)$. Set $T = \tilde{I}_G(\theta, \mathbf{Q}_p)$, and assume $T \neq G$. Notice that $U \subseteq T$. By [Theorem 2.2](#), there exists a unique p -block

B_0 of T such that $B_0 \subseteq \text{Irr}(T|\theta, \mathbf{Q}_p)$ and $B_0^G = B$. Notice that, since the Glauberman correspondence is a bijection and commutes with all Galois action, we have $N_T(D) = \tilde{I}_{N_G(D)}(\mu, \mathbf{Q}_p)$. Hence, by [Theorem 2.2](#), there exists a unique p -block b_0 of $N_T(D)$ such that $b_0 \subseteq \text{Irr}(N_T(D)|\mu, \mathbf{Q}_p)$, and $b_0^{N_G(D)} = b$. Furthermore, since b has defect D , b_0 also has defect D . Hence, b_0^T is defined. By [Theorem 2.4](#), we have $b_0^T \subseteq \text{Irr}(T|\theta, \mathbf{Q}_p)$. Furthermore, b_0^T is a block with $(b_0^T)^G = B$. Hence, by the uniqueness in [Theorem 2.2](#), we have $b_0^T = B_0$. By the minimality of our counterexample, there exists a bijection

$$f_0 : \text{Irr}_0(B_0, \mathbf{Q}_p) \cap \text{Irr}(T|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b_0, \mathbf{Q}_p) \cap \text{Irr}(N_T(D)|\eta, \mathbf{Q}_p)$$

satisfying all of the conditions of the theorem. By [Proposition 2.1](#) and [Theorem 2.2](#), both applied twice, we obtain that

$$\text{Ind}_T^G : \text{Irr}_0(B_0, \mathbf{Q}_p) \cap \text{Irr}(T|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$$

and

$$\text{Ind}_{N_T(D)}^{N_G(D)} : \text{Irr}_0(b_0, \mathbf{Q}_p) \cap \text{Irr}(N_T(D)|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

are bijections. It follows that if we set

$$f := \text{Ind}_{N_T(D)}^{N_G(D)} \circ f_0 \circ (\text{Ind}_T^G)^{-1}$$

then

$$f : \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

is a bijection. It follows from [Proposition 2.1](#) that f satisfies (2), (3), and (4) of the theorem. Let $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, and let B' be the p -block of G such that $\chi \in \text{Irr}(B')$. Let $\chi_0 \in \text{Irr}_0(B_0, \mathbf{Q}_p) \cap \text{Irr}(T|\eta, \mathbf{Q}_p)$ be the unique character such that $\text{Ind}_T^G(\chi_0) = \chi$, and let B'_0 be the block of T such that $\chi_0 \in \text{Irr}(B'_0)$. Let $\psi_0 = f_0(\chi_0) \in \text{Irr}_0(b_0, \mathbf{Q}_p) \cap \text{Irr}(N_T(D)|\eta, \mathbf{Q}_p)$, and let b'_0 be the p -block of $N_T(D)$ such that $\psi_0 \in \text{Irr}(b'_0)$. Finally, let $\psi = \text{Ind}_{N_T(D)}^{N_G(D)}(\psi_0)$, and let b' be the p -block of $N_G(D)$ such that $\psi \in \text{Irr}(b')$. In particular, $f(\chi) = \psi$. It follows from [Theorem 2.2](#) that $b' = b_0^{N_G(D)}$ and that $B_0'^G = B'$, and it follows from the properties of f_0 that $b_0'^T = B'_0$. It follows that

$$b'^G = (b_0'^{N_G(D)})^G = (b_0'^T)^G = B'.$$

Hence f also satisfies (1), contradicting the fact that G is a counterexample to our theorem. Hence, $G = \tilde{I}_G(\theta, \mathbf{Q}_p)$.

