A cohomological proof of Peterson–Kac's theorem on conjugacy of Cartan subalgebras for affine Kac–Moody Lie algebras

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A R T I C L E   I N F O

A B S T R A C T

This paper deals with the problem of conjugacy of Cartan subalgebras for affine Kac–Moody Lie algebras. Unlike the methods used by Peterson and Kac, our approach is entirely cohomological and geometric. It is deeply rooted on the theory of reductive group schemes developed by Demazure and Grothendieck, and on the work of J. Tits on buildings.

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1. Introduction

Chevalley's theorem on the conjugacy of split Cartan subalgebras is one of the cornerstones of the theory of simple finite dimensional Lie algebras over a field of characteristic 0. Indeed, this theorem affords the most elegant proof that the root system is an invariant of the Lie algebra.

The analogous result for symmetrizable Kac–Moody Lie algebras is the celebrated theorem of Peterson and Kac [9] (see also [5] and [6] for detailed proofs). Beyond the finite dimensional case, by far the most important Kac–Moody Lie algebras are the affine ones. These algebras sit at the "border" of finite dimensional Lie theory and they can in fact be viewed as "finite dimensional" (not over the base field but over a Laurent polynomial ring) in the sense of [10]. This approach begs the question as to whether an SGA-inspired proof of conjugacy exists in the affine case. This paper, which builds in [2] and [3], shows that the answer is yes. More precisely, in [7] (the untwisted case) and [2] (general case) conjugacy is established for loop algebras by purely Galois cohomological methods. The step that is missing is extending this result to the "full" Kac–Moody Lie algebra. The central extension presents of course no difficulties, but the introduction of the derivation does. The present paper addresses this issue, thus yielding a new cohomological proof of the conjugacy theorem of Peterson and Kac in the case of affine Kac–Moody Lie algebras.

Even though the conjugacy result is known (due to [9], as mentioned above), it is relevant to point out that the cohomological methods that we are putting forward do have their advantages. Most significantly:

(a) The group under which conjugacy is achieved in our work has a very transparent structure (given in terms of Laurent polynomial points of simply connected group schemes). This is in contrast to the Kac–Moody groups used in [9] which, in the twisted affine case, are quite difficult to "see".

(b) The methods used in [9] are not suitable for attacking the problem of conjugacy in "higher nullity", for example for Extended Affine Lie Algebras (with their centres and derivations). This is definitely not the case for the ideas we present in this paper. Indeed, we believe that the methods exhibited herein will eventually lead to a proof of conjugacy for a class of Lie algebras that includes the Extended Affine Lie Algebras, EALAs for short (the conjugacy "downstairs", namely at the level of the centreless core of the EALA, is still given by [2]. The difficulty, just as in the present paper, lies in how to extend the conjugacy from "downstairs" to "upstairs". That is, to the full EALA where the derivations play an essential role).

2. Affine Kac–Moody Lie algebras

Split case. Let **g** be a split simple finite dimensional Lie algebra over an algebraically closed field k of characteristic 0 and let **Aut(g)** be its automorphism group. If $x, y \in \mathbf{g}$ we denote their product in **g** by $[x, y]$. We also let $R = k[t^{\pm 1}]$ and $L(\mathbf{g}) = \mathbf{g} \otimes_k R$. We still denote the Lie product in $L(\mathbf{g})$ by $[x, y]$ where $x, y \in L(\mathbf{g})$.

The main object under consideration in this paper is the affine (split or twisted) Kac–Moody Lie algebra $\hat{L}$ corresponding to **g**. Any split affine Kac–Moody Lie algebra is of the form (see [4])

$$\hat{L} = \mathbf{g} \otimes_k R \oplus kc \oplus kd.$$

The element c is central and d is a degree derivation for a natural grading of $L(\mathbf{g})$: if $x \in \mathbf{g}$ and $p \in \mathbb{Z}$ then

$$\left[d, x \otimes t^p\right]_{\hat{L}} = px \otimes t^p.$$

If $l_1 = x \otimes t^p$, $l_2 = y \otimes t^q \in L(\mathbf{g})$ are viewed as elements in $\hat{L}$ their Lie product is given by

$$\left[x \otimes t^p, y \otimes t^q\right]_{\hat{L}} = [x, y] \otimes t^{p+q} + p\langle x, y\rangle \delta_{0, p+q} \cdot c$$

where $\langle x, y \rangle$ is the Killing form on **g** and $\delta_{0, p+q}$ is Kronecker's delta.

Twisted case. Let m be a positive integer. Let $S = k[t^{\pm \frac{1}{m}}]$ be the ring of Laurent polynomials in the variable $s = t^{\frac{1}{m}}$ with coefficients in k. Let

$$L(\mathbf{g})_S = L(\mathbf{g}) \otimes_R S$$

be the Lie algebra obtained from the R-Lie algebra $L(\mathbf{g})$ by the base change $R \to S$. Similarly we define Lie algebras[3]

$$\tilde{L}(\mathbf{g})_S = L(\mathbf{g})_S \oplus kc \quad \text{and} \quad \hat{L}(\mathbf{g})_S = L(\mathbf{g})_S \oplus kc \oplus kd.$$

Fix a primitive root of unity $\zeta \in k$ of degree m. The R-automorphism $\zeta^\times : S \to S$ given by $s \mapsto \zeta s$ generates the Galois group $\Gamma = \mathrm{Gal}(S/R)$ which we may identify with the abstract group $\mathbb{Z}/m\mathbb{Z}$ by means of $\zeta^\times$. Note that Γ acts naturally on $\mathbf{Aut}(\mathbf{g})(S) = \mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S)$ and on $L(\mathbf{g})_S = L(\mathbf{g}) \otimes_R S$ through the second factor.

Next, let σ be an automorphism of $\mathbf{g}$ of order m. This gives rise to an S-automorphism of $L(\mathbf{g})_S$ via $x \otimes s \mapsto \sigma(x) \otimes s$ for $x \in \mathbf{g}$, $s \in S$. It then easily follows that the assignment

$$\bar{1} \mapsto z_{\bar{1}} = \sigma^{-1} \in \mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big)$$

gives rise to a cocycle $z = (z_{\bar{i}}) \in Z^1(\Gamma, \mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S))$. This cocycle, in turn, gives rise to a twisted action of Γ on $L(\mathbf{g})_S$. Applying Galois descent formalism we then obtain the Γ-invariant subalgebra

$$L(\mathbf{g}, \sigma) := \big(L(\mathbf{g})_S\big)^\Gamma = \big(L(\mathbf{g}) \otimes_R S\big)^\Gamma.$$

This is a "simple Lie algebra over R" in the sense of [10], which is a twisted form of the "split simple" R-Lie algebra $L(\mathbf{g}) = \mathbf{g} \otimes_k R$. Indeed, S/R is an étale extension and from properties of Galois descent we have

$$L(\mathbf{g}, \sigma) \otimes_R S \simeq L(\mathbf{g})_S = (\mathbf{g} \otimes_k R) \otimes_R S.$$

Note that $L(\mathbf{g}, id) = L(\mathbf{g})$.

For $\bar{i} \in \mathbb{Z}/m\mathbb{Z}$, consider the eigenspace

$$\mathbf{g}_{\bar{i}} = \big\{ x \in \mathbf{g} : \sigma(x) = \zeta^i x \big\}.$$

Simple computations show that

$$L(\mathbf{g}, \sigma) = \bigoplus_{i \in \mathbb{Z}} \mathbf{g}_{\bar{i}} \otimes k\big[t^{\pm 1}\big] s^i.$$

Let

$$\tilde{L}(\mathbf{g}, \sigma) := L(\mathbf{g}, \sigma) \oplus kc \quad \text{and} \quad \hat{L}(\mathbf{g}, \sigma) := L(\mathbf{g}, \sigma) \oplus kc \oplus kd.$$

We give $\hat{L}(\mathbf{g}, \sigma)$ a Lie algebra structure such that c is a central element, d is the degree derivation, i.e. if $x \in \mathbf{g}_{\bar{i}}$ and $p \in \mathbb{Z}$ then

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[3] Unlike $L(\mathbf{g})_S$, these object exist over k but not over S.

$$\left[d, x \otimes t^{\frac{p}{m}} \right] := p x \otimes t^{\frac{p}{m}} \tag{2.0.1}$$

and if $y \otimes t^{\frac{q}{m}} \in L(\mathbf{g}, \sigma)$ we get

$$\left[x \otimes t^{\frac{p}{m}}, y \otimes t^{\frac{q}{m}} \right]_{\hat{L}(\mathbf{g}, \sigma)} = [x, y] \otimes t^{\frac{p+q}{m}} + p \langle x, y \rangle \delta_{0, p+q} \cdot c,$$

where, as before, $\langle x, y \rangle$ is the Killing form on $\mathbf{g}$ and $\delta_{0, p+q}$ is Kronecker's delta.

2.1. Remark. Note that the Lie algebra structure on $\hat{L}(\mathbf{g}, \sigma)$ is induced by that of $\hat{L}(\mathbf{g})_S$ if we view $\hat{L}(\mathbf{g}, \sigma)$ as a subset of $\hat{L}(\mathbf{g})_S$.

2.2. Remark. Let $\hat{\sigma}$ be an automorphism of $\hat{L}(\mathbf{g})_S$ such that $\hat{\sigma}|_{L(\mathbf{g})_S} = \sigma$, $\hat{\sigma}(c) = c$, $\hat{\sigma}(d) = d$. Then $\hat{L}(\mathbf{g}, \sigma) = (\hat{L}(\mathbf{g})_S)^{\hat{\sigma}}$.

Realization Theorem.

(a) *The Lie algebra $\hat{L}(\mathbf{g}, \sigma)$ is an affine Kac–Moody Lie algebra, and every affine Kac–Moody Lie algebra is isomorphic to some $\hat{L}(\mathbf{g}, \sigma)$.*
(b) *$\hat{L}(\mathbf{g}, \sigma) \simeq \hat{L}(\mathbf{g}, \sigma')$ where σ' is a diagram automorphism with respect to some Cartan subalgebra of $\mathbf{g}$.*

Proof. See [4, Theorems 7.4, 8.3 and 8.5]. □

Let $\phi \in \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$. Since $\tilde{L}(\mathbf{g})_S$ is the derived subalgebra of $\hat{L}(\mathbf{g})_S$ the restriction $\phi|_{\tilde{L}(\mathbf{g})_S}$ induces a k-Lie automorphism of $\tilde{L}(\mathbf{g})_S$. Furthermore, passing to the quotient $\tilde{L}(\mathbf{g})_S / kc \simeq L(\mathbf{g})_S$ the automorphism $\phi|_{\tilde{L}(\mathbf{g})_S}$ induces an automorphism of $L(\mathbf{g})_S$. This yields a well-defined morphism

$$\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S) \to \mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S).$$

Similar considerations apply to $\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g}, \sigma))$. The aim of the next few sections is to show that these two morphisms are surjective.

3. *S*-automorphisms of $L(\mathbf{g})_S$

In this section we construct a "simple" system of generators of the automorphism group

$$\mathbf{Aut}(\mathbf{g})(S) = \mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S)$$

which can be easily extended to k-automorphisms of $\hat{L}(\mathbf{g})_S$. We produce our list of generators based on a well-known fact that the group in question is generated by S-points of the corresponding split simple adjoint algebraic group and automorphisms of the corresponding Dynkin diagram.

More precisely, let $\mathbf{G}$ be the split simple simply connected group over k corresponding to $\mathbf{g}$ and let $\bar{\mathbf{G}}$ be the corresponding adjoint group. Choose a maximal split k-torus $\mathbf{T} \subset \mathbf{G}$ and denote its image in $\bar{\mathbf{G}}$ by $\bar{\mathbf{T}}$. The Lie algebra of $\mathbf{T}$ is a Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$. We fix a Borel subgroup $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$.

