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Minkowski decomposition of Okounkov bodies on surfaces



Patrycja Łuszcz-Świdecka^a, David Schmitz^{b,*,1}

^a *Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-438 Kraków, Poland*

^b *Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany*

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ABSTRACT

We prove that the Okounkov body of a big divisor with respect to a general flag on a smooth projective surface whose pseudo-effective cone is rational polyhedral decomposes as the Minkowski sum of finitely many simplices and line segments arising as Okounkov bodies of nef divisors.

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1. Introduction

The construction of Okounkov bodies associated with linear series on a projective variety, which was introduced by Okounkov and was given a theoretical framework in the

* Corresponding author.

E-mail addresses: paczik@gmail.com (P. Łuszcz-Świdecka), schmitzd@mathematik.uni-marburg.de (D. Schmitz).

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seminal papers [7] and [9], recently attracted attention as it encodes plenty of information on geometric properties of line bundles. For example, the volume of a big linear series essentially agrees with the Euclidean volume of its associated Okounkov body.

Okounkov's idea is to assign to a big divisor D on a smooth projective n -dimensional variety X a convex body $\Delta(D)$ in n -dimensional Euclidean space \mathbb{R}^n . The construction, which we sketch in Section 2, depends on the choice of a flag of subvarieties $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$ of codimensions i such that Y_n is a non-singular point on each of the Y_i .

In [9, Theorem B], Lazarsfeld and Mustață prove the existence of a *global Okounkov body*: for a smooth projective variety there is a closed convex cone $\Delta(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ such that the fiber over any big rational class $\xi \in N^1(X)_{\mathbb{R}}$ of the map φ induced by the second projection is equal to $\Delta(\xi)$. Additionally, in order to establish the log-concavity relation

$$\mathrm{vol}_X(D_1 + D_2)^{1/n} \geq \mathrm{vol}_X(D_1)^{1/n} + \mathrm{vol}_X(D_2)^{1/n}$$

for any two big \mathbb{R} -divisors, they deduce from the convexity of the global Okounkov body the inclusion

$$\Delta(D_1) + \Delta(D_2) \subseteq \Delta(D_1 + D_2).$$

Here the left hand side denotes the Minkowski sum of $\Delta(D_1)$ and $\Delta(D_2)$, i.e., the set obtained by pointwise addition, see [9, Corollary 4.12].

In general the above inclusion turns out to be strict (see Example 4.2). However, it would be desirable to know conditions for equality; in particular one would hope to be able to decompose the Okounkov body of any big divisor as the Minkowski sum of “simple” bodies. Specifically, the following questions arise: is there a set Ω of big divisors such that the Okounkov body of any big divisor D with respect to an admissible flag Y_\bullet decomposes as Minkowski sum of the bodies associated with divisors in Ω ? If so, can Ω be chosen to be finite?

An affirmative answer to these questions was given in [10] in the case of the del Pezzo surface X_3 , the blow-up of the projective plane in three non-collinear points, equipped with a certain natural flag. We prove in this paper that the answers to both questions are “yes” for a general admissible flag (see Proposition 2.1) on any smooth projective surface whose pseudo-effective cone is rational polyhedral. For example, this is the case for all del Pezzo surfaces and, more generally, for surfaces with big anticanonical class (see [5, Lemma 3.4]). We will see in the following section that considering nef divisors is sufficient since the Okounkov body of any big divisor is a translate of the body associated with the positive part of its Zariski decomposition.

Theorem. *Let X be a smooth projective surface such that $\overline{\mathrm{Eff}}(X)$ is rational polyhedral, and let $X = Y_0 \supseteq Y_1 \supseteq Y_2 = \{pt\}$ be a general flag. Then there exists a finite set Ω of nef*

\mathbb{Q} -divisors such that for any nef \mathbb{Q} -divisor D there exist non-negative rational numbers $\alpha_P(D)$ such that

$$D = \sum_{P \in \Omega} \alpha_P(D)P \quad \text{and} \quad \Delta_{Y_\bullet}(D) = \sum_{P \in \Omega} \alpha_P(D)\Delta_{Y_\bullet}(P). \quad (1.0.1)$$

Definition. A presentation $D = \sum \alpha_i D_i$ as in (1.0.1) is called a *Minkowski decomposition* of D with respect to the *Minkowski basis* Ω .

The proof, which we present in Section 3, includes the construction of the Minkowski basis Ω as well as an effective method to determine a Minkowski decomposition of any given nef \mathbb{Q} -divisor. It depends on two features distinctive for surfaces, firstly a characterization of Okounkov bodies in terms of intersections with the positive and negative part in the Zariski decomposition due to Lazarsfeld and Mustață, and secondly on the Zariski chamber decomposition of the big cone introduced in [4]. We sketch these results in Section 2.

