The relation type of affine algebras and algebraic varieties

Francesc Planas-Vilanova

Departament de Matemàtica Aplicada 1, Universitat Politècnica de Catalunya, Diagonal 647, ETSEIB, 08028 Barcelona, Catalunya

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We introduce the notion of relation type of an affine algebra and prove that it is well defined by using the Jacobi–Zariski exact sequence of André–Quillen homology. In particular, the relation type is an invariant of an affine algebraic variety. Also as a consequence of the invariance, we show that in order to calculate the relation type of an ideal in a polynomial ring one can reduce the problem to trinomial ideals. When the relation type is at least two, the extreme equidimensional components play no role. This leads to the non-existence of affine algebras of embedding dimension three and relation type two.

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1. Introduction

Let $A = R/I = k[x_1, \ldots, x_n]/I$ be an affine k-algebra, where k is a field, $R = k[x_1, \ldots, x_n]$ is the polynomial ring in the variables $x_1, \ldots, x_n$ over k and $I = (f_1, \ldots, f_s)$ is an ideal of R.

In this note we introduce the following invariant of A: the *relation type of A* is defined as $\mathrm{rt}(A) = \mathrm{rt}(I)$, where $\mathrm{rt}(I)$ stands for the relation type of the ideal I.

Recall that if $\mathbf{R}(I) = R[It] = \oplus_{q \geq 0} I^q t^q$ is the *Rees algebra* of I and $\varphi : S = R[t_1, \ldots, t_s] \to \mathbf{R}(I)$ is the natural graded polynomial presentation sending t_i to $f_i t$, then $L = \ker(\varphi) = \oplus_{q \geq 1} L_q$, referred to as the ideal of *equations* of I, is a graded ideal of S and the relation type of I, denoted by $\mathrm{rt}(I)$, is the least integer $N \geq 1$ such that L is generated by its components of degree at most N. Concerning the equations of an ideal and its relation type, see for instance, and with no pretense of being exhaustive, [6–8,13–15,17–19,21,22], and the references therein.

By means of the Jacobi–Zariski exact sequence of André–Quillen homology, we prove that the definition of $\mathrm{rt}(A)$ does not depend on the presentation of A. In particular, we obtain an invariant for affine algebraic varieties. Concretely, if V is an affine algebraic k-variety, the *relation type* of V is defined as $\mathrm{rt}(V) = \mathrm{rt}(k[V])$, the relation type of its coordinate ring $k[V]$. Another consequence is that in order to calculate the relation type of an ideal in a polynomial ring one can reduce the problem to ideals generated by trinomials, though at the cost of introducing more generators and more variables.

We then study the connection between the equidimensional decomposition of a radical ideal I of $R = k[x_1, \ldots, x_n]$ and its relation type $\mathrm{rt}(I)$. We conclude that the equidimensional components of dimension 1 and n are not relevant whenever the relation type is at least two. As a corollary we obtain a somewhat surprising result, namely, that there are no affine k-algebras of embedding dimension three and relation type two.

Due to the aforementioned result, the examples we provide are essentially focussed on affine space curves. At this point one should emphasize that the explicit calculation of the equations of an ideal is computationally a very expensive task. The reduction to trinomial ideals, unfortunately, does not seem to improve, in general, the approach to the problem. It would be desirable to obtain a wide range of irreducible affine space curves with prescribed relation type. It would also be interesting to understand better the geometric meaning of the relation type of an algebraic variety.

Notice that, to our knowledge, there is at least another notion also named *relation type of an algebra*. Indeed, in [20, Definition 2.7], W.V. Vasconcelos defines the relation type of a standard algebra $A = k[x_1, \ldots, x_n]/I$ as the least integer s such that $I = (I_1, \ldots, I_s)$, where $I \subset (x_1, \ldots, x_n)^2$ is a homogeneous ideal of the polynomial ring $k[x_1, \ldots, x_n]$ over the field k and I_i is the i-th graded component of I. It is clear that both definitions do not coincide, even in the homogeneous case; for instance, take $I = (f)$ a principal ideal of $k[x_1, \ldots, x_n]$ generated by a homogeneous polynomial f of degree $p \geq 2$.

The paper is organized as follows. In Section 2, we set the notations used throughout. In Section 3, we prove the invariance theorem and the reduction to trinomials. Section 4 is devoted to study the effect of an equidimensional decomposition in the computation of the relation type. Finally, in Sections 5 and 6, we give some examples in embedding dimension three.

2. Notations and preliminaries

We begin by setting some notations. Let $I = (f_1, \ldots, f_s)$ be an ideal of a Noetherian ring R and let $\mathbf{R}(I) = R[It] = \oplus_{q \geq 0} I^q t^q$ be its Rees ring. Let $S = R[t_1, \ldots, t_s]$ be the polynomial ring over R and $\varphi : S \to \mathbf{R}(I)$ the graded polynomial presentation of $\mathbf{R}(I)$ sending t_i to $f_i t$. Set $L = \ker(\varphi) = \oplus_{q \geq 1} L_q$, the graded ideal of equations of I. Given $q \geq 1$, let $L\langle q \rangle \subseteq L$ be the ideal generated by the homogeneous equations of I of degree at most q. The relation type of I, denoted by $\mathrm{rt}(I)$, is the least integer $N \geq 1$ such that $L = L\langle N \rangle$. Although L depends on the chosen generating set, $\mathrm{rt}(I)$ does not. Indeed, let $\mathbf{S}(I)$ be the symmetric algebra of I and let $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical graded morphism induced by the identity on degree 1. Let $0 \to Z_1 \to R^s \to I \to 0$ be the presentation associated to $f_1, \ldots, f_s$, where Z_1 stands for the first module of syzygies of I. Applying the symmetric functor, one gets the graded exact sequence

$$0 \to Z_1 S \longrightarrow S = R[t_1, \ldots, t_s] = \mathbf{S}(R^s) \xrightarrow{\Phi} \mathbf{S}(I) \to 0,$$

