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# Groups of infinite rank with finite conjugacy classes of subnormal subgroups <sup>☆</sup>

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## ABSTRACT

A group is called a *V-group* if it has finite conjugacy classes of subnormal subgroups. It is proved here that if  $G$  is a periodic soluble group in which every subnormal subgroup of infinite rank has finitely many conjugates, then  $G$  is a *V-group*, provided that its Hirsch–Plotkin radical has infinite rank. Corresponding results for periodic soluble groups in which every subnormal subgroup of infinite rank has finite index in its normal closure and for those in which every subnormal subgroup of infinite rank is finite over its core, are also obtained. Moreover, it is shown that the assumption on the Hirsch–Plotkin radical can be avoided in the case of periodic groups with nilpotent commutator subgroup.

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## 1. Introduction

A group  $G$  is called a *T-group* if normality in  $G$  is a transitive relation, i.e. if all subnormal subgroups of  $G$  are normal. The structure of soluble *T-groups* has been described

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by W. Gaschütz [12] in the finite case and by D.J.S. Robinson [14] for arbitrary groups. Motivated by a celebrated result of B.H. Neumann [13] on groups with finite conjugacy classes of subgroups and taking as a model the known theory of  $T$ -groups, C. Casolo [2] investigated the class  $V$ , consisting of all groups in which every subnormal subgroup has only finitely many conjugates. Another relevant result of Neumann [13] shows that a group  $G$  has finite commutator subgroup if and only if every subgroup  $X$  of  $G$  has finite index in its normal closure  $X^G$ , and following this idea, Casolo [2] also studied the class  $T^*$  of groups in which every subnormal subgroup has finite index in its normal closure. Finally, the structure of groups  $G$  such that the index  $|X : X_G|$  is finite for every subgroup  $X$  has been considered by Neumann in a joint paper with J. Buckley, J.C. Lennox, H. Smith and J. Wiegold, where it was proved that any locally finite group with this property is abelian-by-finite (see [1]). In this case, the corresponding investigation of the class  $T_*$  of groups in which every subnormal subgroup is finite over its core was carried out in [11].

Recall that a group  $G$  is said to have *finite (Prüfer) rank*  $r = rk(G)$  if every finitely generated subgroup of  $G$  can be generated by at most  $r$  elements, and  $r$  is the least positive integer with such property. The study of the influence on a soluble group of the behavior of its subgroups of infinite rank has been developed in a series of recent papers (see for instance [3–10]); it suggests that in a group of infinite rank the behavior of subgroups of finite rank with respect to an embedding property can be neglected. In particular, it was proved in [7] that if  $G$  is a periodic soluble group of infinite rank in which every subnormal subgroup of infinite rank is normal, then  $G$  is a  $T$ -group. The aim of this paper is to provide a further contribution to this topic, considering periodic groups in which all subnormal subgroups of infinite rank are close to be normal, up to a finite section.

Let  $G$  be a group. We shall say that  $G$  is a  $V^+$ -group if each subnormal subgroup of infinite rank of  $G$  has only finitely many conjugates, or equivalently if the index  $|G : N_G(X)|$  is finite for every subnormal subgroup of infinite rank  $X$  of  $G$ . Similarly,  $G$  will be called a  $T^+$ -group if every subnormal subgroup of infinite rank of  $G$  has finite index in its normal closure. Finally, we will also consider the class  $T_+$  consisting of all groups  $G$  such that the index  $|X : X_G|$  is finite for each subnormal subgroup of infinite rank  $X$  of  $G$ . Of course,  $V$ ,  $T^*$  and  $T_*$  are (proper) subclasses of  $V^+$ ,  $T^+$  and  $T_+$ , respectively, and we will prove that these classes coincide within the universe of periodic soluble groups whose Hirsch–Plotkin radical has infinite rank (recall that the *Hirsch–Plotkin radical* of a group  $G$  is the largest locally nilpotent normal subgroup of  $G$ ). In fact, our first main result is the following.

**Theorem A.** *Let  $G$  be a periodic soluble group whose Hirsch–Plotkin radical has infinite rank. Then:*

- (a) *If  $G$  belongs to the class  $V^+$ , then it is a  $V$ -group.*
- (b) *If  $G$  belongs to the class  $T^+$ , then it is a  $T^*$ -group.*
- (c) *If  $G$  belongs to the class  $T_+$ , then it is a  $T_*$ -group.*

We leave here as an open question whether in the above statement the assumption that the Hirsch–Plotkin radical has infinite rank can be dropped out. It turns out that this is at least possible in the case of periodic metabelian groups. More in general, the following result will be proved.