Let $\Sigma = \Sigma(\mathbf{G}, \mathbf{T})$ be the root system of $\mathbf{G}$ relative to $\mathbf{T}$. The Borel subgroup $\mathbf{B}$ determines an ordering of Σ, hence the system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Fix a Chevalley basis [12]

$$\{H_{\alpha_1}, \ldots, H_{\alpha_n}, X_\alpha, \alpha \in \Sigma\}$$

of $\mathbf{g}$ corresponding to the pair $(\mathbf{T}, \mathbf{B})$. This basis is unique up to signs and automorphisms of $\mathbf{g}$ which preserve $\mathbf{B}$ and $\mathbf{T}$ (see [12, §1, Remark 1]).

Since S is a Euclidean ring, by Steinberg [11] the group $\mathbf{G}(S)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(u) \mid u \in S \rangle$, where $\alpha \in \Sigma$ and

$$x_\alpha(u) = \exp(uX_\alpha) = \sum_{n=0}^{\infty} u^n X_\alpha^n / n! \tag{3.0.1}$$

We recall also that by [12, §10, Cor. (b) after Theorem 29], every automorphism σ of the Dynkin diagram Dyn($\mathbf{G}$) of $\mathbf{G}$ can be extended to an automorphism of $\mathbf{G}$ (and hence of $\bar{\mathbf{G}}$) and $\mathbf{g}$, still denoted by σ, which takes

$$x_\alpha(u) \longrightarrow x_{\sigma(\alpha)}(\varepsilon_\alpha u) \quad \text{and} \quad X_\alpha \longrightarrow \varepsilon_\alpha X_{\sigma(\alpha)}.$$

Here $\varepsilon_\alpha = \pm 1$ and if $\alpha \in \Pi$ then $\varepsilon_\alpha = 1$. Thus we have a natural embedding

$$\mathrm{Aut}\big(\mathrm{Dyn}(\mathbf{G})\big) \hookrightarrow \mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big).$$

The group $\bar{\mathbf{G}}(S)$ acts by S-automorphisms on $L(\mathbf{g})_S$ through the adjoint representations $ad : \bar{\mathbf{G}} \to \mathbf{GL}(L(\mathbf{g})_S)$ and hence we also have a canonical embedding

$$\bar{\mathbf{G}}(S) \hookrightarrow \mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big).$$

As we said before, it is well-known (see [8] for example) that

$$\mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big) = \bar{\mathbf{G}}(S) \rtimes \mathrm{Aut}\big(\mathrm{Dyn}(\mathbf{G})\big).$$

For later use we need one more fact.

3.1. Proposition. *Let $f : \mathbf{G} \to \bar{\mathbf{G}}$ be the canonical morphism. The group $\bar{\mathbf{G}}(S)$ is generated by the root subgroups $f(U_\alpha)$, $\alpha \in \Sigma$, and $\bar{\mathbf{T}}(S)$.*

Proof. Let $\mathbf{Z} \subset \mathbf{G}$ be the centre of $\mathbf{G}$. The exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{G} \longrightarrow \bar{\mathbf{G}} \longrightarrow 1$$

gives rise to an exact sequence in Galois cohomology

$$f\big(\mathbf{G}(S)\big) \hookrightarrow \bar{\mathbf{G}}(S) \longrightarrow \mathrm{Ker}\big[H^1(S, \mathbf{Z}) \to H^1(S, \mathbf{G})\big] \longrightarrow 1.$$

Since $H^1(S, \mathbf{Z}) \to H^1(S, \mathbf{G})$ factors through

$$H^1(S, \mathbf{Z}) \longrightarrow H^1(S, \mathbf{T}) \longrightarrow H^1(S, \mathbf{G})$$

and since $H^1(S, \mathbf{T}) = 1$ (because Pic $S = 1$) we obtain

$$f\big(\mathbf{G}(S)\big) \hookrightarrow \bar{\mathbf{G}}(S) \longrightarrow H^1(S, \mathbf{Z}) \longrightarrow 1. \tag{3.1.1}$$

Similar considerations applied to

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{T} \longrightarrow \bar{\mathbf{T}} \longrightarrow 1$$

show that

$$f\big(\mathbf{T}(S)\big) \hookrightarrow \bar{\mathbf{T}}(S) \longrightarrow H^1(S, \mathbf{Z}) \longrightarrow 1. \tag{3.1.2}$$

The result now follows from (3.1.1) and (3.1.2). □

3.2. Corollary. *One has*

$$\mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big) = \big\langle \mathrm{Aut}\big(\mathrm{Dyn}(\mathbf{G})\big), U_\alpha, \alpha \in \Sigma, \bar{\mathbf{T}}(S)\big\rangle.$$

4. *k*-automorphisms of $L(\mathbf{g})_S$

We keep the above notation. Recall that for any algebra $\mathfrak{A}$ over a field k the centroid of $\mathfrak{A}$ is

$$\mathrm{Ctrd}(\mathfrak{A}) = \big\{ \chi \in \mathrm{End}_k(\mathfrak{A}) \,\big|\, \chi(a \cdot b) = a \cdot \chi(b) = \chi(a) \cdot b \text{ for all } a, b \in \mathfrak{A} \big\}.$$

It is easy to check that if $\chi_1, \chi_2 \in \mathrm{Ctrd}(\mathfrak{A})$ then both linear operators $\chi_1 \circ \chi_2$ and $\chi_1 + \chi_2$ are contained in $\mathrm{Ctrd}(\mathfrak{A})$ as well. Thus, $\mathrm{Ctrd}(\mathfrak{A})$ is a unital associative subalgebra of $\mathrm{End}_k(\mathfrak{A})$. It is also well-known that the centroid is commutative whenever $\mathfrak{A}$ is perfect.

Example. Consider the k-Lie algebra $\mathfrak{A} = L(\mathbf{g})_S$. For any $s \in S$ the linear k-operator $\chi_s : L(\mathbf{g})_S \to L(\mathbf{g})_S$ given by $x \to xs$ satisfy

$$\chi_s\big([x, y]\big) = \big[x, \chi_s(y)\big] = \big[\chi_s(x), y\big],$$

hence $\chi_s \in \mathrm{Ctrd}(L(\mathbf{g})_S)$. Conversely, it is known (see [1, Lemma 4.2]) that every element in $\mathrm{Ctrd}(L(\mathbf{g})_S)$ is of the form χ_s. Thus,

$$\mathrm{Ctrd}\big(L(\mathbf{g})_S\big) = \{\chi_s \mid s \in S\} \simeq S.$$

4.1. Proposition. *(See [8, Proposition 1].)* *One has*

$$\mathrm{Aut}_{k\text{-}Lie}\big(L(\mathbf{g})_S\big) \simeq \mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big) \rtimes \mathrm{Aut}_k\big(\mathrm{Ctrd}\big(L(\mathbf{g})_S\big)\big)$$

$$\simeq \mathrm{Aut}_{S\text{-}Lie}\big(L(\mathbf{g})_S\big) \rtimes \mathrm{Aut}_k(S).$$

4.2. Corollary. *One has*

$$\mathrm{Aut}_{k\text{-}Lie}\big(L(\mathbf{g})_S\big) = \big\langle \mathrm{Aut}_k(S), \mathrm{Aut}\big(\mathrm{Dyn}(\mathbf{G})\big), U_\alpha, \alpha \in \Sigma, \bar{\mathbf{T}}(S)\big\rangle.$$

Proof. This follows from Corollary 3.2 and Proposition 4.1. □

5. Automorphisms of $\tilde{L}(\mathbf{g})_S$

We remind the reader that the centre of $\tilde{L}(\mathbf{g})_S$ is the k-span of c and that $\tilde{L}(\mathbf{g})_S = L(\mathbf{g})_S \oplus kc$. Since any automorphism ϕ of $\tilde{L}(\mathbf{g})_S$ takes the centre into itself we have a natural (projection) mapping

$$\mu : \tilde{L}(\mathbf{g})_S \to \tilde{L}(\mathbf{g})_S / kc \simeq L(\mathbf{g})_S$$

which induces the mapping

$$\lambda : \mathrm{Aut}_{k\text{-}Lie}\big(\tilde{L}(\mathbf{g})_S\big) \to \mathrm{Aut}_{k\text{-}Lie}\big(L(\mathbf{g})_S\big)$$

given by $\phi \to \phi'$ where $\phi'(x) = \mu(\phi(x))$ for all $x \in L(\mathbf{g})_S$. In the last formula we view x as an element of $\tilde{L}(\mathbf{g})_S$ through the embedding $L(\mathbf{g})_S \hookrightarrow \tilde{L}(\mathbf{g})_S$.

5.1. Remark. It is straightforward to check that ϕ' is indeed an automorphism of $L(\mathbf{g})_S$.

5.2. Proposition. *The mapping λ is an isomorphism.*

Proof. See [8, Proposition 4]. $\square$

In what follows if $\phi \in \mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S)$ we denote its (unique) lifting to $\mathrm{Aut}_{k\text{-}Lie}(\tilde{L}(\mathbf{g})_S)$ by $\tilde{\phi}$.

5.3. Remark. For later use we need an explicit formula for lifts of automorphisms of $L(\mathbf{g})_S$ induced by some "special" points in $\bar{\mathbf{T}}(S)$ (those which are not in the image of $\mathbf{T}(S) \to \bar{\mathbf{T}}(S)$). More precisely, the fundamental coweights give rise to the decomposition $\bar{\mathbf{T}} \simeq \mathbf{G}_{m,S} \times \cdots \times \mathbf{G}_{m,S}$. As usual, we have the decomposition $\bar{\mathbf{T}}(S) \simeq \bar{\mathbf{T}}(k) \times \mathrm{Hom}(\mathbf{G}_m, \bar{\mathbf{T}})$. The second factor in the last decomposition is the cocharacter lattice of $\bar{\mathbf{T}}$ and its elements correspond (under the adjoint action) to the subgroup in $\mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S)$ isomorphic to $\mathrm{Hom}(Q, \mathbb{Z})$ where Q is the corresponding root lattice: if $\phi \in \mathrm{Hom}(Q, \mathbb{Z})$ it induces an S-automorphism of $L(\mathbf{g})_S$ (still denoted by ϕ) given by

$$X_\alpha \to X_\alpha \otimes s^{\phi(\alpha)}, \qquad H_{\alpha_i} \to H_{\alpha_i}.$$

It is straightforward to check the mapping $\tilde{\phi} : \tilde{L}(\mathbf{g})_S \to \tilde{L}(\mathbf{g})_S$ given by

$$H_\alpha \to H_\alpha + \phi(\alpha)\langle X_\alpha, X_{-\alpha}\rangle \cdot c, \qquad H_\alpha \otimes s^p \to H_\alpha \otimes s^p$$

if $p \neq 0$ and

$$X_\alpha \otimes s^p \to X_\alpha \otimes s^{p+\phi(\alpha)}$$

is an automorphism of $\tilde{L}(\mathbf{g})_S$, hence it is the (unique) lift of ϕ.