where $Z_1 S$ is the ideal of S generated by the elements of Z_1 regarded as linear forms of S. Thus $\varphi = \alpha \circ \Phi : S \to \mathbf{R}(I)$ and $Z_1 S$ can be interpreted as the ideal $L\langle 1 \rangle$ of linear equations of I. For each $q \geq 2$, set $E(I)_q = \ker(\alpha_q)/I \cdot \ker(\alpha_{q-1})$ and call it the module of *effective q-relations* of I. One can show that the so-called module of *fresh generators in degree q*, $(L/L\langle q-1 \rangle)_q = L_q/S_1 L_{q-1}$, is isomorphic to $E(I)_q$ (see, e.g., [22, before Definition 1.9] and [17, Theorem 2.4]). In particular, $L_q/S_1 L_{q-1}$ does not depend on the presentation of I. Furthermore, $\mathrm{rt}(I)$ can be thought as $\mathrm{rt}(I) = \min\{r \geq 1 \mid E(I)_q = 0 \text{ for all } q \geq r+1\}$. It follows that $\mathrm{rt}(I)$ is independent too of the presentation of I and that of $\mathbf{R}(I)$.

Observe that $\mathrm{rt}(I) = 1$ if and only if $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is an isomorphism. In such a case I is said to be of *linear type*. If $\alpha_2 : \mathbf{S}_2(I) \to I^2$ is an isomorphism, I is said to be *syzygetic*. It is well known that ideals generated by a regular sequence (more in general, by a d-sequence) are of linear type (see, e.g., [8] or [7]). We recall too that the relation type is a local invariant, in other words, $\mathrm{rt}(I) = \sup\{\mathrm{rt}(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} = \sup\{\mathrm{rt}(I_{\mathfrak{m}}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}$ (see, e.g., [17, Example 3.2]; here Max stands for the set of maximal ideals). Moreover, it is enough to localize at primes $\mathfrak{p}$ or maximals $\mathfrak{m}$ that contain I (considering that the improper ideal $I = R$ has relation type 1).

More generally, let R be a Noetherian ring and let $U = \oplus_{q \geq 0} U_q$ be a standard R-algebra, i.e., $U_0 = R$ and U is an R-algebra finitely generated by elements $f_1, \ldots, f_s$ of U_1. Let $\mathbf{S}(U_1)$ be the symmetric algebra of U_1 and $\alpha : \mathbf{S}(U_1) \to U$ be the canonical graded morphism induced by the identity on U_1. For $q \geq 2$, define $E(U)_q = \ker(\alpha_q)/U_1 \cdot \ker(\alpha_{q-1})$. If $\varphi : S = R[t_1, \ldots, t_s] \to U$ is the graded polynomial presentation of U sending t_i to f_i and $L = \ker(\varphi)$, then $E(U)_q \cong L_q/S_1 L_{q-1}$, for all $q \geq 2$ [17, Theorem 2.4]. The relation type of U can be defined as $\mathrm{rt}(U) = \min\{r \geq 1 \mid E(U)_q = 0 \text{ for all } q \geq r+1\}$, i.e., the maximum degree appearing in a minimal generating

set of equations of U. For $U = \mathbf{R}(I)$ one has $E(\mathbf{R}(I))_q = E(I)_q$ and $\mathrm{rt}(\mathbf{R}(I)) = \mathrm{rt}(I)$, recovering the definitions above.

The double expression of $E(U)_q$ as $\ker(\alpha_q)/U_1 \cdot \ker(\alpha_{q-1})$ and $L_q/S_1 L_{q-1}$ has an advantage; while the first is canonical, the second is easier to deal with. To prove that the relation type of an ideal of a polynomial ring solely depends on the quotient ring, we use a third expression. Indeed, for each $q \geq 2$, there exists a graded isomorphism of R-modules $E(U)_q \cong H_1(R, U, R)_q$, where $H_1(R, U, R) = \oplus_{q \geq 0} H_1(R, U, R)_q$ stands for the first André–Quillen homology group associated to the homomorphism of commutative rings $R \to U$ and the U-module $R = U/(\oplus_{q > 0} U_q)$ (see [1] referring to the homology of commutative rings and particularly [17, Remark 2.3] for this result).

3. The invariance of the relation type

We begin by proving the invariance with respect to the quotient of the relation type of an ideal in a polynomial ring.

Theorem 3.1. *Let k be a field and $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be variables over k. Let I be an ideal of $R = k[x_1, \ldots, x_n]$ and let J be an ideal of $S = k[y_1, \ldots, y_m]$. Suppose that there exists an isomorphism of k-algebras $R/I \cong S/J$. Then $\mathrm{rt}(I) = \mathrm{rt}(J)$.*

Proof. Let $\mathbf{G}(I) = \mathbf{R}(I)/I\mathbf{R}(I) = \oplus_{q \geq 0} I^q/I^{q+1}$ be the associated graded ring of I and let $\mathbf{G}(J)$ be the associated graded ring of J. Using [10, Exercise 13, Chapter V, § 5], one deduces that there exists a graded isomorphism of R/I-algebras:

$$\mathbf{G}(I)[y_1, \ldots, y_m] \cong \mathbf{G}(J)[x_1, \ldots, x_n].$$

Consider the natural augmentation morphisms. They induce the following commutative diagram of homomorphisms of rings:

$$
\begin{array}{ccccc}
R/I & \longrightarrow & \mathbf{G}(I)[y_1, \ldots, y_m] & \longrightarrow & R/I \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
S/J & \longrightarrow & \mathbf{G}(J)[x_1, \ldots, x_n] & \longrightarrow & S/J.
\end{array}
$$

This induces the graded isomorphism of homology groups:

$$H_1(R/I, \mathbf{G}(I)[y_1, \ldots, y_m], R/I) \cong H_1(S/J, \mathbf{G}(J)[x_1, \ldots, x_n], S/J).$$

Applying the Jacobi–Zariski exact sequence of André–Quillen homology associated to the ring homomorphisms:

$$R/I \to \mathbf{G}(I) \to \mathbf{G}(I)[y_1, \ldots, y_m]$$

and the $\mathbf{G}(I)[y_1, \ldots, y_m]$-module R/I, for each $q \geq 2$, one gets the isomorphism

$$E(\mathbf{G}(I))_q \cong H_1(R/I, \mathbf{G}(I), R/I)_q \cong H_1(R/I, \mathbf{G}(I)[y_1, \ldots, y_m], R/I)_q.$$