Throughout this paper we work over the complex numbers.

2. Okounkov bodies on surfaces

In this section we first give a quick review of Okounkov's construction in arbitrary dimension (we refer to [9] for details), and then turn to additional features known in the case of surfaces.

As mentioned in the introduction, one assigns to a big divisor D on a smooth projective n -dimensional variety X a convex body $\Delta(D)$ in \mathbb{R}^n . The construction depends on the choice of a flag on X , i.e., a sequence $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$ of subvarieties Y_i of codimension i . A flag is *admissible* if Y_n is a non-singular point on each of the Y_i . To an admissible flag, one assigns a function

$$\nu_{Y_\bullet} : H^0(X, \mathcal{O}_X(D)) \rightarrow \mathbb{Z}^n,$$

by mapping a section $s \in H^0(X, \mathcal{O}_X(D))$ to the tuple $(\nu_1(s), \dots, \nu_n(s))$ where $\nu_1(s) := \text{ord}_{Y_1}(s)$, $\nu_2(s)$ is given by the order of vanishing along Y_2 of the section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$ determined by s , and so forth up to $\nu_n(s)$. Repeating this construction for integral multiples of D , we define the Okounkov body $\Delta(D) = \Delta_{Y_\bullet}(D)$ to be the closed convex hull of the set

$$S(D) := \bigcup_{k \geq 0} \left\{ \frac{1}{k} \nu_{Y_\bullet}(s) \mid s \in H^0(X, \mathcal{O}_X(kD)) \right\}.$$

Note that although the number of image vectors (ν_1, \dots, ν_n) is equal to the dimension of $H^0(X, \mathcal{O}_X(kD))$ for each k , the convex body $\Delta(D)$ need not be polyhedral (see [9, Section 6.3]). By [9, Proposition 4.1], numerically equivalent divisors have identical

Okounkov bodies and for any positive integer p we have the scaling $\Delta(pD) = \frac{1}{p}\Delta(D)$, so we can assign an Okounkov body to big rational classes in the Néron–Severi vector space $N^1(X)_{\mathbb{R}}$. For non-rational classes this is not so straightforward. Instead, it follows from the existence of global Okounkov bodies [9, Theorem B]: There is a closed convex cone $\Delta(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ such that the fiber over any big rational class $\xi \in N^1(X)_{\mathbb{R}}$ of the map φ induced by the second projection is equal to $\Delta(\xi)$. Consequently, the Okounkov body of a big real class is defined as its fiber under φ . Additionally, since the image of $\Delta(X)$ under φ is the pseudo-effective cone $\overline{\text{Eff}}(X)$, the construction can be extended to pseudo-effective real classes.

From the existence of the global Okounkov body on X many interesting properties of the volume function $\text{vol}_X : \text{Big}(X) \rightarrow \mathbb{R}$ can quite easily be proved. For example, the log-concavity relation

$$\text{vol}_X(D_1 + D_2)^{1/n} \geq \text{vol}_X(D_1)^{1/n} + \text{vol}_X(D_2)^{1/n}$$

for any two big \mathbb{R} -divisors is a consequence of the Brunn–Minkowski theorem: from the convexity of the global Okounkov body we obtain the inclusion

$$\Delta(D_1) + \Delta(D_2) \subseteq \Delta(D_1 + D_2)$$

with the Minkowski sum on the left hand side (see [9, Corollary 4.12]).

For the remainder of this section, let X be a smooth projective surface with an admissible flag

$$X \supseteq C \supseteq \{p\}$$

on it. Any pseudo-effective (rational) divisor D on X has a Zariski decomposition

$$D = P_D + N_D,$$

where P_D is nef, and N_D is effective, orthogonal to P_D , and if it is not the zero-divisor, it has negative definite intersection matrix. Define

$$\mu_C(D) := \sup\{t \mid D - tC \text{ effective}\}$$

and consider the functions

$$\alpha, \beta : [0, \mu_C(D)] \rightarrow \mathbb{R},$$

with

$$\begin{aligned} \alpha(x) &= \text{ord}_p(N_{D-xC}), \quad \text{and} \\ \beta(x) &= \text{ord}_p(N_{D-xC}) + (C \cdot (P_{D-xC})). \end{aligned}$$

Then by [9, Theorem 6.4], α and β are the upper and lower boundary functions for $\Delta(D)$, respectively. Concretely,

$$\Delta(D) = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \mu_C(D), \alpha(x) \leq y \leq \beta(x)\}.$$

This characterization of the Okounkov body on surfaces has many pleasant consequences. For example in [1, Proposition 2.2] it is used to prove that for any big and nef divisor on a surface there exists a flag such that the corresponding Okounkov body is rational polyhedral. The following proposition shows that in the situation of the theorem, in order to determine the Okounkov body of a big divisor D it is sufficient to know the positive part of the divisors $D - tC$ for $0 \leq t \leq \mu_C(D)$.