**Theorem B.** *Let  $G$  be a periodic group of infinite rank with nilpotent commutator subgroup. Then:*

- (a) *If  $G$  belongs to the class  $V^+$ , then it is a  $V$ -group.*
- (b) *If  $G$  belongs to the class  $T^+$ , then it is a  $T^*$ -group.*
- (c) *If  $G$  belongs to the class  $T_+$ , then it is a  $T_*$ -group.*

An example will show that there exists a metabelian non-periodic group of infinite rank in the class  $V^+ \cap T^+ \cap T_+$  which does not belong to  $V \cup T^* \cup T_*$ . Therefore Theorem B cannot be extended to the case of non-periodic groups.

Most of our notation is standard and can be found in [15].

## 2. Proofs

Let  $G$  be a group, and let  $H$  and  $K$  be normal subgroups of  $G$  such that  $H \cap K = \{1\}$ . It is well-known that if  $X$  is any subgroup of  $G$  such that  $X \cap \langle H, K \rangle = \{1\}$ , then  $X = XH \cap XK$ . This property can be easily extended in the following way, which is useful for our purposes.

**Lemma 1.** *Let  $G$  be a group, and let  $X$  be a subgroup of  $G$ . If  $H$  and  $K$  are normal subgroups of  $G$  such that  $H \cap K$  is finite and contains  $X \cap HK$ , then the index  $|XH \cap XK : X|$  is finite.*

**Proof.** Consider the factor group  $\bar{G} = G/H \cap K$ . As

$$X(H \cap K) \cap HK = (H \cap K)(X \cap HK) = H \cap K,$$

we have

$$\bar{X} \cap \bar{H} \bar{K} = \bar{H} \cap \bar{K} = \{1\}.$$

Then  $\bar{X} \bar{H} \cap \bar{X} \bar{K} = \bar{X}$ , so that  $XH \cap XK = X(H \cap K)$ , and hence the index  $|XH \cap XK : X|$  is finite.  $\square$

Our next result shows that in any group  $G$  the normal closure of an arbitrary finite soluble subnormal subgroup cannot be infinite and of finite rank.

**Lemma 2.** *Let  $G$  be a group and let  $X$  be a finite soluble subnormal subgroup of  $G$ . If the normal closure  $X^G$  of  $X$  in  $G$  is infinite, then it has infinite rank.*

**Proof.** Assume for a contradiction that the statement is false, and choose a counterexample  $G$  containing a finite soluble subnormal subgroup  $X$  of smallest order such that  $X^G$  is infinite but has finite rank. Clearly,  $X$  contains a normal subgroup  $Y$  of prime index  $p$ , and by the minimal assumption on the order of  $X$ , we have that the normal subgroup  $Y^G$  is finite. Replacing  $G$  by the factor group  $G/Y^G$ , we may suppose without loss of generality that  $X$  has order  $p$ . Then all abelian subgroups of the locally finite  $p$ -group  $X^G$  satisfy the minimal condition, and so  $X^G$  is a Černikov group (see [15, Part 1, Theorem 3.32]). As all finite subgroups of  $X^G$  are subnormal, it follows that  $X^G$  is nilpotent. Then  $X^G$  has finite exponent, and hence it is even finite. This contradiction proves the lemma.  $\square$

Recall that a group  $G$  is said to be an  $FC$ -group if every element of  $G$  has only finitely many conjugates. It follows from the well-known Dietzmann's Lemma that a periodic group has the  $FC$ -property if and only if it is covered by finite normal subgroups; the same result also shows that a finite subgroup of a group  $G$  has finite index in its normal closure if and only if it has finitely many conjugates in  $G$ .

**Lemma 3.** *Let  $G$  be a group of infinite rank in the class  $V^+ \cup T^+$ . If  $X$  is a finite soluble subnormal subgroup of  $G$ , then the normal closure  $X^G$  of  $X$  in  $G$  is finite.*

**Proof.** Assume for a contradiction that the statement is false, and choose a counterexample  $G$  containing a finite subnormal subgroup  $X$  of smallest possible order such that the normal closure  $X^G$  is infinite. As  $X$  is soluble, its commutator subgroup  $X'$  has a finite normal closure in  $G$ , and hence the normalizer  $N_G(X')$  has finite index in  $G$ . Let  $K$  be the core of  $N_G(X')$  in  $G$ . Then  $KX$  is a subnormal subgroup of finite index of  $G$ , so that it is likewise a group of infinite rank in the class  $T^+ \cup V^+$ . Clearly,  $X/X'$  has infinitely many conjugates in  $KX/X'$ , so that  $KX/X'$  is also a counterexample and hence it follows from our minimal choice that  $X$  is abelian. Thus each proper subgroup of  $X$  has finitely many conjugates in  $G$ . In particular,  $X$  cannot be generated by two proper subgroups and so it is a  $p$ -group for some prime number  $p$ . Moreover, the normalizer  $N_G(X^p)$  has finite index in  $G$ , so that the same argument used above shows that  $X^p = \{1\}$  and  $|X| = p$ .