6. Automorphisms of split affine Kac–Moody Lie algebras

Since $\tilde{L}(\mathbf{g})_S = [\hat{L}(\mathbf{g})_S, \hat{L}(\mathbf{g})_S]$ we have a natural (restriction) mapping

$$\tau : \mathrm{Aut}_{k\text{-}Lie}\big(\hat{L}(\mathbf{g})_S\big) \to \mathrm{Aut}_{k\text{-}Lie}\big(\tilde{L}(\mathbf{g})_S\big).$$

6.1. Proposition. *The mapping τ is surjective.*

Proof. By Proposition 5.2 and Corollary 4.2 the group $\mathrm{Aut}_{k\text{-}Lie}(\tilde{L}(\mathbf{g})_S)$ has the distinguished system of generators $\{\tilde{\phi}\}$ where

$$\phi \in \mathrm{Aut}\big(\mathrm{Dyn}(\mathbf{G})\big), \bar{\mathbf{T}}(S), \mathrm{Aut}_k(S), U_\alpha, \quad \alpha \in \Sigma.$$

We want to construct a mapping $\hat{\phi} : \hat{L}(\mathbf{g})_S \to \hat{L}(\mathbf{g})_S$ which preserves the identity

$$\big[d, x \otimes t^{\frac{p}{m}}\big]_{\hat{L}} = px \otimes t^{\frac{p}{m}}$$

for all $x \in \mathbf{g}$ and whose restriction to $\tilde{L}(\mathbf{g})_S$ coincides with $\tilde{\phi}$. These two properties would imply that $\hat{\phi}$ is an automorphism of $\hat{L}(\mathbf{g})_S$ lifting $\tilde{\phi}$.

If $\phi \in U_\alpha$ is unipotent we define $\hat{\phi}$, as usual, through the exponential map. If $\phi \in \mathrm{Aut}(\mathrm{Dyn}(\mathbf{G}))$ we put $\hat{\phi}(d) = d$. If ϕ is as in Remark 5.3 we extend it by $d \to d - X$ where $X \in \mathbf{h}$ is the unique element such that $[X, X_\alpha] = \phi(\alpha) X_\alpha$ for all roots $\alpha \in \Sigma$. Note that automorphisms of $L(\mathbf{g})_S$ given by points in $\bar{\mathbf{T}}(k)$ are in the image of $\mathbf{T}(k) \to \bar{\mathbf{T}}(k)$ and hence they are generated by unipotent elements. Lastly, if $\phi \in \mathrm{Aut}_k(S)$ is of the form $s \to as^{-1}$ where $a \in k^\times$ (resp. $s \to as$) we extend $\tilde{\phi}$ by $\hat{\phi}(d) = -d$ (resp. $\hat{\phi}(d) = d$). We leave it to the reader to verify that in all cases $\hat{\phi}$ preserves the above identity and hence $\hat{\phi}$ is an automorphism of $\hat{L}(\mathbf{g})_S$. $\square$

6.2. Proposition. *One has* $\mathrm{Ker}\,\tau \simeq V$ *where* $V = \mathrm{Hom}_k(kd, kc)$.

Proof. We first embed $V \hookrightarrow \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$. Let $v \in V$. Recall that any element $x \in \hat{L}(\mathbf{g})_S$ can be written uniquely in the form $x = x' + ad$ where $x' \in \tilde{L}(\mathbf{g})_S$ and $a \in k$. We define $\hat{v} : \hat{L}(\mathbf{g})_S \to \hat{L}(\mathbf{g})_S$ by $x \to x + v(ad)$. One checks that $\hat{v}$ is an automorphism of $\hat{L}(\mathbf{g})_S$ and thus the required embedding is given by $v \to \hat{v}$.

Since $\hat{v}(x') = x'$ for all $x' \in \tilde{L}$ we have $\hat{v} \in \mathrm{Ker}\,\tau$. Conversely, let $\psi \in \mathrm{Ker}\,\tau$. Then $\psi(x) = x$ for all $x \in \tilde{L}(\mathbf{g})_S$. We need to show that $\psi(d) = ac + d$ where $a \in k$. Let $\psi(d) = x' + ac + bd$ where $a, b \in k$ and $x' \in L(\mathbf{g})_S$. Since $[d, X_\alpha]_{\hat{L}(\mathbf{g})_S} = 0$ we get

$$\left[\psi(d), \psi(X_\alpha)\right]_{\hat{L}(\mathbf{g})_S} = 0.$$

Substituting $\psi(d) = x' + ac + bd$ we obtain

$$\left[x' + ac + bd, X_\alpha\right]_{\hat{L}(\mathbf{g})_S} = 0$$

or $[x', X_\alpha]_{\tilde{L}(\mathbf{g})_S} = 0$. Since this is true for all roots $\alpha \in \Sigma$, the element x' commutes with $\mathbf{g}$ and this can happen if and only if $x' = 0$.

It remains to show that $b = 1$. To see this we can argue similarly by considering the equality

$$\left[d, X_\alpha \otimes t^{\frac{1}{m}}\right]_{\hat{L}(\mathbf{g})_S} = X_\alpha \otimes t^{\frac{1}{m}}$$

and applying ψ. $\square$

6.3. Corollary. *The sequence of groups*

$$1 \longrightarrow V \longrightarrow \mathrm{Aut}_{k\text{-}Lie}\left(\hat{L}(\mathbf{g})_S\right) \xrightarrow{\lambda \circ \tau} \mathrm{Aut}_{k\text{-}Lie}\left(L(\mathbf{g})_S\right) \longrightarrow 1 \tag{6.3.1}$$

is exact.

7. Automorphism group of twisted affine Kac–Moody Lie algebras

We keep the notation introduced in Section 2. In particular, we fix an integer m and a primitive root of unity $\zeta = \zeta_m \in k$ of degree m. Consider the k-automorphism $\zeta^\times : S \to S$ such that $s \to \zeta s$ which we view as a k-automorphism of $L(\mathbf{g})_S$ through the embedding

$$\mathrm{Aut}_k(S) \hookrightarrow \mathrm{Aut}_{k\text{-}Lie}\left(L(\mathbf{g})_S\right) \simeq \mathrm{Aut}_{S\text{-}Lie}\left(L(\mathbf{g})_S\right) \rtimes \mathrm{Aut}_k(S)$$

(see Proposition 4.1). As it is explained in Section 6 we then get the automorphism $\hat{\zeta}^\times$ (resp. $\tilde{\zeta}^\times$) of $\hat{L}(\mathbf{g})_S$ (resp. $\tilde{L}(\mathbf{g})_S$) given by

$$x \otimes s^i + ac + bd \longrightarrow x \otimes \zeta^i s^i + ac + bd$$

where $a, b \in k$ and $x \in \mathbf{g}$.

Consider now the abstract group $\Gamma = \mathbb{Z}/m\mathbb{Z}$ (which can be identified with $\mathrm{Gal}(S/R)$ as already explained) and define its action on $\hat{L}(\mathbf{g})_S$ (resp. $\tilde{L}(\mathbf{g})_S, L(\mathbf{g})_S$) with the use of $\hat{\zeta}^\times$ (resp. $\tilde{\zeta}^\times, \zeta^\times$). More precisely, for every $l \in \hat{L}(\mathbf{g})_S$ we let $\bar{i}(l) := (\hat{\zeta}^\times)^i(l)$. Similarly, we define the action of Γ on $\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$ by

$$\bar{i} : \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S) \longrightarrow \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S), \qquad x \rightarrow (\hat{\zeta}^\times)^i x (\hat{\zeta}^\times)^{-i}.$$

Therefore, $\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$ can be viewed as a Γ-set. Along the same lines one defines the action of Γ on $\mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S)$ and $\mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S)$ with the use of $\zeta^\times$. It is easy to see that Γ acts trivially on the subgroup $V \subset \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$ introduced in Proposition 6.2. Thus, (6.3.1) can be viewed as an exact sequence of Γ-groups.

We next choose an element $\pi \in \mathrm{Aut}(\mathrm{Dyn}(\mathbf{G})) \subset \mathrm{Aut}_k(\mathbf{g})$ of order m (clearly, m can take value $1, 2$ or 3 only). Like before, we have the corresponding automorphism $\hat{\pi}$ of $\hat{L}(\mathbf{g})_S$ given by

$$x \otimes s^i + ac + bd \longrightarrow \pi(x) \otimes s^i + ac + bd$$

where $a, b \in k$ and $x \in \mathbf{g}$.

Note that $\hat{\zeta}^\times \hat{\pi} = \hat{\pi} \hat{\zeta}^\times$. It then easily follows that the assignment

$$\bar{1} \rightarrow z_{\bar{1}} = \hat{\pi}^{-1} \in \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$$

gives rise to a cocycle $z = (z_{\bar{i}}) \in Z^1(\Gamma, \mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S))$.

This cocycle, in turn, gives rise to a (new) twisted action of Γ on $\hat{L}(\mathbf{g})_S$ and $\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S)$. Analogous considerations (with the use of π) are applied to $\mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S)$ and $L(\mathbf{g})_S$. For future reference note that $\hat{\pi}$ commutes with elements in V, hence the twisted action of Γ on V is still trivial. From now on we view (6.3.1) as an exact sequence of Γ-groups, the action of Γ being the twisted action.

7.1. Remark. As we noticed before, the invariant subalgebra

$$\mathcal{L} = L(\mathbf{g}, \pi) = \left(L(\mathbf{g})_S\right)^\Gamma = \left((\mathbf{g} \otimes_k R) \otimes_R S\right)^\Gamma$$

is a simple Lie algebra over R, a twisted form of a split Lie algebra $\mathbf{g} \otimes_k R$. The same cohomological formalism also yields that

$$\mathrm{Aut}_{R\text{-}Lie}(\mathcal{L}) \simeq \left(\mathrm{Aut}_{S\text{-}Lie}(L(\mathbf{g})_S)\right)^\Gamma. \tag{7.1.1}$$

7.2. Remark. It is worth mentioning that the canonical embedding

$$\iota : \left(\mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S)\right)^\Gamma \hookrightarrow \mathrm{Aut}_{k\text{-}Lie}\left((L(\mathbf{g})_S)^\Gamma\right) = \mathrm{Aut}_{k\text{-}Lie}(\mathcal{L})$$

$$\simeq \mathrm{Aut}_{R\text{-}Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R),$$

where the last isomorphism can be established in the same way as in Proposition 4.1, is not necessarily surjective in general case. Indeed, one checks that if $m = 3$ then the k-automorphism of R given by $t \rightarrow t^{-1}$ and viewed as an element of $\mathrm{Aut}_{k\text{-}Lie}(\mathcal{L}) \simeq \mathrm{Aut}_{R\text{-}Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R)$ is not in $\mathrm{Im}\,\iota$. However, (7.1.1) implies that the group $\mathrm{Aut}_{R\text{-}Lie}(\mathcal{L})$ is in the image of ι.

7.3. Remark. The k-Lie algebra $\hat{\mathcal{L}} = (\hat{L}(\mathbf{g})_S)^\Gamma$ is a twisted affine Kac–Moody Lie algebra. Conversely, by the Realization Theorem every twisted affine Kac–Moody Lie algebra can be obtained in such a way.

7.4. Lemma. *One has $H^1(\Gamma, V) = 1$.*

Proof. Since Γ is cyclic of order m acting trivially on $V \simeq k$ it follows that

$$Z^1(\Gamma, V) = \{x \in k \mid mx = 0\} = 0$$

as required. $\square$

The long exact cohomological sequence associated to (6.3.1) together with Lemma 7.4 imply the following.

