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Quasiphantom categories on a family of surfaces isogenous to a higher product



Hyun Kyu Kim^{a,*}, Yun-Hwan Kim^b, Kyoung-Seog Lee^c

^a School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

^b Department of Mathematical Sciences, Seoul National University, GwanAkRo 1, Gwanak-Gu, Seoul 08826, Republic of Korea

^c Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Republic of Korea

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ABSTRACT

We construct exceptional collections of line bundles of maximal length 4 on $S = (C \times D)/G$ which is a surface isogenous to a higher product with $p_g = q = 0$ where $G = G(32, 27)$ is a finite group of order 32 having number 27 in the list of Magma library. From these exceptional collections, we obtain new examples of quasiphantom categories as their orthogonal complements.

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1. Introduction

Quasiphantom categories are surprising new subcategories in the derived categories of algebraic varieties first discovered by Böhning, Bothmer and Sosna in [7]. Their discovery provides new perspectives on the study of derived categories of algebraic varieties and recently many examples of quasiphantom categories were constructed by many authors

* Corresponding author.

E-mail addresses: hkim@kias.re.kr, hyunkyu87@gmail.com (H. Kim), yunttang@snu.ac.kr (Y.-H. Kim), kyoungseog02@gmail.com (K.-S. Lee).

(see [1,6,7,10,11,15,16,18–20,22–24] for more details). However their structures are quite mysterious and we do not know whether every surface of general type with $p_g = q = 0$ has a quasiphantom category in its derived category.

A surface S which is isomorphic to $(C \times D)/G$ where C, D are curves of genus ≥ 2 and G is a finite group acting on $C \times D$ freely is called a surface isogenous to a higher product. Surfaces isogenous to a higher product are interesting and important classes of surfaces of general type. They play an important role in the study of moduli spaces of general type surfaces. Bauer, Catanese and Grunewald classified surfaces isogenous to a higher product of unmixed type with $p_g = q = 0$ in [4]. In particular, they proved that the possible list of groups G is $\mathbb{Z}_2^3, \mathbb{Z}_2^4, \mathbb{Z}_3^2, \mathbb{Z}_5^2, \mathbb{Z}_2 \times D_4, S_4, \mathbb{Z}_2 \times S_4, G(16, 3), G(32, 27), A_5$.

In [11,19,22–24], the authors constructed quasiphantom categories in derived categories of surfaces isogenous to a higher product except $G = G(32, 27), A_5$ cases. It is very natural to expect that derived categories of all surfaces isogenous to a higher product with $p_g = q = 0$ have quasiphantom categories in their derived categories. However quasiphantom categories in derived categories of surfaces had not been constructed for $G = G(32, 27), A_5$ cases.

In this paper we construct exceptional collections of line bundles of maximal length 4 on $S = (C \times D)/G$ which is a surface isogenous to a higher product with $p_g = q = 0$ and G is $G(32, 27)$. From these exceptional collections we can obtain new examples of quasiphantom categories. They are obtained as the orthogonal complements of the exceptional collections of maximal length 4 on S .

Notations. We will work over \mathbb{C} . Derived category of a variety will mean the bounded derived category of coherent sheaves on the variety. In this paper, G denotes a finite group, $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$ denotes the character group of G , and $G(32, 27)$ means the finite group of order 32 having number 27 in the list in Magma library.

2. Preliminaries

In this section we recall several definitions and facts which we will use later.

2.1. Surfaces isogenous to a higher product with $p_g = q = 0$

We review the basic theory of surfaces isogenous to a higher product with $p_g = q = 0$.

Definition 2.1. [2, Definition 2.1] A surface S of general type is said to be isogenous to a higher product if S is isomorphic to $(C \times D)/G$, where C and D are curves of genus at least 2, and G is a finite group acting freely on $C \times D$. We call S of unmixed type if G acts diagonally on $C \times D$.

Bauer, Catanese and Grunewald classified surfaces isogenous to a higher product with $p_g = q = 0$ in [4]. Their strategy was to classify all groups acting freely on product of

two curves with some specified conditions. Let $S = (C \times D)/G$ be a surface isogenous to a higher product of unmixed type with $p_g = q = 0$. Consider the two quotient maps $C \rightarrow C/G$ and $D \rightarrow D/G$. Since $q = 0$, we see that $C/G \cong \mathbb{P}^1 \cong D/G$. These quotient maps give several group theoretic data. We recall several terminologies following [4].

Definition 2.2. [4] Let G be a group and r be a natural number with $r \geq 2$.