Analogously,

$$E(\mathbf{G}(J))_q \cong H_1(S/J, \mathbf{G}(J), S/J)_q \cong H_1(S/J, \mathbf{G}(J)[x_1, \ldots, x_n], S/J)_q.$$

Thus, for each $q \geq 2$, we have an isomorphism $E(\mathbf{G}(I))_q \cong E(\mathbf{G}(J))_q$. In particular, $\mathrm{rt}(\mathbf{G}(I)) = \mathrm{rt}(\mathbf{G}(J))$. It is known that $\mathrm{rt}(\mathbf{G}(I)) = \mathrm{rt}(\mathbf{R}(I))$ (see [17, Proposition 3.3] or [6, p. 268]). Therefore the relation type of $\mathbf{R}(I)$ is equal to the relation type of $\mathbf{R}(J)$, i.e., $\mathrm{rt}(I) = \mathrm{rt}(J)$. $\square$

Definition 3.2. Let $A = R/I = k[x_1, \ldots, x_n]/I$ be an affine k-algebra. We define the *relation type* of A as $\mathrm{rt}(A) = \mathrm{rt}(I)$, where $\mathrm{rt}(I)$ is the relation type of the ideal I of $R = k[x_1, \ldots, x_n]$. If V is an affine algebraic k-variety, the *relation type* of V is defined as $\mathrm{rt}(V) = \mathrm{rt}(k[V])$, the relation type of its coordinate ring $k[V]$. By Theorem 3.1, $\mathrm{rt}(A)$ and $\mathrm{rt}(V)$ are well-defined.

Remark 3.3. Let I be an ideal of a Noetherian ring R and let J be an ideal of a Noetherian ring S. Suppose that $R/I \cong S/J$. Then one cannot deduce that $\mathrm{rt}(I)$ is equal to $\mathrm{rt}(J)$.

Example 3.4. Consider the Neile's semicubical parabola $x^3 - y^2 = 0$ in the complex plane $\mathbb{C}^2$ and its coordinate ring $R = \mathbb{C}[x,y]/(x^3 - y^2) = \mathbb{C}[\bar{x}, \bar{y}]$, where $\bar{x}$ and $\bar{y}$ stand for the classes of x and y in R. Let $\mathfrak{m} = (\bar{x} - 1, \bar{y} - 1)$ and $\mathfrak{n} = (\bar{x}, \bar{y})$ be the maximal ideals of R corresponding to the regular point $(1,1)$ and to the origin, respectively. Although the quotient rings $R/\mathfrak{m}$ and $R/\mathfrak{n}$ are isomorphic (to $\mathbb{C}$), $\mathrm{rt}(\mathfrak{m}) = 1$ whereas $\mathrm{rt}(\mathfrak{n}) = 2$. Note here that $\mathfrak{m} = (x-1, y-1)/(x^3 - y^2)$ and $\mathfrak{n} = (x, y)/(x^3 - y^2)$, where $(x-1, y-1)$ and (x, y) are two ideals of the polynomial ring $\mathbb{C}[x,y]$ with $\mathbb{C}[x,y]/(x-1, y-1) \cong \mathbb{C}[x,y]/(x,y)$. In fact, $\mathrm{rt}(x-1, y-1) = 1$ and $\mathrm{rt}(x,y) = 1$, since they are generated by a regular sequence. However, $\mathfrak{m}$ and $\mathfrak{n}$ are ideals of R, which is not a polynomial ring, and although $R/\mathfrak{m} \cong \mathbb{C}[x,y]/(x-1, y-1) \cong \mathbb{C}[x,y]/(x,y) \cong R/\mathfrak{n}$, we cannot deduce that their relation types coincide.

Proof. Since $x^3 - y^2$ is irreducible, R is a domain. From $(x+y)(y-1) = (x^2 + x + y)(x-1) - (x^3 - y^2)$ one deduces that $\mathfrak{m}R_\mathfrak{m} = (\bar{x} - 1)R_\mathfrak{m}$ is locally generated by a regular sequence, so $\mathrm{rt}(\mathfrak{m}) = 1$ (see, e.g., [8, Corollary 3.7]). On the other hand, in R, we have the strict inclusion of colon ideals $(\bar{x} : \bar{y}) \subsetneq \bar{x}\mathfrak{n} : \bar{y}^2 = R$. By [17, Proposition 4.5], $\mathrm{rt}(\mathfrak{n}) = 2$. $\square$

Following an idea of Eisenbud and Sturmfels in [4, p. 1] and as a consequence of Theorem 3.1, we show that in order to calculate the relation type of an ideal of a polynomial

ring one can suppose that the ideal is generated by trinomials, that is, polynomials with at most three terms.

Proposition 3.5. *Let k be a field and $x_1, \ldots, x_n$ be variables over k. Let $I = (f_1, \ldots, f_s)$ be an ideal of $R = k[x_1, \ldots, x_n]$. Then there exist a polynomial ring $S = R[y_1, \ldots, y_r]$ and a surjective homomorphism of R-algebras $\sigma : S \to R$ such that $\sigma^{-1}(I)$ is an ideal of S generated by trinomials and such that $\mathrm{rt}(I) = \mathrm{rt}(\sigma^{-1}(I))$.*

Proof. Suppose that $f_1 = g_1 + \ldots + g_m$, where g_i are monomial terms. If $m \geq 4$, take y_1 a variable over R, set $R_1 = R[y_1]$ and let $\rho_1 : R_1 \to R$ be such that $\rho_1(y_1) = g_{m-1} + g_m$. Clearly $\ker(\rho_1) = (y_1 - (g_{m-1} + g_m))$. Let $J_1 = (y_1 - (g_{m-1} + g_m), g_1 + \ldots + g_{m-2} + y_1, f_2, \ldots, f_s)$. Then $\rho_1(J_1) = I$ and, since $\ker(\rho_1) \subset J_1$, $\rho_1^{-1}(I) = J_1$. In particular, ρ_1 induces an isomorphism of k-algebras $R_1/J_1 \cong R/I$ and, by Theorem 3.1, $\mathrm{rt}(J_1) = \mathrm{rt}(I)$.