7.5. Theorem. *The following sequence*

$$1 \longrightarrow V \longrightarrow \left(\mathrm{Aut}_{k\text{-}Lie}\left(\hat{L}(\mathbf{g})_S\right)\right)^\Gamma \xrightarrow{\nu} \left(\mathrm{Aut}_{k\text{-}Lie}\left(L(\mathbf{g})_S\right)\right)^\Gamma \longrightarrow 1$$

is exact. In particular, the group $\mathrm{Aut}_{R\text{-}Lie}(\mathcal{L})$ is in the image of the canonical mapping

$$\mathrm{Aut}_{k\text{-}Lie}(\hat{\mathcal{L}}) \longrightarrow \mathrm{Aut}_{k\text{-}Lie}(\mathcal{L}) \simeq \mathrm{Aut}_{R\text{-}Lie}(\mathcal{L}) \rtimes \mathrm{Aut}_k(R).$$

Proof. The first assertion is clear. As for the second one, note that as in Remark 7.2 we have the canonical embedding

$$\left(\mathrm{Aut}_{k\text{-}Lie}\left(\hat{L}(\mathbf{g})_S\right)\right)^\Gamma \hookrightarrow \mathrm{Aut}_{k\text{-}Lie}\left(\left(\hat{L}(\mathbf{g})_S\right)^\Gamma\right) = \mathrm{Aut}_{k\text{-}Lie}(\hat{\mathcal{L}})$$

and the commutative diagram

$$
\begin{array}{ccc}
(\mathrm{Aut}_{k\text{-}Lie}(\hat{L}(\mathbf{g})_S))^\Gamma & \xrightarrow{\ \nu\ } & (\mathrm{Aut}_{k\text{-}Lie}(L(\mathbf{g})_S))^\Gamma \\
\downarrow & & \downarrow \\
\mathrm{Aut}_{k\text{-}Lie}(\hat{\mathcal{L}}) & \longrightarrow & \mathrm{Aut}_{k\text{-}Lie}(\mathcal{L})
\end{array}
$$

Then surjectivity of ν and Remark 7.2 yield the result. $\square$

8. Some properties of affine Kac–Moody Lie algebras

Henceforth we fix a simple finite dimensional Lie algebra $\mathbf{g}$ and a (diagram) automorphism σ of finite order m. For brevity, we will write $\hat{\mathcal{L}}$ and $(\tilde{\mathcal{L}}, \mathcal{L})$ for $\hat{L}(\mathbf{g}, \sigma)$ and $(\tilde{L}(\mathbf{g}, \sigma), L(\mathbf{g}, \sigma))$ respectively.

For all $l_1, l_2 \in \mathcal{L}$ one has

$$[l_1, l_2] - [l_1, l_2]_{\hat{\mathcal{L}}} = ac \tag{8.0.1}$$

for some scalar $a \in k$. Using (2.0.1) it is also easy to see that for all $y \in \mathcal{L}$ one has

$$[d, yt^n]_{\hat{\mathcal{L}}} = mnyt^n + [d, y]_{\hat{\mathcal{L}}}t^n. \tag{8.0.2}$$

8.1. Remark. Recall that $\mathcal{L}$ has a natural R-module structure: If $y = x \otimes t^{\frac{p}{m}} \in \mathcal{L}$ then

$$yt := x \otimes t^{\frac{p}{m}+1} = x \otimes t^{\frac{p+m}{m}} \in \mathcal{L}.$$

Therefore since $[d, y]_{\hat{\mathcal{L}}}$ is contained in $\mathcal{L}$ the expression $[d, y]_{\hat{\mathcal{L}}} t^n$ is meaningful.

Henceforth we will denote by **G** the simple simply connected group scheme over R corresponding to $\mathcal{L}$.

The infinite dimensional Lie algebra $\hat{\mathcal{L}}$ admits a unique (up to nonzero scalar) invariant nondegenerate bilinear form $(\cdot, \cdot)$. Its restriction to $\mathcal{L} \subset \hat{\mathcal{L}}$ is nondegenerate (see [4, 7.5.1 and 8.3.8]) and we have

$$(c, c) = (d, d) = 0, \qquad 0 \neq (c, d) = \beta \in k^{\times}$$

and

$$(c, l) = (d, l) = 0 \quad \text{for all } l \in \mathcal{L}.$$

8.2. Remark. It is known that a nondegenerate invariant bilinear form on $\hat{\mathcal{L}}$ is unique up to nonzero scalar. We may view $\hat{\mathcal{L}}$ as a subalgebra in the split Kac–Moody Lie algebra $\hat{L}(\mathbf{g})_S$. The last one also admits a nondegenerate invariant bilinear form and it is known that its restriction to $\hat{\mathcal{L}}$ is nondegenerate. Hence this restriction is proportional to the form $(\cdot, \cdot)$.

Let $\mathbf{h}_{\bar{0}}$ be a Cartan subalgebra of the Lie algebra $\mathbf{g}_{\bar{0}}$.

8.3. Lemma. *The centralizer of $\mathbf{h}_{\bar{0}}$ in $\mathbf{g}$ is a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$.*

Proof. See [4, Lemma 8.1]. □

The algebra $\mathcal{H} = \mathbf{h}_{\bar{0}} \oplus kc \oplus kd$ plays the role of Cartan subalgebra for $\hat{\mathcal{L}}$. With respect to $\mathcal{H}$ our algebra $\hat{\mathcal{L}}$ admits a root space decomposition. The roots are of two types: anisotropic (real) or isotropic (imaginary). This terminology comes from transferring the form to $\mathcal{H}^*$ and computing the "length" of the roots.

The *core* $\tilde{\mathcal{L}}$ of $\hat{\mathcal{L}}$ is the subalgebra generated by all the anisotropic root spaces. In our case we have $\tilde{\mathcal{L}} = \mathcal{L} \oplus kc$. The correct way to recover $\mathcal{L}$ inside $\hat{\mathcal{L}}$ is as its core modulo its centre.[4]

If $\mathfrak{m} \subset \hat{\mathcal{L}}$ is an abelian subalgebra and $\alpha \in \mathfrak{m}^* = \mathrm{Hom}(\mathfrak{m}, k)$ we denote the corresponding eigenspace in $\hat{\mathcal{L}}$ (with respect to the adjoint representation of $\hat{\mathcal{L}}$) by $\hat{\mathcal{L}}_{\alpha}$. Thus,

$$\hat{\mathcal{L}}_{\alpha} = \left\{ l \in \hat{\mathcal{L}} \mid [x, l]_{\hat{\mathcal{L}}} = \alpha(x)l \text{ for all } x \in \mathfrak{m} \right\}.$$

The subalgebra $\mathfrak{m}$ is called *diagonalizable* in $\hat{\mathcal{L}}$ if

$$\hat{\mathcal{L}} = \bigoplus_{\alpha \in \mathfrak{m}^*} \hat{\mathcal{L}}_{\alpha}.$$

Every diagonalizable subalgebra of $\mathfrak{m} \subset \hat{\mathcal{L}}$ is necessarily abelian. We say that $\mathfrak{m}$ is a maximal (abelian) diagonalizable subalgebra (MAD) if it is not properly contained in a larger diagonalizable subalgebra of $\hat{\mathcal{L}}$.

---

[4] In nullity one the core coincides with the derived algebra, but this is not necessarily true in higher nullities.

8.4. Remark. Every MAD of $\hat{\mathcal{L}}$ contains the centre kc of $\hat{\mathcal{L}}$.

8.5. Example. The subalgebra $\mathcal{H}$ is a MAD in $\hat{\mathcal{L}}$ (see [4, Theorem 8.5]).

Our aim is to show that an arbitrary maximal diagonalizable subalgebra $\mathfrak{m} \subset \hat{\mathcal{L}}$ is conjugate to $\mathcal{H}$ under an element of $\mathrm{Aut}_k(\hat{\mathcal{L}})$. For future reference we record the following facts:

8.6. Theorem.

(a) *Every diagonalizable subalgebra in $\mathcal{L}$ is contained in a MAD of $\mathcal{L}$ and all MADs of $\mathcal{L}$ are conjugate. More precisely, let* **G** *be the simple simply connected group scheme over R corresponding to $\mathcal{L}$. Then for any MAD $\mathfrak{m}$ of $\mathcal{L}$ there exists $g \in \mathbf{G}(R)$ such that $Ad(g)(\mathfrak{m}) = \mathbf{h}_{\bar{0}}$.*
(b) *There exists a natural bijection between MADs of $\tilde{\mathcal{L}}$ and MADs of $\mathcal{L}$. Every diagonalizable subalgebra in $\tilde{\mathcal{L}}$ is contained in a MAD of $\tilde{\mathcal{L}}$. All MADs of $\tilde{\mathcal{L}}$ are conjugate by elements in $Ad(\mathbf{G}(R)) \subset \mathrm{Aut}_k(\mathcal{L}) \simeq \mathrm{Aut}_k(\tilde{\mathcal{L}})$.*
(c) *The image of the canonical map $\mathrm{Aut}_k(\hat{\mathcal{L}}) \to \mathrm{Aut}_k(\tilde{\mathcal{L}}) \simeq \mathrm{Aut}_k(\mathcal{L})$ obtained by restriction to the derived subalgebra $\tilde{\mathcal{L}}$ contains $\mathrm{Aut}_{R\text{-}Lie}(\mathcal{L})$.*

Proof. (a) From the explicit realization of $\mathcal{L}$ one knows that $\mathbf{h}_{\bar{0}}$ is a MAD of $\mathcal{L}$. Now (a) follows from [2].

(b) The correspondence follows from the fact that every MAD of $\tilde{\mathcal{L}}$ contains kc. A MAD $\tilde{\mathfrak{m}}$ of $\tilde{\mathcal{L}}$ is necessarily of the form $\mathfrak{m} \oplus kc$ for some MAD $\mathfrak{m}$ of $\mathcal{L}$ and conversely. The canonical map $\mathrm{Aut}_k(\tilde{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$ is an isomorphism by Proposition 5.2.

(c) This was established in Theorem 7.5. □

8.7. Lemma. *If $\mathfrak{m} \subset \hat{\mathcal{L}}$ is a MAD of $\hat{\mathcal{L}}$ then $\mathfrak{m} \not\subset \tilde{\mathcal{L}}$.*

Proof. Assume that $\mathfrak{m} \subset \tilde{\mathcal{L}}$. By Theorem 8.6(b), there exists a MAD $\mathfrak{m}'$ of $\tilde{\mathcal{L}}$ containing $\mathfrak{m}$. Applying again Theorem 8.6 we may assume that up to conjugation by an element of $\mathrm{Aut}_k(\hat{\mathcal{L}})$, in fact of $\hat{\mathbf{G}}(R)$, we have $\mathfrak{m} \subset \mathfrak{m}' = \mathbf{h}_{\bar{0}} \oplus kc$. Then $\mathfrak{m}$ is a proper subalgebra of the MAD $\mathcal{H}$ of $\hat{\mathcal{L}}$ and this contradicts the maximality of $\mathfrak{m}$. □

In the next three sections we are going to prove some preliminary results related to a subalgebra $\hat{A}$ of the twisted affine Kac–Moody Lie algebra $\hat{\mathcal{L}}$ which satisfies the following two conditions:

a) *$\hat{A}$ is of the form $\hat{A} = A \oplus kc \oplus kd$, where A is an R-subalgebra of $\mathcal{L}$ such that $A \otimes_R K$ is a semisimple Lie algebra over K where $K = k(t)$ is the fraction field of R.*
b) *The restriction to $\hat{A}$ of the nondegenerate invariant bilinear form $(-, -)$ of $\hat{\mathcal{L}}$ is nondegenerate.*

In particular, all these results will be valid for $\hat{A} = \hat{\mathcal{L}}$.

9. Weights of semisimple operators and their properties

Let $x = x' + d \in \hat{A}$ where $x' \in A$. It induces a k-linear operator

$$ad(x) : \hat{A} \to \hat{A}, \qquad y \to ad(x)(y) = [x, y]_{\hat{A}}.$$

We say that x is a k-diagonalizable element of $\hat{A}$ if $\hat{A}$ has a k-basis consisting of eigenvectors of $ad(x)$. Throughout we assume that $x' \neq 0$ and that x is k-diagonalizable.

