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Fusion rules for the vertex operator algebra $V_{L_2}^{A_4}$

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The fusion rules for vertex operator algebra $V_{L_2}^{A_4}$ are determined.

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1. Introduction

The classification of rational vertex operator algebras with central charge $c = 1$ has advanced a lot during the last few years. Let $V = \bigoplus_{n\geq 0} V_n$ be a rational vertex operator algebra with $c = 1$ and $\dim V_0 = 1$. If $V_1 \neq 0$ then V is a lattice vertex operator algebra [22]. If $V_1 = 0$ and $\dim V_4 \geq 3$, V is isomorphic to V_L^+ where L is a rank one positive definite even lattice [37,9–11]. The remaining problem in the classification of rational vertex operator algebras with central charge $c = 1$ is the characterization of $V_{L_2}^G$, where $L_2 = \mathbb{Z}\alpha$ with $(\alpha, \alpha) = 2$, G is a subgroup of $SO(3)$ isomorphic to A_4, S_4, A_5. The vertex operator algebra $V_{L_2}^G$ has not been fully understood. In the case $G = A_4$, the rationality, C_2-cofiniteness and classification of irreducible modules of $V_{\mathbb{Z}\alpha}^{A_4}$ have been established in [12]. In this paper, we determine the fusion rules for $V_{L_2}^{A_4}$. A characterization of $V_{L_2}^{A_4}$ has been recently given in [13].

One important tool in the determination of fusion rules is the quantum dimension of a module over a vertex operator algebra which has been studied systematically in [15]. For a rational, C_2-cofinite, self-dual vertex operator algebra of CFT type, quantum dimensions of its irreducible modules have nice properties. In particular, the product of quantum dimensions of two modules is equal to the quantum dimension of the fusion product of the modules. It turns out that this is very helpful in determining fusion rules. It has been proved in [12] that the vertex operator algebra is rational, C_2-cofinite, self-dual vertex operator algebra of CFT type. So we can apply the results in [15] on quantum dimensions to the vertex operator algebra $V_{L_2}^{A_4}$. The fusion rules for the most cases can be determined by using the quantum dimensions. For some fusion rules involving irreducible $V_{L_2}^{A_4}$-modules occurring in some twisted sectors, we need to find out the corresponding S-matrix and use Verlinde formula to determine the remaining fusion rules.

The paper is organized as follows: In Section 2, we give some basic definitions. In Section 3, we recall the vertex operator algebra $V_{L_2}^{A_4}$ and give the realization of all irreducible modules of $V_{L_2}^{A_4}$. We compute the quantum dimensions of the irreducible $V_{L_2}^{A_4}$-modules in Section 4. The fusion rules for irreducible $V_{L_2}^{A_4}$-modules are obtained in Section 5. The portion of S-matrix that we need is listed in Appendix A.

2. Basics

Let $(V, Y, 1, \omega)$ be a vertex operator algebra (see [28]) and g an automorphism of V of finite order T. Denote the decomposition of V into eigenspaces of g as:

$$V = \bigoplus_{r\in\mathbb{Z}/T\mathbb{Z}} V^r,$$

where $V^r = \{v \in V \mid gv = e^{2\pi ir/T} v\}$. Now we recall notions of twisted modules for vertex operator algebras. Let $W\{z\}$ denote the space of W-valued formal series in arbitrary complex powers of z for a vector space W.

Definition 2.1. A *weak g-twisted V-module M* is a vector space with a linear map

$$Y_M : V \to (\operatorname{End} M)\{z\}$$
$$v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M)$$

which satisfies the following: for all $0 \le r \le T-1$, $u \in V^r$, $v \in V$, $w \in M$,

$$Y_M(u,z) = \sum_{n\in \frac{r}{T}+\mathbb{Z}} u_n z^{-n-1},$$
$$u_l w = 0 \quad \text{for } l \gg 0,$$
$$Y_M(\mathbf{1},z) = Id_M,$$
$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(v,z_2)Y_M(u,z_1)$$
$$z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r/T}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M\big(Y(u,z_0)v,z_2\big),$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Definition 2.2. A *g-twisted V-module* is a weak g-twisted V-module M which carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$ where $L(0)$ is one of the coefficient operators of $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. That is, we have $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. Moreover we require that $\dim M_\lambda$ is finite and for fixed λ, $M_{\frac{n}{T}+\lambda} = 0$ for all small enough integers n.

Definition 2.3. An *admissible g-twisted V-module* $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)$ is a $\frac{1}{T}\mathbb{Z}_+$-graded weak g-twisted module such that $u_m M(n) \subset M(\operatorname{wt} u - m - 1 + n)$ for homogeneous $u \in V$ and $m, n \in \frac{1}{T}\mathbb{Z}$.

If $g = Id_V$ we have the notions of weak, ordinary and admissible V-modules [18].

Definition 2.4. A vertex operator algebra V is called *g-rational* if the admissible g-twisted module category is semisimple. V is called *rational* if V is 1-rational.

The following lemma about g-rational vertex operator algebras is well known [18].

Lemma 2.5. *If V is g-rational and M is an irreducible admissible g-twisted V-module, then*

(1) *M is a g-twisted V-module and there exists a number $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M_{\lambda+n}$ where $M_\lambda \neq 0$. The number λ is called the conformal weight of M;*
(2) *There are only finitely many irreducible admissible g-twisted V-modules up to isomorphism.*

Definition 2.6. We say that a vertex operator algebra V is C_2-*cofinite* if $V/C_2(V)$ is finite dimensional, where $C_2(V) = \langle v_{-2}u \mid v, u \in V\rangle$.

Remark 2.7. If V is a vertex operator algebra satisfying C_2-cofinite property, V has only finitely many irreducible admissible modules up to isomorphism [18,31].

Definition 2.8. Let $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)$ be an admissible g-twisted V-module, the *contragredient module* M' is defined as follows:

$$M' = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n)^*,$$

where $M(n)^* = \mathrm{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$. The vertex operator $Y_{M'}(v, z)$ is defined for $v \in V$ via

$$\langle Y_{M'}(v, z)f, u\rangle = \langle f, Y_M\big(e^{zL(1)}\big(-z^{-2}\big)^{L(0)}v, z^{-1}\big)u\rangle,$$

where $\langle f, w\rangle = f(w)$ is the natural paring $M' \times M \to \mathbb{C}$.

Remark 2.9. 1. $(M', Y_{M'})$ is an admissible g^{-1}-twisted V-module [27].

2. We can also define the contragredient module M' for a g-twisted V-module M. In this case, M' is a g^{-1}-twisted V-module. Moreover, M is irreducible if and only if M' is irreducible.

Now we review the notions of intertwining operators and fusion rules from [27].

Definition 2.10. Let (V, Y) be a vertex operator algebra and let (W^1, Y^1), (W^2, Y^2) and (W^3, Y^3) be V-modules. An *intertwining operator* of type $\binom{W^3}{W^1\ W^2}$ is a linear map

$$I(\cdot, z) : W^1 \to \mathrm{Hom}\big(W^2, W^3\big)\{z\}$$
$$u \to I(u, z) = \sum_{n\in\mathbb{Q}} u_n z^{-n-1}$$

satisfying:

(1) for any $u \in W^1$ and $v \in W^2$, $u_n v = 0$ for n sufficiently large;
(2) $I(L(-1)v, z) = (\frac{d}{dz})I(v, z)$;
(3) (Jacobi identity) for any $u \in V$, $v \in W^1$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y^1(u, z_1)I(v, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)I(v, z_2)Y^3(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)I\big(Y^2(u, z_0)v, z_2\big).$$

The space of all intertwining operators of type $\binom{W^3}{W^1\ W^2}$ is denoted by

$$I_V\begin{pmatrix} & W^3 & \\ W^1 & & W^2\end{pmatrix}.$$

Let $N_{W^1,W^2}^{W^3} = \dim I_V\binom{W^3}{W^1\ W^2}$. These integers $N_{W^1,W^2}^{W^3}$ are usually called the *fusion rules*.

Definition 2.11. Let V be a vertex operator algebra, and W^1, W^2 be two V-modules. A module (W, I), where $I \in I_V\binom{W}{W^1\ W^2}$, is called a *tensor product* (or fusion product) of W^1 and W^2 if for any V-module M and $\mathcal{Y} \in I_V\binom{M}{W^1\ W^2}$, there is a unique V-module homomorphism $f : W \to M$, such that $\mathcal{Y} = f \circ I$. As usual, we denote (W, I) by $W^1 \boxtimes_V W^2$.

The basic result is that the fusion product exists if V is rational. It is well known that if V is rational, for any two irreducible V-modules W^1, W^2,

$$W^1 \boxtimes_V W^2 = \sum_W N_{W^1,W^2}^{W} W$$

where W runs over the set of equivalence classes of irreducible V-modules.

It is well known that fusion rules have the following symmetric property [27].

Proposition 2.12. *Let W^i $(i = 1, 2, 3)$ be V-modules. Then*

$$N_{W^1,W^2}^{W^3} = N_{W^2,W^1}^{W^3}, \qquad N_{W^1,W^2}^{W^3} = N_{W^1,(W^3)'}^{(W^2)'}.$$

Now we recall some notions about quantum dimensions.

Definition 2.13. Let $M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M_{\lambda+n}$ be a g-twisted V-module, the *formal character* of M is defined as

$$\operatorname{ch}_q M = \operatorname{tr}_M q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n\in\frac{1}{T}\mathbb{Z}_+} (\dim M_{\lambda+n}) q^n,$$

where c is the central charge of the vertex operator algebra V and λ is the conformal weight of M.

It is proved [36,19] that $\operatorname{ch}_q M$ converges to a holomorphic function in the domain $|q| < 1$. We denote the holomorphic function $\operatorname{ch}_q M$ by $Z_M(\tau)$. Here and below, τ is in the upper half plane $\mathbb{H}$ and $q = e^{2\pi i\tau}$.

Let $M^0, \cdots, M^d$ be the inequivalent irreducible V-modules with corresponding conformal weights λ_i and $M^0 \cong V$. Define

$$Z_i(u,v,\tau) = \operatorname{tr}_{M^i} e^{2\pi i(v(0))+(v,u)/2} q^{L(0)+u(0)+(u,u)/2-c/24}$$

for $u, v \in V_1$ such that $u(0)$ and $v(0)$ act semisimply on M^i. Notice that if $u, v = 0$, $Z_i(u,v,\tau) = Z_i(\tau)$. Then we have the following theorem [32,36,16,19]:

Theorem 2.14. *Let V be a rational, C_2-cofinite vertex operator algebra of CFT type. Assume $u, v \in V_1$ such that u, v span an abelian Lie subalgebra of V_1 and $u(0)$ and $v(0)$ act semisimply on M^i, $0 \le i \le d$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. Then $Z_i(u,v,\tau)$ converges to a holomorphic function in the upper half plane and*

$$Z_i(u,v,\gamma\tau) = \sum_{j=0}^{d} \gamma_{i,j} Z_j(au+bv, cu+dv, \tau),$$

where $\gamma\tau = \frac{a\tau+b}{c\tau+d}$ and $\gamma_{i,j} \in \mathbb{C}$ are given in [36] and independent of vectors u, v.

Remark 2.15. If $V_1 = 0$, then $u = v = 0$. So

$$Z_i(\gamma\tau) = \sum_{j=0}^{d} \gamma_{i,j} Z_j(\tau).$$

Definition 2.16. In the case $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in Theorem 2.14, we have

$$Z_i\left(u, v, -\frac{1}{\tau}\right) = \sum_{j=0}^{d} S_{i,j} Z_j(-v, u, \tau).$$

The matrix $S = (S_{i,j})_{i,j=0}^{d}$ is called an *S-matrix*. As we mentioned in Theorem 2.14 the S-matrix is uniquely determined by the irreducible V-modules $M^0, \cdots, M^d$.

The following theorem will play an important role in the last section [34,29].

Theorem 2.17. *Let V be a rational and C_2-cofinite simple vertex operator algebra of CFT type and assume $V \cong V'$. Let $S = (S_{i,j})_{i,j=0}^{d}$ be the S-matrix as defined above. Then*

(1) *$(S^{-1})_{i,j} = S_{i,j'} = S_{i',j}$, and $S_{i',j'} = S_{i,j}$;*
(2) *S is symmetric and $S^2 = (\delta_{i,j'})$;*
(3) *$N_{i,j}^{k} = \sum_{s=0}^{d} \frac{S_{i,s} S_{j,s} S_{s,k}^{-1}}{S_{0,s}}$.*

We need the concept of quantum dimensions from [15].