Among all counterexamples  $G$  containing a subnormal subgroup  $X$  of prime order  $p$  which is not almost normal, choose one for which the defect  $k$  of  $X$  in  $G$  is minimal. The normal closure  $X^G$  of  $X$  has infinite rank by Lemma 2, and  $X$  has defect  $k - 1$  in  $X^G$ , so that  $X$  has only finitely many conjugates in  $X^G$ . Then  $X^G$  is covered by its finite normal subgroups, and hence it is an  $FC$ -group. As  $X^G$  is a locally finite  $p$ -group, it follows that

$$X^G = \bigcup_{n \in \mathbb{N}} Z_n(X^G)$$

(see for instance [16, Corollary 1.15]). Obviously, there exists a positive integer  $c$  such that  $X$  is contained in  $Z_c(X^G)$ , and hence  $X^G = Z_c(X^G)$  is nilpotent. Moreover, the centralizer  $C_{X^G}(X)$  has finite index in  $X^G$ , and so it contains two abelian subgroups  $A_1$  and  $A_2$  of infinite rank such that

$$\langle A_1, A_2 \rangle = A_1 \times A_2$$

and

$$\langle A_1, A_2 \rangle \cap X = \{1\}.$$

Since  $A_1$  and  $A_2$  are subnormal subgroups of  $G$  centralizing  $X$ , also the products  $XA_1$  and  $XA_2$  are subnormal in  $G$ . If  $G$  is a  $T^+$ -group, it follows that the indices  $|(XA_1)^G : XA_1|$  and  $|(XA_2)^G : XA_2|$  are finite, while if  $G$  belongs to the class  $V^+$  we obtain that  $XA_1$  and  $XA_2$  have finitely many conjugates. In both cases, the subgroup

$$X = XA_1 \cap XA_2$$

has finitely many conjugates in  $G$ , and this last contradiction completes the proof of the lemma.  $\square$

As a consequence of Lemma 3, it can be observed that the equality

$$V^+ \cap T_* = T^+ \cap T_* = V \cap T^*$$

holds at least within the universe of soluble groups of infinite rank. In fact, we have:

**Corollary 4.** *Let  $G$  be a soluble  $T_*$ -group of infinite rank. Then:*

- (a) *If  $G$  belongs to the class  $V^+$ , then it is a  $V$ -group.*
- (b) *If  $G$  belongs to the class  $T^+$ , then it is a  $T^*$ -group.*

**Proof.** Let  $X$  be any subnormal subgroup of finite rank of  $G$ . Then the index  $|X : X_G|$  is finite and  $G/X_G$  is a group of infinite rank in the class  $V^+ \cup T^+$ , so that it follows from Lemma 3 that  $X^G/X_G$  is finite. Therefore both indices  $|G : N_G(X)|$  and  $|X^G : X|$  are finite, and the statement is proved.  $\square$

**Lemma 5.** *Let  $G$  be a  $V^+$ -group containing an abelian subnormal subgroup of infinite rank. Then every soluble subnormal subgroup of  $G$  has finitely many conjugates.*

**Proof.** Clearly, the group  $G$  contains an abelian subnormal subgroup  $A$  of infinite rank, which is either free abelian or a direct product of a collection of subgroups of prime order. Let  $X$  be any soluble subnormal subgroup of finite rank of  $G$ . Observe that, if  $A$

is torsion-free, the subgroup  $X \cap A$  is finitely generated. Then in any case  $A$  contains two subgroups  $U$  and  $V$  of infinite rank such that

$$\langle U, V \rangle = U \times V$$

and

$$X \cap \langle U, V \rangle = \{1\}.$$

The intersection  $N_G(U) \cap N_G(V)$  is a subgroup of finite index of  $G$ , and hence also its core  $K$  has finite index in  $G$ . The subgroup  $Y = X \cap K$  is subnormal in  $G$  and normalizes both  $U$  and  $V$ , so that the products  $YU$  and  $YV$  are subnormal subgroups of infinite rank of  $G$ . Thus  $YU$  and  $YV$  have finitely many conjugates in  $G$ , and so also the conjugacy class of