If $m = 4$, $g_1 + \ldots + g_{m-2} + y_1$ is already a trinomial. Suppose that $m > 4$. Recursively, for each $i = 2, \ldots, m - 3$, take a new variable y_i over R_{i-1}, set $R_i = R_{i-1}[y_i]$ and let $\rho_i : R_i \to R_{i-1}$ be such that $\rho_i(y_i) = g_{m-i} + y_{i-1}$. Then $\ker(\rho_i) = (y_i - (g_{m-i} + y_{i-1}))$. Let

$$J_i = (y_1 - (g_{m-1} + g_m), \ldots, y_i - (g_{m-i} + y_{i-1}), g_1 + \ldots + g_{m-i-1} + y_i, f_2, \ldots, f_s).$$

One has $\rho_i(J_i) = J_{i-1}$ and $\rho_i^{-1}(J_{i-1}) = J_i$. Therefore, ρ_i induces an isomorphism of k-algebras $R_i/J_i \cong R_{i-1}/J_{i-1}$ and $\mathrm{rt}(J_i) = \mathrm{rt}(J_{i-1})$. Set $S_1 = R_{m-3} = R[y_1, \ldots, y_{m-3}]$ and let $\sigma_1 = \rho_{m-3} \circ \ldots \circ \rho_1$, where $\sigma_1 : S_1 \to R$ is a surjective homomorphism of R-algebras. Set $J = J_{m-3}$, where

$$J = J_{m-3} = (y_1 - (g_{m-1} + g_m), \ldots, y_{m-3} - (g_3 + y_{m-2}), g_1 + g_2 + y_{m-3}, f_2, \ldots, f_s).$$

Observe that $\sigma_1(J) = I$ and $\sigma_1^{-1}(I) = J$. Thus $S_1/J = R_{m-3}/J_{m-3} \cong R_{m-4}/J_{m-4}$ and $\mathrm{rt}(J) = \mathrm{rt}(J_{m-4})$, which is equal to $\mathrm{rt}(I)$. Note that in J we have replaced the polynomial f_1 by $m - 2$ trinomials by introducing $m - 3$ new variables. To finish, proceed recursively with the rest of the generators of I. $\square$

Remark 3.6. An easy refinement of the argument above allows us to suppose that the final ideal $\sigma^{-1}(I)$ is generated by polynomials of the following kind: either monomials, or binomials with one of the two terms being linear, or trinomials with all the three terms being linear.

Remark 3.7. From the proof of Proposition 3.5 one obtains an effective way to reduce a polynomial ideal to a trinomial ideal preserving the relation type at the same time. However, its interest seems more theoretical than practical due to its cost in introducing more generators and more variables.

4. Equidimensional decomposition and relation type

We start this section with some easy, but clarifying examples. From now on, $R = k[x_1, \ldots, x_n]$ will be a polynomial ring in n variables $x_1, \ldots, x_n$ over a field k, I will be a proper ideal of R and $A = R/I$. Clearly if $R = k[x]$, then R is a principal ideal domain, and every proper ideal I of R is principal generated by a nonzero divisor, hence I is of linear type and $\mathrm{rt}(A) = 1$. In two variables we have the following simple example of a family of algebras with unbounded relation type.

Example 4.1. Let $p \geq 1$ and $I = (x^p, y^p, x^{p-1}y)$ in $R = k[x, y]$. Set $A = R/I = k[x, y]/I$. Then $\mathrm{rt}(I) = p$ and $\mathrm{rt}(A) = p$.

Proof. By [17, Example 3.2], we can localize at $\mathfrak{m} = (x, y)$ in order to calculate the relation type of I. The result then follows from [15, Example 5.1]. □

However, the example above is not reduced. Allowing an arbitrary number of variables, we give the following example of a family of reduced algebras with unbounded relation type.

Example 4.2. Let $p \geq 1$ and $n = 2p$. Let I be the monomial ideal of $R = k[x_1, \ldots, x_n]$ generated by the quadratic square free monomials $x_1x_2, \ldots, x_{n-1}x_n, x_nx_1$. Set $A = R/I$. Then $\mathrm{rt}(I) = p$ and $\mathrm{rt}(A) = p$. Observe that I is the edge ideal $I(\mathcal{G})$ associated to the graph $\mathcal{G}$, where $\mathcal{G}$ is a cycle of even length $n = 2p$.

Proof. The result follows from [23, Section 3]. □

Before proceeding, we recall a concept very relevant to the purpose of this section. Given an irredundant primary decomposition of I, let $I_{i,j}$ be the primary components of I of a given height $i \geq 1$, for $j = 1, \ldots, r_i$. Set $I_i = I_{i,1} \cap \ldots \cap I_{i,r_i}$ and call it the i-th *equidimensional component of I*. Note that, for $i > \mathrm{height}(I)$, the $I_{i,j}$ (and hence I_i) are not uniquely defined (see [21, Definition 3.2.3]). To our convenience, let us write $I_i = R$ if I has no primary components of height i. Henceforth, I can be expressed as $I = I_1 \cap \ldots \cap I_n$. Such a representation will be called an *equidimensional decomposition* of I (associated to the given irredundant primary decomposition of I). Note that, since the given primary decomposition is irredundant, the equidimensional decomposition is irredundant in the following sense: either $I_i = R$, or else I_i is an unmixed ideal of height i such that $I_1 \cap \ldots \cap I_{i-1} \cap I_{i+1} \cap \ldots \cap I_n \nsubseteq I_i$. If I is radical, then each I_i is either R, or else an unmixed radical ideal of height i. With these notations, we start with the following easy example.

Example 4.3. Let I be a proper radical ideal of $R = k[x, y]$. Then $\mathrm{rt}(I) = 1$. In particular, an affine k-algebra of embedding dimension at most 2 has relation type 1.