Definition 2.18. Let V be a vertex operator algebra and M a g-twisted V-module such that $Z_V(\tau)$ and $Z_M(\tau)$ exist. The quantum dimension of M over V is defined as

$$\mathrm{qdim}_V M = \lim_{y\to 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where y is real and positive.

Remark 2.19. Assume V is a simple, rational and C_2-cofinite vertex operator algebra of CFT type with $V \cong V'$. Let M^i be as before where $M^0 \cong V$. Also assume $\lambda_0 = 0$ and $\lambda_i > 0$, $\forall i \neq 0$. Then $\mathrm{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}$ [15].

From now on, we assume V is a rational, C_2-cofinite vertex operator algebra of CFT type with $V \cong V'$. Let $M^0 \cong V, M^1, \cdots, M^d$ denote all inequivalent irreducible V-modules. Moreover, we assume the conformal weights λ_i of M^i are positive for all $i > 0$. It is proved in [12] that $V_{L_2}^{A_4}$ satisfies all the assumptions.

Recall that simple module M^i is called a *simple current* if $M^i \boxtimes M^j$ is simple for all $j = 0, \cdots, d$. Here are some results on quantum dimensions [15].

Proposition 2.20. *Let V be a vertex operator algebra as before. Then*

(1) $\mathrm{qdim}_V M^i \geq 1$, $\forall i = 0, \cdots, d$.
(2) *For any $i, j = 0, \cdots, d$,*

$$\mathrm{qdim}_V \left(M^i \boxtimes M^j\right) = \mathrm{qdim}_V M^i \cdot \mathrm{qdim}_V M^j.$$

(3) *A V-module M is a simple current if and only if* $\mathrm{qdim}_V M = 1$.

Theorem 2.21. *Let V be a rational and C_2-cofinite simple vertex operator algebra, G a finite subgroup of $Aut(V)$. Also assume that V is g-rational and the conformal weight of any irreducible g-twisted V-module is positive except for V itself for all $g \in G$. Then* $\mathrm{qdim}_{V^G} V$ *exists and equals to $|G|$.*

3. The vertex operator algebra $V_{L_2}^{A_4}$

Now we first briefly review the construction of rank one lattice vertex operator algebra from [28]. Then we recall some related results about V_L^+ and $V_{L_2}^{A_4}$ from [1–4,23–25,14, 28,12]. In the last part of this section, we also give the realization of all irreducible $V_{L_2}^{A_4}$-modules.

3.1. Construction of the vertex operator algebra $V_{L_2}^{A_4}$

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one, i.e., $(\alpha, \alpha) = 2k$ for some positive integer k. Set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and extend $(\cdot, \cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\mathbb{C}[\mathfrak{h}]$ be the group algebra of $\mathfrak{h}$ with a basis $\{e^\lambda \mid \lambda \in \mathfrak{h}\}$.

The dual lattice L° of L is

$$L^\circ = \{\lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z}\} = \frac{1}{2k} L.$$

Then $L^\circ = \bigcup_{i=-k+1}^{k} (L + \lambda_i)$ is the coset decomposition with $\lambda_i = \frac{i}{2k}\alpha$. Set $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$. Then $V_{L+\lambda_i}$ for $i = -k+1, \cdots, k$ are all the inequivalent irreducible modules for V_L [5,28,6].

Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K$ be the corresponding Heisenberg algebra such that

$$[\alpha(m), \alpha(n)] = 2km\delta_{m+n,0}K \quad \text{and} \quad [K, \hat{\mathfrak{h}}] = 0$$

for any $m, n \in \mathbb{Z}$, where $\alpha(m) = \alpha \otimes t^m$. Then $\hat{\mathfrak{h}}_{\geq 0} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K$ is a subalgebra of $\hat{\mathfrak{h}}$ and the group algebra $\mathbb{C}[\mathfrak{h}]$ becomes a $\hat{\mathfrak{h}}_{\geq 0}$-module by the action $\alpha(m) \cdot e^\lambda = (\lambda, \alpha)\delta_{m,0}e^\lambda$ and $K \cdot e^\lambda = e^\lambda$ for any $\lambda \in \mathfrak{h}$ and $m \geq 0$. We denote by

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_{\geq 0})} \mathbb{C}e^\lambda$$

the $\hat{\mathfrak{h}}$-module induced from $\hat{\mathfrak{h}}_{\geq 0}$-module $\mathbb{C}e^\lambda$. Set $M(1) = M(1, 0)$. Then there exists a linear map $Y : M(1) \to \mathrm{End}M(1)[[z, z^{-1}]]$ such that $(M(1), Y, \mathbf{1}, \omega)$ carries a simple vertex operator algebra structure and $M(1, \lambda)$ becomes an irreducible $M(1)$-module for $\lambda \in \mathfrak{h}$ [28]. Let $\mathbb{C}[L]$ be the group algebra of L with a basis e^α for $\alpha \in L$. The lattice vertex operator algebra associated to L is given by

$$V_L = M(1) \otimes \mathbb{C}[L].$$

Let θ be a linear isomorphism of $V_{\mathfrak{h}}$ defined by

$$\theta(\alpha(-n_1) \cdots \alpha(-n_k) \otimes e^\lambda) = (-1)^k \alpha(-n_1) \cdots \alpha(-n_k) \otimes e^{-\lambda},$$

for $n \in \mathbb{Z}_+$ and $\lambda \in \mathfrak{h}$. Then θ induces automorphisms of V_L and $M(1)$. For a θ-invariant subspace W of $V_{\mathfrak{h}} = M(1) \otimes \mathbb{C}[\mathfrak{h}]$, we denote the ± 1-eigenspaces of W for θ by $W^\pm$. Then $(V_L^+, Y, \mathbf{1}, \omega)$ and $(M(1)^+, Y, \mathbf{1}, \omega)$ are vertex operator algebras.

Now we recall the construction of θ-twisted V_L-modules [28,7]. Let $\hat{\mathfrak{h}}[-1] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be a Lie algebra with the commutation relation

$$[\alpha \otimes t^m, \alpha \otimes t^n] = m\delta_{m+n,0}(\alpha, \alpha)K \quad \text{and} \quad [K, \hat{\mathfrak{h}}[-1]] = 0$$

for $m, n \in 1/2 + \mathbb{Z}$. Then there is a one-dimensional module for $\hat{\mathfrak{h}}[-1]_+ = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t] \oplus \mathbb{C}K$, which could be identified with $\mathbb{C}$, by the action

$$(\alpha \otimes t^m) \cdot 1 = 0 \quad \text{and} \quad K \cdot 1 = 1 \quad \text{for } m \in 1/2 + \mathbb{N}.$$

Set $M(1)(\theta)$ the induced $\hat{\mathfrak{h}}[-1]$-module:

$$M(1)(\theta) = U(\hat{\mathfrak{h}})[-1] \otimes_{U(\hat{\mathfrak{h}}[-1]_+)} \mathbb{C}.$$

Let χ_s be a character of $L/2L$ such that $\chi_s(\alpha) = (-1)^s$ for $s = 0, 1$ and $T_{\chi_s} = \mathbb{C}$ the irreducible $L/2L$-module with character χ_s. Then $V_L^{T_s} = M(1)(\theta) \otimes T_{\chi_s}$ is an irreducible θ-twisted V_L-module. We denote the ± 1-eigenspaces of $V_L^{T_s}$ under θ by $(V_L^{T_s})^\pm$. Then we have the following result:

Theorem 3.1. *Any irreducible V_L^+-module is isomorphic to one of the following modules:*

$$V_L^\pm, \qquad V_{L+\lambda_i} \quad (1 \le i \le k-1), \qquad V_{L+\lambda_k}^\pm, \qquad \left(V_L^{T_s}\right)^\pm.$$

Let $L_2 = \mathbb{Z}\alpha$ be the rank one positive-definite even lattice such that $(\alpha, \alpha) = 2$ and V_{L_2} the associated simple rational vertex operator algebra. Then $(V_{L_2})_1 \cong sl_2(\mathbb{C})$ and $(V_{L_2})_1$ has an orthonormal basis:

$$x^1 = \frac{1}{\sqrt{2}}\alpha(-1)\mathbf{1}, \qquad x^2 = \frac{1}{\sqrt{2}}\left(e^\alpha + e^{-\alpha}\right), \qquad x^3 = \frac{i}{\sqrt{2}}\left(e^\alpha - e^{-\alpha}\right).$$

For $x \in (V_{L_2})_1$ we also use $x(n)$ for x_n for $n \in \mathbb{Z}$. Let $\sigma, \tau_i \in Aut(V_{L_2})$, $i = 1, 2, 3$ be such that

$$\sigma\left(x^1, x^2, x^3\right) = \left(x^1, x^2, x^3\right) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\tau_1\left(x^1, x^2, x^3\right) = \left(x^1, x^2, x^3\right) \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix},$$

$$\tau_2\left(x^1, x^2, x^3\right) = \left(x^1, x^2, x^3\right) \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix},$$

$$\tau_3\left(x^1, x^2, x^3\right) = \left(x^1, x^2, x^3\right) \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Then σ and τ_i, $i = 1, 2, 3$, generate a finite subgroup of $Aut(V_{L_2})$ isomorphic to the alternating group A_4. We simply denote this group by A_4. It is easy to check that the subgroup K generated by τ_i, $i = 1, 2, 3$, is a normal subgroup of A_4 of order 4. Let $\beta = 2\alpha$. The following result can be found in [8].

Lemma 3.2. *We have $V_{L_2}^K = V_{\mathbb{Z}\beta}^+$ and $V_{L_2}^{A_4} = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$.*

By [21], there is a decomposition

$$V_{\mathbb{Z}\beta}^+ = \left(V_{\mathbb{Z}\beta}^+\right)^0 \oplus \left(V_{\mathbb{Z}\beta}^+\right)^1 \oplus \left(V_{\mathbb{Z}\beta}^+\right)^2 \tag{3.1}$$

where $(V_{\mathbb{Z}\beta}^+)^0 = (V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ is a simple vertex operator algebra and $(V_{\mathbb{Z}\beta}^+)^i$ is an irreducible $(V_{\mathbb{Z}\beta}^+)^0$-module, $i = 1, 2$. Similarly, as a $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module, we have

$$V_{\mathbb{Z}+\frac{1}{4}\beta} = V_{\mathbb{Z}+\frac{1}{4}\beta}^0 \oplus V_{\mathbb{Z}+\frac{1}{4}\beta}^1 \oplus V_{\mathbb{Z}+\frac{1}{4}\beta}^2 \tag{3.2}$$

such that $V_{\mathbb{Z}+\frac{1}{4}\beta}^i$ is irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module, $i = 0, 1, 2$ [26]. The details of the realization of $V_{\mathbb{Z}+\frac{1}{4}\beta}^i$ will be provided in the next subsection.

Let $W_{\sigma^i,1}$, $W_{\sigma^i,2}$ be the two irreducible σ^i-twisted modules of $V_{\mathbb{Z}\beta}^+$, $i = 1, 2$ [12]. Then each $W_{\sigma^i,j}$ is a direct sum of irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-submodules $W_{\sigma^i,j}^k$ for $k = 0, 1, 2$. There are exactly 21 irreducible modules of $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ which is listed as following [12]:

$$\left\{\left(V_{\mathbb{Z}\beta}^+\right)^m, V_{\mathbb{Z}\beta}^-, V_{\mathbb{Z}\beta+\frac{1}{8}\beta}, V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, W_{\sigma^i,j}^k, V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^n \;\middle|\; m, n, k = 0, 1, 2;\; i, j = 1, 2\right\}. \tag{3.3}$$

Here $(V_{\mathbb{Z}\beta}^+)^m$ is the eigenspace of σ with eigenvalue $e^{\frac{2\pi i m}{3}}$.