$$Y = YU \cap YV$$

is finite. Thus the normalizer  $N_G(Y)$  has finite index in  $G$ , and so also its core  $L$  has finite index. Consider the subnormal subgroup  $H = XL$  of  $G$ . Since  $H/Y$  is a  $V^+$ -group of infinite rank and  $X/Y$  is a finite soluble subnormal subgroup of  $H/Y$ , it follows from [Lemma 3](#) that  $X/Y$  has finitely many conjugates in  $H/Y$ . Therefore  $X$  has finitely many conjugates in  $H$ , and so also in  $G$ .  $\square$

Corresponding results can be proved for the class of periodic  $T^+$ -groups and for that of  $T_+$ -groups.

**Lemma 6.** *Let  $G$  be a periodic  $T^+$ -group containing an abelian subnormal subgroup of infinite rank. Then  $G$  is a  $T^*$ -group.*

**Proof.** Let  $A$  be an abelian subnormal subgroup of infinite rank of  $G$ . Then the index  $|A^G : A|$  is finite, so that  $A^G$  is abelian-by-finite and hence it contains an abelian characteristic subgroup  $B$  of finite index. Clearly, the socle  $S$  of  $B$  is a normal subgroup of infinite rank of  $G$ , and it is a direct product of subgroups of prime order. Let  $X$  be any subnormal subgroup of finite rank of  $G$ , and let  $S_0$  be a subgroup of  $S$  such that

$$S = (X \cap S) \times S_0.$$

Then  $S_0$  has infinite rank, and so it has finite index in its normal closure  $N = S_0^G$ . It follows that the intersection  $X \cap N$  is finite. Clearly,  $N$  contains subgroups of infinite rank  $U$  and  $V$  such that

$$N/X \cap N = (U/X \cap N) \times (V/X \cap N).$$

In particular, the intersection  $U \cap V = X \cap N$  is finite. Put  $H = U^G$  and  $K = V^G$ . Then  $H \cap K$  is finite and

$$X \cap HK \leq X \cap N \leq H \cap K,$$

so that it follows from [Lemma 1](#) that the index  $|XH \cap XK : X|$  is finite. On the other hand,  $XH$  and  $XK$  are subnormal subgroups of infinite rank of  $G$ , so that each of them has finite index in its normal closure and hence also the index  $|X^G : X|$  is finite. Therefore  $G$  is a  $T^*$ -group.  $\square$

**Lemma 7.** *Let  $G$  be a  $T_+$ -group containing an abelian subnormal subgroup of infinite rank. Then  $G$  is a  $T_*$ -group.*

**Proof.** Clearly, the group  $G$  contains an abelian subnormal subgroup  $A$  of infinite rank, which is either free abelian or a direct product of a collection of subgroups of prime order. Let  $X$  be any soluble subnormal subgroup of finite rank of  $G$ . Clearly, if  $A$  is torsion-free, the subgroup  $X \cap A$  is finitely generated. Then in any case  $A$  contains two subgroups  $U$  and  $V$  of infinite rank such that

$$\langle U, V \rangle = U \times V$$

and

$$X \cap \langle U, V \rangle = \{1\}.$$

As  $G$  is a  $T_+$ -group, the indices  $|U : U_G|$  and  $|V : V_G|$  are finite, so that  $U_G$  and  $V_G$  have infinite rank. It follows that the subnormal subgroups  $XU_G$  and  $XV_G$  are finite over their cores in  $G$ , and hence also

$$X = XU_G \cap XV_G$$

is finite over its core  $X_G$ . Therefore  $G$  is a  $T_*$ -group.  $\square$

The next lemma collects some known results on the Fitting subgroup of a group in the class  $V \cup T^* \cup T_*$  (see [\[2, Lemma 3.1\]](#) and [\[11, Corollary 3.2\]](#)).

**Lemma 8.** *Let  $G$  be a group in the class  $V \cup T^* \cup T_*$ . Then the Fitting subgroup of  $G$  is nilpotent.*

In the case of  $(V^+ \cup T^+ \cup T_+)$ -groups, the following weak extension can be obtained.