Proof. Let $I = I_1 \cap I_2$ an equidimensional decomposition associated to an irredundant primary decomposition of I, where either $I_1 = R$, or else $I_1 = I_{1,1} \cap \ldots \cap I_{1,r_1} = (g_1) \cap \ldots \cap (g_{r_1}) = (g)$ is a principal ideal, with g_j irreducible and $g = g_1 \cdots g_{r_1}$ (see [11, Exercise 20.3]); moreover, either $I_2 = R$, or else $I_2 = I_{2,1} \cap \ldots \cap I_{2,r_2}$, where $I_{2,j} \in \text{Max}(R)$, for $j = 1, \ldots, r_2$. If $I_1 = R$, then $I = I_2 \neq R$. For each $\mathfrak{m} \in \text{Max}(R) \setminus \{I_{2,1}, \ldots, I_{2,r_2}\}$, then $I_\mathfrak{m} = R_\mathfrak{m}$. If $\mathfrak{m} \in \{I_{2,1}, \ldots, I_{2,r_2}\}$, then $I_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$, which is generated by a regular sequence. Thus $\text{rt}(I) = \sup\{\text{rt}(I_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\} = 1$. Suppose that $I_1 \neq R$. Then $I = (g) \cap I_2$. If $I_2 = R$, then $I = (g)$ and $\text{rt}(I) = 1$. Suppose that $I_2 \neq R$. Then $g \notin I_{2,j}$, due to the irredundancy of the primary decomposition of I. Take $\mathfrak{m}$ a maximal ideal of R containing I. If $\mathfrak{m}$ contains g (and hence $\mathfrak{m} \neq I_{2,j}$ for all $j = 1, \ldots, r_2$), then $I_\mathfrak{m} = (g)R_\mathfrak{m}$. If $\mathfrak{m} = I_{2,j}$, then $I_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$. So $\text{rt}(I) = \sup\{\text{rt}(I_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R), \mathfrak{m} \supseteq I\} = 1$. $\square$

The next result shows that, in embedding dimension $n \geq 3$ and relation type at least 2, the equidimensional components of dimension 1 and n are irrelevant with respect to the relation type.

Proposition 4.4. *Let I be a proper radical ideal of $R = k[x_1, \ldots, x_n]$, $n \geq 3$. Let $I = I_1 \cap \ldots \cap I_n$ be an equidimensional decomposition of I. Then either $\text{rt}(I) = 1$, or $\text{rt}(I) = \text{rt}(I_2 \cap \ldots \cap I_{n-1})$, where $I_i \neq R$ for some $i = 2, \ldots, n-1$. (Both cases may occur simultaneously.)*

Proof. To simplify notations, set $L = I_2 \cap \ldots \cap I_n$, so that $I = I_1 \cap L$. If $I_1 = R$, then $L \neq R$ and $I = L$. Suppose that $I_1 \neq R$. Since R is a unique factorization domain, then $I_1 = (g)$ (see the proof of Example 4.3). If $L = R$, then $I = (g)$. Suppose that $L \neq R$. Write $I_i = I_{i,1} \cap \ldots \cap I_{i,r_i}$ for all $i = 2, \ldots, n$, with $I_i \neq R$. Then $g \notin I_{i,j}$, for all $j = 1, \ldots, r_i$. Indeed, if $g \in I_{i,j}$, then the given primary decomposition of I would be redundant. Take now $ag \in I = I_1 \cap L = (g) \cap L$, with $a \in R$. Since $g \notin I_{i,j}$, for all $i = 2, \ldots, n$, with $I_i \neq R$, we deduce that $a \in I_{i,j}$, because $I_{i,j}$ is prime. Therefore $a \in I_i$, for all $i = 2, \ldots, n$, so that $ag \in g(I_2 \cap \ldots \cap I_n) = gL$ and $I = gL$.

In conclusion, either $I = L$, or $I = (g)$, or $I = gL$, with $L \neq R$. We know that L and gL have the same relation type (see, e.g., the characterization of the relation type in terms of the André–Quillen homology, [17, Remark 2.3]). Therefore, either $\text{rt}(I) = 1$, or $\text{rt}(I) = \text{rt}(L)$, where $L = I_2 \cap \ldots \cap I_n \neq R$.

Suppose that $\text{rt}(I) = \text{rt}(L)$, with $L \neq R$. If $I_n = R$, then $L = I_2 \cap \ldots \cap I_{n-1}$ and $\text{rt}(I) = \text{rt}(I_2 \cap \ldots \cap I_{n-1})$, where $I_i \neq R$ for some $i = 2, \ldots, n-1$, and we are done.

Suppose that $\text{rt}(I) = \text{rt}(L)$, with $L \neq R$, and that $I_n \neq R$. Write $I_n = I_{n,1} \cap \ldots \cap I_{n,r_n}$, where $I_{n,j}$ are maximal ideals of R. Set $J = I_2 \cap \ldots \cap I_{n-1}$. Thus $L = I_2 \cap \ldots \cap I_{n-1} \cap I_n = J \cap I_n$. If $J = R$, then $L = I_n$ and $\text{rt}(I) = \text{rt}(L) = \text{rt}(I_n) = \sup\{\text{rt}((I_n)_\mathfrak{m}) \mid \mathfrak{m} \in \{I_{n,1}, \ldots, I_{n,r_n}\}\} = 1$, because in this case, $(I_n)_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$, which is generated by a regular sequence. Suppose that $J \neq R$. For each $\mathfrak{m} \in \text{Max}(R) \setminus \{I_{n,1} \ldots, I_{n,r_n}\}$,

$L_\mathfrak{m} = J_\mathfrak{m} \cap (I_n)_\mathfrak{m} = J_\mathfrak{m}$, because $(I_n)_\mathfrak{m} = R_\mathfrak{m}$. If $\mathfrak{m} \in \{I_{n,1}, \dots, I_{n,r_n}\}$, then $J \nsubseteq \mathfrak{m}$, so $J_\mathfrak{m} = R_\mathfrak{m}$, $L_\mathfrak{m} = J_\mathfrak{m} \cap (I_n)_\mathfrak{m} = \mathfrak{m} R_\mathfrak{m}$ and $\mathrm{rt}(L_\mathfrak{m}) = 1$. Thus,

$$\mathrm{rt}(L) = \sup\{\mathrm{rt}(L_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)\} = \sup\{\mathrm{rt}(L_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R) \setminus \{I_{n,1}\dots, I_{n,r_n}\}\}$$
$$= \sup\{\mathrm{rt}(J_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R) \setminus \{I_{n,1}\dots, I_{n,r_n}\}\}$$
$$= \sup\{\mathrm{rt}(J_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)\} = \mathrm{rt}(J).$$