3.2. Realizations of the irreducible $V_{L_2}^{A_4}$-modules

Let σ, τ_i and x^i, $i = 1, 2, 3$ be as before. Set

$$h = \frac{1}{3\sqrt{6}}\left(x^1 + x^2 - x^3\right),$$
$$y^1 = \frac{1}{\sqrt{3}}\left(x^1 + \frac{-1 + \sqrt{3}i}{2}x^2 + \frac{1 + \sqrt{3}i}{2}x^3\right),$$
$$y^2 = \frac{1}{\sqrt{3}}\left(x^1 + \frac{-1 - \sqrt{3}i}{2}x^2 + \frac{1 - \sqrt{3}i}{2}x^3\right).$$

Then

$$L(n)h = \delta_{n,0}h, \qquad h(n)h = \frac{1}{18}\delta_{n,1}\mathbf{1}, \quad n \in \mathbb{Z},$$
$$h(0)y^1 = \frac{1}{3}y^1, \qquad h(0)y^2 = -\frac{1}{3}y^2, \qquad y^1(0)y^2 = 6h.$$

(see [12]). It follows that $h(0)$ acts semisimply on V_{L_2} with rational eigenvalues. So $e^{2\pi i h(0)}$ is an automorphism of V_{L_2} of finite order [8,31]. Since

$$e^{2\pi i h(0)}h = h, \qquad e^{2\pi i h(0)}y^1 = \frac{-1 + \sqrt{3}i}{2}y^1, \qquad e^{2\pi i h(0)}y^2 = \frac{-1 - \sqrt{3}i}{2}y^2,$$

it is easy to see that

$$e^{2\pi i h(0)} = \sigma. \tag{3.4}$$

The action of the group generated by σ, τ_i, $i = 1, 2, 3$ on V_{L_2} is isomorphic to alternating group A_4. Actually, $\sigma = e^{2\pi i h(0)}$ and $\tau_j = e^{\pi i x_j(0)}$ $(j = 1, 2, 3)$ also act on $V_{\mathbb{Z}\frac{1}{2}\alpha} = M(1) \otimes \mathbb{C}[\frac{1}{2}\mathbb{Z}\alpha]$, where the action of the group $\langle \sigma, \tau_i \mid i = 1, 2, 3\rangle$ on $V_{\mathbb{Z}\frac{1}{2}\alpha}$ is isomorphic to $SL(2, 3)$, the special linear group of degree 2 over a field of three elements. Thus by the quantum Galois theory [21],

$$V_{\mathbb{Z}\frac{\alpha}{2}} \cong \bigoplus_{\chi} V_\chi \otimes W_\chi \tag{3.5}$$

where χ runs over all irreducible characters of $SL(2, 3)$. The irreducible representations of the group $SL(2, 3)$ are well known: three 1-dimensional, one 3-dimensional and three 2-dimensional irreducible representations. We denote them by U_1^k, U_3 and U_2^k, $k = 0, 1, 2$ respectively, where the subindex i is the dimension of the module and the upper indices distinguish the irreducible modules of the same dimension. The irreducible modules with the same dimension can be distinguished by the eigenvalues of the action of σ: $\sigma|_{U_1^k} = e^{\frac{2\pi ik}{3}}$, σ has eigenvalues $e^{\frac{2\pi i}{6}+\frac{2\pi ik}{3}}$ and $e^{-\frac{2\pi i}{6}+\frac{2\pi ik}{3}}$ on U_2^k and the eigenvalues of σ on U_3 are the three cube roots of unity.

The set of scalar matrices of $SL(2, 3)$ is a normal subgroup isomorphic to $\mathbb{Z}_2$ and $A_4 \cong SL(2, 3)/\mathbb{Z}_2$. The group A_4 has three 1-dimensional and one 3-dimensional irreducible modules. Thus V_{L_2} and $V_{L_2+\frac{1}{2}\alpha}$ can be decomposed as $V_{L_2}^{A_4}$-modules [26]:

$$V_{L_2} = \left(V_{\mathbb{Z}\beta}^+\right)^0 \otimes U_1^0 \oplus \left(V_{\mathbb{Z}\beta}^+\right)^1 \otimes U_1^1 \oplus \left(V_{\mathbb{Z}\beta}^+\right)^2 \otimes U_1^2 \oplus V_{\mathbb{Z}\beta}^- \otimes U_3,$$

$$V_{L_2+\frac{1}{2}\alpha} = V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0 \otimes U_2^0 \oplus V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1 \otimes U_2^1 \oplus V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2 \otimes U_2^2.$$

Some of those $V_{L_2}^{A_4}$-modules listed in (3.3) can be realized differently by considering the orbifold vertex operator algebra $V_{L_2}^{\langle\sigma\rangle}$.

Proposition 3.3. *Let g be an automorphism of V_{L_2} of order $T \neq 1$. Then there exists some vector $u \in (V_{L_2})_1$, such that $g = e^{2\pi i u(0)}$.*

Proof. The vertex operator algebra V_{L_2} is isomorphic to the affine vertex operator algebra associated to the simple Lie algebra $sl_2(\mathbb{C})$ of level 1. We know that $Aut(V_{L_2}) \cong Aut(sl_2(\mathbb{C}))$. The restriction of g on $(V_{L_2})_1$ is also an isomorphism of order T, which has an eigenspaces decomposition.

Claim: There exists a unique (up to a scalar) nonzero vector $a \in (V_{L_2})_1$ such that $ga = a$.

Note that $(a, b) = -a(1)b$ defines a nondegenerate symmetric invariant bilinear form on $(V_{L_2})_1$, since $(V_{L_2})_1$ is isomorphic to $sl_2(\mathbb{C})$ which has a unique nondegenerate symmetric invariant bilinear form up to a constant. This implies that $(ga, gb) = (a, b)$ for all $a, b \in (V_{L_2})_1$ and $(V_{L_2})_1^\rho$ and $(V_{L_2})_1^{\bar{\rho}}$ have the same dimensions where $(V_{L_2})_1^\rho$ is the eigenspace of g on $(V_{L_2})_1$ with eigenvalue ρ which is a root of unity.

First, $g|_{(V_{L_2})_1}$ is not a constant. Since $(V_{L_2})_1$ is a simple Lie algebra we see that $(V_{L_2})_1$ is spanned by $a(0)b$ for $a, b \in (V_{L_2})_1$. If g acts on $(V_{L_2})_1$ as a constant ρ. Then g also acts as ρ^2. This forces $\rho = 1$, a contradiction.

If there are exactly two eigenvalues ρ_1, ρ_2 of g on $(V_{L_2})_1$, we deduce that $\rho_1 = \pm 1$, $\rho_2 = \mp 1$. Otherwise $\rho_1 = \overline{\rho_2} \neq \pm 1$ and $(V_{L_2})_1$ has even dimension, a contradiction. Without loss of generality, assume $\rho_1 = 1$, $\rho_2 = -1$. If the eigenspace of g with eigenvalue 1 is two dimensional, then the eigenspace of g on $[(V_{L_2})_1, (V_{L_2})_1] = (V_{L_2})_1$ with eigenvalue 1 is one dimensional, a contradiction.

The only case left is that ρ_1, ρ_2, ρ_3 are three distinct eigenvalues of g on $(V_{L_2})_1$. Assume that $\rho_1 = \overline{\rho_2}$. Using the fact that $[(V_{L_2})_1, (V_{L_2})_1] = (V_{L_2})_1$ we see that $\rho_3 = 1$ and each eigenspace is one dimensional. The claim is proved.

Let $a \in (V_{L_2})_1$ be an eigenvector of g with eigenvalue 1. Consider the Jordan decomposition of $a = a_s + a_n$, where a_s and a_n are the semisimple part and nilpotent part of a. It is easy to see that a is not nilpotent due to the eigenspace decomposition, and a_s is also a fixed point of g since a is a fixed point of g. Since the fixed point space is 1 dimensional, $a = a_s$, which acts semisimply on $(V_{L_2})_1$. The structure of $sl_2(\mathbb{C})$ tells us that there exists γa, for some $\gamma \in \mathbb{C}^*$, such that $(\gamma a, \gamma a) = 2$, $[\gamma a, e^{\gamma a}] = 2e^{\gamma a}$, $[\gamma a, e^{-\gamma a}] = -2e^{-\gamma a}$, $g(e^{\gamma a}) = e^{2\pi i \frac{r}{T}} e^{\gamma a}$ and $g(e^{-\gamma a}) = e^{-2\pi i \frac{r}{T}} e^{-\gamma a}$. It is clear that $g = e^{\pi i \frac{r}{T} \gamma a(0)}$, i.e. $u = \frac{r}{2T} \gamma a$. $\square$

Remark 3.4. The group $SO(3)$ is the connected compact subgroup of $Aut(V_{L_2})$, whose discrete subgroup are the cyclic group Z_n, the dihedral group D_n, A_4, S_4 and A_5. The above proposition indicates that the orbifold vertex operator algebra $V_{L_2}^{Z_n} \cong V_{\mathbb{Z}n\alpha}$. One could also get $V_{L_2}^{D_n} \cong V_{\mathbb{Z}n\alpha}^+$.

Remark 3.5. It is worthy to point out that for any $g \in Aut(V_{L_2})$ of finite order T, the g-twisted module category is equivalent to the category of ordinary modules. Thus V_{L_2} is g-rational for any such g. Following from Theorem 2.21, $\mathrm{qdim}_{V_{L_2}^{A_4}} V_{L_2} = o(A_4) = 12$.

In our case, we have $V_{L_2}^{\langle\sigma\rangle} \cong V_{\mathbb{Z}\gamma} \cong V_{\mathbb{Z}3\alpha}$, $(\gamma, \gamma) = 18$, i.e. $\gamma = 18h$. One immediately gets that $V_{L_2} \cong V_{\mathbb{Z}\gamma} \oplus V_{\mathbb{Z}\gamma + \frac{1}{3}\gamma} \oplus V_{\mathbb{Z}\gamma + \frac{2}{3}\gamma}$ and $V_{L_2 + \frac{1}{2}\alpha} \cong V_{\mathbb{Z}\gamma + \frac{1}{6}\gamma} \oplus V_{\mathbb{Z}\gamma + \frac{1}{2}\gamma} \oplus V_{\mathbb{Z}\gamma - \frac{1}{6}\gamma}$ due to the eigenvalues of the σ action on $V_{\mathbb{Z}\frac{1}{2}\alpha}$. The eigenvalues of σ on V_{L_2} and $V_{L_2 + \frac{1}{2}\alpha}$ (see Eq. (3.5)) give us the following proposition.

Proposition 3.6. *As $V_{L_2}^{A_4}$-modules, we have the following identifications:*

$$V_{\mathbb{Z}\gamma} \cong \left(V_{\mathbb{Z}\beta}^+\right)^0 + V_{\mathbb{Z}\beta}^-,$$

$$V_{\mathbb{Z}\gamma + \frac{1}{3}\gamma} \cong \left(V_{\mathbb{Z}\beta}^+\right)^1 + V_{\mathbb{Z}\beta}^-,$$

$$V_{\mathbb{Z}\gamma + \frac{2}{3}\gamma} \cong \left(V_{\mathbb{Z}\beta}^+\right)^2 + V_{\mathbb{Z}\beta}^-,$$

$$V_{\mathbb{Z}\gamma + \frac{1}{6}\gamma} = V_{\mathbb{Z}\beta + \frac{1}{4}\beta}^0 + V_{\mathbb{Z}\beta + \frac{1}{4}\beta}^1,$$

$$V_{\mathbb{Z}\gamma+\frac{1}{2}\gamma}=V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta}+V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta},$$
$$V_{\mathbb{Z}\gamma-\frac{1}{6}\gamma}=V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta}+V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta}.$$

Now we briefly review the irreducible $V_{L_2}^{A_4}$-modules which are constructed from the σ^i-twisted V_{L_2}-modules. Let $W^1\cong V_{L_2}$, $W^2\cong V_{L_2+\frac{1}{2}\alpha}$. Set

$$w^1=e^{\frac{\alpha}{2}}+\frac{(\sqrt{3}-1)(1+i)}{2}e^{-\frac{\alpha}{2}},\qquad w^2=\frac{1}{\sqrt{2}}\big[(\sqrt{3}-1)e^{\frac{\alpha}{2}}-(1+i)e^{-\frac{\alpha}{2}}\big].$$

For any $u\in(V_{L_2})_1$ such that $g=e^{2\pi iu(0)}$ is an automorphism of V_{L_2} of finite order, define

$$\Delta(u,z)=z^{h(0)}\exp\left(\sum_{k=1}^{\infty}\frac{u(k)}{-k}(-z)^{-k}\right).$$

It is proved in [30] that $(W^{i,T_1},Y_g(\cdot,z))=(W^i,Y(\Delta(u,z)\cdot,z))$ are irreducible g-twisted modules of V_{L_2}, $i=1,2$. The σ^i-twisted V_{L_2}-modules were constructed in [12] following this idea, where the twisted vertex operator was also determined.