**Lemma 9.** *Let  $G$  be a periodic group of infinite rank in the class  $V^+ \cup T^+ \cup T_+$ , and let  $F$  be the Fitting subgroup of  $G$ . Then all Sylow subgroups of  $F$  are nilpotent.*

**Proof.** Let  $p$  be a prime number, and let  $P$  be the unique Sylow  $p$ -subgroup of  $F$ . Consider any subnormal subgroup of finite rank  $X$  of  $P$ . Then  $X$  is a Černikov group, so that it contains a divisible abelian normal subgroup  $J$  of finite index, and  $X = JE$  for a suitable finite subgroup  $E$ . Let  $K$  be a nilpotent normal subgroup of  $P$ . Then  $JK$  is nilpotent, and hence  $[J, K] = \{1\}$  (see [15, Part 1, Lemma 3.13]). As  $P$  is generated by its nilpotent normal subgroups, it follows that  $J$  is contained in  $Z(P)$ , and in particular  $X/X_P$  is finite. Moreover, if the normal subgroup  $E^P$  of  $P$  is finite, we also have that  $X$  has finite index in  $X^P = JE^P$  and that the index  $|P : N_P(X)|$  is finite. Therefore an application of Lemma 3 yields that  $P$  belongs to the class  $V \cup T^* \cup T_*$ , and hence it is nilpotent by Lemma 8. The statement is proved.  $\square$

We are now in a position to prove Theorem A. Recall that for any group  $G$ , the set of prime numbers which are orders of elements of  $G$  is usually denoted by  $\pi(G)$ .

**Proof of Theorem A.** (a) Assume for a contradiction that the statement is false, and choose a counterexample  $G$  containing a subnormal subgroup  $X$  of smallest possible derived length such that the conjugacy class of  $X$  in  $G$  is infinite. Then the commutator subgroup  $X'$  of  $X$  has finitely many conjugates in  $G$ , and the same argument used in the first part of the proof of Lemma 3 shows that  $X$  must be abelian, so that it is contained in the Hirsch–Plotkin radical  $H$  of  $G$ . Suppose first that there exists a prime number  $p$  such that the Sylow  $p$ -subgroup  $P$  of  $H$  has infinite rank, and let  $F$  be the Fitting subgroup of  $P$ . As  $C_P(F) \leq F$ , the subgroup  $F$  cannot be a Černikov group, because every periodic group of automorphisms of a Černikov group is likewise a Černikov group (see [15, Part 1, Theorem 3.29]), and hence it has finite rank. Moreover, the subgroup  $F$  is nilpotent by Lemma 9. If  $A$  is a maximal abelian normal subgroup of  $F$ , then  $C_F(A) = A$ , so that  $A$  has infinite rank and  $G$  is a  $V$ -group by Lemma 5. This contradiction shows that every Sylow subgroup of  $H$  has finite rank.

Since the group  $H$  has infinite rank, it can be decomposed into a direct product  $H = H_1 \times H_2$  of two subgroups of infinite rank  $H_1$  and  $H_2$  such that

$$\pi(H_1) \cap \pi(H_2) = \emptyset.$$

For  $i = 1, 2$  we may now consider a further direct decomposition

$$H_i = H_{i,1} \times H_{i,2},$$

where the factors are coprime and have infinite rank. Put  $\pi_i = \pi(H_i)$ , and let  $X_i$  the  $\pi_i$ -component of  $X$  ( $i = 1, 2$ ). If  $i \neq j$ , the products  $X_i H_{j,1}$  and  $X_i H_{j,2}$  are subnormal subgroups of infinite rank of  $G$ , so that they have finitely many conjugates and hence also the conjugacy class of

$$X_i = X_i H_{j,1} \cap X_i H_{j,2}$$



is finite. It follows that  $X = X_1X_2$  has likewise finitely many conjugates in  $G$ , and this contradiction completes the proof of part (a) of the theorem.

(b) Assume for a contradiction that the statement is false, and choose a counterexample  $G$  containing a subnormal subgroup  $X$  of smallest possible derived length such that the index  $|X^G : X|$  is infinite. Then the commutator subgroup  $Y = X'$  of  $X$  has finite index in its normal closure  $Y^G$ , and so also the index  $|XY^G : X|$  is finite. Suppose that  $X$  is not abelian. Since  $X$  has finite rank, it is clear that also  $Y^G$  has finite rank, and so the Hirsch–Plotkin radical of  $G/Y^G$  has infinite rank. Then the minimal assumption on the derived length of  $X$  yields that  $XY^G/Y^G$  has finite index in its normal closure in  $G/Y^G$ , so that  $XY^G$  has finite index in  $X^G$  and hence also the index  $|X^G : X|$  is finite. This contradiction shows that  $X$  must be abelian, so that it is contained in the Hirsch–Plotkin radical of  $G$ . The proof can now be completed using the argument introduced in the proof of part (a).