For any scalar $w \in k$ we let

$$\hat{A}_w = \big\{ y \in \hat{A} \mid [x, y]_{\hat{A}} = wy \big\}.$$

We say that w is a *weight* (= eigenvalue) of $ad(x)$ if $\hat{A}_w \neq 0$. More generally, if O is a diagonalizable linear operator of a vector space V over k (of main interest to us are the vector spaces $\hat{A}, \tilde{A} = A \oplus kc, A$) and if w is its eigenvalue following standard practice we will denote by $V_w \subset V$ the corresponding eigenspace of O.

9.1. Lemma.

(a) If w is a nonzero weight of $ad(x)$ then $\hat{A}_w \subset \tilde{A}$.
(b) $\hat{A}_0 = \tilde{A}_0 \oplus \langle x \rangle$.

Proof. Clearly we have $[\hat{A}, \hat{A}] \subset \tilde{A}$ and this implies $ad(x)(\tilde{A}) \subset \tilde{A}$. It then follows that the linear operator $ad(x)|_{\tilde{A}}$ is k-diagonalizable. Let $\tilde{A} = \bigoplus \tilde{A}_{w'}$ where the sum is taken over all weights of $ad(x)|_{\tilde{A}}$. Since $x \in \hat{A}_0$ and since $\hat{A} = \langle x \rangle \oplus \tilde{A}$ we conclude that

$$\hat{A} = \langle x, \tilde{A}_0 \rangle \oplus \left(\bigoplus_{w' \neq 0} \tilde{A}_{w'} \right),$$

so that the result follows. $\square$

The operator $ad(x)|_{\tilde{A}}$ maps the centre $\langle c \rangle = kc$ of $\tilde{A}$ into itself, hence it induces a linear operator O_x of $A \simeq \tilde{A}/kc$ which is also k-diagonalizable. The last isomorphism is induced by a natural (projection) mapping $\lambda : \tilde{A} \to A$. If $w \neq 0$ the restriction of λ to $\tilde{A}_w$ is injective (because $\tilde{A}_w$ does not contain kc). Since $\tilde{A} = \bigoplus_w \tilde{A}_w$ it then follows that

$$\lambda|_{\tilde{A}_w} : \tilde{A}_w \longrightarrow A_w$$

is an isomorphism for $w \neq 0$. Thus the three linear operators $ad(x)$, $ad(x)|_{\tilde{A}}$ and O_x have the same nonzero weights.

9.2. Lemma. *Let $w \neq 0$ be a weight of O_x and let $n \in \mathbb{Z}$. Then $w + mn$ is also a weight of O_x and $A_{w+mn} = t^n A_w$.*

Proof. Assume $y \in A_w \subset A$, hence $O_x(y) = wy$. Let us show that $yt^n \in A_{w+mn}$. We have

$$O_x(yt^n) = \lambda(ad(x)(yt^n)) = \lambda([x, yt^n]_{\hat{A}}). \tag{9.2.1}$$

Substituting $x = x' + d$ we get

$$[x, yt^n]_{\hat{A}} = [x', yt^n]_{\hat{A}} + [d, yt^n]_{\hat{A}}.$$

Applying (8.0.1) and (8.0.2) we get that the right hand side is equal to

$$[x', y]t^n + ac + [d, y]_{\hat{A}} t^n + mnyt^n$$

where $a \in k$ is some scalar. Substituting this into (9.2.1) we get

$$O_x(yt^n) = \lambda([x', y]t^n + ac + [d, y]_{\hat{A}} t^n + mnyt^n)$$
$$= [x', y]t^n + \lambda([d, y]_{\hat{A}} t^n) + mnyt^n.$$

By (8.0.1) there exists $b \in k$ such that

$$[x', y]t^n = ([x', y]_{\hat{A}} + bc)t^n.$$

Here we view $[x', y]t^n$ as an element in $\hat{A}$. Therefore

$$O_x(yt^n) = mnyt^n + \lambda(([x', y]_{\hat{A}} + bc)t^n + [d, y]_{\hat{A}}t^n)$$
$$= mnyt^n + \lambda(([x, y]_{\hat{A}} + bc)t^n).$$

We now note that by construction $[x, y]_{\hat{A}} + bc$ is contained in $A \subset \tilde{A}$. Hence

$$\lambda(([x, y]_{\hat{A}} + bc)t^n) = \lambda([x, y]_{\hat{A}} + bc)t^n = \lambda([x, y]_{\hat{A}})t^n.$$

Since $\lambda([x, y]_{\hat{A}}) = O_x(y) = wy$ we finally get

$$O_x(yt^n) = mnyt^n + wyt^n = (w + mn)yt^n.$$

Thus we have showed that $A_w t^n \subset A_{w+nm}$. By symmetry $A_{w+nm}t^{-n} \subset A_w$ and we are done. □

We now consider the case $w = 0$.

9.3. Lemma. *Assume that* $\dim \tilde{A}_0 > 1$ *and* $n \in \mathbb{Z}$. *Then* mn *is a weight of* $ad(x)$.

Proof. Since $\dim \tilde{A}_0 > 1$ there exists nonzero $y \in A$ such that $[x, y]_{\hat{A}} = 0$. Then the same computations as above show that $[x, yt^n]_{\tilde{A}} = mnyt^n$. □

Our next aim is to show that if w is a weight of $ad(x)$ so is $-w$. We remind the reader that $\hat{A}$ is equipped with the nondegenerate invariant bilinear form $(-, -)$. Hence for all $y, z \in \hat{A}$ one has

$$([x, y]_{\hat{A}}, z) = -(y, [x, z]_{\hat{A}}). \tag{9.3.1}$$

9.4. Lemma. *If* w *is a weight of* $ad(x)$ *then so is* $-w$.

Proof. If $w = 0$ there is nothing to prove. Assume $w \neq 0$. Consider the root space decomposition

$$\hat{A} = \bigoplus_{w'} \hat{A}_{w'}.$$

It suffices to show that for any two weights w_1, w_2 of $ad(x)$ such that $w_1 + w_2 \neq 0$ the subspaces $\hat{A}_{w_1}$ and $\hat{A}_{w_2}$ are orthogonal to each other. Indeed, the last implies that if $-w$ were not a weight then every element in $\hat{A}_w$ would be orthogonal to all elements in $\hat{A}$, which is impossible.

Let $y \in \hat{A}_{w_1}$ and $z \in \hat{A}_{w_2}$. Applying (9.3.1) we have

$$w_1(y, z) = ([x, y]_{\hat{A}}, z) = -(y, [x, z]_{\hat{A}}) = -w_2(y, z).$$

Since $w_1 \neq -w_2$ we conclude $(y, z) = 0$. □

Now we switch our interest to the operator O_x and its weight subspaces. Since the nonzero weights of $ad(x)$, $ad(x)|_{\bar{A}}$ and O_x are the same we obtain, by Lemmas 9.2 and 9.3, that for every weight w of O_x all elements in the set

$$\{w + mn \mid n \in \mathbb{Z}\}$$

are also weights of O_x. We call this set of weights by *w-series*. Recall that by Lemma 9.2 we have

$$A_{w+mn} = A_w t^n.$$

9.5. Lemma. *Let w be a weight of O_x and let $A_w R$ be the R-span of A_w in A. Then the natural map $\nu : A_w \otimes_k R \to A_w R$ given by $l \otimes t^n \mapsto lt^n$ is an isomorphism of k-vector spaces.*

Proof. Clearly, the sum $\sum_n A_{w+mn}$ of vector subspaces A_{w+mn} in A is a direct sum. Hence

$$A_w R = \sum_n A_w t^n = \sum_n A_{w+mn} = \bigoplus_n A_{w+mn}. \tag{9.5.1}$$

Fix a k-basis $\{e_i\}$ of A_w. Then $\{e_i \otimes t^j\}$ is a k-basis of $A_w \otimes_k R$. Since

$$\nu\big(e_i \otimes t^n\big) = e_i t^n \in A_{w+mn}$$

the injectivity of ν easily follows from (9.5.1). The surjectivity is also obvious. $\square$

Notation. We will denote the R-span $A_w R$ by $A_{\{w\}}$.

By our construction $A_{\{w\}}$ is an R-submodule of A and

$$A = \bigoplus_w A_{\{w\}} \tag{9.5.2}$$

where the sum is taken over fixed representatives of weight series.

9.6. Corollary. $\dim_k A_w < \infty$.

Proof. Indeed, by the above lemma we have

$$\dim_k A_w = \operatorname{rank}_R (A_w \otimes_k R) = \operatorname{rank}_R A_w R = \operatorname{rank}_R A_{\{w\}} \leqslant \operatorname{rank}_R A < \infty,$$

as required. $\square$

9.7. Corollary. *There are finitely many weight series.*

Proof. This follows from the fact that A is a free R-module of finite rank. $\square$

9.8. Lemma. *Let w_1, w_2 be weights of O_x. Then $[A_{w_1}, A_{w_2}] \subset A_{w_1+w_2}$.*

Proof. This is straightforward to check. $\square$

10. Weight zero subspace

10.1. Theorem. $A_0 \neq 0$.

Proof. Assume that $A_0 = 0$. Then, by Lemma 9.2, $A_{mn} = 0$ for all $n \in \mathbb{Z}$. It follows that for any weight w, any integer n and all $y \in A_w$, $z \in A_{-w+mn}$ we have $[y, z] = 0$. Indeed

$$[A_w, A_{-w+mn}] \subset A_{w+(-w)+mn} = A_{mn} = 0. \tag{10.1.1}$$

For $y \in A$ the operator $ad(y) : A \to A$ may be viewed as a k-operator or as an R-operator. When we deal with the Killing form $\langle -, - \rangle$ on the R-Lie algebra A we will view $ad(y)$ as an R-operator of A.

10.2. Lemma. *Let w_1, w_2 be weights of $ad(x)$ such that $\{w_1\} \neq \{-w_2\}$. Then for any integer n and all $y \in A_{w_1}$ and $z \in A_{w_2+mn}$ we have $\langle y, z \rangle = 0$.*

Proof. Let w be a weight of $ad(x)$. By our condition we have $\{w\} \neq \{w + w_1 + w_2\}$. Since $(ad(y) \circ ad(z))(A_{\{w\}}) \subset A_{\{w+w_1+w_2\}}$, in any R-basis of A corresponding to the decomposition (9.5.2) the operator $ad(y) \circ ad(z)$ has zeroes on the diagonal, hence $\text{Tr}(ad(y) \circ ad(z)) = 0$. □

10.3. Lemma. *Let w be a weight of $ad(x)$, n be an integer and let $y \in A_w$. Assume that $ad(y)$ viewed as an R-operator of A is nilpotent. Then for every $z \in A_{-w+mn}$ we have $\langle y, z \rangle = 0$.*

Proof. Indeed, let l be such that $(ad(y))^l = 0$. Since by (10.1.1), $ad(y)$ and $ad(z)$ are commuting operators we have

$$\big(ad(y) \circ ad(z)\big)^l = \big(ad(y)\big)^l \circ \big(ad(z)\big)^l = 0.$$

Therefore $ad(y) \circ ad(z)$ is nilpotent and this implies its trace is zero. □

Since the Killing form is nondegenerate, it follows immediately from the above two lemmas that for every nonzero element $y \in A_w$ the operator $ad(y)$ is not nilpotent. Recall that by Lemma 9.8 we have $ad(y)(A_{w'}) \subset A_{w+w'}$. Hence taking into consideration Corollary 9.7 we conclude that there exits a weight w' and a positive integer l such that

$$ad(y)(A_{\{w'\}}) \neq 0, \quad \big(ad(y) \circ ad(y)\big)(A_{\{w'\}}) \neq 0, \quad \ldots, \quad \big(ad(y)\big)^l(A_{\{w'\}}) \neq 0$$

and $(ad(y))^l(A_{\{w'\}}) \subset A_{\{w'\}}$. We may assume that l is the smallest positive integer satisfying these conditions. Then all consecutive scalars

$$w', \ w' + w, \ w' + 2w, \ \ldots, \ w' + lw \tag{10.3.1}$$

are weights of $ad(x)$, $\{w' + iw\} \neq \{w' + (i+1)w\}$ for $i < l$ and $\{w'\} = \{w' + lw\}$. In particular, we automatically get that lw is an integer (divisible by m) which in turn implies that w is a rational number.