For the σ-twisted V_{L_2}-modules,

$$\Delta(h,z)L(-2)\mathbf{1}=L(-2)\mathbf{1}+z^{-1}h(-1)\mathbf{1}+\frac{1}{36}z^{-2}\mathbf{1},$$
$$Y_\sigma(h,z)=Y\left(h+\frac{1}{18}z^{-1},z\right),$$
$$Y_\sigma\big(y^1,z\big)=z^{\frac{1}{3}}Y\big(y^1,z\big),$$
$$Y_\sigma\big(y^2,z\big)=z^{-\frac{1}{3}}Y\big(y^2,z\big).$$

For the σ^2-twisted V_{L_2}-modules,

$$\Delta(-h,z)L(-2)\mathbf{1}=L(-2)\mathbf{1}-z^{-1}h(-1)\mathbf{1}+\frac{1}{36}z^{-2}\mathbf{1},$$
$$Y_{\sigma^2}(h,z)=Y\left(-h+\frac{1}{18}z^{-1},z\right),$$
$$Y_{\sigma^2}\big(y^1,z\big)=z^{-\frac{1}{3}}Y\big(y^1,z\big),$$
$$Y_{\sigma^2}\big(y^2,z\big)=z^{\frac{1}{3}}Y\big(y^2,z\big).$$

The irreducible $V_{L_2}^{A_4}$-modules $W^k_{\sigma^i,j}$ are realized in the σ^i-twisted V_{L_2}-module [12]. The following table gives the conformal weight of each $W^k_{\sigma^i,j}$ [12]:

	$W^0_{\sigma,1}$	$W^1_{\sigma,1}$	$W^2_{\sigma,1}$	$W^0_{\sigma,2}$	$W^1_{\sigma,2}$	$W^2_{\sigma,2}$
$L(0)$	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
Lowest weight vector	$\mathbf{1}$	y^2	y^1	w^2	w^1	$y^2(-2)w^2$

	$W^0_{\sigma^2,1}$	$W^1_{\sigma^2,1}$	$W^2_{\sigma^2,1}$	$W^0_{\sigma^2,2}$	$W^1_{\sigma^2,2}$	$W^2_{\sigma^2,2}$
$L(0)$	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
Lowest weight vector	$\mathbf{1}$	y^1	y^2	w^1	w^2	$y^1(-2)w^1$

Proposition 3.7. *As $V_{L_2}^{A_4}$-modules, we have the following identification:*

$$V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma} \cong W^0_{\sigma,1}, \qquad V_{\mathbb{Z}\gamma-\frac{5}{18}\gamma} \cong W^1_{\sigma,1}, \qquad V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma} \cong W^2_{\sigma,1},$$
$$V_{\mathbb{Z}\gamma-\frac{1}{9}\gamma} \cong W^0_{\sigma,2}, \qquad V_{\mathbb{Z}\gamma+\frac{2}{9}\gamma} \cong W^1_{\sigma,2}, \qquad V_{\mathbb{Z}\gamma-\frac{4}{9}\gamma} \cong W^2_{\sigma,2},$$
$$V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma} \cong W^0_{\sigma^2,1}, \qquad V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma} \cong W^1_{\sigma^2,1}, \qquad V_{\mathbb{Z}\gamma-\frac{7}{18}\gamma} \cong W^2_{\sigma^2,1},$$
$$V_{\mathbb{Z}\gamma+\frac{1}{9}\gamma} \cong W^0_{\sigma^2,2}, \qquad V_{\mathbb{Z}\gamma-\frac{2}{9}\gamma} \cong W^1_{\sigma^2,2}, \qquad V_{\mathbb{Z}\gamma+\frac{4}{9}\gamma} \cong W^2_{\sigma^2,2}.$$

Proof. We only prove the first isomorphism. We know that the $V_{L_2}^{A_4}$-module in V_{L_2} generated by y^1 is isomorphic to $(V_{\mathbb{Z}\beta}^+)^1 \subset V_{\mathbb{Z}\gamma+\frac{1}{3}\gamma}$. Since the conformal weight of $W^0_{\sigma,1}$ is $\frac{1}{36}$, $W^0_{\sigma,1} \cong V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma}$ or $V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma}$. The twisted vertex operator $Y_\sigma(y^1,z)$ would help us to determine which is the right isomorphism. We have

$$Y_\sigma\left(y^1,z\right)\mathbf{1} = z^{\frac{1}{3}}Y\left(y^1,z\right)\mathbf{1} \subset W^2_{\sigma,1},$$

where the conformal weight of $W^2_{\sigma,1} = \frac{49}{36}$. On the other hand, the fusion rules among irreducible $V_{\mathbb{Z}\gamma}$-modules are as following:

$$V_{\mathbb{Z}\gamma+\frac{1}{3}\gamma} \boxtimes V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma} = V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma}, \qquad V_{\mathbb{Z}\gamma+\frac{1}{3}\gamma} \boxtimes V_{\mathbb{Z}\gamma-\frac{1}{18}\gamma} = V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma}.$$

We already mentioned that $y^1 \in (V_{\mathbb{Z}\beta}^+)^1 \subset V_{\mathbb{Z}\gamma+\frac{1}{3}\gamma}$. Comparing the conformal weights of $W^2_{\sigma,1}$, $V_{\mathbb{Z}\gamma+\frac{7}{18}\gamma}$ and $V_{\mathbb{Z}\gamma+\frac{5}{18}\gamma}$ tells us that $W^0_{\sigma,1} \cong V_{\mathbb{Z}\gamma+\frac{1}{18}\gamma}$.

Other isomorphisms could be proved using a similar argument. □

4. Quantum dimensions of irreducible $V_{L_2}^{A_4}$-modules

In this section, we determine the quantum dimensions of all irreducible $V_{L_2}^{A_4}$-modules. We first investigate properties of quantum dimensions of irreducible twisted $V_{L_2}^{A_4}$-modules.

Let V be a vertex operator algebra and let $g = e^{2\pi i h(0)}$ be an automorphism of V of finite order where $h \in V_1$ such that $h(0)$ acts on V semisimply. Let M be an irreducible V-module. By [17,30] we see that $(M^g, Y_g(\cdot,z)) = (M, Y_M(\Delta(h,z)\cdot,z))$ is a g-twisted V-module, where $\Delta(h,z) = z^{h(0)}\exp(\sum_{k=1}^{\infty}\frac{h(k)}{-k}(-z)^{-k})$.

Proposition 4.1. *Let V be a rational, C_2-cofinite vertex operator algebra with central charge c and $M^0, \cdots, M^d$ all of the inequivalent irreducible V-modules with $M^0 \cong V$ and the corresponding conformal weights $\lambda_i > 0$ for $0 < i \leq d$. Let g be as defined. Then* $\operatorname{qdim} M^i = \operatorname{qdim}(M^i)^g$, $0 \leq i \leq d$.

Proof. The q-character of $(M^j)^g$ are given by

$$\operatorname{ch}_q\bigl(M^j\bigr)^g = \operatorname{tr}_{M^j} q^{L(0)+h(0)+(h,h)/2-c/24} = Z_{M^i}(h, 0, \tau).$$

Thus the quantum dimension of $(M^j)^g$ can be computed:

$$\begin{aligned}
\operatorname{qdim}\bigl(M^j\bigr)^g &= \lim_{y \to 0} \frac{Z_j(h, 0, iy)}{Z_V(iy)} \\
&= \lim_{\tau \to i\infty} \frac{Z_j(h, 0, -\frac{1}{\tau})}{Z_V(-\frac{1}{\tau})} \\
&= \lim_{\tau \to i\infty} \frac{\sum_k S_{j,k} Z_k(0, h, \tau)}{\sum_k S_{0,k} Z_k(0, h, \tau)} \\
&= \lim_{q \to 0} \frac{\sum_k S_{j,k} \operatorname{tr}_{M^k} e^{2\pi i h(0)} q^{L(0)-c/24}}{\sum_k S_{0,k} \operatorname{tr}_{M^k} e^{2\pi i h(0)} q^{L(0)-c/24}} \\
&= S_{j,0}/S_{0,0}
\end{aligned}$$

where the last equation follows from the conformal weight $\lambda_i > 0$ for $0 < i \leq d$. Remark 2.19 asserts that

$$\operatorname{qdim}\bigl(M^i\bigr)^g = \operatorname{qdim} M^i. \qquad \square$$

Let M be an irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-module. For simplicity, from now on we denote the quantum dimension of M over $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$ by $\operatorname{qdim} M$ instead of $\operatorname{qdim}_{(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}} M$.

Theorem 4.2. *The quantum dimensions for all irreducible $(V_{\mathbb{Z}\beta}^+)^{\langle\sigma\rangle}$-modules are given by the following tables:*

	$(V_{\mathbb{Z}\beta}^+)^0$	$(V_{\mathbb{Z}\beta}^+)^1$	$(V_{\mathbb{Z}\beta}^+)^2$	$V_{\mathbb{Z}\beta}^-$	$V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$
ω	0	4	4	1	$\frac{1}{16}$	$\frac{9}{16}$
qdim	1	1	1	3	6	6

	$W_{\sigma,1}^0$	$W_{\sigma,1}^1$	$W_{\sigma,1}^2$	$W_{\sigma,2}^0$	$W_{\sigma,2}^1$	$W_{\sigma,2}^2$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
qdim	4	4	4	4	4	4

	$W^0_{\sigma^2,1}$	$W^1_{\sigma^2,1}$	$W^2_{\sigma^2,1}$	$W^0_{\sigma^2,2}$	$W^1_{\sigma^2,2}$	$W^2_{\sigma^2,2}$
ω	$\frac{1}{36}$	$\frac{25}{36}$	$\frac{49}{36}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{16}{9}$
qdim	4	4	4	4	4	4

	$V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta}$	$V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta}$	$V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta}$
ω	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{9}{4}$
qdim	2	2	2

Proof. 1) We know from Remark 3.5 that

$$\operatorname{qdim} V_{L_2} = |A_4| = 12.$$

It is also obvious that

$$V_{L_2} = V^+_{\mathbb{Z}\beta} \oplus V^-_{\mathbb{Z}\beta} \oplus V^+_{\mathbb{Z}\beta+\frac{\beta}{2}} \oplus V^-_{\mathbb{Z}\beta+\frac{\beta}{2}},$$

where as $V^{A_4}_{L_2}$-modules $V^+_{\mathbb{Z}\beta} = (V^+_{\mathbb{Z}\beta})^0 \oplus (V^+_{\mathbb{Z}\beta})^1 \oplus (V^+_{\mathbb{Z}\beta})^2$ and $V^-_{\mathbb{Z}\beta} \cong V^+_{\mathbb{Z}\beta+\frac{\beta}{2}} \cong V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$. Since $V^-_{\mathbb{Z}\beta}$, $V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$, $V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$ are all simple currents of $V^+_{\mathbb{Z}\beta}$, one gets

$$\operatorname{qdim} V^-_{\mathbb{Z}\beta} = \operatorname{qdim} V^+_{\mathbb{Z}\beta} = 3.$$

From (3.1), we have the decomposition as irreducible $(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$-modules:

$$V^+_{\mathbb{Z}\beta} = \left(V^+_{\mathbb{Z}\beta}\right)^0 \oplus \left(V^+_{\mathbb{Z}\beta}\right)^1 \oplus \left(V^+_{\mathbb{Z}\beta}\right)^2$$

where $(V^+_{\mathbb{Z}\beta})^0 = (V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$ with $\operatorname{qdim}(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle} = 1$. Since $(V^+_{\mathbb{Z}\beta})^1$ and $(V^+_{\mathbb{Z}\beta})^2$ are irreducible $(V^+_{\mathbb{Z}\beta})^{\langle\sigma\rangle}$-modules, by Proposition 2.20(1), we get

$$\operatorname{qdim}\left(V^+_{\mathbb{Z}\beta}\right)^1 = \operatorname{qdim}\left(V^+_{\mathbb{Z}\beta}\right)^2 = 1.$$

2) By [20], every irreducible V_L-module is a simple current. Thus from Proposition 2.20 we get

$$\operatorname{qdim}_{V_{\mathbb{Z}\beta}} V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = 1 \quad \text{and} \quad \operatorname{qdim}_{V^+_{\mathbb{Z}\beta}} V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = 2.$$

Hence

$$\operatorname{qdim} V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = \operatorname{qdim} V_{\mathbb{Z}\beta} = 6.$$