(c) Assume for a contradiction that the statement is false, and choose a counterexample  $G$  containing a subnormal subgroup  $X$  of smallest possible derived length such that the index  $|X : X_G|$  is infinite. Consider the commutator subgroup  $Y = X'$  of  $X$ , and the factor group  $\bar{G} = G/Y_G$ . Clearly,  $Y_G$  has finite rank, and hence the Hirsch–Plotkin radical of  $G/Y_G$  has infinite rank. Thus it follows from the assumption on the derived length of  $X$  that the index  $|Y : Y_G|$  is finite, so that  $\bar{X} = X/Y_G$  has finite commutator subgroup. As  $G$  is a  $T_+$ -group, the subgroup  $X$  has finite rank, and hence the center  $\bar{C} = Z(\bar{X})$  has finite index in  $\bar{X}$ . Then  $C$  is a subnormal subgroup of finite index of  $X$ , and so the index  $|C : C_G|$  must be infinite. Therefore also the pair  $(\bar{G}, \bar{C})$  is a counterexample, and again our minimal choice yields that  $X$  has to be abelian, so that  $X$  lies in the Hirsch–Plotkin radical of  $G$ . Also in this case, the proof can now be completed as part (a).  $\square$

The proof of [Theorem B](#) essentially depends on the following lemma.

**Lemma 10.** *Let  $G$  be a periodic group, and let  $L$  be a nilpotent normal subgroup of finite rank of  $G$  such that the set  $\pi(G)$  is infinite. If  $G/Z(L)$  contains an abelian subgroup of infinite rank, then there exist a sequence  $(S_n)_{n \in \mathbb{N}}$  of subgroups of  $G$  and a sequence  $(p_n)_{n \in \mathbb{N}}$  of pairwise different primes in  $\pi(L)$  satisfying the following conditions:*

- (a) *every  $S_n$  contains  $Z = Z(L)$ , and  $S_n/Z$  is a finite abelian group of prime exponent;*
- (b)  *$rk(S_n) < rk(S_{n+1})$  for all positive integers  $n$ ;*
- (c)  *$[S_m, L_{p_n}] = \{1\}$  for all  $m$  and  $n$ .*

**Proof.** Put  $C_p = C_G(L_p)$  for each prime number  $p \in \pi(L)$ . As  $L_p$  is a nilpotent  $p$ -group of finite rank, it is central-by-finite, so that  $G/C_p$  is finite and hence  $C_p/Z$  contains an abelian subgroup of infinite rank. Moreover, there exists a positive integer  $k$ , depending on the rank and on the nilpotency class of  $L$  only, such that  $rk(G/C_p) \leq k$  for all  $p$ .

Let  $p_1$  be a prime in  $\pi(L)$ . Clearly,  $C_{p_1}/Z$  contains a finite abelian subgroup  $X_1/Z$  of prime exponent such that  $rk(X_1/Z) > k$ . As  $X_1/Z$  is finite, there exists an infinite subset  $\pi_1$  of  $\pi(L)$  such that

$$C_{X_1}(L_p) = C_{X_1}(L_q) = S_1$$

for all  $p, q$  in  $\pi_1$ . Then  $rk(S_1) \geq 1$  and  $[S_1, L_{p_1}] = \{1\}$ . Suppose now that  $S_1, \dots, S_h$  are subgroups of  $G$  and  $p_1, \dots, p_h$  are pairwise different primes in  $\pi(L)$  for which the following conditions hold:

- $Z$  is contained in  $S_j$ , and  $S_j/Z$  is a finite abelian group of prime exponent for all  $j \leq h$ ;
- $rk(S_1) < rk(S_2) < \dots < rk(S_h)$ ;
- $[\langle S_1, \dots, S_h \rangle, L_{p_j}] = \{1\}$  for all  $j \leq h$ ;
- there exists an infinite subset  $\pi_h$  of  $\pi(L)$  such that

$$[\langle S_1, \dots, S_h \rangle, L_p] = \{1\}$$

for all  $p \in \pi_h$ .