Set $M = I_G(\theta)$. Then $M \trianglelefteq G$ and G/M is isomorphic to some subgroup of $\text{Gal}(\mathbf{Q}_p(\theta)/\mathbf{Q}_p)$, so, in particular, G/M is abelian. Now, by [Theorem 2.3](#), we have that $\text{Irr}(G|\theta)$ is a single p -block, that D is a Sylow p -subgroup of M , and that $\text{Irr}(N_G(D)|\mu)$ is a single p -block. It follows that $B = \text{Irr}(G|\theta)$ and that $b = \text{Irr}(N_G(D)|\mu)$. Furthermore, $\text{Irr}_0(B)$ is obtained by inducing the characters of degree prime to p of $\text{Irr}(M|\theta)$ to G , and similarly, $\text{Irr}_0(b)$ is obtained by inducing the characters of degree prime to p of $\text{Irr}(N_M(D)|\mu)$ to $N_G(D)$. Similar results hold for all the Galois conjugates of these blocks. It follows that if $U_1 \subseteq D$ is a normal subgroup of G and $\eta_1 \in \text{Irr}(U_1)$ with $\eta_1(1) \neq 1$, then $p \mid \eta_1(1)$ and

$$\text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta_1, \mathbf{Q}_p) = \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta_1, \mathbf{Q}_p) = \emptyset.$$

In particular, it follows that $\eta(1) = 1$, as otherwise the bijection claimed by the theorem trivially exists.

We now note that if S is any subgroup of G and $ND \subseteq S$ then $O_{p'}(S) = N$. This can be seen as follows. Suppose $N \neq O_{p'}(S)$. Since clearly $N \subseteq O_{p'}(S)$, it follows that there exists some $g \in O_{p'}(S)$ such that $g \notin N$. Since G is p -solvable, this implies that g acts non-trivially on some p -chief factor of G . Since G/M is abelian, we know that g acts non-trivially on some p -chief factor of G contained in M . By our assumption S contains a Sylow p -subgroup of M , so that S covers all p -chief factors of G contained in M . Hence, g acts non-trivially on some p -chief factor of S . Therefore $g \notin O_{p'}(S)$. This contradiction shows that $O_{p'}(S) = N$.

In an analogous way, we have that if S is a subgroup of $N_G(D)$ and $C_N(D)D \subseteq S$ then $O_{p'}(S) = C_N(D)$. Indeed,

$$C_N(D) \subseteq O_{p'}(S) \subseteq O_{p'}(S)N \subseteq O_{p'}(SN) = N$$

with the last equality by the previous paragraph. Hence, we have $O_{p'}(S) = N \cap S = C_N(D)$, as desired.

Suppose that $\tilde{I}_G(\eta, \mathbf{Q}_p) \neq G$. Set $I = \tilde{I}_G(\eta, \mathbf{Q}_p)$. Then $ND \subseteq I$, and $C_N(D)D \subseteq N_I(D)$. By [Theorem 2.3](#), we have that $\text{Irr}(I|\theta)$ is a single p -block, that $\text{Irr}(N_I(D)|\mu)$ is a single p -block, and that D is a defect group of both blocks. We set $B_1 = \text{Irr}(I|\theta)$ and $b_1 = \text{Irr}(N_I(D)|\mu)$. We know that b_1^I is defined, and, by [Theorem 2.4](#), we have that $b_1^I = B_1$. By our choice of counterexample, we know that the theorem holds for I , so that there exists a bijection

$$f_1 : \text{Irr}_0(B_1, \mathbf{Q}_p) \cap \text{Irr}(I|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b_1, \mathbf{Q}_p) \cap \text{Irr}(N_I(D)|\eta, \mathbf{Q}_p)$$

satisfying all of the conditions of the theorem. By [Proposition 2.1](#) applied twice, we obtain that

$$\text{Ind}_I^G : \text{Irr}(I|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(G|\eta, \mathbf{Q}_p)$$