Proposition 2.1. *If the pseudo-effective cone $\overline{\text{Eff}}(X)$ is rational polyhedral and $X \supseteq C \supseteq p$ is a general admissible flag, then C is big and nef as a divisor, and*

$$\alpha(x) = 0, \quad \beta(x) = C \cdot P_{D-xC}$$

for all $0 \leq x \leq \mu_C(D)$.

Proof. If $\overline{\text{Eff}}(X)$ on X is rational polyhedral, then in particular there are only finitely many irreducible curves E on X with self-intersection $E^2 \leq 0$. Therefore, in a general flag $X \supseteq C \supseteq p$ the irreducible curve C has positive self-intersection, so it is big and nef as a divisor. Furthermore, p is a non-singular point on C , which does not lie on any curve with negative self-intersection. Now by definition, the negative part N_{D-xC} in the Zariski decomposition of $D - xC$ either is the zero-divisor, or has negative definite intersection matrix. In the latter case, its support consists of curves with negative self-intersection, so in either case we have $\text{ord}_p(N_{D-xC}) = 0$ for all x . \square

Example 2.2. For any $0 \leq t \leq 1$ the class $C - tC$ is nef and effective, hence $P_{C-tC} = C - tC$. So by the proposition, $\Delta(C)$ is the simplex of height C^2 and length 1.

Remark. By [10, Corollary 2.2] the Okounkov body of a big divisor D with respect to a flag $X \supseteq C \supseteq p$ such that C is not a component of N_D is a translate by the vector $(0, \text{ord}_p(N_D))$. In particular, by the above proof, for a general flag on a surface with rational polyhedral pseudo-effective cone, the Okounkov bodies of any big divisor and of its positive part coincide.

Recall that by the main result of [4] on a smooth projective surface there exists a locally finite decomposition of $\text{Big}(X)$ into locally polyhedral subcones, the so-called *Zariski chambers*, such that

- the support of negative parts of divisors is constant on each chamber,
- the volume function $\text{vol}_X(\cdot)$ varies polynomially on the chambers, and
- on the interior of each chamber the augmented base loci \mathbb{B}_+ are constant.

The basic idea of [4] is to consider for a big and nef divisor P the set

$$\Sigma_P := \{D \in \text{Big}(X) \mid \text{Neg}(D) = \text{Null}(P)\},$$

where $\text{Neg}(D)$ denotes the support of N_D and $\text{Null}(P)$ is the set of irreducible curves orthogonal to P with respect to the intersection product. These sets give a decomposition of $\text{Big}(X)$ obviously satisfying the first property in the above list, while proving the remaining properties as well as local finiteness still requires quite an effort. For an explicit description of chambers, passing to closures in [4, Proposition 1.10] we obtain the identity

$$\overline{\Sigma}_P = \text{convex hull}(\text{Nef}(X) \cap \text{Null}(P)^\perp, \text{Null}(P)), \quad (2.2.1)$$

from which we deduce the following useful statement about positive parts.

Proposition 2.3. *Let P be a big and nef divisor on X with corresponding Zariski chamber Σ_P . Then for all $D_1, D_2 \in \overline{\Sigma}_P$ we have*

$$P_{D_1+D_2} = P_{D_1} + P_{D_2},$$

i.e., the positive parts of the Zariski decompositions vary linearly on the closure of each Zariski chamber.

Proof. Let $D_1 = P_1 + \sum_{i=1}^s \alpha_i N_i$ and $D_2 = P_2 + \sum_{i=1}^s \beta_i N_i$ be representations corresponding to (2.2.1) with $\alpha_i, \beta_i \geq 0$, $N_i \in \text{Null}(P)$, and P_1, P_2 nef. Clearly, $P_1 + P_2$ is nef and has intersection product zero with the N_i . Furthermore, the divisor $\sum_{i=1}^s (\alpha_i + \beta_i) N_i$ is effective and has negative definite intersection matrix. Thus

$$D_1 + D_2 = (P_1 + P_2) + \sum_{i=1}^s (\alpha_i + \beta_i) N_i$$

is the Zariski decomposition. \square

3. Minkowski decomposition

In this section we prove the main theorem. Fix throughout a general admissible flag $Y_\bullet : X \supseteq C \supseteq p$ on a smooth projective surface X .