Let  $p_{h+1}$  be a prime number in  $\pi_h \setminus \{p_1, \dots, p_h\}$ , and put

$$C = C_{p_1} \cap \dots \cap C_{p_h} \cap C_{p_{h+1}}.$$

Choose a positive integer  $m > rk(S_h)$ . Then  $C/Z$  contains a finite abelian subgroup  $X_{h+1}/Z$  of prime exponent such that  $rk(X_{h+1}/Z) \geq k + m$ . As  $X_{h+1}/Z$  is finite, there exists an infinite subset  $\pi_{h+1}$  of  $\pi_h$  such that

$$C_{X_{h+1}}(L_p) = C_{X_{h+1}}(L_q) = S_{h+1}$$

for all  $p, q$  in  $\pi_{h+1}$ . Then

$$rk(S_h) < m \leq rk(S_{h+1}/Z) \leq rk(S_{h+1}).$$

Moreover,

$$[\langle S_1, \dots, S_h \rangle, L_{p_{h+1}}] = \{1\}$$

because  $p_{h+1} \in \pi_h$ , and hence

$$[\langle S_1, \dots, S_h, S_{h+1} \rangle, L_{p_j}] = \{1\}$$

for all  $j \leq h + 1$ . It is also clear that

$$[\langle S_1, \dots, S_h, S_{h+1} \rangle, L_p] = \{1\}$$

for all  $p \in \pi_{h+1}$ . The iteration of this method gives the required sequences  $(S_n)_{n \in \mathbb{N}}$  and  $(p_n)_{n \in \mathbb{N}}$ , and so the lemma is proved.  $\square$

**Corollary 11.** *Let  $G$  be a periodic group, and let  $L$  be a nilpotent normal subgroup of finite rank of  $G$  such that the set  $\pi(L)$  is infinite. If  $G/L$  is an abelian group of infinite rank, then there exists an infinite subset  $\pi$  of  $\pi(L)$  such that the factor group  $G/L_{\pi'}$  contains a nilpotent normal subgroup of infinite rank.*

**Proof.** Since  $G$  is soluble, the factor group  $G/Z(L)$  contains an abelian subgroup of infinite rank, and hence it follows from Lemma 10 that there exist a sequence  $(S_n)_{n \in \mathbb{N}}$  of subgroups of  $G$  and a sequence  $(p_n)_{n \in \mathbb{N}}$  of pairwise different primes in  $\pi(L)$  such that:

- $Z = Z(L)$  is contained in every  $S_n$ , and  $S_n/Z$  is a finite abelian group of prime exponent;
- $rk(S_n) < rk(S_{n+1})$  for all  $n$ ;
- $[S_m, L_{p_n}] = \{1\}$  for all  $m$  and  $n$ .

Consider the infinite set  $\pi = \{p_n \mid n \in \mathbb{N}\}$ . As the group  $G/L$  is abelian, the centralizer

$$C/L_{\pi'} = C_{G/L_{\pi'}}(L/L_{\pi'})$$

is a nilpotent normal subgroup of  $G/L_{\pi'}$ . Moreover, the subgroup of infinite rank  $\langle S_n \mid n \in \mathbb{N} \rangle$  is contained in  $C$ , and hence  $C/L_{\pi'}$  has infinite rank.  $\square$

**Proof of Theorem B.** (a) Assume for a contradiction that the statement is false. As in the proof of Theorem A(a), it can be proved that there exists a counterexample  $G$  containing an abelian subnormal subgroup  $X$  with infinitely many conjugates. Then  $L = XG'$  is a nilpotent normal subgroup of  $G$ , and so it follows from Theorem A(a) that  $L$  must have finite rank. Let  $p$  be any prime in the set  $\pi(X)$ . As  $L/L_{p'}$  is a nilpotent  $p$ -group of finite rank, the factor group  $G/C_G(L/L_{p'})$  is finite, and hence  $C_{G/L_{p'}}(L/L_{p'})$  is a nilpotent normal subgroup of infinite rank of  $G/L_{p'}$ . Thus  $G/L_{p'}$  is a  $V$ -group by Theorem A(a), so that  $X_p L_{p'}$  has finitely many conjugates in  $G$ , and hence also the conjugacy class of  $X_p$  in  $G$  is finite. Therefore the normalizer of each primary component of  $X$  has finite index in  $G$ , and so the set