- (1) An r -tuple $T = [g_1, \dots, g_r] \in G^r$ is called a spherical system of generators of G if g_1, \dots, g_r is a system of generators of G and $g_1 \cdots g_r = 1$.
- (2) Let $A = [m_1, \dots, m_r]$ be an r -tuple of natural numbers with $2 \leq m_1 \leq \dots \leq m_r$ then a spherical system of generators $T = [g_1, \dots, g_r]$ is said to have type $A = [m_1, \dots, m_r]$ if there is a permutation $\tau \in S_r$ such that $\text{ord}(g_i) = m_{\tau(i)}$ for all i .
- (3) The stabilizer set $\Sigma(T)$ of a spherical system of generators $T = [g_1, \dots, g_r]$ is defined as follows:

$$\Sigma(T) = \bigcup_{g \in G} \bigcup_{j \in \mathbb{Z}} \bigcup_{i=1}^r \{gg_i^j g^{-1}\} = \{gg_i^j g^{-1} \mid g \in G, j \in \mathbb{Z}, i = 1, \dots, r\}.$$

- (4) Let T_1, T_2 be a pair of spherical systems of generators (T_1, T_2) of G is called disjoint if $\Sigma(T_1) \cap \Sigma(T_2) = \{1\}$.

Definition 2.3. [4, Definition 1.1] An unmixed ramification structure of type (A_1, A_2) for G is a disjoint pair (T_1, T_2) of spherical systems of generators of G , such that T_1 has type A_1 and T_2 has type A_2 . We define $\mathcal{B}(G; A_1, A_2)$ to be the set of unmixed ramification structures of type (A_1, A_2) for G .

Whenever we have a surface $S = (C \times D)/G$ we have an unmixed ramification structure of type (A_1, A_2) . Moreover we see that an unmixed ramification structure gives a surface isogenous to a higher product by the following proposition. See [4] for more details.

Proposition 2.4. [4, Proposition 2.5] Let G be a finite group and A_1, A_2 be two tuples of natural numbers. Then for any ramification structure $\mathcal{T} \in \mathcal{B}(G; A_1, A_2)$, there is a surface isogenous to a higher product of unmixed type with $G(S) = G$ and $\mathcal{T}(S) = \mathcal{T}$.

2.2. Automorphisms of curves

We review several results about automorphisms of curves and their invariants. First, let us recall Lefschetz fixed point formula.

Theorem 2.5 (Lefschetz fixed point formula). [9, Corollary 12.3] Let $g \in G$ be a non-trivial automorphism of an algebraic curve C of genus $g \geq 2$, and let χ_{K_C} the character of the action of G on $H^0(C, K_C)$. Then we have

$$\chi_{K_C}(g) + \overline{\chi_{K_C}(g)} = 2 - |\text{Fix}(g)|.$$

In particular, $\chi_{K_C}(g) = 1 - \frac{1}{2}|\text{Fix}(g)|$ when all characters are real-valued.

Beauville studied theta characteristics on curves with involution in [5]. His results are the main tools of our investigation of involution invariant line bundles. Let C be a curve and σ be the involution on C . Let B be the quotient curve $C/\langle\sigma\rangle$, $\pi: C \rightarrow B$ be the quotient map and $R \subset C$ be the set of ramification points. This double covering corresponds to a line bundle ρ on B such that $\rho^2 = \mathcal{O}_B(\pi_*R)$. See [5] for more details. Beauville obtained the following result which tells us which σ -invariant line bundles on C comes from B .

Lemma 2.6. [5] *Consider the map $\phi: \mathbb{Z}^R \rightarrow \text{Pic}(C)$ which maps $r \in R$ to $\mathcal{O}_C(r)$. Its image lies in the subgroup $\text{Pic}(C)^\sigma$ of σ -invariant line bundles. We get a short exact sequence*

$$0 \rightarrow \mathbb{Z}/2 \rightarrow (\mathbb{Z}/2)^R \rightarrow \text{Pic}(C)^\sigma / \pi^* \text{Pic}(B) \rightarrow 0,$$

and the kernel is generated by $(1, \dots, 1)$.

And he obtains the following result which tells us how to compute the sheaf cohomology groups of invariant theta characteristics as follows.