and

$$\text{Ind}_{N_I(D)}^{N_G(D)} : \text{Irr}(N_I(D)|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

are bijections. Now $\text{Irr}(B_1, \mathbf{Q}_p) = \text{Irr}(I|\theta, \mathbf{Q}_p)$, $\text{Irr}(B, \mathbf{Q}_p) = \text{Irr}(G|\theta, \mathbf{Q}_p)$, $\text{Irr}(b_1, \mathbf{Q}_p) = \text{Irr}(N_I(D)|\mu, \mathbf{Q}_p)$ and $\text{Irr}(b, \mathbf{Q}_p) = \text{Irr}(N_G(D)|\mu, \mathbf{Q}_p)$, so we may restrict the previous bijections to new bijections (which we still denote with the same symbols)

$$\text{Ind}_I^G : \text{Irr}(B_1, \mathbf{Q}_p) \cap \text{Irr}(I|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$$

and

$$\text{Ind}_{N_I(D)}^{N_G(D)} : \text{Irr}(b_1, \mathbf{Q}_p) \cap \text{Irr}(N_I(D)|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p).$$

Since these bijections preserve codegrees and the defect group of all the involved blocks is D , it follows that we can further restrict our bijections and obtain new bijections (which again we denote with the same symbols)

$$\text{Ind}_I^G : \text{Irr}_0(B_1, \mathbf{Q}_p) \cap \text{Irr}(I|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$$

and

$$\text{Ind}_{N_I(D)}^{N_G(D)} : \text{Irr}_0(b_1, \mathbf{Q}_p) \cap \text{Irr}(N_I(D)|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p).$$

It follows that, if we set

$$f := \text{Ind}_{N_I(D)}^{N_G(D)} \circ f_1 \circ (\text{Ind}_I^G)^{-1},$$

then

$$f : \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

is a bijection. Furthermore, it follows from [Proposition 2.1](#) and the properties of f_1 that f satisfies conditions (2), (3), and (4) of the theorem. Let $\chi \in \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)$, and let B' be the p -block of G such that $\chi \in \text{Irr}(B')$. Let $\chi_1 \in \text{Irr}_0(B_1, \mathbf{Q}_p) \cap \text{Irr}(I|\eta, \mathbf{Q}_p)$ be the unique character such that $\text{Ind}_I^G(\chi_1) = \chi$, and let B'_1 be the block of I such that $\chi_1 \in \text{Irr}(B'_1)$. Let $\psi_1 = f_1(\chi_1) \in \text{Irr}_0(b_1, \mathbf{Q}_p) \cap \text{Irr}(N_I(D)|\eta, \mathbf{Q}_p)$, and let b'_1 be the p -block of $N_I(D)$ such that $\psi_1 \in \text{Irr}(b'_1)$. Finally, let $\psi = \text{Ind}_{N_I(D)}^{N_G(D)}(\psi_1)$, and let b' be the p -block of $N_G(D)$ such that $\psi \in \text{Irr}(b')$. In particular, $f(\chi) = \psi$. Let $\theta' \in \text{Irr}(N)$ be contained in $\text{Res}_N^I(\chi_1)$. Of course, θ' is also contained in $\text{Res}_N^G(\chi)$. Since $G = \tilde{I}_G(\theta, \mathbf{Q}_p)$, θ and θ' are Galois conjugate over \mathbf{Q}_p . Then θ' is D -invariant, and by [Theorem 2.3](#), $B' = \text{Irr}(G|\theta')$ and $B'_1 = \text{Irr}(I, \theta')$. By (1) of f_1 , we know that b'_1 is the Brauer correspondent to B'_1 . Let $\mu' \in \text{Irr}(C_N(D))$ be the Glauberman correspondent to θ' under the action of D . By [Theorem 2.4](#), we then have $b'_1 \subseteq \text{Irr}(N_I(D)|\mu')$. It then follows that $b' \subseteq \text{Irr}(N_G(D)|\mu')$. It follows from the properties of the Glauberman correspondence that the orbit of μ' under the action of $N_G(D)$ on $\text{Irr}(C_N(D))$ is contained in the orbit of μ' under the action of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ on $\text{Irr}(C_N(D))$. Hence, by [Theorem 2.3](#), $\text{Irr}(N_G(D)|\mu')$ is a single block and $b' = \text{Irr}(N_G(D)|\mu')$. By [Theorem 2.4](#), it follows that b' is the Brauer correspondent to B' . Hence, (1) holds. Since G is a counterexample to our theorem, this is a contradiction. Hence, $G = \tilde{I}_G(\eta, \mathbf{Q}_p)$. It follows that all G -conjugates of η have the same kernel. The minimality of our counterexample then implies that η is a faithful character. In particular, U is cyclic.