As stated in the introduction, the starting point for this investigation was the observation from [9] that for any two pseudo-effective divisors D_1, D_2 we have the inclusion

$$\Delta(D_1) + \Delta(D_2) \subseteq \Delta(D_1 + D_2).$$

This inclusion turns out to be strict in general. We refer to [10] for examples.

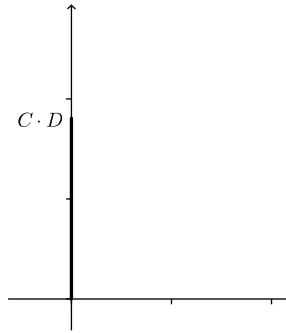


Fig. 1. The Okounkov body $\Delta(D)$.

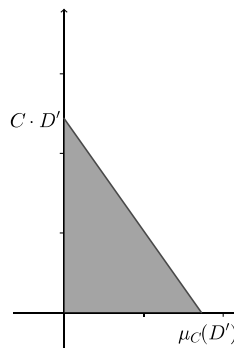


Fig. 2. The Okounkov body $\Delta(D')$.

On the other hand, one observes that the Okounkov body of a pseudo-effective divisor D with respect to Y_\bullet can always be decomposed as the Minkowski sum of finitely many simplices and line segments. ($\Delta(D)$ is the area of the upper right quadrant bounded by the piecewise linear, concave function β .) The question then is: do these elementary “building blocks” come up as Okounkov bodies themselves? As the theorem shows, the answer is “yes”.

Before we prove the theorem, let us consider candidates for a Minkowski basis, i.e., nef divisors whose Okounkov bodies are of one of the elementary types mentioned above.

- For a nef divisor D with $D^2 = 0$, for positive t none of the divisors $D - tC$ is effective since C by [Proposition 2.1](#) is big and nef being the curve in a general admissible flag. Therefore, $\mu_C(D) = 0$, and $\Delta(D)$ is the vertical line segment of length $C \cdot D$ (see [Fig. 1](#)).
- If for a big and nef divisor D' all the classes $D' - tC$ for $0 < t < \mu_C(D')$ lie in the same Zariski chamber then by [Proposition 2.3](#) the positive parts $P_{D'-tC}$ vary linearly with t . Consequently, $\Delta(D')$ is the simplex of height $C \cdot D'$ and length $\mu_C(D')$ (see [Fig. 2](#)).

We now turn to the proof of the theorem. It consists of two parts: we first construct the set Ω and then show how to find the presentation of any big and nef divisor D in terms of elements of Ω which yields the Minkowski decomposition of D .

Remark. Effective representations of a nef divisor in terms of the Minkowski basis are not unique. It is possible that such a representation is *not* a Minkowski decomposition (see [Example 4.2](#)). This is why the second part of the proof is important as it shows how to pick the right decomposition.

3.1. Construction of a Minkowski basis

In the Zariski chamber decomposition of the big cone $\text{Big}(X)$ we assign to each chamber an element of Ω as follows. Writing $\{N_1, \dots, N_s\}$ for the set of curves in the support of negative parts of divisors in a chamber Σ , we define the “corresponding Minkowski basis element” M as follows: Consider the linear subspace of $N^1(X)_{\mathbb{R}}$ spanned by C together with the classes of the curves N_i . Its intersection with the subspace $N_1^{\perp} \cap \dots \cap N_s^{\perp}$ is a rational line, spanned by some integral divisor $M = dC + \sum \alpha_i N_i$. We will argue that d and the α_i all have the same signs, and we conclude that either M or $-M$ is nef.

The intersection matrix S of the divisor $\sum N_i$ is negative definite with non-negative entries outside the diagonal. By the auxiliary result [[4, Lemma 4.1](#)] (see also [[2, Lemma A.1](#)]), the inverse matrix S^{-1} has only negative entries. Therefore, and since $CN_i \geq 0$, the solution to the system of the equations

$$S \cdot (\alpha_1, \dots, \alpha_s)^t = -d(CN_1, \dots, CN_s)^t \quad (3.0.1)$$

for fixed d is a vector $(\alpha_1, \dots, \alpha_s)$ whose entries have the same sign as d . Fix a positive integral solution and set $M = dC + \sum \alpha_i N_i$. Note that since M lies in $N_1^{\perp} \cap \dots \cap N_s^{\perp}$ it is nef by the positivity of its coefficients and the nefness of C . Furthermore, it lies in the closure of Σ , or more concretely in the closure of the face $\overline{\Sigma} \cap \text{Nef}(X)$.