3) Recall that $V^{\langle\sigma\rangle}_{L_2} = V_{\mathbb{Z}\gamma}$, where $(\gamma,\gamma) = 18$. We get $\operatorname{qdim} V_{\mathbb{Z}\gamma} = 4$, since $\operatorname{qdim} V_{L_2} = 12$ and $o(\sigma) = 3$. Notice that by Proposition 3.6,

$$V_{\mathbb{Z}\gamma+\frac{1}{6}\gamma} = V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta} + V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta},$$
$$V_{\mathbb{Z}\gamma+\frac{1}{2}\gamma} = V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta} + V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta},$$
$$V_{\mathbb{Z}\gamma-\frac{1}{6}\gamma} = V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta} + V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta},$$

where $\operatorname{qdim} V_{\mathbb{Z}\gamma+\mu} = 4$, for any $\mu = \pm\frac{1}{6}\gamma, \frac{1}{2}\gamma$. It is easy to determine

$$\operatorname{qdim} V^0_{\mathbb{Z}\beta+\frac{1}{4}\beta} = \operatorname{qdim} V^1_{\mathbb{Z}\beta+\frac{1}{4}\beta} = \operatorname{qdim} V^2_{\mathbb{Z}\beta+\frac{1}{4}\beta} = 2.$$

4) We have $\operatorname{qdim} V_{L_2} = \operatorname{qdim} V_{L_2+\frac{\alpha}{2}} = 12$. By Proposition 2.20, we have

$$\operatorname{qdim}_V W_{\sigma^i,j} = 12, \quad i,j = 1,2.$$

Consider the action of $(V^+_{\mathbb{Z}\beta})^k$ on $W^k_{\sigma^i,j}$, $i,j = 1,2$; $k = 0,1,2$. Use the similar argument in 3), we can prove $\operatorname{qdim} W^k_{\sigma^i,j}$ are the same for all $i,j = 1,2$; $k = 0,1,2$. Therefore we get $\operatorname{qdim} W^k_{\sigma^i,j} = 4$ for $i,j = 1,2$; $k = 0,1,2$. $\square$

Remark 4.3. Let V be a vertex operator algebra with only finitely many irreducible modules, the global dimension is defined as $\operatorname{glob}(V) = \sum_{M\in Irr(V)} \operatorname{qdim}(M)^2$ [15]. Assume G is a finite subgroup of $Aut(G)$, it is conjectured that $|G|^2 \operatorname{glob}(V) = \operatorname{glob} V^G$, which was derived in the frame work of conformal nets [35]. The vertex operator algebra version is still open. However, the quantum dimensions above verify this conjecture, which gives us more evidence to believe the conjecture is true.

5. Fusion rules

In this section, we find fusion rules for irreducible $V_{L_2}^{A_4}$-modules. Quantum dimensions play an important role in determining fusions. We also need the Verlinde formula to deal with these fusion rules that involve with twisted modules. We first list all fusion products results, then we give the proof.

Let W^1, W^2, W^3 be irreducible $V_{L_2}^{A_4}$-modules. For simplicity, in the following, the space of all intertwining operators of type $\binom{W^3}{W^1\ W^2}$ is denoted by $I\binom{W^3}{W^1\ W^2}$, instead of $I_{V_{L_2}^{A_4}}\binom{W^3}{W^1\ W^2}$. The fusion product of W^1 and W^2 is denoted by $W^1 \boxtimes W^2$, instead of $W^1 \boxtimes_{V_{L_2}^{A_4}} W^2$.

To determine fusion products of $W^j_{\sigma^m,i} \boxtimes W^l_{\sigma^m,k}$, $i,k,m = 1,2$, $j,l = 0,1,2$, we first need to find out certain entries of the S-matrix.

Lemma 5.1. *The entries of the S-matrix that involve with irreducible twisted modules of $V_{L_2}^{A_4}$ are given as the table in Appendix A.*

Proof. For convenience, we denote the irreducible $V_{L_2}^{A_4}$-modules by M^i, $i = 0,1,\cdots,20$ as following:

$(V_{\mathbb{Z}\beta}^+)^0$	$(V_{\mathbb{Z}\beta}^+)^1$	$(V_{\mathbb{Z}\beta}^+)^2$	$V_{\mathbb{Z}\beta}^-$	$V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$	$V_{\mathbb{Z}\beta+\frac{3}{8}\beta}$
M^0	M^1	M^2	M^3	M^4	M^5

$W_{\sigma,1}^0$	$W_{\sigma,1}^1$	$W_{\sigma,1}^2$	$W_{\sigma,2}^0$	$W_{\sigma,2}^1$	$W_{\sigma,2}^2$
M^6	M^7	M^8	M^9	M^{10}	M^{11}

$W_{\sigma^2,1}^0$	$W_{\sigma^2,1}^1$	$W_{\sigma^2,1}^2$	$W_{\sigma^2,2}^0$	$W_{\sigma^2,2}^1$	$W_{\sigma^2,2}^2$
M^{12}	M^{13}	M^{14}	M^{15}	M^{16}	M^{17}

$V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^0$	$V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^1$	$V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^2$
M^{18}	M^{19}	M^{20}

First we consider the vertex operator algebra $V_{\mathbb{Z}\gamma}$ where $(\gamma,\gamma)=18$. Its irreducible modules are $V_{\mathbb{Z}\gamma+\lambda_k}$, where $\lambda_k=\frac{k}{18}\gamma$ and $k=0,\cdots,17$. By page 106 [33], we see that

$$Z_{V_{\mathbb{Z}\gamma+\lambda_i}}\left(-\frac{1}{\tau}\right)=\sum_{j=0}^{17}\frac{1}{\sqrt{18}}e^{-2\pi i(\lambda_i,\lambda_j)}Z_{V_{\mathbb{Z}\gamma+\lambda_j}}(\tau).$$

Thus the entries of S-matrix $(S_{\lambda_k,\lambda_l})$ for $V_{\mathbb{Z}\gamma}$ is given by

$$S_{\lambda_k,\lambda_l}=\frac{1}{\sqrt{18}}e^{-2\pi i(\lambda_k,\lambda_l)},\quad k,l=0,1,\cdots,17.$$

Denote the S-matrix for the vertex operator algebra $V_{L_2}^{A_4}$ by $(S_{i,j})$. By the identifications given Proposition 3.7 and the S-matrix of $V_{\mathbb{Z}\gamma}$, it is easy to see that $S_{i,0}=\frac{1}{\sqrt{18}}$, $i=6,\cdots,17$. By Remark 2.19 and quantum dimensions listed in Theorem 4.2, we have

$$\operatorname{qdim} M^i=\frac{S_{i,0}}{S_{0,0}}=4,\quad i=6,7,\cdots,17,$$

which implies $S_{0,0}=\frac{1}{4\sqrt{18}}$. Hence we have $S_{i,0}=\frac{\operatorname{qdim} M^i}{4\sqrt{18}}$ for $i=0,1,\cdots,20$. Applying the quantum dimensions as listed in Theorem 4.2, we get the first column of the table.

Now let $M^j\cong V_{\mathbb{Z}\gamma+\lambda_l}$. If M^i is not a submodule of $V_{\mathbb{Z}\gamma+\lambda_k}$ for all λ_k, then $S_{i,j}=0$. Otherwise, M^i is a submodule of $V_{\mathbb{Z}\gamma+\lambda_{k_s}}$ for some $k_1,\cdots,k_r$ and M^i is not a submodule of $V_{\mathbb{Z}\gamma+\lambda_p}$ for all $\lambda_p\neq\lambda_{k_s}$, $\forall s=1,\cdots,r$. In this case, $S_{ij}=\sum_{s=1}^r S_{\lambda_l,\lambda_{k_s}}$.

In this way, we can get the entries of the S-matrix as listed in the table in Appendix A. □

Theorem 5.2. *The fusion rules for all irreducible $V_{L_2}^{A_4}$-modules are given as following (here $\bar{n}$ is remainder when dividing n by 3 for $n\in\mathbb{Z}$):*

$$\left(V_{\mathbb{Z}\beta}^+\right)^i\boxtimes\left(V_{\mathbb{Z}\beta}^+\right)^j=\left(V_{\mathbb{Z}\beta}^+\right)^{\overline{i+j}},\quad i,j=0,1,2, \tag{5.1}$$

$$\left(V_{\mathbb{Z}\beta}^{+}\right)^{i} \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{j} = V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{\overline{i+j}}, \quad i,j=0,1,2, \tag{5.2}$$

$$\left(V_{\mathbb{Z}\beta}^{+}\right)^{i} \boxtimes V_{\mathbb{Z}\beta}^{-} = V_{\mathbb{Z}\beta}^{-}, \quad i=0,1,2, \tag{5.3}$$

$$\begin{aligned} \left(V_{\mathbb{Z}\beta}^{+}\right)^{k} \boxtimes W_{\sigma^i,i}^{l} &= W_{\sigma^i,i}^{\overline{l-k}}, \quad i=1,2;\ k,l=0,1,2 \\ \left(V_{\mathbb{Z}\beta}^{+}\right)^{k} \boxtimes W_{\sigma^i,3-i}^{l} &= W_{\sigma^i,3-i}^{\overline{k+l}}, \quad i=1,2;\ k,l=0,1,2, \end{aligned} \tag{5.4}$$

$$\left(V_{\mathbb{Z}\beta}^{+}\right)^{i} \boxtimes V_{\mathbb{Z}\beta+\frac{j}{8}\beta} = V_{\mathbb{Z}\beta+\frac{j}{8}\beta}, \quad i=0,1,2;\ j=1,3, \tag{5.5}$$

$$V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{i} \boxtimes V_{\mathbb{Z}\beta}^{-} = V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{2}, \quad i=0,1,2, \tag{5.6}$$

$$V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{i} \boxtimes V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{j} = V_{\mathbb{Z}\beta}^{-} \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{\overline{i+j}}, \quad i,j=0,1,2, \tag{5.7}$$

$$\begin{aligned} V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{k} \boxtimes W_{\sigma^i,3-i}^{l} &= W_{\sigma^i,i}^{\overline{2l-k}} \oplus W_{\sigma^i,i}^{\overline{2l-k+1}}, \quad i=1,2;\ k,l=0,1,2, \\ V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{k} \boxtimes W_{\sigma^i,i}^{l} &= W_{\sigma^i,3-i}^{\overline{k-l}} \oplus W_{\sigma^i,3-i}^{\overline{k-l+1}}, \quad i=1,2;\ k,l=0,1,2, \end{aligned} \tag{5.8}$$

$$V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{k} \boxtimes V_{\mathbb{Z}\beta+\frac{l}{8}\beta} = V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, \quad k=0,1,2;\ l=1,3, \tag{5.9}$$

$$V_{\mathbb{Z}\beta}^{-} \boxtimes V_{\mathbb{Z}\beta}^{-} = \left(V_{\mathbb{Z}\beta}^{+}\right)^{0} \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{1} \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{2} \oplus 2V_{\mathbb{Z}\beta}^{-}, \tag{5.10}$$

$$V_{\mathbb{Z}\beta}^{-} \boxtimes W_{\sigma^i,j}^{k} = W_{\sigma^i,j}^{0} \oplus W_{\sigma^i,j}^{1} \oplus W_{\sigma^i,j}^{2}, \quad i,j=1,2;\ k=0,1,2, \tag{5.11}$$

$$\begin{aligned} V_{\mathbb{Z}\beta}^{-} \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta} &= V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus 2V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, \\ V_{\mathbb{Z}\beta}^{-} \boxtimes V_{\mathbb{Z}\beta+\frac{3}{8}\beta} &= 2V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} V_{\mathbb{Z}\beta+\frac{r}{8}\beta} \boxtimes V_{\mathbb{Z}\beta+\frac{r}{8}\beta} = {} & V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{2} \\ & \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{0} \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{1} \oplus \left(V_{\mathbb{Z}\beta}^{+}\right)^{2} \\ & \oplus V_{\mathbb{Z}\beta}^{-} \oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus 2V_{\mathbb{Z}\beta+\frac{3\beta}{8}}, \quad r=1,3, \end{aligned} \tag{5.13}$$

$$\begin{aligned} V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \boxtimes V_{\mathbb{Z}\beta+\frac{3}{8}\beta} = {} & V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{0} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{1} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{4}}^{2} \\ & \oplus 2V_{\mathbb{Z}\beta}^{-} \oplus 2V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus 2V_{\mathbb{Z}\beta+\frac{3\beta}{8}}, \end{aligned} \tag{5.14}$$

$$W_{\sigma^i,1}^{k} \boxtimes W_{\sigma^i,1}^{l} = W_{\sigma^{3-i},1}^{0} \oplus W_{\sigma^{3-i},1}^{1} \oplus W_{\sigma^{3-i},1}^{2} \oplus W_{\sigma^{3-i},2}^{\overline{-i(k+l)}}, \tag{5.15}$$

$$W_{\sigma^i,2}^{k} \boxtimes W_{\sigma^i,2}^{l} = W_{\sigma^{3-i},1}^{0} \oplus W_{\sigma^{3-i},1}^{1} \oplus W_{\sigma^{3-i},1}^{2} \oplus W_{\sigma^{3-i},2}^{\overline{1+i(k+l)}}, \tag{5.16}$$

$$W_{\sigma^i,1}^{k} \boxtimes W_{\sigma^i,2}^{l} = \bigoplus_{k=0}^{2} W_{\sigma^{3-i},2}^{k} \oplus W_{\sigma^{3-i},2}^{\overline{i(l-k)}}, \tag{5.17}$$

$$W^k_{\sigma^i,j} \boxtimes V_{\mathbb{Z}\beta+\frac{s}{8}\beta} = W^0_{\sigma^i,1} \oplus W^1_{\sigma^i,1} \oplus W^2_{\sigma^i,1} \oplus W^0_{\sigma^i,2} \oplus W^1_{\sigma^i,2} \oplus W^2_{\sigma^i,2},$$
$$i,j = 1,2;\ k = 0,1,2;\ s = 1,3. \tag{5.18}$$