Therefore, $\mathrm{rt}(I) = \mathrm{rt}(L) = \mathrm{rt}(J) = \mathrm{rt}(I_2 \cap \dots \cap I_{n-1})$, where $I_i \neq R$ for some $i = 2, \dots, n-1$. $\quad \square$

Proposition 4.5. *Let I be a proper radical ideal of $R = k[x, y, z]$. Then either $\mathrm{rt}(I) = 1$, or $\mathrm{rt}(I) = \mathrm{rt}(I_2)$, where $I_2 \neq R$, the second equidimensional component of I, is syzygetic. Moreover, either $\mathrm{rt}(I) = 1$, or else $\mathrm{rt}(I) \geq 3$.*

Proof. By Proposition 4.4, we can suppose that $\mathrm{rt}(I) = \mathrm{rt}(I_2)$, where $I_2 = I_{2,1} \cap \dots \cap I_{2,r_2} \neq R$, with $I_{2,j}$ prime ideals of R of height 2. In particular, I_2 is generically a complete intersection (i.e., a complete intersection localized at each minimal prime), and a perfect ideal of projective dimension 1. Hence I_2 is syzygetic, i.e., $E(I_2)_2 = \ker(\alpha_2 : \mathbf{S}_2(I_2) \to I_2^2) = 0$ (see the subsequent Remark to [8, Proposition 2.7]).

If I_2 is of linear type, then $\mathrm{rt}(I) = \mathrm{rt}(I_2) = 1$ and we are done. Suppose that I_2 is not of linear type. Then $\mathrm{rt}(I_2) > 2$. Indeed, if $\mathrm{rt}(I_2) \leq 2$, since $\mathrm{rt}(I_2) = \min\{r \geq 1 \mid E(I_2)_q = 0$ for all $q \geq r + 1\}$, then $E(I_2)_q = 0$ for all $q \geq 3$. Since $E(I_2)_2 = 0$, then it would follow that $\mathrm{rt}(I_2) = 1$, a contradiction. Therefore $\mathrm{rt}(I) = \mathrm{rt}(I_2) \geq 3$. (Note that we do not affirm that I is syzygetic.) $\quad \square$

As an immediate consequence we have the following result.

Corollary 4.6. *There do not exist affine k-algebras of embedding dimension 3 and relation type 2. There do not exist affine algebraic k-varieties of embedding dimension 3 and relation type 2.*

5. Examples in embedding dimension three

In this section we focus our attention on affine k-algebras of embedding dimension three. Our purpose is to give illustrative examples of affine algebras with different relation types. So now, $R = k[x, y, z]$ will be a polynomial ring in three variables x, y, z over a field k, I will be a proper radical ideal of R and $A = R/I$. According to Proposition 4.5, we can suppose that I is an irredundant intersection of prime ideals of height 2. In particular, I is a perfect ideal of projective dimension 1. Suppose that $I = (f_1, \dots, f_s)$ is minimally generated by $s \geq 2$ elements and that

$$0 \longrightarrow R^{s-1} \overset{\eta}{\longrightarrow} R^s \longrightarrow I \longrightarrow 0$$

is a presentation of I. By the theorem of Hilbert–Burch, there exists an element $g \in R$, $g \neq 0$, such that $I = g I_{s-1}(\eta)$, where $I_{s-1}(\eta)$ is the determinantal ideal generated by the $(s-1) \times (s-1)$ minors of the $s \times (s-1)$ matrix η (see, e.g., [2, Theorem 1.4.16]). In particular, $\mathrm{rt}(I) = \mathrm{rt}(I_{s-1}(\eta))$. Therefore, in terms of the computation of the relation type, we can directly suppose that $I = I_{s-1}(\eta)$.

On taking the symmetric functor in the short exact sequence above one gets:

$$0 \longrightarrow L\langle 1 \rangle \longrightarrow S = \mathbf{S}(R^s) = R[t_1, \ldots, t_s] \xrightarrow{\Phi} \mathbf{S}(I) \longrightarrow 0,$$

where $L\langle 1 \rangle$ is the defining ideal of $\mathbf{S}(I)$. Recall the notation in Section 2, where $L = \ker(\varphi)$ is the ideal of equations of I given by the polynomial presentation $\varphi : \alpha \circ \Phi : S \to \mathbf{R}(I)$ and $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is the canonical morphism. If we denote by $[t_1, \ldots, t_s]$ the $1 \times s$ matrix of entries t_i, then $L\langle 1 \rangle = (g_1, \ldots, g_s)S$, with $[g_1, \ldots, g_s] = [t_1, \ldots, t_s] \cdot \eta$. One can write this last expression as

$$[g_1, \ldots, g_s] = [t_1, \ldots, t_s] \cdot \eta = [x, y, z] \cdot B(\eta),$$

where $B(\eta)$ is a $3 \times s$ matrix of linear forms in the variables t_i. This matrix is called a *Jacobian dual* matrix of η. One can show that the determinantal ideal $I_3(B(\eta))$ generated by the 3×3 minors of $B(\eta)$, is included in L, the ideal of equations of I (see [21, Proposition 7.2.3]).

The ideal I is said to have the *expected equations* if $L = (L\langle 1 \rangle, I_3(B(\eta)))$ (see [22, Definition 1.8]; see [19, Introduction and Section 3.1], where these ideas were first stated; see also [13,14]). Most of the results on the expected equations of an ideal I of a ring R, suppose that either R is a Noetherian local ring, or else R is a standard graded and I is a homogeneous ideal.

Although in our case I is not homogeneous, we can use these techniques as a first approach to obtain the equations of I and its relation type, as the next example shows.