Remark. Note that in the above construction different chambers can have the same corresponding Minkowski basis element. For example, on the del Pezzo surface X_2 with standard basis H, E_1, E_2 and with a flag such that C has class $H = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ the chambers Σ_H , Σ_{2H-E_1} , and Σ_{2H-E_2} have $M = H$.

Note also that the corresponding basis element to the nef cone is always C .

We can now describe the Minkowski basis Ω : it consists of the divisors M_{Σ} constructed above together with one integral representative for each ray of the nef cone not contained in $\text{Big}(X)$.

Note that since $\overline{\text{Eff}}(X)$ is rational polyhedral the set Ω is finite. Note furthermore that the divisors in Ω have Okounkov bodies which cannot be decomposed as Minkowski sums, i.e., in a sense the set Ω is minimal: By construction, for all $0 < t < \mu_C(M) = d$

the class $M_\Sigma - tC$ lies in the cone spanned by $\text{Nef}(X) \cap N_1^\perp \cap \dots \cap N_s^\perp$ and the N_1, \dots, N_s , i.e., in the closure of the Zariski chamber Σ . Therefore, the positive part of $M_\Sigma - tC$ varies linearly, so $\Delta(M_\Sigma)$ is the simplex of height $C \cdot M$ and length d , whereas the other basis elements D_i lie in the boundary of $\overline{\text{Eff}}(X)$, so $\mu_C(D_i) = 0$ which means that the corresponding Okounkov body is the vertical line segment of length $C \cdot D_i$.

3.2. Algorithmic construction of Minkowski decompositions

To complete the proof we now describe how to find the Minkowski decomposition of a given nef divisor D .

If D is not big, then $D^2 = 0$ and Ω contains some positive multiple $D' = \beta D$. Thus

$$\Delta(D) = \frac{1}{\beta} \Delta(D'),$$

and we are done.

Otherwise, consider the Zariski chamber Σ corresponding to the big and nef divisor D . Let M be the corresponding Minkowski basis element and set

$$\tau := \sup\{t \mid D - tM \text{ nef}\}.$$

Since nefness is defined by finitely many linear conditions, τ is rational. The nef \mathbb{Q} -divisor $D' := D - \tau M$ lies on the boundary of the face $\text{Nef}(X) \cap \text{Null}(D)$. If $D' = 0$, we are done.

Otherwise, we claim that

$$\Delta(D) = \tau \Delta(M) + \Delta(D'), \quad (3.0.2)$$

so $\Delta(D)$ decomposes into the elementary part $\tau \Delta(M)$ and the Okounkov body of the divisor D' .

For the proof we first note that by construction of the Minkowski basis, M lies, like D' , on the boundary of the Zariski chamber Σ . Furthermore, as we have seen above, the divisors $M - tC$ lie in the closure of the chamber Σ for $0 < t < \mu_C(M)$. Thus by [Proposition 2.3](#) we have

$$P_{D-xC} = P_{D'} + P_{\tau M-xC} = D' + P_{\tau M-xC}$$

for $0 \leq x \leq \mu_C(\tau M)$.

For the remaining $\mu_C(\tau M) \leq x \leq \mu_C(D)$ let \widetilde{M} denote the divisor $\tau M - \mu_C(\tau M)C$ (which is just $\tau(\sum \alpha_i N_i)$ in the above notation). We claim that for any $t > 0$ we have the inclusions

$$\text{supp}(\widetilde{M}) \subseteq \text{Null}(D) \subseteq \text{Null}(D') = \mathbb{B}_+(D') \subseteq \mathbb{B}_+(D' - tC) = \text{Null}(P_{D'-tC}).$$

The two equalities are given by [6, Examples 1.10 and 1.11] respectively. The first inclusion is clear since the N_i are contained in $\text{Null}(D)$. The second one follows from the fact that D' is contained in the boundary of the face of the nef cone containing D , while the last inclusion is a direct consequence of the fact that subtracting a nef divisor can only augment the base locus.

Note that in general for a big divisor E with Zariski decomposition $E = P_E + N_E$ and an effective divisor F with support contained in $\text{Null}(P_E)$ the decomposition

$$E + F = P_E + (N_E + F)$$

is the Zariski decomposition: P_E is nef, has trivial intersection with all components of $(N_E + F)$, and the latter divisor has negative definite intersection matrix. In other words, adding an effective divisor F with support contained in $\text{Null}(P_D)$ does not alter the positive part.