For $k,l = 0,1,2$, $r,i = 1,2$,

$$W^k_{\sigma,r} \boxtimes W^l_{\sigma^2,r} = \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{r(l-k)}} \oplus V^-_{\mathbb{Z}\beta} \oplus V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}, \tag{5.19}$$

$$W^k_{\sigma,r} \boxtimes W^l_{\sigma^2,3-r} = (V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^{\overline{r(-k-l)}} \oplus (V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^{\overline{r(-k-l+1)}} \oplus V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}. \tag{5.20}$$

Proof. *(5.1), (5.2), (5.4) and (5.8)* are obvious by Proposition 3.6 and fusion rules for irreducible $V^+_{\mathbb{Z}\beta}$-modules and $V_{\mathbb{Z}\gamma}$-modules.

Proof of (5.3): By Proposition 2.20, each irreducible module with quantum dimension 1 is a simple current. Thus the right hand side should be one irreducible module with quantum dimension 3 while $V^-_{\mathbb{Z}\beta}$ is the only irreducible module with such quantum dimension.

Proof of (5.5): By fusion rules for irreducible $V^+_{\mathbb{Z}\beta}$-modules, $I_{V^+_{\mathbb{Z}\beta}}\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{j}{8}\beta} \\ V^+_{\mathbb{Z}\beta}\ V_{\mathbb{Z}\beta+\frac{j}{8}\beta}\end{pmatrix} \neq 0$. Since $(V^+_{\mathbb{Z}\beta})^i \subset V^+_{\mathbb{Z}\beta}$ is a simple current of $(V^+_{\mathbb{Z}\beta})^0$, we get the desired fusion rule.

Proof of (5.6): First by fusion rules for $V^+_{\mathbb{Z}\beta}$-modules, we have

$$V^-_{\mathbb{Z}\beta} \boxtimes_{V^+_{\mathbb{Z}\beta}} V_{\mathbb{Z}\beta+\frac{\beta}{4}} = V_{\mathbb{Z}\beta+\frac{\beta}{4}}$$

Let $\mathcal{Y}(\cdot,z)$ be the intertwining operator of type $\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ V^-_{\mathbb{Z}\beta}\ V_{\mathbb{Z}\beta+\frac{\beta}{4}}\end{pmatrix}$. For a fixed $v \in V^i_{\mathbb{Z}\beta+\frac{\beta}{4}}$, consider $\mathcal{Y}(u,z)v$, $u \in V^-_{\mathbb{Z}\beta}$. Then $\langle u_i v \mid u \in V^-_{\mathbb{Z}\beta}, i \in \mathbb{Z}\rangle = V^+_{\mathbb{Z}\beta+\frac{\beta}{4}}$. Thus we get fusion product for irreducible $V^{A_4}_{L_2}$-modules as follows:

$$V^-_{\mathbb{Z}\beta} \boxtimes V^i_{\mathbb{Z}\beta+\frac{\beta}{4}} = V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^2_{\mathbb{Z}\beta+\frac{\beta}{4}}, \quad i = 0,1,2.$$

Proof of (5.7): From (5.6), we see that $I\begin{pmatrix} V^j_{\mathbb{Z}\beta+\frac{\beta}{4}} \\ V^i_{\mathbb{Z}\beta+\frac{\beta}{4}}\ V^-_{\mathbb{Z}\beta}\end{pmatrix} \neq 0$, $i,j = 0,1,2$. Since $(V^1_{\mathbb{Z}\beta+\frac{\beta}{4}})' = V^2_{\mathbb{Z}\beta+\frac{\beta}{4}}$, $(V^-_{\mathbb{Z}\beta})' = V^-_{\mathbb{Z}\beta}$ and $(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}})' = V^0_{\mathbb{Z}\beta+\frac{\beta}{4}}$. We obtain

$$I\begin{pmatrix} V^-_{\mathbb{Z}\beta} \\ V^i_{\mathbb{Z}\beta+\frac{\beta}{4}}\ \ V^j_{\mathbb{Z}\beta+\frac{\beta}{4}}\end{pmatrix} \neq 0, \quad i,j = 0,1,2.$$

By Proposition 2.12, $I\begin{pmatrix} (V^+_{\mathbb{Z}\beta})^{\overline{i+j}} \\ V^i_{\mathbb{Z}\beta+\frac{\beta}{4}}\ V^j_{\mathbb{Z}\beta+\frac{\beta}{4}}\end{pmatrix} \neq 0$. By counting quantum dimensions, we get (5.7).

Proof of (5.9): Since $V^i_{\mathbb{Z}\beta+\frac{\beta}{4}} \subset V_{\mathbb{Z}\beta+\frac{\beta}{4}}$ and $V_{\mathbb{Z}\beta+\frac{\beta}{4}}$ is an irreducible $V^+_{\mathbb{Z}\beta}$-module. By fusion rules of irreducible $V^+_{\mathbb{Z}\beta}$-modules, we get $I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{r}{8}\beta} & \\ V^i_{\mathbb{Z}\beta+\frac{\beta}{4}} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix} \neq 0$, for $r = 1, 3$. By counting quantum dimensions of both sides, we get the desired fusion product.

Proof of (5.10): First by (5.3) and Proposition 2.12, we get $I\begin{pmatrix} & (V^+_{\mathbb{Z}\beta})^i & \\ V^-_{\mathbb{Z}\beta} & & V^-_{\mathbb{Z}\beta}\end{pmatrix} \neq 0$, for $i = 0, 1, 2$. By fusion rules for irreducible V^+_L-modules [2], we get $I_{V^+_{\mathbb{Z}\beta}}\begin{pmatrix} & V^{\mp}_{\mathbb{Z}\beta+\frac{1}{2}\beta} & \\ V^-_{\mathbb{Z}\beta} & & V^{\pm}_{\mathbb{Z}\beta+\frac{1}{2}\beta}\end{pmatrix} \neq 0$. Using the identifications

$$V^-_{\mathbb{Z}\beta} \cong V^+_{\mathbb{Z}\beta+\frac{1}{2}\beta} \cong V^-_{\mathbb{Z}\beta+\frac{1}{2}\beta}$$

give $I\begin{pmatrix} & V^-_{\mathbb{Z}\beta} & \\ V^-_{\mathbb{Z}\beta} & & V^-_{\mathbb{Z}\beta}\end{pmatrix} \neq 0$. So it suffices to prove that $I\begin{pmatrix} & V^-_{\mathbb{Z}\beta} & \\ V^-_{\mathbb{Z}\beta} & & V^-_{\mathbb{Z}\beta}\end{pmatrix} = 2$. Let $\mathcal{Y}_1(\cdot, z)$, $\mathcal{Y}_2(\cdot, z)$ be the standard intertwining operators of types $\begin{pmatrix} & V^-_{\mathbb{Z}\beta} & \\ V^-_{\mathbb{Z}\beta} & & V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}\end{pmatrix}$ and $\begin{pmatrix} & V^-_{\mathbb{Z}\beta} & \\ V^-_{\mathbb{Z}\beta} & & V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}\end{pmatrix}$ respectively (see [20,2]). Note that $e^{\beta} - e^{-\beta} \in V^-_{\mathbb{Z}\beta}$, $e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}} \in V^+_{\mathbb{Z}\beta+\frac{\beta}{2}}$ and $e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}} \in V^-_{\mathbb{Z}\beta+\frac{\beta}{2}}$. Then we have

$$\begin{aligned}
&\mathcal{Y}_1\big(\beta(-1)\mathbf{1}, z\big)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) \\
&\quad = 4\big(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}}\big)z^{-1} + \beta(-1)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big)z^0 + \text{higher power terms of } z, \\
&\mathcal{Y}_1\big(\beta(-1)\mathbf{1}, z\big)\big(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}}\big) \\
&\quad = 4\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big)z^{-1} + \beta(-1)\big(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}}\big)z^0 + \text{higher power terms of } z.
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathcal{Y}_1\big(e^{\beta} - e^{-\beta}, z\big)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) \\
&\quad = Y\big(e^{\beta}, z\big)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) - Y\big(e^{-\beta}, z\big)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) \\
&\quad = E^-(-\beta, z)E^+(-\beta, z)e^{\beta}z^{\beta}\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) \\
&\qquad - E^-(\beta, z)E^+(\beta, z)e^{-\beta}z^{-\beta}\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big) \\
&\quad = E^-(-\beta, z)\big(z^4e^{\frac{3\beta}{2}} + z^{-4}e^{\frac{\beta}{2}}\big) - E^-(\beta, z)\big(z^{-4}e^{-\frac{\beta}{2}} + z^4e^{-\frac{3\beta}{2}}\big) \\
&\quad = \exp\left(\sum_{n<0} \frac{-\beta(n)}{n} z^n\right)\big(z^4e^{\frac{3\beta}{2}} + z^{-4}e^{\frac{\beta}{2}}\big) \\
&\qquad - \exp\left(\sum_{n<0} \frac{\beta(n)}{n} z^n\right)\big(z^{-4}e^{-\frac{\beta}{2}} + z^4e^{-\frac{3\beta}{2}}\big) \\
&\quad = \beta(-1)\big(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}}\big)z^{-4} + \text{higher power terms of } z,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{Y}_2\big(e^{\beta}-e^{-\beta},z\big)\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)\\
&\quad= Y\big(e^{\beta},z\big)\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)-Y\big(e^{-\beta},z\big)\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)\\
&\quad= E^{-}(-\beta,z)E^{+}(-\beta,z)e^{\beta}z^{\beta}\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)\\
&\qquad - E^{-}(\beta,z)E^{+}(\beta,z)e^{-\beta}z^{-\beta}\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)\\
&\quad= E^{-}(-\beta,z)\big(z^{4}e^{\frac{3\beta}{2}}-z^{-4}e^{\frac{\beta}{2}}\big)-E^{-}(\beta,z)\big(z^{-4}e^{-\frac{\beta}{2}}-z^{4}e^{-\frac{3\beta}{2}}\big)\\
&\quad= \exp\Bigg(\sum_{n<0}\frac{-\beta(n)}{n}z^{n}\Bigg)\big(z^{4}e^{\frac{3\beta}{2}}-z^{-4}e^{\frac{\beta}{2}}\big)\\
&\qquad - \exp\Bigg(\sum_{n<0}\frac{\beta(n)}{n}z^{n}\Bigg)\big(z^{-4}e^{-\frac{\beta}{2}}-z^{4}e^{-\frac{3\beta}{2}}\big)\\
&\quad= -\beta(-1)\big(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\big)z^{-4}+\text{higher power terms of } z.
\end{aligned}$$

From the above computations we see immediately that $\mathcal{Y}_1(\cdot,z)$ and $\mathcal{Y}_2(\cdot,z)$ are linearly independent. Thus we obtain $N_{V_{\mathbb{Z}\beta}^{-}V_{\mathbb{Z}\beta}^{-}}^{V_{\mathbb{Z}\beta}^{-}}\geq 2$. By counting quantum dimensions as listed in Theorem 4.2, we get $N_{V_{\mathbb{Z}\beta}^{-}V_{\mathbb{Z}\beta}^{-}}^{V_{\mathbb{Z}\beta}^{-}}=2$ and hence we proved (5.10).

Proof of (5.11): This is clear by fusion rules for irreducible $V_{\mathbb{Z}\gamma}$-modules and the identification in Propositions 3.6 and 3.7.