Example 5.1. Let $\mathbb{N}$ be the set of positive integers and let $n = (n_1, n_2, n_3)$, $m = (m_1, m_2, m_3) \in \mathbb{N}^3$, with $\gcd(n) = 1$ and $\gcd(m) = 1$. Let $\mathfrak{p}_n$ be the kernel of the k-homomorphism $R = k[x, y, z] \to k[t]$ which sends x, y and z, to t^{n_1}, t^{n_2} and t^{n_3}, respectively. Similarly, one defines $\mathfrak{p}_m$. Clearly, $\mathfrak{p}_n$ and $\mathfrak{p}_m$ are prime. It is known that they are either a complete intersection, or else an almost complete intersection, and in particular, of relation type 1 (see, e.g., [10, Example V.3.13, f)] and [7, Theorem 4.8]). Fix now $n = (3, 4, 5)$ and $m = (3, 4, 3r)$, for some $r \geq 3$ and let $I = \mathfrak{p}_n \cap \mathfrak{p}_m$. Using SINGULAR [3], one gets the minimal system of generators for I:

$$f_1 = x^4 - y^3, \ f_2 = -x^{r+1}z + x^r y^2 + xz^2 - y^2 z,$$

$$f_3 = -x^{r+3} + x^r yz + x^3 z - yz^2 \text{ and } f_4 = -x^{r+2}y + x^r z^2 + x^2 yz - z^3,$$

and the presentation $0 \longrightarrow R^3 \xrightarrow{\eta} R^4 \longrightarrow I \longrightarrow 0$, with

$$\eta = \begin{pmatrix} z - x^r & 0 & 0 \\ -y & z & -x^2 \\ -x & -y & z \\ 0 & x & -y \end{pmatrix}.$$

In other words, I is generated by the 3×3 minors of the 4×3 matrix η. Thus $L\langle 1 \rangle$ is generated by g_1, g_2, g_3, where $[g_1, g_2, g_3] = [t_1, \ldots, t_4] \cdot \eta = [x, y, z] \cdot B(\eta)$, and $B(\eta)$ is the Jacobian dual of η. Concretely,

$$B(\eta) = \begin{pmatrix} -t_3 - x^{r-1} t_1 & t_4 & -xt_2 \\ -t_2 & -t_3 & -t_4 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Using SINGULAR [3] again, one deduces that $L = \ker(\varphi)$ is generated by g_1, g_2, g_3 and $\det(B)$. In particular, $\mathrm{rt}(I) = 3$. Note that, as in [19, Theorem 3.1.1], I has the *expected equations*, three of degree 1 and exactly one of degree 3, though in the aforementioned result, R is supposed to be a regular local ring and I is supposed to be a prime ideal, whereas in our case, $R = k[x, y, z]$ and I is not prime nor homogeneous.

The next example shows that all the cases arising in the proof of Proposition 4.5 can occur.

Example 5.2. Let us consider the intersection of two ideals of Herzog–Northcott type. Recall that an ideal of Herzog–Northcott type of $R = k[x, y, z]$ is the determinantal ideal J generated by the 2×2 minors of a 2×3 matrix with rows $x^{a_1}, y^{a_2}, z^{a_3}$ and $y^{b_2}, z^{b_3}, x^{b_1}$, where (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in \mathbb{N}^3$. In other words, $J = (x^{c_1} - y^{b_2} z^{a_3}, y^{c_2} - x^{a_1} z^{b_3}, z^{c_3} - x^{b_1} y^{a_2})$, where $c_i = a_i + b_i$, for $i = 1, 2, 3$. It is known that such an ideal J is of linear type and, if k has characteristic zero (or large enough), then J is radical (see [16, Remark 6.4 and Theorem 9.1]). Moreover, J is prime if and only if $\gcd(m(J)) = 1$, where $m(J) = (c_2 c_3 - a_2 b_3, c_1 c_3 - a_3 b_1, c_1 c_2 - a_1 b_2)$ (see [16, Definition 7.1, Remark 7.2 and Theorem 7.8]).

Consider now the following four ideals of Herzog–Northcott type: $J_1 = (x^3 - yz, y^3 - x^2 z, z^2 - xy^2)$, $J_2 = (x^3 - yz, y^2 - xz, z^2 - x^2 y)$, $J_3 = (x^2 - y^2 z, y^3 - xz, z^2 - xy)$ and $J_4 = (x^2 - yz, y^2 - xz^2, z^3 - xy)$. Observe that $m(J_1) = (4, 5, 7)$, $m(J_2) = (3, 4, 5)$, $m(J_3) = (5, 3, 4)$ and $m(J_4) = (4, 5, 3)$. In particular, J_i are prime ideals and $J_i \cap J_j$ are Cohen–Macaulay ideals of projective dimension 1, for $i, j = 1, 2, 3, 4$. SINGULAR [3] shows that $J_1 \cap J_2$ is a complete intersection and that $J_1 \cap J_4$ is an almost complete intersection, in particular, $J_1 \cap J_2$ and $J_1 \cap J_4$ are ideals of linear type. However, $J_1 \cap J_3$ is locally at (x, y, z) minimally generated by four elements. Therefore $J_1 \cap J_3$ is not of linear type (see [8, Proposition 2.4]; see also [18, Theorem 5.1]). By Proposition 4.5, $J_1 \cap J_3$ is syzygetic and has relation type at least 3.

6. The relation type of irreducible affine space curves

In this section, we give some examples where $I = \mathfrak{p}$ is a prime ideal of height 2 in $R = k[x, y, z]$. Alternatively, these are examples of irreducible affine algebraic curves V in the three-dimensional affine k-space $\mathbb{A}^3(k)$. We start with monomial curves.

Example 6.1. Let $m = (m_1, m_2, m_3) \in \mathbb{N}^3$ with $\gcd(m_1, m_2, m_3) = 1$. Let $V \subset \mathbb{A}^3(k)$ be the parametrized curve $V = \{(\lambda^{m_1}, \lambda^{m_2}, \lambda^{m_3}) \in \mathbb{A}^3(k) \mid \lambda \in k\}$. Suppose that k is infinite. Then $\mathrm{rt}(V) = 1$.