Taking in the above consideration E and F to be $D - xC$ and \widetilde{M} respectively, we obtain the identity

$$P_{D-xC} = P_{D'-(x-\mu_C(\tau M))C}$$

for $\mu_C(\tau M) \leq x \leq \mu_C(D)$. Putting the two decompositions of positive parts together, we get

$$\beta_D(x) = \begin{cases} \beta_{\tau M}(x) + C \cdot D', & 0 \leq x \leq \mu_C(\tau M), \\ \beta_{D'}(x - \mu_C(\tau M)), & \mu_C(\tau M) \leq x \leq \mu_C(D), \end{cases}$$

which amounts to the claimed identity (3.0.2).

Repeat the above procedure with the divisor D' . This is possible because if D' is big and nef, it defines a Zariski chamber Σ with $M_\Sigma \neq M$, which can be seen as follows: if it were not the case, we would have $\text{Null}(D') \subseteq \text{Null}(M)$, but then it follows from $D = M + D'$ that $\text{Null}(D') \subseteq \text{Null}(D)$, which is impossible. The algorithm terminates after at most ρ steps, since in every step the dimension of the face of the nef cone in which D lies decreases. Eventually, we end up with either 0 or a divisor spanning an extremal ray of the nef cone. Such a divisor has a multiple in Ω , and we are done.

Note that in order to determine the Minkowski decomposition of a given divisor D it is not necessary to know the whole Minkowski basis of X . Instead in every step the necessary basis element can be found based on knowledge of the intersection matrix of $\text{Null}(D)$ alone. In fact, the algorithm can be implemented for automated computation, provided the intersection matrix of C together with the negative curves on X is known.

4. Del Pezzo surfaces

On a del Pezzo surface X the pseudo-effective cone is rational polyhedral by the cone theorem. Concretely, it is spanned by rational curves of self-intersection -1 . The surface

X is either \mathbb{P}^2 , its blow-up X_r in up to 8 general points, or $\mathbb{P}^1 \times \mathbb{P}^1$. A complete list of the (-1) -curves on the X_r is well known [11, 8 Chapt. IV] (cf. [3, Theorem 3.1] for an elementary proof): they are the exceptional curves E_1, \dots, E_r together with the strict transforms of

- lines through two of the p_i ,
- irreducible conics through five of the p_i , if $r \geq 5$,
- irreducible cubics through six of the p_i with a double point in one of them, if $r \geq 7$,
- irreducible quartics through the eight points p_i with a double point in three of them, if $r \geq 8$,
- irreducible quintics through the eight points p_i with a double point in six of them, if $r \geq 8$,
- irreducible sextics through the eight points p_i with a double point in seven of them, and a triple point in one of them, if $r \geq 8$.

A general flag on X_r consists of an irreducible curve C with a general point p on it where C is the strict transform of an irreducible member of the class $\mathcal{O}_{\mathbb{P}^2}(k)$ for some $k > 0$. We consider the case $k = 1$ (the others work analogously) and write as usual H for the class of C . Let us construct a Minkowski basis for X_r . Starting with any chamber Σ , we consider $\text{Neg}(\Sigma) = \{N_1, \dots, N_s\}$, the support of negative parts of the divisors in Σ . Its intersection matrix, being negative definite with diagonal entries -1 , can have only zero entries outside the diagonal. In particular, we can immediately read off the basis element $M(\Sigma)$ from the system of Eqs. (3.0.1): Setting $d = 1$, we obtain $\alpha_i = N_i \cdot H$ for all i , hence we have

$$M(\Sigma) = H + \sum_{i=1}^s (N_i \cdot H) N_i.$$

Let us determine the Okounkov bodies of this Minkowski basis element. It is clear that $\mu_H(M(\Sigma)) = 1$, since $\sum_{i=1}^s (N_i \cdot H) N_i$ lies on the boundary of $\overline{\text{Eff}}(X_r)$. On the other hand, setting

$$\lambda := H \cdot \left(H + \sum_{i=1}^s (N_i \cdot H) N_i \right),$$

by the argumentation in the proof of the theorem, $\Delta(M(\Sigma))$ is the simplex of height λ and length 1, which we denote by $\Delta(\lambda, 1)$. The remaining elements of Ω are curves E with self-intersection $E^2 = 0$. As we have seen above, their Okounkov body is the vertical line segment of length $H \cdot E$. The following statement thus is a direct consequence of the theorem.

Table 1Zariski chambers and corresponding Minkowski basis elements on X_6 .