Thus, under our assumption $A_0 = 0$ we have proved that all weights of $ad(x)$ are rational numbers. We now choose (in a unique way) representatives $w_1, \ldots, w_s$ of all weight series such that $0 < w_i < m$ and up to renumbering we may assume that

$$0 < w_1 < w_2 < \cdots < w_s < m.$$

10.4. Remark. Recall that for any weight w_i, the scalar $-w_i$ is also a weight. Since $0 < -w_i + m < m$ the representative of the weight series $\{-w_i\}$ is $m - w_i$. Then the inequality $m - w_i \geqslant w_1$ implies $m - w_1 \geqslant w_i$. Hence out of necessity we have $w_s = m - w_1$.

We now apply the observation (10.3.1) to the weight $w = w_1$. Let $w' = w_i$ be as in (10.3.1). Choose the integer $j \geqslant 0$ such that $w_i + jw_1$, $w_i + (j+1)w_1$ are weights and $w_i + jw_1 < m$, but $w_i + (j+1)w_1 \geqslant m$. We note that since m is not a weight of $ad(x)$ we automatically obtain $w_i + (j+1)w_1 > m$. Furthermore, we have $w_i + jw_1 \leqslant w_s = m - w_1$ (because $w_i + jw_1$ is a weight of $ad(x)$). This implies

$$m < w_i + (j+1)w_1 \leqslant w_s + w_1 = m - w_1 + w_1 = m$$

– a contradiction that completes the proof of the theorem. $\square$

11. A lower bound of dimensions of MADs in $\hat{\mathcal{L}}$

11.1. Theorem. *Let* $\mathfrak{m} \subset \hat{\mathcal{L}}$ *be a MAD. Then* $\dim \mathfrak{m} \geqslant 3$.

By Lemma 8.7, $\mathfrak{m}$ contains an element x of the form $x = x' + d$ where $x' \in \mathcal{L}$ and it also contains c. Since x and c generate a subspace of $\mathfrak{m}$ of dimension 2 the statement of the theorem is equivalent to $\langle x, c \rangle \neq \mathfrak{m}$.

Assume the contrary: $\langle x, c \rangle = \mathfrak{m}$. Since $\mathfrak{m}$ is k-diagonalizable we have the weight space decomposition

$$\hat{\mathcal{L}} = \bigoplus_\alpha \hat{\mathcal{L}}_\alpha$$

where the sum is taken over linear mappings $\alpha \in \mathfrak{m}^* = \mathrm{Hom}(\mathfrak{m}, k)$. To find a contradiction we first make some simple observations about the structure of the corresponding eigenspace $\hat{\mathcal{L}}_0$.

If $\hat{\mathcal{L}}_\alpha \neq 0$, it easily follows that $\alpha(c) = 0$ (because c is in the centre of $\hat{\mathcal{L}}$). Then α is determined uniquely by the value $w = \alpha(x)$ and so instead of $\hat{\mathcal{L}}_\alpha$ we will write $\hat{\mathcal{L}}_w$.

Recall that by Theorem 10.1, $\mathcal{L}_0 \neq 0$. Our aim is first to show that $\mathcal{L}_0$ contains a nonzero element y such that the adjoint operator $ad(y)$ of $\mathcal{L}$ is k-diagonalizable. We will next see that y necessarily commutes with x viewed as an element in $\hat{\mathcal{L}}$ and that it is k-diagonalizable in $\hat{\mathcal{L}}$ as well. It then follows that the subspace in $\hat{\mathcal{L}}$ spanned by c, x and y is a commutative k-diagonalizable subalgebra and this contradicts the fact that $\mathfrak{m}$ is a MAD.

11.2. Lemma. *Let* $y \in \mathcal{L}$ *be nonzero such that* $O_x(y) = 0$. *Then* $[x, y]_{\hat{\mathcal{L}}} = 0$.

Proof. Assume that $[x, y]_{\hat{\mathcal{L}}} = bc \neq 0$. Then

$$\left(x, [x, y]_{\hat{\mathcal{L}}}\right) = (x, bc) = \left(x' + d, bc\right) = (d, bc) = \beta b \neq 0.$$

On the other hand, since the form is invariant we get

$$\left(x, [x, y]_{\hat{\mathcal{L}}}\right) = \left([x, x]_{\hat{\mathcal{L}}}, y\right) = (0, y) = 0$$

– a contradiction which completes the proof. $\square$

11.3. Lemma. *Assume that* $y \in \mathcal{L}_0$ *is nonzero and that the adjoint operator* $ad(y)$ *of* $\mathcal{L}$ *is k-diagonalizable. Then* $ad(y)$ *viewed as an operator of* $\hat{\mathcal{L}}$ *is also k-diagonalizable.*

Proof. Choose a k-basis $\{e_i\}$ of $\mathcal{L}$ consisting of eigenvectors of $ad(y)$. Thus we have $[y, e_i] = u_i e_i$ where $u_i \in k$ and hence

$$[y, e_i]_{\hat{\mathcal{L}}} = u_i e_i + b_i c$$

where $b_i \in k$.

Case 1: Suppose first that $u_i \neq 0$. Let

$$\tilde{e}_i = e_i + \frac{b_i}{u_i} \cdot c \in \tilde{\mathcal{L}}.$$

Then we have

$$[y, \tilde{e}_i]_{\hat{\mathcal{L}}} = [y, e_i]_{\hat{\mathcal{L}}} = u_i e_i + b_i c = u_i \tilde{e}_i$$

and therefore $\tilde{e}_i$ is an eigenvector of the operator $ad(y) : \hat{\mathcal{L}} \to \hat{\mathcal{L}}$.

Case 2: Let now $u_i = 0$. Then $[y, e_i]_{\hat{\mathcal{L}}} = b_i c$ and we claim that $b_i = 0$. Indeed, we have

$$\big(x, [y, e_i]_{\hat{\mathcal{L}}}\big) = \big([x, y]_{\hat{\mathcal{L}}}, e_i\big) = (0, e_i) = 0$$

and on the other hand

$$\big(x, [y, e_i]_{\hat{A}}\big) = (x, b_i c) = \big(x' + d, b_i c\big) = (d, b_i c) = \beta b_i.$$

It follows that $b_i = 0$ and thus $\tilde{e}_i = e_i$ is an eigenvector of $ad(y)$.

Summarizing, replacing e_i by $\tilde{e}_i$ we see that the set $\{\tilde{e}_i\} \cup \{c, x\}$ is a k-basis of $\hat{\mathcal{L}}$ consisting of eigenvectors of $ad(y)$. $\quad\square$

11.4. Proposition. *The subalgebra $\mathcal{L}_0$ contains an element y such that the operator $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable.*

Proof. We split the proof in three steps.

Step 1: Assume first that there exists $y \in \mathcal{L}_0$ which as an element in $\mathcal{L}_K = \mathcal{L} \otimes_R K$ is semisimple. We claim that our operator $ad(y)$ is k-diagonalizable. Indeed, choose representatives $w_1 = 0, w_2, \ldots, w_l$ of the weight series of $ad(x)$. The sets $\mathcal{L}_{w_1}, \ldots, \mathcal{L}_{w_l}$ are vector spaces over k of finite dimension, by Lemma 9.6, and they are stable with respect to $ad(y)$ (because $y \in \mathcal{L}_0$). In each k-vector space $\mathcal{L}_{w_i}$ choose a Jordan basis

$$\{e_{ij}, \ j = 1, \ldots, l_i\}$$

of the operator $ad(y)|_{\mathcal{L}_{w_i}}$. Then the set

$$\{e_{ij}, \ i = 1, \ldots, l, \ j = 1, \ldots, l_i\} \tag{11.4.1}$$

is an R-basis of $\mathcal{L}$, by Lemma 9.5 and the decomposition given in (9.5.2). It follows that the matrix of the operator $ad(y)$ viewed as a K-operator of $\mathcal{L} \otimes_R K$ is a block diagonal matrix whose blocks corresponds to the matrices of $ad(y)|_{\mathcal{L}_{w_i}}$ in the basis $\{e_{ij}\}$. Hence (11.4.1) is a Jordan basis for $ad(y)$ viewed as an operator on $\mathcal{L} \otimes_R K$. Since y is a semisimple element of $\mathcal{L} \otimes_R K$ all matrices of $ad(y)|_{\mathcal{L}_{w_i}}$ are diagonal and this in turn implies that $ad(y)$ is k-diagonalizable operator of $\mathcal{L}$.

Step 2: We next consider the case when all elements in $\mathcal{L}_0$ viewed as elements of the R-algebra $\mathcal{L}$ are nilpotent. Then $\mathcal{L}_0$, being finite dimensional, is a nilpotent Lie algebra over k. In particular its centre is nontrivial since $\mathcal{L}_0 \neq 0$. Let $c \in \mathcal{L}_0$ be a nonzero central element of $\mathcal{L}_0$. For any $z \in \mathcal{L}_0$ the operators $ad(c)$ and $ad(z)$ of $\mathcal{L}$ commute. Then $ad(z) \circ ad(c)$ is nilpotent, hence $\langle c, z \rangle = 0$. Furthermore, by Lemma 10.2 $\langle c, z \rangle = 0$ for any $z \in \mathcal{L}_{w_i}$, $w_i \neq 0$. Thus $c \neq 0$ is in the radical of the Killing form of $\mathcal{L}$ – a contradiction.

Step 3: Assume now that $\mathcal{L}_0$ contains an element y which as an element of $\mathcal{L}_K$ has nontrivial semisimple part y_s. Let us first show that $y_s \in \mathcal{L}_{\{0\}} \otimes_R K$ and then that $y_s \in \mathcal{L}_0$. By Step 1, the last would complete the proof of the proposition.

By decomposition (9.5.2) applied to $A = \mathcal{L}$ we may write y_s as a sum

$$y_s = y_1 + y_2 + \cdots + y_l$$

where $y_i \in \mathcal{L}_{\{w_i\}} \otimes_R K$. In Step 1 we showed that in an appropriate R-basis (11.4.1) of $\mathcal{L}$ the matrix of $ad(y)$ is block diagonal whose blocks correspond to the Jordan matrices of $ad(y)|_{\mathcal{L}_{w_i}} : \mathcal{L}_{w_i} \to \mathcal{L}_{w_i}$. It follows that the semisimple part of $ad(y)$ is also a block diagonal matrix whose blocks are semisimple parts of $ad(y)|_{A_{w_i}}$.

Since $\mathcal{L}_K$ is a semisimple Lie algebra over a perfect field we get that $ad(y_s) = ad(y)_s$. Hence for all weights w_i we have

$$[y_s, \mathcal{L}_{w_i}] \subset \mathcal{L}_{w_i}. \tag{11.4.2}$$

On the other hand, for any $u \in \mathcal{L}_{w_i}$ we have

$$ad(y_s)(u) = [y_1, u] + [y_2, u] + \cdots + [y_l, u].$$

Since $[y_j, u] \in \mathcal{L}_{\{w_i + w_j\}} \otimes_R K$, it follows that $ad(y_s)(u) \in \mathcal{L}_{\{w_i\}}$ if and only if $[y_2, u] = \cdots = [y_l, u] = 0$. Since this is true for all i and all $u \in \mathcal{L}_{w_i}$ and since the kernel of the adjoint representation of $\mathcal{L}_K$ is trivial we obtain $y_2 = \cdots = y_l = 0$. Therefore $y_s \in \mathcal{L}_{\{0\}} \otimes_R K$.