We have $U \subseteq O_p(M) \subseteq D \subseteq M$. Suppose that $U \neq O_p(M)$. Set $L = O_p(M)$. Let $E = \text{Irr}(L|\eta, \mathbf{Q}_p)$. The group $M \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ acts on E . We let Ω be the set of $M \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ -orbits on E . Let $\omega \in \Omega$. Suppose first that D does not fix any element of ω . Then, for every $\eta_1 \in \omega$, we have that

$$\text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta_1, \mathbf{Q}_p) = \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta_1, \mathbf{Q}_p) = \emptyset,$$

since both the M -orbit and the $N \cap M$ -orbit of η_1 have size divisible by p . We let Ω_0 be the set of all elements of Ω which contain an element fixed by D . Let $\omega \in \Omega_0$. Glauberman's Lemma tells us that the elements of ω which are fixed by D form a single $N_M(D) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ -orbit. Indeed, it is clear that they are a union of $N_M(D) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ -orbits. Suppose that $\phi_1, \phi_2 \in \omega$ are fixed by D . Then, there exists some $\sigma \in \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ such that ϕ_1 and $\sigma\phi_2$ are in the same M -orbit. Let $g \in M$ be such that $\phi_1 = {}^g(\sigma\phi_2)$. Then D and gD are Sylow p -subgroup of the stabilizer $I_M(\phi_1)$ of ϕ_1 in M . Hence, there is some $s \in I_M(\phi_1)$ such that $D = {}^{sg}D$. This implies that $sg \in N_M(D)$, and $\phi_1 = {}^{sg}(\sigma\phi_2)$. Hence, the elements of ω which are fixed by D form a single $N_M(D) \times \text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ -orbit. For each $\omega \in \Omega_0$, we choose some $\phi_\omega \in \omega$ which is fixed by D . We now have that

$$\text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) = \bigcup_{\omega \in \Omega_0} (\text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\phi_\omega, \mathbf{Q}_p))$$

and

$$\text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p) = \bigcup_{\omega \in \Omega_0} (\text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\phi_\omega, \mathbf{Q}_p)),$$

both unions being disjoint. By the minimality of our counterexample, we have bijections satisfying the conditions of the theorem for each ϕ_ω , and putting these together we obtain a bijection that contradicts the fact that we have a counterexample to the theorem. Hence, $U = O_p(M)$.

Set $Q = O_{p',p}(M) \cap D$, so that $Q \in \text{Syl}_p(O_{p',p}(M))$. Suppose that Q is not a normal subgroup of G . We have that $N_G(D) \subseteq N_G(Q) \neq G$. Notice that U is a normal p -subgroup of M , so that we have $U \subseteq Q \subseteq N_G(Q)$. Let $b_0 = b^{N_G(Q)}$ so that $b_0^G = B$. Let $\mu_0 \in \text{Irr}(C_N(Q))$ be the Glauberman correspondent to θ under the action of Q . By properties of the Glauberman correspondence, we have that μ is the Glauberman correspondent to μ_0 under the action of D . By Theorem 2.4, we have $b_0 \subseteq \text{Irr}(N_G(Q)|\mu_0)$, and, by Theorem 2.3, we have that $\text{Irr}(N_G(Q)|\mu_0)$ is a single p -block. Hence, $b_0 = \text{Irr}(N_G(Q)|\mu_0)$. By the minimality of our counterexample, there is a bijection