$\text{Neg}(\Sigma)$	$M(\Sigma)$
E_1, \dots, E_s	H
$L_{1,2}, \dots, L_{1,1+s}, E_{s+1}, \dots, E_{s+t}$	$(s+1)H - sE_1 - E_2 - \dots - E_s$
$L_{1,2}, L_{1,3}, L_{2,3}, E_4, \dots, E_{4+t}$	$4H - 2E_1 - 2E_2 - 2E_3$
$C_1, L_{2,3}, \dots, L_{2,2+s}, (E_1)$	$(5+s)H - (2+s)E_2 - 3E_3 - \dots - 3E_{s+2} - 2E_{s+3} - \dots - 2E_6$
$C_1, L_{2,3}, L_{2,4}, L_{3,4}, (E_1)$	$8H - 4E_2 - 4E_3 - 4E_4 - 2E_5 - \dots - 2E_6$
$C_1, C_2, L_{3,4}, \dots, L_{3,3+s}$	$(9+s)H - 2E_1 - 2E_2 - (s+4)E_3 - 5E_4 - \dots - 5E_{3+s} - 4E_{4+s} - \dots - 4E_r$
$C_1, C_2, L_{3,4}, L_{3,5}, L_{4,5}$	$12H - 2E_1 - 2E_2 - 6E_3 - 6E_4 - 6E_5 - 4E_6$

Proposition 4.1. *On a del Pezzo surface X_r , for any big divisor $D \subseteq X_r$ the function $\beta(x)$ bounding the Okounkov body is piecewise linear with integer slope on each linear piece.*

For a concrete calculation, consider the del Pezzo surface X_6 . Up to permutation of the E_i , we have the possible supports for Zariski chambers with corresponding basis elements displayed in Table 1 in the standard basis H, E_1, \dots, E_r , with $L_{i,j}$, C_1 , C_2 denoting the (-1) -curves coming from lines and conics, respectively.

The additional Minkowski basis elements (corresponding to non-big nef classes) are the strict transforms of

- lines through one of the p_i ,
- irreducible conics through four of the p_i .

We thus get the following elementary bodies as building blocks for the Okounkov body of any big divisor on X_6 :

$$\Delta(1,1), \dots, \Delta(12,1), \Delta(1,0), \Delta(2,0).$$

Example 4.2. Consider the divisor $D = 7H - 2E_1 - E_2 - 3E_3 - 2E_4 - 2E_5$ on X_6 .

- For $D_1 = D = 7H - 2E_1 - E_2 - 3E_3 - 2E_4 - 2E_5$, we find $\text{Null}(D) = \{E_6\}$, so $M(D) = H$. With $\tau = 2$ we get $D_2 = 5H - 2E_1 - E_2 - 3E_3 - 2E_4 - 2E_5$.
- Now, $\text{Null}(D_2) = \{C_6, L_{1,3}, L_{3,4}, L_{3,5}, E_6\}$, so $M(D) = 8H - 3E_1 - 2E_2 - 5E_3 - 3E_4 - 3E_5$. With $\tau = \frac{1}{2}$ we get $D_3 = H - \frac{1}{2}E_1 - \frac{1}{2}E_3 - \frac{1}{2}E_4 - \frac{1}{2}E_5$.
- Then $D_3^2 = 0$, so we are done.

Consequently, the Okounkov body of D is given as the Minkowski sum

$$\Delta(D) = \Delta(2,2) + \frac{1}{2}\Delta(8,1) + \Delta(1,0)$$

depicted in Fig. 3.

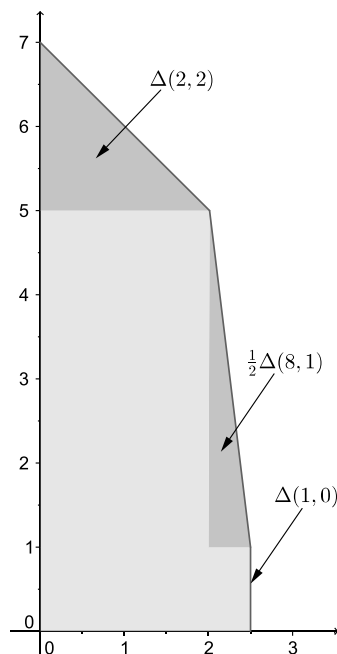


Fig. 3. The Okounkov body $\Delta(D)$ as a Minkowski sum.

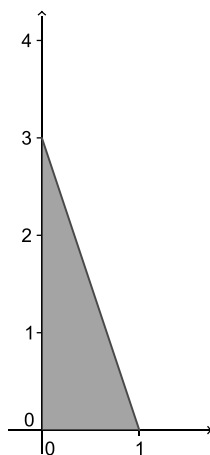
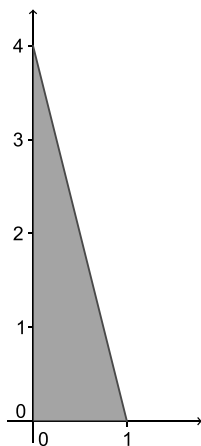


Fig. 4. $\Delta(3H - 2E_1 - E_2 - E_3)$.