Proof. Let $\mathfrak{p} \subset R = k[x, y, z]$ be the kernel of the k-homomorphism $R = k[x, y, z] \to k[t]$ sending x, y, z to t^{m_1}, t^{m_2}, t^{m_3}, respectively. According to [5, Lemma 3.4], if $\gcd(m_1, m_2, m_3) = 1$, then $V = \mathfrak{V}(\mathfrak{p})$ is the affine algebraic k-variety defined by the ideal $\mathfrak{p} \subset R = k[x, y, z]$. Moreover if k is infinite, the vanishing ideal of V is $\mathfrak{I}(V) = \mathfrak{p}$ (see, e.g., [24, Corollary 7.1.12]). Thus $k[V] = k[x, y, z]/\mathfrak{I}(V) = k[x, y, z]/\mathfrak{p}$, where $\mathfrak{p}$ is either a complete intersection ideal, or else an almost complete intersection, hence $\mathfrak{p}$ is of linear type ([10, Example V.3.13, f)] and [7, Theorem 4.8]). In particular, $\mathrm{rt}(\mathfrak{p}) = 1$ and $\mathrm{rt}(V) = 1$. □

The following example is taken from [19, Section 3.1]. It also shows that the relation type may depend on the characteristic of the ground field.

Example 6.2. Let $\mathfrak{p} \subset R = k[x, y, z]$ be the kernel of the k-homomorphism $\psi : R = k[x, y, z] \to k[t]$ which sends x, y and z to t^6, t^8 and $t^{10} + t^{11}$, respectively. Let $V = \mathfrak{V}(\mathfrak{p}) \subset \mathbb{A}^3(k)$ be the affine algebraic curve defined by $\mathfrak{p}$. If k is infinite, then $\mathrm{rt}(V) = 3$. If $k = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{rt}(V) = 1$.

Proof. Let $W = \{(\lambda^6, \lambda^8, \lambda^{10} + \lambda^{11}) \mid \lambda \in k\}$. For all $f \in \mathfrak{p}$, $f(\lambda^6, \lambda^8, \lambda^{10} + \lambda^{11}) = \psi(f)(\lambda) = 0$ and $W \subseteq V = \mathfrak{V}(\mathfrak{p})$. In particular, $\mathfrak{p} \subseteq \mathfrak{I}(\mathfrak{V}(\mathfrak{p})) = \mathfrak{I}(V) \subseteq \mathfrak{I}(W)$. Let $f \in \mathfrak{I}(W)$, i.e., $f(\lambda^6, \lambda^8, \lambda^{10} + \lambda^{11}) = 0$, for all $\lambda \in k$. Set $g(t) = f(t^6, t^8, t^{10} + t^{11}) = \psi(f)$. Then $g(\lambda) = 0$, for all $\lambda \in k$. Suppose that k is infinite. Then $g(t) = 0$ and $f \in \mathfrak{p}$. Thus $\mathfrak{I}(W) = \mathfrak{p}$, so $\mathfrak{I}(V) = \mathfrak{p}$. Using SINGULAR, one deduces that $\mathfrak{p}$ is minimally generated by four elements and that $\mathfrak{p}$ has the expected equations (see also [19, Proposition 3.1.1]). Thus $\mathrm{rt}(\mathfrak{p}) = 3$ and $\mathrm{rt}(V) = 3$. Suppose that $k = \mathbb{Z}/2\mathbb{Z}$. Clearly $(0, 0, 0)$ and $(1, 1, 0)$ are in $W \subseteq V = \mathfrak{V}(\mathfrak{p})$. Moreover $f_1 = x^4 + y^3$ and $f_2 = x^2 y + xy^2 + z^2$, say, are easily seen to be in $\mathfrak{p}$, so in $\mathfrak{I}(V)$. Since f_1 does not vanish on $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$ and $(1, 0, 1)$ and f_2 does not vanish on $(0, 0, 1)$ and $(1, 1, 1)$, it follows that these six points are not in V. Hence $W = V = \{(0, 0, 0), (1, 1, 0)\}$ and $\mathfrak{I}(W) = \mathfrak{I}(V)$. Clearly $x + y, z \in \mathfrak{I}(V) \setminus \mathfrak{p}$ and $\mathfrak{p} \subsetneq \mathfrak{I}(V)$. In fact, $\mathfrak{I}(V) = (x + y, z)$, which is generated by a regular sequence, so $\mathfrak{I}(V)$ is of linear type. □

The next curve has relation type 3 too (if k is infinite). Moreover, when $k = \mathbb{C}$, it is known to be a set-theoretically complete intersection (see [9, Example 3.7]).

Example 6.3. Let $\mathfrak{p} \subset R = k[x, y, z]$ be the kernel of the k-homomorphism $R = k[x, y, z] \to k[t]$ which sends x, y and z to t^6, $t^7 + t^{10}$ and t^8, respectively. Let $V = \mathfrak{V}(\mathfrak{p}) \subset \mathbb{A}^3(k)$ be the affine algebraic curve defined by $\mathfrak{p}$. If k is infinite, then $\mathrm{rt}(V) = 3$. If $k = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{rt}(V) = 1$.

Proof. The proof is analogous to the former one. To deduce the equations of $\mathfrak{I}(V)$ one can use SINGULAR (see [19, Proposition 3.1.1 and Example 3.2.1], and [9, Example 3.7]). $\quad\square$

Remark 6.4. In the examples above (6.1, 6.2 or 6.3) we have obtained irreducible affine algebraic space curves with relation types 1 and 3. What are the geometric properties that make them have different relation types? What is the geometric meaning of the relation type of an irreducible affine algebraic space curve?

Closing Remark 6.5. Although the examples above do not exceed four generators, they have costly computations. To obtain a general procedure to find the equations of prime ideals minimally generated by an arbitrary number of elements seems a very difficult task. For instance, one could ask for the equations, or just the relation type, of the prime ideals defined by Moh in [12], a question to which we do not have an answer at the present moment. We intend to pursue this problem in future work.

Acknowledgments

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