$$f_1 : \text{Irr}_0(b_0, \mathbf{Q}_p) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

satisfying the conditions of the theorem. Since $QN \trianglelefteq G$, we have $N_G(Q)N = G$, and by Theorem 3.2 there exists a bijection

$$f_2 : \text{Irr}(G|\theta, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}(N_G(Q)|\mu_0, \mathbf{Q}_p) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p)$$

satisfying all of the conditions of that theorem. We know that $B = \text{Irr}(G|\theta)$, that $b_0 = \text{Irr}(N_G(Q)|\mu_0)$, and that $b = \text{Irr}(N_G(D)|\mu)$. It follows that

$$\text{Irr}(B, \mathbf{Q}_p) = \text{Irr}(G|\theta, \mathbf{Q}_p),$$

$$\text{Irr}(b_0, \mathbf{Q}_p) = \text{Irr}(N_G(Q)|\mu_0, \mathbf{Q}_p),$$

$$\text{Irr}(b, \mathbf{Q}_p) = \text{Irr}(N_G(D)|\mu, \mathbf{Q}_p).$$

In particular, all relevant blocks have defect D , and, by the properties listed in Theorem 3.2 the restriction of f_2 to

$$f_3 : \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b_0, \mathbf{Q}_p) \cap \text{Irr}(N_G(Q)|\eta, \mathbf{Q}_p)$$

is a bijection. Then, we may set $f = f_1 f_3$ and

$$f : \text{Irr}_0(B, \mathbf{Q}_p) \cap \text{Irr}(G|\eta, \mathbf{Q}_p) \rightarrow \text{Irr}_0(b, \mathbf{Q}_p) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p)$$

is a bijection. Furthermore f satisfies (2), (3) and (4) of the theorem. In addition, by Theorem 3.2 and the properties of f_1

$$f(\text{Irr}_0(B) \cap \text{Irr}(G|\eta, \mathbf{Q}_p)) = \text{Irr}_0(b) \cap \text{Irr}(N_G(D)|\eta, \mathbf{Q}_p),$$

which, together with condition (3), implies that f satisfies (1). This is a contradiction, and shows that $Q \trianglelefteq G$.

It now follows that $Q = O_p(M) = U$. In particular, Q is cyclic. By the celebrated Hall–Higman Lemma, see for example [9, Lemma 7.1], we have

$$C_M(O_{p',p}(M)/O_{p'}(M)) = O_{p',p}(M).$$

Since Q is a cyclic group, $\text{Aut}(Q)$ is an abelian group, and since Q is a Sylow p -subgroup of $O_{p',p}(M)$ this implies that $M/O_{p',p}(M)$ is abelian. Therefore, the Sylow p -subgroup of $M/O_{p',p}(M)$ is a normal subgroup, hence, it is trivial by the definition of $O_{p',p}(M)$. It follows that p does not divide $[M : O_{p',p}(M)]$. Hence, $Q = D$. Now $G = N_G(D)$ and f can be taken to be the identity map. This yields a final contradiction, and completes the proof of the theorem. \square

Corollary 4.3. *The strengthened Alperin–McKay conjecture holds for all finite p -solvable groups.*

Proof. Let G be a finite p -solvable group, and let D be a p -subgroup of G . Then, the blocks of G of defect D can be split into orbits under the action of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. These orbits correspond under the Brauer correspondence to orbits under the same group on the p -blocks of $N_G(D)$ of defect D . Apply Theorem 4.2 with $U = 1$ and $\eta = 1_U$ to each orbit, and construct a bijection by putting together these individual bijections. The corollary follows. \square

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