Note on the other hand that we have the identity

$$D = (3H - 2E_1 - E_2 - E_3) + (4H - 2E_3 - 2E_4 - 2E_5)$$

and both summands are Minkowski basis elements. Clearly, this representation cannot be a Minkowski decomposition (see [Figs. 4 and 5](#)).

Fig. 5. $\Delta(4H - 2E_3 - 2E_4 - 2E_5)$.

5. Non-del-Pezzo examples

- For a simple non-del-Pezzo example, let $\pi : X \rightarrow \mathbb{P}^2$ be the blow-up of 3 points on a line with exceptional divisors E_1, E_2, E_3 . Choose C general in the class $H := \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and $p \in C$ a general point. This gives a flag as above. The pseudo-effective cone is spanned by the exceptional divisors together with the class $D := H - E_1 - E_2 - E_3$ of the strict transform of the line joining the blown up points. We have 12 Zariski chambers: the nef chamber, the 7 chambers belonging to principal submatrices of the intersection matrix of $E_1 + E_2 + E_3$, the one corresponding to D , and three chambers with support D together with one of the exceptional divisors. The corresponding Minkowski basis element is H for the first 8 chambers, $3H - E_1 - E_2 - E_3$ for the 9th, and $2H - E_i - E_j$ for last three. The remaining elements of Ω are $H - E_1, H - E_2, H - E_3$. Let's calculate the decomposition for the arbitrarily chosen divisor $P = 15H - 3E_1 - 3E_2 - E_3$.

- The divisor P is ample, so $M = H$; with $\tau = 8$ we get $P_1 = 7H - 3E_1 - 3E_2 - E_3$.
- Now, $\text{Null}(P_1) = D$, so $M_\Sigma = 3H - E_1 - E_2 - E_3$; with $\tau = 1$ we get $P_2 = 4H - 2E_1 - 2E_2$.
- In the next step, $\text{Null}(P_2) = \{D, E_3\}$, so $M_\Sigma = 2H - E_1 - E_2$; with $\tau = 2$, we get $P_3 = 0$, and we are done.

Thus we get the decomposition

$$P = 8 \cdot H + (3H - E_1 - E_2 - E_3) + 2 \cdot (2H - E_1 - E_2)$$

with corresponding Minkowski decomposition of the Okounkov body

$$\begin{aligned} \Delta(P) &= 8\Delta(H) + \Delta(3H - E_1 - E_2 - E_3) + 2\Delta(2H - E_1 - E_2) \\ &= \Delta(8, 8) + \Delta(3, 1) + \Delta(4, 2). \end{aligned}$$

2. (K3-surface)

For an example of a surface which is not a blow-up of \mathbb{P}^2 let us consider a K3-surface. As Kovács proves in [8], for any $1 \leq \rho \leq 19$ there exists a K3-surface X with Picard number ρ whose pseudo-effective cone is rational polyhedral, spanned by the classes of finitely many rational (-2) -curves. We consider a certain K3-surface of this type: It was proved in [2, Proposition 3.3] that there exists a K3-surface X with Picard number 3 such that the pseudo-effective cone is spanned by three (-2) -curves L_1 , L_2 , D forming a hyperplane section $L_1 + L_2 + D$ such that L_1 and L_2 are lines and D is an irreducible conic. The hyperplane section $L_1 + L_2 + D$ has intersection matrix

$$\begin{pmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}.$$

Therefore, the Zariski chamber decomposition consists of five chambers, namely the nef chamber, one chamber corresponding to each of the (-2) -curves D , L_1 , L_2 , and one chamber with support $L_1 + L_2$. Pick C to be an irreducible curve with class $L_1 + L_2 + D$, i.e., a general hyperplane section, and p to be a point in C not on L_1 , L_2 , and D . Then the Minkowski basis elements corresponding to the above list of chambers are C , $3L_1 + 2L_2 + 2D$, $2L_1 + 3L_2 + 2D$, $L_1 + L_2 + 2D$, and $2L_1 + 2L_2 + D$. In addition, the Minkowski basis Ω contains the curves $L_1 + D$ and $L_2 + D$ of self-intersection zero. Thus, by the theorem, the building blocks of Okounkov bodies of nef divisors on X are

$$\Delta(4, 1), \Delta(9, 2), \Delta(6, 1), \Delta(3, 0).$$

In particular, in contrast to the del Pezzo case, the slope of a linear piece of the bounding function β need not be integral for K3-surfaces.

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