It remains to show that $y_s \in \mathcal{L}_0$. We may write y_s in the form

$$y_s = \frac{1}{g(t)} \big(u_0 \otimes 1 + u_1 \otimes t + \cdots + u_m \otimes t^m \big)$$

where $u_0, \ldots, u_l \in \mathcal{L}_0$ and $g(t) = g_0 + g_1 t + \cdots + g_n t^n$ is a polynomial with coefficients $g_0, \ldots, g_n$ in k with $g_n \neq 0$. The above equality can be rewritten in the form

$$g_0 y_s + g_1 y_s \otimes t + \cdots + g_n y_s \otimes t^n = u_0 \otimes 1 + \cdots + u_m \otimes t^m. \tag{11.4.3}$$

Consider an arbitrary index i and let $u \in \mathcal{L}_{w_i}$. Recall that by (11.4.2) we have

$$ad(y_s)(\mathcal{L}_{w_i}) \subset \mathcal{L}_{w_i}.$$

Applying both sides of (11.4.3) to u and comparing $\mathcal{L}_{w_i + n}$-components we conclude that $[g_n y_s, u] = [u_n, u]$. Since this is true for all u and all i and since the adjoint representation of $\mathcal{L}_K$ has trivial kernel we obtain $g_n y_s = u_n$. Since $g_n \neq 0$ we get $y_s = u_n / g_n \in \mathcal{L}_0$. $\quad\square$

Now we can easily finish the proof of Theorem 11.1. Suppose the contrary. Then $\dim(\mathfrak{m}) < 3$ and hence by Lemma 8.7 we have $\mathfrak{m} = \langle c, x' + d \rangle$ with $x' \in \mathcal{L}$. Consider the operator O_x on $\mathcal{L}$. By Theorem 10.1 we have $\mathcal{L}_0 \neq 0$. By Propositions 11.4 and 11.3 there exists a nonzero k-diagonalizable

element $y \in \mathcal{L}_0$. Clearly, y is not contained in $\mathfrak{m}$. Furthermore, by Lemma 11.2, y viewed as an element of $\hat{\mathcal{L}}$ commutes with $\mathfrak{m}$ and by Lemma 11.3 it is k-diagonalizable in $\hat{\mathcal{L}}$. It follows that the subspace $\mathfrak{m}_1 = \mathfrak{m} \oplus \langle y \rangle$ is an abelian k-diagonalizable subalgebra of $\hat{\mathcal{L}}$. But this contradicts maximality of $\mathfrak{m}$.

12. All MADs are conjugate

12.1. Theorem. *Let $\hat{\mathbf{G}}(R)$ be the preimage of $\{Ad(g): g \in \mathbf{G}(R)\}$ under the canonical map $\mathrm{Aut}_k(\hat{\mathcal{L}}) \to \mathrm{Aut}_k(\mathcal{L})$. Then all MADs of $\hat{\mathcal{L}}$ are conjugate under $\hat{\mathbf{G}}(R)$ to the subalgebra $\mathcal{H}$ in 8.5.*

Proof. Let $\mathfrak{m}$ be a MAD of $\hat{\mathcal{L}}$. By Lemma 8.7, $\mathfrak{m} \not\subset \tilde{\mathcal{L}}$. Fix a vector $x = x' + d \in \mathfrak{m}$ where $x' \in \mathcal{L}$ and let $\mathfrak{m}' = \mathfrak{m} \cap \mathcal{L}$. Thus we have $\mathfrak{m} = \langle x, c, \mathfrak{m}' \rangle$. Note that $\mathfrak{m}' \neq 0$, by Theorem 11.1. Furthermore, since $\mathfrak{m}'$ is k-diagonalizable in $\mathcal{L}$, without loss of generality we may assume that $\mathfrak{m}' \subset \mathbf{h}_{\bar{0}}$ given that by Theorem 8.6(b) there exists $g \in \mathbf{G}(R)$ such that $Ad(g)(\mathfrak{m}') \subset \mathbf{h}_{\bar{0}}$ and that by Theorem 7.5 g has lifting to $\mathrm{Aut}_{k\text{-}Lie}(\hat{\mathcal{L}})$.

Consider the weight space decomposition

$$\mathcal{L} = \bigoplus_i L_{\alpha_i} \tag{12.1.1}$$

with respect to the k-diagonalizable subalgebra $\mathfrak{m}'$ of $\mathcal{L}$ where $\alpha_i \in (\mathfrak{m}')^*$ and as usual

$$L_{\alpha_i} = \left\{ z \in \mathcal{L} \mid [t, z] = \alpha_i(t)z \text{ for all } t \in \mathfrak{m}' \right\}.$$

12.2. Lemma. *L_{α_i} is invariant with respect to the operator O_x.*

Proof. The k-linear operator O_x commutes with $ad(t)$ for all $t \in \mathfrak{m}'$ (because x and $\mathfrak{m}'$ commute in $\hat{\mathcal{L}}$), so the result follows. □

12.3. Lemma. *We have $x' \in L_0$.*

Proof. By our construction $\mathfrak{m}'$ is contained in $\mathbf{h}_0$, hence d commutes with the elements of $\mathfrak{m}'$. But x also commutes with the elements of $\mathfrak{m}'$ and so does $x' = x - d$. □

$L_0 = C_{\mathcal{L}}(\mathfrak{m}')$, being the Lie algebra of the reductive group scheme $C_{\mathbf{G}}(\mathfrak{m}')$ (see [2]), is of the form $L_0 = \mathbf{z} \oplus A$ where $\mathbf{z}$ and A are the Lie algebras of the central torus of $C_{\mathbf{G}}(\mathfrak{m}')$ and its semisimple part respectively. Our next goal is to show that $A = 0$.

Suppose this is not true. To get a contradiction we will show that the subset $\hat{A} = A \oplus kc \oplus kd \subset \hat{\mathcal{L}}$ is a subalgebra satisfying conditions a) and b) stated at the end of Section 8 and that it is stable with respect to $ad(x)$. This, in turn, will allow us to construct an element $y \in A$ which viewed as an element of $\hat{\mathcal{L}}$ commutes with x and $\mathfrak{m}'$ and is k-diagonalizable. The last, of course, contradicts the maximality of $\mathfrak{m}$.

Let $\mathbf{H}$ denote the simple simply connected Chevalley-Demazure algebraic k-group corresponding to $\mathbf{g}$. Since $\mathbf{G}$ is split over S we have

$$\mathbf{H}_S = \mathbf{H} \times_k S \simeq \mathbf{G}_S = \mathbf{G} \times_R S.$$

Let $C_{\mathbf{g}}(\mathfrak{m}') = \mathbf{t} \oplus \mathbf{r}$ where $\mathbf{t}$ is the Lie algebra of the central torus of the reductive k-group $C_{\mathbf{H}}(\mathfrak{m}')$ and $\mathbf{r}$ is the Lie algebra of its semisimple part. Since centralizers commute with base change, we obtain that

$$\mathbf{t}_S = \mathbf{t} \otimes_k S = \mathbf{z} \otimes_R S = \mathbf{z}_S, \qquad \mathbf{r}_S = \mathbf{r} \times_k S = A \otimes_R S = A_S.$$

12.4. Lemma. *We have $ad(d)(A) \subset A$ and in particular $\hat{A}$ is a subalgebra of $\hat{\mathcal{L}}$.*

Proof. Since $\mathbf{r}$ consists of "constant" elements we have $[d, \mathbf{r}]_{\hat{L}(\mathbf{g})_S} = 0$, and this implies that $[d, \mathbf{r}_S]_{\hat{L}(\mathbf{g})_S} \subset \mathbf{r}_S$. Also, viewing $\mathcal{L}$ as a subalgebra of $\hat{L}(\mathbf{g})_S$ we have $[d, \mathcal{L}]_{\hat{\mathcal{L}}} \subset \mathcal{L}$. Furthermore, S/R is faithfully flat, hence $A = A_S \cap \mathcal{L} = \mathbf{r}_S \cap \mathcal{L}$. Since both subalgebras $\mathbf{r}_S$ and $\mathcal{L}$ are stable with respect to $ad(d)$, so is their intersection. $\square$

12.5. Lemma. *The restriction of the nondegenerate invariant bilinear form $(\cdot, \cdot)$ on $\hat{\mathcal{L}}$ to L_0 is nondegenerate.*

Proof. We mentioned before that the restriction of $(\cdot, \cdot)$ to $\mathcal{L}$ is nondegenerate. Hence in view of decomposition (12.1.1) it suffices to show that for all $a \in L_0$ and $b \in L_{\alpha_i}$ with $\alpha_i \neq 0$ we have $(a, b) = 0$.

Let $l \in \mathfrak{m}'$ be such that $\alpha_i(l) \neq 0$. Using the invariance of $(\cdot, \cdot)$ we get

$$\alpha_i(l)(a, b) = \big(a, \alpha_i(l)b\big) = \big(a, [l, b]\big) = \big([a, l], b\big) = 0.$$

Hence $(a, b) = 0$ as required. $\square$

12.6. Lemma. *The restriction of $(\cdot, \cdot)$ to A is nondegenerate.*

Proof. By Lemma 12.5 it is enough to show that $\mathbf{z}$ and A are orthogonal in $\hat{\mathcal{L}}$. Moreover, viewing $\mathbf{z}$ and A as subalgebras of the split affine Kac–Moody Lie algebra $\hat{L}(\mathbf{g})_S$ and using Remark 8.2 we conclude that it suffices to verify that $\mathbf{z}_S = \mathbf{t}_S$ and $A_S = \mathbf{r}_S$ are orthogonal in $\hat{L}(\mathbf{g})_S$.

Let $a \in \mathbf{t}$ and $b \in \mathbf{r}$. We know that

$$\big(at^{\frac{i}{m}}, bt^{\frac{j}{m}}\big) = \langle a, b \rangle \delta_{i+j, 0}$$

where $\langle \cdot, \cdot \rangle$ is a Killing form of $\mathbf{g}$. Since $\mathbf{r}$ is a semisimple algebra we have $\mathbf{r} = [\mathbf{r}, \mathbf{r}]$. It follows that we can write b in the form $b = \sum [a_i, b_i]$ for some $a_i, b_i \in \mathbf{r}$. Using the facts that $\mathbf{t}$ and $\mathbf{r}$ commute and that the Killing form is invariant we have

$$\langle a, b \rangle = \Big\langle a, \sum [a_i, b_i] \Big\rangle = \sum \langle [a, a_i], b_i \rangle = \sum \langle 0, b_i \rangle = 0.$$

Thus $(at^{\frac{i}{m}}, bt^{\frac{j}{m}}) = 0$. $\square$

12.7. Remark. It follows immediately from Lemma 12.6 that the restriction to $\hat{A}$ of the nondegenerate invariant bilinear form $(-, -)$ is nondegenerate. Indeed, we have $\hat{A} = A \oplus \langle c, d \rangle$. We know that the restriction of our form to $\langle c, d \rangle$ is nondegenerate. By the lemma its restriction to A is also nondegenerate. Since A and $\langle c, d \rangle$ are orthogonal to each other our assertion follows.