Proof of (5.12): Since

$$V_{\mathbb{Z}\beta+\frac{\beta}{8}}\cong V_{\mathbb{Z}\beta}^{T_2,+}\cong V_{\mathbb{Z}\beta}^{T_1,+},\qquad V_{\mathbb{Z}\beta+\frac{3\beta}{8}}\cong V_{\mathbb{Z}\beta}^{T_2,-}\cong V_{\mathbb{Z}\beta}^{T_1,-},\qquad V_{\mathbb{Z}\beta}^{-}\cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{-}\cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{+}$$

as irreducible $V_{L_2}^{A_4}$-modules [12], it follows from the fusion rules of irreducible $V_{\mathbb{Z}\beta}^{+}$-modules [2] that $I\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{i}{8}\beta} \\ V_{\mathbb{Z}\beta}^{-}\ V_{\mathbb{Z}\beta+\frac{j}{8}\beta}\end{pmatrix}\neq 0$, $i,j=1,3$, $i\neq j$. It suffices to prove that $I\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{i}{8}\beta} \\ V_{\mathbb{Z}\beta}^{-}\ V_{\mathbb{Z}\beta+\frac{j}{8}\beta}\end{pmatrix}=2$ for $i,j=1,3$, $i\neq j$. First we prove $I\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{3}{8}\beta} \\ V_{\mathbb{Z}\beta}^{-}\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}=2$. Let $\mathcal{Y}_1(\cdot,z)\in I_{V_{\mathbb{Z}\beta}^{+}}\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{3\beta}{8}} \\ V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{+}\ V_{\mathbb{Z}\beta+\frac{\beta}{8}}\end{pmatrix}$, $\mathcal{Y}_2(\cdot,z)\in I_{V_{\mathbb{Z}\beta}^{+}}\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{3\beta}{8}} \\ V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{-}\ V_{\mathbb{Z}\beta+\frac{\beta}{8}}\end{pmatrix}$. Note that $e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}\in V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{+}$, $e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}\in V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{-}$, $e^{\frac{\beta}{8}},e^{-\frac{7\beta}{8}}\in V_{\mathbb{Z}\beta+\frac{\beta}{8}}$. Considering $\mathcal{Y}_1(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}},z)e^{\frac{\beta}{8}}$, $\mathcal{Y}_2(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}},z)e^{\frac{\beta}{8}}$, $\mathcal{Y}_1(e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}},z)e^{-\frac{7\beta}{8}}$, $\mathcal{Y}_2(e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}},z)e^{-\frac{7\beta}{8}}$ and applying similar argument as in the proof of (5.10), we can prove $N_{V_{\mathbb{Z}\beta}^{-}V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{3}{8}\beta}}=2$. Similarly, $N_{V_{\mathbb{Z}\beta}^{-}V_{\mathbb{Z}\beta+\frac{3}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}=2$.

Proof of (5.13): Case 1: $r=1$. From (5.5), (5.9) and Proposition 2.12, we get

$$I\begin{pmatrix} V_{\mathbb{Z}\beta+\frac{1}{4}\beta}^{i} \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\quad V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}\neq 0,\qquad I\begin{pmatrix} (V_{\mathbb{Z}\beta}^{+})^{j} \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\quad V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}\neq 0,\quad i,j=0,1,2.$$

So it is sufficient to prove $N_{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}=2$ and $N_{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{3}{8}\beta}}=2$ using the quantum dimensions.

Note from [12] that there are isomorphisms of irreducible $V_{L_2}^{A_4}$-modules:

$$V_{\mathbb{Z}\beta+\frac{\beta}{8}}\cong V_{\mathbb{Z}\beta}^{T_1,+}\cong V_{\mathbb{Z}\beta}^{T_2,+},\qquad V_{\mathbb{Z}\beta}^{-}\cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{-}\cong V_{\mathbb{Z}\beta+\frac{\beta}{2}}^{+}.$$

By fusion rules of irreducible $V_{\mathbb{Z}\beta}^{+}$-modules [2], we get

$$I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}\neq 0,\qquad I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{3}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}\neq 0.$$

Let $T=T^1\oplus T^2$ be the direct sum of irreducible $\mathbb{C}[\mathbb{Z}\beta]$-modules T^1 and T^2, and define a linear isomorphism $\psi\in\operatorname{End}T$ by $\psi(t_1)=t_2$, $\psi(t_2)=t_1$, where t_i is a basis of T^i for $i=1,2$. For $\lambda\in(\mathbb{Z}\beta)^\circ$, we write $\lambda=r\beta/8+m\beta$ for $-3\le r\le 4$ and $m\in\mathbb{Z}$, and define $\psi_\lambda\in\operatorname{End}T$ by $\psi_\lambda=e_{m\alpha}\psi^r$. By fusion rules for irreducible $V_{\mathbb{Z}\beta}^{+}$-modules [1], we have $I_{V_{\mathbb{Z}\beta}^{+}}\begin{pmatrix} & V_{\mathbb{Z}\beta}^{T_i,+} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta}^{T_j,+}\end{pmatrix}\neq 0$, $i,j=1,2$. Let $\mathcal{Y}_{ij}(\cdot,z)\in I_{V_{\mathbb{Z}\beta}^{+}}\begin{pmatrix} & V_{\mathbb{Z}\beta}^{T_i,+} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta}^{T_j,+}\end{pmatrix}$. Note that the intertwining operator is given by

$$\mathcal{Y}(u,z)=\mathcal{Y}^\theta(u,z)\otimes\psi_\lambda\quad\text{for }\lambda\in(\mathbb{Z}\beta)^\circ\text{ and }u\in M(1,\lambda)$$

where $\mathcal{Y}^\theta(e_\lambda,z)=2^{-\langle\lambda,\lambda\rangle}z^{-\frac{\langle\lambda,\lambda\rangle}{2}}\exp(\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(-n)}{n}z^n)\exp(-\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(n)}{n}z^n)$.

For $1\otimes e_{\beta+\frac{1}{8}\beta}\in V_{\mathbb{Z}\beta+\frac{1}{8}\beta}=M(1)\otimes\mathbb{C}[\mathbb{Z}\beta+\frac{1}{8}\beta]$, $t_1\in T_1$ and $t_2\in T_2$, we have

$$\begin{aligned}\mathcal{Y}_{21}(1\otimes e_{\beta+\frac{1}{8}\beta},z)t_1&=\mathcal{Y}_{21}^\theta(e_{\frac{9}{8}\beta},z)\psi_{\frac{9}{8}\beta}t_1\\&=\mathcal{Y}_{21}^\theta(e_{\frac{9}{8}\beta},z)e_\beta\psi_{\frac{1}{8}\beta}t_1\\&=\mathcal{Y}_{21}^\theta(e_{\frac{9}{8}\beta},z)e_\beta t_2\\&=-\mathcal{Y}_{21}^\theta(e_{\frac{9}{8}\beta},z)t_2\\&=-2^{-\frac{81}{8}}z^{-\frac{81}{16}}\exp\left(-\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(n)}{n}z^n\right)t_2,\end{aligned}\tag{5.21}$$

$$\begin{aligned}\mathcal{Y}_{12}(1\otimes e_{\frac{9}{8}\beta},z)t_2&=\mathcal{Y}_{12}^\theta(e_{\frac{9}{8}\beta},z)\psi_{\frac{1}{8}\beta}t_2\\&=\mathcal{Y}_{12}^\theta(e_{\frac{9}{8}\beta},z)e_\beta\psi t_2\\&=\mathcal{Y}_{12}^\theta(e_{\frac{9}{8}\beta},z)e_\beta t_1\\&=\mathcal{Y}_{12}^\theta(e_{\frac{9}{8}\beta},z)t_1\\&=2^{-\frac{81}{8}}z^{-\frac{81}{16}}\exp\left(-\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(n)}{n}z^n\right)t_1.\end{aligned}\tag{5.22}$$

We also have

$$\begin{aligned}
\mathcal{Y}_{21}(1\otimes e_{\frac{1}{8}\beta},z)t_1 &= \mathcal{Y}_{21}^{\theta}(e_{\frac{1}{8}\beta},z)\psi_{\frac{1}{8}\beta}t_1\\
&= \mathcal{Y}_{21}^{\theta}(e_{\frac{1}{8}\beta},z)\psi t_1\\
&= \mathcal{Y}_{21}^{\theta}(e_{\frac{1}{8}\beta},z)t_2\\
&= -\mathcal{Y}_{21}^{\theta}(e_{\frac{1}{8}\beta},z)t_2\\
&= 2^{-\frac{1}{8}}z^{-\frac{1}{16}}\exp\Bigg(-\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(n)}{n}z^n\Bigg)t_2,
\end{aligned}\tag{5.23}$$

$$\begin{aligned}
\mathcal{Y}_{12}(1\otimes e_{\frac{1}{8}\beta},z)t_2 &= \mathcal{Y}_{12}^{\theta}(e_{\frac{1}{8}\beta},z)\psi_{\frac{1}{8}\beta}t_2\\
&= \mathcal{Y}_{12}^{\theta}(e_{\frac{1}{8}\beta},z)\psi t_2\\
&= \mathcal{Y}_{12}^{\theta}(e_{\frac{1}{8}\beta},z)t_1\\
&= 2^{-\frac{1}{8}}z^{-\frac{1}{16}}\exp\Bigg(-\sum_{n\in 1/2+\mathbb{N}}\frac{\lambda(n)}{n}z^n\Bigg)t_1.
\end{aligned}\tag{5.24}$$

So $\mathcal{Y}_{12}(\cdot,z),\mathcal{Y}_{21}(\cdot,z)\in I\left(\begin{smallmatrix} & V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{smallmatrix}\right)$ are linearly independent and $N_{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}\geq 2$. By a similar argument, we can prove that $N_{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}^{V_{\mathbb{Z}\beta+\frac{3}{8}\beta}}\geq 2$. Counting quantum dimensions of modules in the fusion product then asserts

$$I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}=2 \quad\text{and}\quad I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{3}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\end{pmatrix}=2.$$

Case 2: $r=3$. The proof is similar to that of case 1. This finishes the proof of (5.13).

Proof of (5.14): By (5.9), (5.12) and Proposition 2.12, we have

$$I\begin{pmatrix} & V^{i}_{\mathbb{Z}\beta+\frac{1}{4}\beta} & \\ V_{\mathbb{Z}\beta+\frac{\beta}{8}} & & V_{\mathbb{Z}\beta+\frac{3\beta}{8}}\end{pmatrix}\neq 0,\quad i=0,1,2;\qquad I\begin{pmatrix} & V^{-}_{\mathbb{Z}\beta} & \\ V_{\mathbb{Z}\beta+\frac{\beta}{8}} & & V_{\mathbb{Z}\beta+\frac{3\beta}{8}}\end{pmatrix}\neq 0.$$

Note that we have the following isomorphism of irreducible $V_{\mathbb{Z}\alpha}^{A_4}$-modules [12]:

$$V_{\mathbb{Z}\beta+\frac{\beta}{8}}\cong V_{\mathbb{Z}\beta}^{T_1,+}\cong V_{\mathbb{Z}\beta}^{T_2,+},\qquad V_{\mathbb{Z}\beta+\frac{3\beta}{8}}\cong V_{\mathbb{Z}\beta}^{T_1,-}\cong V_{\mathbb{Z}\beta}^{T_2,-}.$$

Eq. (5.13) indicates that

$$I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{3}{8}\beta}\end{pmatrix}=2,\qquad I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{3}{8}\beta} & \\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta} & & V_{\mathbb{Z}\beta+\frac{3}{8}\beta}\end{pmatrix}=2.$$

Proof of (5.15), (5.16) and (5.17): We can prove these fusion products by applying Proposition 2.17 and Lemma 5.1.

Proof of (5.18): We only give a proof of

$$W_{\sigma,1}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = W_{\sigma,1}^0 \oplus W_{\sigma,1}^1 \oplus W_{\sigma,1}^2 \oplus W_{\sigma,2}^0 \oplus W_{\sigma,2}^1 \oplus W_{\sigma,2}^2$$

here and proofs for the other cases are similar.

First we prove that $I\binom{W}{W_{\sigma,1}^0\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}} = 0$ for any irreducible $V_{L_2}^{A_4}$-module W appearing in the untwisted $V_{\mathbb{Z}\beta}^+$-modules. Otherwise, there is some W such that $I\binom{W}{W_{\sigma,1}^0\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}} \neq 0$. By Proposition 2.12, we obtain $I\binom{W_{\sigma^2,1}^0}{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\ W'} \neq 0$. The fusion products $V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \boxtimes W'$ for all such W have been known already. It is easy to see that $I\binom{W_{\sigma^2,1}^0}{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\ W'} = 0$ for all such W, which is a contradiction.