$$\{N_G(X_p) \mid p \in \pi(X)\}$$

must be infinite. Choose an infinite subset  $\pi$  of  $\pi(X)$  such that

$$N_G(X_p) \neq N_G(X_q)$$

for all different primes  $p, q$  in  $\pi$ , and put

$$Y = \langle X_p \mid p \in \pi \rangle = \operatorname{Dr}_{p \in \pi} X_p.$$

Then

$$N_G(Y) = \bigcap_{p \in \pi} N_G(X_p),$$

and so the index  $|G : N_G(Y)|$  is infinite. Moreover,  $Y$  is a characteristic subgroup of  $YL_{\pi'}$ , and hence replacing  $G$  by  $G/L_{\pi'}$ , it can be assumed without loss of generality that  $\pi(L) = \pi$ . Application of [Corollary 11](#) yields that  $\pi$  contains an infinite subset  $\pi_0$  such that  $G/L_{\pi'_0}$  has a nilpotent normal subgroup of infinite rank. Thus  $G/L_{\pi'_0}$  is a  $V$ -group again by [Theorem A\(a\)](#), and so  $X_{\pi_0}L_{\pi'_0}$  has finitely many conjugates in  $G$ . It follows that also the conjugacy class of  $X_{\pi_0}$  in  $G$  is finite, which is impossible because

$$N_G(X_{\pi_0}) = \bigcap_{p \in \pi_0} N_G(X_p)$$

has infinite index in  $G$ . This contradiction proves statement (a).

(b) Assume for a contradiction that the statement is false. As in the proof of [Theorem A\(b\)](#), there exists a counterexample  $G$  containing an abelian subnormal subgroup  $X$  such that the index  $|X^G : X|$  is infinite. Then  $L = XG'$  is a nilpotent normal subgroup of  $G$ , and hence it has finite rank by [Theorem A\(b\)](#). If  $p$  is any prime in the set  $\pi(X)$ , the group  $G/L_{p'}$  contains the nilpotent normal subgroup of infinite rank  $C_{G/L_{p'}}(L/L_{p'})$ . Then  $G/L_{p'}$  is a  $T^*$ -group. It follows that  $X_pL_{p'}$  has finite index in its normal closure in  $G$ , and so also the index  $|X_p^G : X_p|$  is finite. Therefore each primary component of  $X$  has finite index in its normal closure in  $G$ . Thus the set  $\pi$  of all primes  $p$  in  $\pi(X)$  such that  $X_p$  is not normal in  $G$  must be infinite. Also in this case, an application of [Corollary 11](#) yields that  $\pi$  contains an infinite subset  $\pi_0$  such that  $G/L_{\pi'_0}$  has a nilpotent normal subgroup of infinite rank. Then  $G/L_{\pi'_0}$  is a  $T^*$ -group by [Theorem A\(b\)](#), and in particular  $XL_{\pi'_0}$  has finite index in its normal closure in  $G$ , a contradiction, because  $\pi_0$  is infinite and  $X_p$  is not normal in  $G$  for each  $p \in \pi_0$ .

(c) Assume for a contradiction that the statement is false. As in the proof of [Theorem A\(c\)](#), we can choose a counterexample  $G$  containing an abelian subnormal subgroup  $X$  such that the index  $|X : X_G|$  is infinite. Then  $L = XG'$  is a nilpotent normal subgroup of  $G$ , and hence it has finite rank. As in the proofs of the previous statements, it can be shown that  $G/L_{p'}$  is a  $T_*$ -group for each prime  $p \in \pi(X)$ , and hence that every primary component of  $X$  is normal-by-finite in  $G$ . In particular, the set  $\pi(X)$  is infinite. Moreover, the argument used in the proof of statement (b) can be adapted to prove that  $\pi(X)$  contains an infinite subset  $\pi_0$  such that  $G/L_{\pi'_0}$  is a  $T_*$ -group, but  $X_p$  is not normal in  $G$  for each  $p \in \pi_0$ . This contradiction completes the proof of the theorem.  $\square$

The following example was constructed in [7], and proves also that in the statement of [Theorem B](#) the periodicity assumption cannot be dropped out.

Let  $A$  be the additive group of rational numbers, and for each prime number  $p$  let  $x_p$  be the automorphism of  $A$  defined by setting  $ax_p = pa$  for each element  $a$  of  $A$ . Then

$$X = \operatorname{Dr}_p \langle x_p \rangle$$

is a free abelian subgroup of infinite rank of the full automorphism group of  $A$ , and the semidirect product

$$G = X \ltimes A$$

is a torsion-free metabelian group. Clearly,  $A$  is a minimal normal subgroup of  $G$ , and so  $G$  does not belong to the class  $V \cup T^* \cup T_*$ . On the other hand, it is easy to show that any subnormal subgroup of  $G$  either is contained in  $A$  or contains  $A$ ; in particular, all subnormal subgroups of infinite rank of  $G$  contain  $A$ , so that they are normal in  $G$ , and  $G$  lies in the class  $V^+ \cap T^+ \cap T_+$ .

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