12.8. Lemma. *The k-subspace $A \subset \mathcal{L}$ is invariant with respect to O_x.*

Proof. Let $a \in A$. We need to verify that

$$[x, a]_{\hat{\mathcal{L}}} \in A \oplus kc \subset \hat{\mathcal{L}}.$$

But $[d, A]_{\hat{\mathcal{L}}} \subset A + kc$ by Lemma 12.4. We also have

$$\big[x', A\big]_{\hat{\mathcal{L}}} \subset A \oplus kc$$

(because $x' \in L_0$, by Lemma 12.3, and A viewed as a subalgebra in L_0 is an ideal). Since $x = x' + d$ the result follows. $\quad\square$

According to Lemma 12.3 we can write $x' = x_0' + x_1'$ where $x_0' \in \mathbf{z}$ and $x_1' \in A$.

12.9. Lemma. *We have $O_x|_A = O_{x_1'+d}|_A$. In particular, the operator $O_{x_1'+d}|_A$ of A is k-diagonalizable.*

Proof. By Lemma 12.8, we have $O_x(A) \subset A$. Since O_x is k-diagonalizable (as an operator of $\mathcal{L}$), so is the operator $O_x|_A$ of A. Therefore the last assertion of the lemma follows from the first one.

Let now $a \in A$. Using the fact that x_0' and a commute in $\mathcal{L}$ we have

$$\left[x', a\right]_{\hat{\mathcal{L}}} = \left[x_0', a\right]_{\hat{\mathcal{L}}} + \left[x_1', a\right]_{\hat{\mathcal{L}}} = \left[x_1', a\right]_{\hat{\mathcal{L}}} + bc$$

for some $b \in k$. Thus $O_x(a) = O_{x_1'+d}(a)$. $\quad\square$

12.10. Lemma. *The operator $ad(x_1' + d) : \hat{A} \to \hat{A}$ is k-diagonalizable.*

Proof. Since by Lemma 12.9 $O_{x_1'+d}|_A : A \to A$ is k-diagonalizable we can apply the same arguments as in Lemma 11.3. $\quad\square$

Now we can produce the required element y. It follows from Lemma 12.6 that the Lie algebra $\hat{A}$ satisfies all the conditions stated at the end of Section 8. By Lemma 12.10, $ad(x_1' + d)$ is k-diagonalizable operator of $\hat{A}$. Hence arguing as in Theorem 11.1 we see that there exists a nonzero $y \in A$ such that $[y, x_1' + d]_{\hat{\mathcal{L}}} = 0$ and $ad(y)$ is a k-diagonalizable operator on $\hat{A}$. Then by Lemma 12.9 we have $O_x(y) = O_{x_1'+d}(y) = 0$ and hence, by Lemma 11.2, x and y commute in $\hat{\mathcal{L}}$.

According to our plan it remains to show that y is k-diagonalizable in $\hat{\mathcal{L}}$. To see this we need

12.11. Lemma. *Let $z \in \mathfrak{m}'$. Then $[z, y]_{\hat{\mathcal{L}}} = 0$.*

Proof. Since $y \in A \subset C_{\mathcal{L}}(\mathfrak{m}')$ we have $[z, y]_{\mathcal{L}} = 0$. Then $[z, y]_{\hat{\mathcal{L}}} = bc$ for some $b \in k$. It follows

$$0 = (0, y) = \left([x, z]_{\hat{\mathcal{L}}}, y\right) = \left(x, [z, y]_{\hat{\mathcal{L}}}\right) = (x' + d, bc) = (d, bc) = \beta b.$$

This yields $b = 0$ as desired. $\quad\square$

12.12. Proposition. *The operator $ad(y) : \hat{\mathcal{L}} \to \hat{\mathcal{L}}$ is k-diagonalizable.*

Proof. According to Lemma 11.3, it suffices to prove that $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable. Since y viewed as an element of A is semisimple it is still semisimple viewed as an element of $\mathcal{L}$. In particular, the R-operator $ad(y) : \mathcal{L} \to \mathcal{L}$ is also semisimple.

Recall that we have the decomposition of $\mathcal{L}$ into the direct sum of the weight spaces with respect to O_x:

$$\mathcal{L} = \bigoplus_w \mathcal{L}_w = \bigoplus_i \bigoplus_n \mathcal{L}_{w_i+mn} = \bigoplus_i \mathcal{L}_{\{w_i\}}.$$

Since y and x commute in $\hat{\mathcal{L}}$, for all weights w we have $ad(y)(\mathcal{L}_w) \subset \mathcal{L}_w$. If we choose any k-basis of $\mathcal{L}_w$ it is still an R-basis of $\mathcal{L}_{\{w\}} = \mathcal{L}_w \otimes_k R$ and in this basis the R-operator $ad(y)|_{\mathcal{L}_{\{w\}}}$ and the k-operator $ad(y)|_{\mathcal{L}_w}$ have the same matrices. Since the R-operator $ad(y)|_{\mathcal{L}_{\{w\}}}$ is semisimple, so is $ad(y)|_{\mathcal{L}_w}$, i.e. $ad(y)|_{\mathcal{L}_w}$ is a k-diagonalizable operator. Thus $ad(y) : \mathcal{L} \to \mathcal{L}$ is k-diagonalizable. $\quad\square$

Summarizing, assuming $A \neq 0$ we have constructed the k-diagonalizable element

$$y \notin \mathfrak{m} = \langle \mathfrak{m}', x, c \rangle$$

in $\hat{\mathcal{L}}$ which commutes with $\mathfrak{m}'$ and x in $\hat{\mathcal{L}}$. Then the subalgebra $\langle \mathfrak{m}, y \rangle$ in $\hat{\mathcal{L}}$ is commutative and k-diagonalizable which is impossible since $\mathfrak{m}$ is a MAD. Thus A is necessarily trivial and this implies $C_{\mathcal{L}}(\mathfrak{m}')$ is the Lie algebra of the R-torus $C_{\mathbf{G}}(\mathfrak{m}')$, in particular $C_{\mathcal{L}}(\mathfrak{m}')$ is abelian.

Note that $x' \in C_{\mathcal{L}}(\mathfrak{m}')$, by Lemma 12.3, and that $\mathbf{h}_{\bar{0}} \subset C_{\mathcal{L}}(\mathfrak{m}')$ (because $\mathfrak{m}' \subset \mathbf{h}_{\bar{0}}$, by construction). Since $C_{\mathcal{L}}(\mathfrak{m}')$ is abelian and since $x = x' + d$ it follows that $ad(x)(\mathbf{h}_{\bar{0}}) = 0$. Hence $\langle \mathbf{h}_{\bar{0}}, x, c \rangle$ is a commutative k-diagonalizable subalgebra in $\hat{\mathcal{L}}$. But it contains our MAD $\mathfrak{m}$. Therefore $\mathfrak{m} = \langle \mathbf{h}_{\bar{0}}, x, c \rangle$. To finish the proof of Theorem 12.1 it now suffices to show that $x' \in \mathbf{h}_{\bar{0}}$. For that, in turn, we may view x' as an element of $L(\mathbf{g})_S$ and it suffices to show that $x' \in \mathbf{h}$ because $\mathbf{h} \cap \mathcal{L} = \mathbf{h}_{\bar{0}}$.

12.13. Lemma. $x' \in \mathbf{h}$.

Proof. Consider the root space decomposition of $\mathbf{g}$ with respect to the Cartan subalgebra $\mathbf{h}$:

$$\mathbf{g} = \mathbf{h} \oplus \left(\bigoplus_{\alpha \neq 0} \mathbf{g}_{\alpha} \right).$$

Every k-subspace $\mathbf{g}_{\alpha}$ has dimension 1. Choose a nonzero elements $X_{\alpha} \in \mathbf{g}_{\alpha}$. It follows from $\mathfrak{m}' = \mathbf{h}_{\bar{0}}$ that $C_{L(\mathbf{g})_S}(\mathfrak{m}') = \mathbf{h}_S$. Thus $x' \in \mathbf{h}_S$. Then $\mathbf{g}_{\alpha} \otimes_k S$ is stable with respect to $ad(x')$ and clearly it is stable with respect to $ad(d)$. Hence it is also stable with respect to O_x.

Arguing as in Lemma 9.2 one can easily see that the operator O_x, viewed as an operator of $L(\mathbf{g})_S$, is k-diagonalizable. Since $\mathbf{g}_{\alpha} \otimes_k S$ is stable with respect to O_x, it is the direct sum of its weight subspaces. Hence

$$\mathbf{g}_{\alpha} \otimes_k S = \bigoplus_{w} \left(L(\mathbf{g})_S \right)_{\{w\}}$$

where $\{w\} = \{w + j/m \mid j \in \mathbb{Z}\}$ is the weight series corresponding to w. But $\mathbf{g}_{\alpha} \otimes_k S$ has rank 1 as an S-module. This implies that in the above decomposition we have only one weight series $\{w\}$ for some weight w of O_x.

We next note that automatically we have $\dim_k (L(\mathbf{g})_S)_w = 1$. Any its nonzero vector which is a generator of the S-module $\mathbf{g}_{\alpha} \otimes_k S$ is of the form $X_{\alpha} t^{\frac{j}{m}}$. It follows from Lemma 9.2 that $\mathbf{g}_{\alpha} = \langle X_{\alpha} \rangle$ is also a weight subspace of O_x. Thus for every root α we have

$$[x, X_{\alpha}]_{\hat{L}(\mathbf{g})_S} = [x' + d, X_{\alpha}]_{\hat{L}(\mathbf{g})_S} = [x', X_{\alpha}] = b_{\alpha} X_{\alpha}$$

for some scalar $b_{\alpha} \in k$. Since $x' \in \mathbf{h}_S$ this can happen if and only if $x' \in \mathbf{h}$. $\square$

By the previous lemma we have $x' \in \mathbf{h}_{\bar{0}}$, hence

$$\mathfrak{m} = \langle \mathbf{h}_{\bar{0}}, c, d \rangle = \mathcal{H}.$$

The proof of Theorem 12.1 is complete. $\square$

References

 [1] B. Allison, S. Berman, A. Pianzola, Covering algebras II: Loop algebras of Kac–Moody Lie algebras, J. Reine Angew. Math. (Crelle) 571 (2004) 39–71.
 [2] V. Chernousov, P. Gille, A. Pianzola, Conjugacy theorems for loop reductive group schemes and Lie algebras, arXiv: 1109.5236.
 [3] P. Gille, A. Pianzola, Torsors, reductive group schemes and extended affine Lie algebras, Mem. Amer. Math. Soc. 226 (2013), # 1063.
 [4] V. Kac, Infinite Dimensional Lie Algebras, third edition, Cambridge University Press, Cambridge, 1990.
 [5] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Springer-Verlag, 2002.
 [6] R.V. Moody, A. Pianzola, Lie Algebras with Triangular Decomposition, John Wiley, New York, 1995.
 [7] A. Pianzola, Locally trivial principal homogeneous spaces and conjugacy theorems for Lie algebras, J. Algebra 275 (2) (2004) 600–614.
 [8] A. Pianzola, Automorphisms of toroidal Lie algebras and their central quotients, J. Algebra Appl. 1 (2002) 113–121.
 [9] D.H. Peterson, V. Kac, Infinite flag varieties and conjugacy theorems, Proc. Natl. Acad. Sci. USA 80 (1983) 1778–1782.
[10] M. Demazure, A. Grothendieck, Schémas en groupes, in: Séminaire de Géométrie algébrique de l'I.H.E.S., in: Lecture Notes in Math., vol. 151–153, Springer, 1970.
[11] R. Steinberg, Générateurs, relations et revétements des groupes algébriques, in: Colloque sur la théorie des groupes algébriques, Bruxelles, 1962, Gauthier-Villars, Paris, 1962.
[12] R. Steinberg, Lectures on Chevalley Groups, Yale University, 1967.