Now we show that $I\binom{W_{\sigma^2,i}^j}{W_{\sigma,1}^0\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}} = 0$, for all $i = 1, 2$, $j = 0, 1, 2$. Otherwise, if there exists some $i_0 \in \{1, 2\}$, $j_0 \in \{0, 1, 2\}$ such that $I\binom{W_{\sigma^2,i_0}^{j_0}}{W_{\sigma,1}^0\ V_{\mathbb{Z}\beta+\frac{1}{8}\beta}} \neq 0$. Since $(V_{\mathbb{Z}\beta+\frac{1}{8}\beta})' = V_{\mathbb{Z}\beta+\frac{1}{8}\beta}$ and $(W_{\sigma^2,i_0}^{j_0})' = W_{\sigma,i_0}^{j_0}$, we see that $I\binom{V_{\mathbb{Z}\beta+\frac{1}{8}\beta}}{W_{\sigma,1}^0\ W_{\sigma,i_0}^{j_0}} \neq 0$ by Proposition 2.12, which contradicts with (5.15) or (5.16).

Thus we have $W_{\sigma,1}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = \bigoplus_{p,q} m_{p,q} W_{\sigma,p}^q$ where $m_{p,q}$ are integers. Assume that $m_{p,q} \neq 0$ for some $p \in \{1, 2\}$, $q \in \{0, 1, 2\}$, then by (5.4) and (5.5) we have $m_{p0} = m_{p1} = m_{p2} \neq 0$. Assume that $m_{3-p,k} = 0$ for all $k = 0, 1, 2$. Then by quantum dimensions of each module, we get

$$W_{\sigma,1}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = 2W_{\sigma,p}^0 \oplus 2W_{\sigma,p}^1 \oplus 2W_{\sigma,p}^2. \tag{5.25}$$

By (5.25) and (5.8) we obtain

$$(V_{\mathbb{Z}\beta+\frac{1}{4}\beta})^0 \boxtimes \left(W_{\sigma,1}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\right) = 4W_{\sigma,2}^0 \oplus 4W_{\sigma,2}^1 \oplus 4W_{\sigma,2}^2. \tag{5.26}$$

But by associativity of fusion product and (5.9) we have

$$\begin{aligned} V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0 \boxtimes \left(V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \boxtimes W_{\sigma,1}^0\right) &= \left(V_{\mathbb{Z}\beta+\frac{\beta}{4}}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta}\right) \boxtimes W_{\sigma,1}^0 \\ &= (V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}) \boxtimes W_{\sigma,1}^0 \\ &= 2W_{\sigma,p}^0 \oplus 2W_{\sigma,p}^1 \oplus 2W_{\sigma,p}^2 \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta} \boxtimes W_{\sigma,1}^0, \end{aligned}$$

a contradiction with (5.26). Hence there exists some $l = 0, 1, 2$ such that $m_{3-p,l} \neq 0$, then we also have $m_{3-p,0} = m_{3-p,1} = m_{3-p,2} \neq 0$ by applying (5.4). By counting quantum dimensions of both sides, we see that

$$W_{\sigma,1}^0 \boxtimes V_{\mathbb{Z}\beta+\frac{1}{8}\beta} = W_{\sigma,1}^0 \oplus W_{\sigma,1}^1 \oplus W_{\sigma,1}^2 \oplus W_{\sigma,2}^0 \oplus W_{\sigma,2}^1 \oplus W_{\sigma,2}^2.$$

Proof of (5.19): Since $(W^0_{\sigma,1})' = W^0_{\sigma^2,1}$, by Proposition 2.12, we get

$$I\begin{pmatrix} & (V^+_{\mathbb{Z}\beta})^0 & \\ W^0_{\sigma,1} & & W^0_{\sigma^2,1}\end{pmatrix} \neq 0.$$

By (5.11), (5.18) and Proposition 2.12, we obtain

$$I\begin{pmatrix} & V^-_{\mathbb{Z}\beta} & \\ W^0_{\sigma^2,1} & & W^0_{\sigma,1}\end{pmatrix} \neq 0, \qquad I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{r}{8}} & \\ W^0_{\sigma,1} & & W^0_{\sigma^2,1}\end{pmatrix} \neq 0, \quad r = 1, 3.$$

Thus

$$W^0_{\sigma,1} \boxtimes W^0_{\sigma^2,1} = \left(V^+_{\mathbb{Z}\beta}\right)^0 \oplus V^-_{\mathbb{Z}\beta} \oplus V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta} \tag{5.27}$$

by counting the quantum dimensions.

From (5.4), for $k, l = 0, 1, 2$, we have

$$W^k_{\sigma,1} = \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{-k}} \boxtimes W^0_{\sigma,1}, \qquad W^l_{\sigma^2,1} = \left(V^+_{\mathbb{Z}\beta}\right)^l \boxtimes W^0_{\sigma^2,1}.$$

So

$$\begin{aligned} W^k_{\sigma,1} \boxtimes W^l_{\sigma^2,1} &= \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{-k}} \boxtimes \left(V^+_{\mathbb{Z}\beta}\right)^l \boxtimes \left(W^0_{\sigma,1} \boxtimes W^0_{\sigma^2,1}\right) \\ &= \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{l-k}} \boxtimes \left(\left(V^+_{\mathbb{Z}\beta}\right)^0 \oplus V^-_{\mathbb{Z}\beta} + V_{\mathbb{Z}\beta+\frac{\beta}{8}} + V_{\mathbb{Z}\beta+\frac{3}{8}\beta}\right) \\ &= \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{l-k}} \oplus V^-_{\mathbb{Z}\beta} \oplus V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}. \end{aligned} \tag{5.28}$$

Similarly we can prove that

$$W^k_{\sigma,2} \boxtimes W^l_{\sigma^2,2} = \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{l-k}} \oplus V^-_{\mathbb{Z}\beta} \oplus V_{\mathbb{Z}\beta+\frac{1}{8}\beta} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}.$$

This finishes the proof of (5.19).

Proof of (5.20): From (5.8), we have

$$I\begin{pmatrix} & W^0_{\sigma,2} & \\ W^0_{\sigma,1} & & V^0_{\mathbb{Z}\beta+\frac{\beta}{4}}\end{pmatrix} \neq 0, \qquad I\begin{pmatrix} & W^0_{\sigma,2} & \\ W^0_{\sigma,1} & & V^2_{\mathbb{Z}\beta+\frac{\beta}{4}}\end{pmatrix} \neq 0.$$

Since $(W^0_{\sigma,2})' = W^0_{\sigma^2,2}$, $\left(V^0_{\mathbb{Z}\beta+\frac{\beta}{4}}\right)' = V^0_{\mathbb{Z}\beta+\frac{\beta}{4}}$ and $\left(V^2_{\mathbb{Z}\beta+\frac{\beta}{4}}\right)' = V^1_{\mathbb{Z}\beta+\frac{\beta}{4}}$, by Proposition 2.12 we obtain

$$I\begin{pmatrix} & V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} & \\ W^0_{\sigma,1} & & W^0_{\sigma^2,2}\end{pmatrix} \neq 0, \qquad I\begin{pmatrix} & V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} & \\ W^0_{\sigma,1} & & W^0_{\sigma^2,2}\end{pmatrix} \neq 0.$$

By (5.18), for $r = 1, 2$, $k = 0, 1, 2$, $s = 1, 3$,

$$I\begin{pmatrix} & W^k_{\sigma,r} & \\ W^0_{\sigma,1} & & V_{\mathbb{Z}\beta+\frac{s}{8}\beta}\end{pmatrix} \neq 0.$$

Since $(V_{\mathbb{Z}\beta+\frac{s}{8}\beta})' = V_{\mathbb{Z}\beta+\frac{s}{8}\beta}$, we obtain $I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{s}{8}\beta} & \\ W^0_{\sigma,1} & & W^k_{\sigma^2,r}\end{pmatrix} \neq 0$. In particular,

$$I\begin{pmatrix} & V_{\mathbb{Z}\beta+\frac{s}{8}\beta} & \\ W^0_{\sigma,1} & & W^0_{\sigma^2,r}\end{pmatrix} \neq 0, \quad s = 1, 3.$$

By counting quantum dimensions, we obtain

$$W^0_{\sigma,1} \boxtimes W^0_{\sigma^2,2} = V^0_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V^1_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}.$$

From (5.4) we have

$$W^k_{\sigma,1} = \left(V^+_{\mathbb{Z}\beta}\right)^{-k} \boxtimes W^0_{\sigma,1}, \qquad W^l_{\sigma^2,2} = \left(V^+_{\mathbb{Z}\beta}\right)^{-l} \boxtimes W^0_{\sigma^2,2}.$$

Thus

$$\begin{aligned} W^k_{\sigma,1} \boxtimes W^l_{\sigma^2,2} &= \left(V^+_{\mathbb{Z}\beta}\right)^{\overline{-k-l}} \boxtimes \left(W^0_{\sigma,1} \boxtimes W^0_{\sigma^2,2}\right) \\ &= V^{\overline{-k-l}}_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta\oplus\frac{\beta}{4}})^{\overline{-k-l+l}} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}. \end{aligned}$$

Similarly, we can show

$$W^k_{\sigma,2} \boxtimes W^l_{\sigma^2,1} = V^{\overline{k+l}}_{\mathbb{Z}\beta+\frac{\beta}{4}} \oplus (V_{\mathbb{Z}\beta\oplus\frac{\beta}{4}})^{\overline{k+l+l}} \oplus V_{\mathbb{Z}\beta+\frac{\beta}{8}} \oplus V_{\mathbb{Z}\beta+\frac{3}{8}\beta}.$$

Thus (5.20) holds. □

Appendix A

The following is the part of the S-matrix for irreducible $V_{L_2}^{A_4}$-modules that we need (see Lemma 5.1 and its proof):

$\sqrt{18}S_{i,j}$	0	6	7	8	9	10	11
0	$\frac{1}{4}$	1	1	1	1	1	1
1	$\frac{1}{4}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
2	$\frac{1}{4}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
3	$\frac{3}{4}$	0	0	0	0	0	0
4	$\frac{3}{2}$	0	0	0	0	0	0
5	$\frac{3}{2}$	0	0	0	0	0	0
6	1	$e^{-\frac{\pi i}{9}}$	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{7\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$
7	1	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{7\pi i}{9}}$	$e^{-\frac{\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$
8	1	$e^{-\frac{7\pi i}{9}}$	$e^{-\frac{\pi i}{9}}$	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$
9	1	$e^{\frac{2\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$
10	1	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$
11	1	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$
12	1	$e^{\frac{\pi i}{9}}$	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{7\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$
13	1	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{7\pi i}{9}}$	$e^{\frac{\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$
14	1	$e^{\frac{7\pi i}{9}}$	$e^{\frac{\pi i}{9}}$	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$
15	1	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$
16	1	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$
17	1	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$
18	$\frac{1}{2}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{\pi i}{3}}$	$e^{\frac{\pi i}{3}}$	$e^{\frac{\pi i}{3}}$
19	$\frac{1}{2}$	1	1	1	-1	-1	-1
20	$\frac{1}{2}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$

$\sqrt{18}S_{i,j}$	12	13	14	15	16	17
0	1	1	1	1	1	1
1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
2	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	$e^{\frac{\pi i}{9}}$	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{7\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$
7	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{7\pi i}{9}}$	$e^{\frac{\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$
8	$e^{\frac{7\pi i}{9}}$	$e^{\frac{\pi i}{9}}$	$e^{-\frac{5\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$
9	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$
10	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$
11	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{-\frac{2\pi i}{9}}$	$e^{\frac{4\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$
12	$e^{-\frac{\pi i}{9}}$	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{7\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$
13	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{7\pi i}{9}}$	$e^{-\frac{\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$
14	$e^{-\frac{7\pi i}{9}}$	$e^{-\frac{\pi i}{9}}$	$e^{\frac{5\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$
15	$e^{\frac{2\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$
16	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{8\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$
17	$e^{\frac{8\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{\frac{2\pi i}{9}}$	$e^{-\frac{4\pi i}{9}}$	$e^{\frac{8\pi i}{9}}$
18	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$	$e^{-\frac{\pi i}{3}}$
19	1	1	1	-1	-1	-1
20	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{\pi i}{3}}$	$e^{\frac{\pi i}{3}}$	$e^{\frac{\pi i}{3}}$

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