Proposition 2.7. [5] *Let κ be a σ -invariant theta characteristic on C . Then*

- (1) $\kappa \cong \pi^*L(E)$ for some $L \in \text{Pic}(B)$ and $E \subset R$ with $L^2 \cong K_B \otimes \rho(-\pi_*E)$. If another pair (L', E') satisfies $\kappa \cong \pi^*L'(E')$, we have $(L', E') = (L, E)$ or $(L', E') = (K_B \otimes L^{-1}, R - E)$.
- (2) $h^0(\kappa) = h^0(L) + h^1(L)$, and the parity of κ is equal to $\deg(L) - b + 1 \pmod{2}$ where b is the genus of B .

Finally we recall the following result of Dolgachev which enables us to construct G -equivariant line bundles on $C \times D$ using G -invariant line bundles on C and D .

Proposition 2.8. [12] *Let X be a smooth projective variety and let G be a finite group acting on X . There is a well-known exact sequence*

$$0 \rightarrow \widehat{G} \rightarrow \text{Pic}^G(X) \rightarrow \text{Pic}(X)^G \rightarrow H^2(G, \mathbb{C}^*),$$

and the last homomorphism is surjective when X is a curve.

Table 1

The number of fixed points for T_1 .

Conjugacy class	1	g_5	g_4	g_4g_5	$g_2g_3g_5$	g_2	g_2g_3	g_3g_4	g_2g_5	g_3	g_1	$g_1g_2g_3$	g_1g_2	g_1g_3
#fixed points	∞	8	0	0	0	0	8	0	8	0	4	0	0	4

3. Derived categories of surfaces isogenous to a higher product with $G = G(32, 27)$

Let $S = (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$. It is easy to see that the maximal possible length of an exceptional collection is less than or equal to 4 (see [11,19,22–24] for more details). In this section we construct exceptional collections of line bundles of maximal length 4 on $S = (C \times D)/G$ where $G = G(32, 27)$. The method of construction is similar to other cases. We can construct G -invariant ineffective theta characteristics on C . However there is no G -equivariant ineffective line bundle of degree 8 on D . In order to overcome this situation we need to show that there are enough characters of G to construct exceptional collection of length 4. To do this we use computer algebra system Magma [8] and group theoretic properties of G . See Appendix A and Appendix B for more details. In particular, see Appendix A for the character table of G .

3.1. Equivariant geometry of C and D

Let g_1, g_2, \dots, g_5 be generators of G for the presentation of G as in Appendix A. The unmixed ramification structure $\mathcal{T} = (T_1, T_2)$ corresponds to $S = (C \times D)/G$ is described in [4]. They computed that $\mathcal{T} = (T_1, T_2)$ is of type $([2, 2, 2, 4], [2, 2, 4, 4])$ on $G(32, 27)$ which is equivalent to

$$T_1 = [g_1g_4g_5, g_2g_3g_4g_5, g_2g_4g_5, g_1g_3g_4], \quad T_2 = [g_2g_3g_4, g_2, g_1g_2g_3g_5, g_1g_2]$$

([4], see also [3] and Appendix A). We are going to construct the desired line bundles from the above unmixed ramification structure. We can compute the representation of $H^0(C, K_C)$ by the Lefschetz fixed point formula. For T_1 , the numbers of fixed points are given by Table 1. (Note that 1 fixes every point in C .)

Therefore we get the character χ_{K_C} of the action of G on $H^0(C, K_C)$. The value of χ_{K_C} at the identity class is the genus 5. At any non-trivial conjugacy class of $g \in G(32, 27)$, the value of χ_{K_C} at g is given by

$$\chi_{K_C}(g) = \frac{1}{2}(2 - |\text{Fix}(g)|)$$

since every character of $G(32, 27)$ is a real-valued function; See Remark A.3. Thus the values of the character χ_{K_C} at the fourteen conjugacy classes ordered as above are as following Table 2.

Table 2The values of χ_{K_C} .

Conjugacy class	1	g_5	g_4	g_4g_5	$g_2g_3g_5$	g_2	g_2g_3	g_3g_4	g_2g_5	g_3	g_1	$g_1g_2g_3$	g_1g_2	g_1g_3
χ_{K_C}	5	-3	1	1	1	1	-3	1	-3	1	-1	1	1	-1

Table 3The number of fixed points for T_2 .

Conjugacy class	1	g_5	g_4	g_4g_5	$g_2g_3g_5$	g_2	g_2g_3	g_3g_4	g_2g_5	g_3	g_1	$g_1g_2g_3$	g_1g_2	g_1g_3
#fixed points	∞	0	8	8	8	8	0	0	0	0	0	4	4	0

Table 4The values of χ_{K_D} .

Conjugacy class	1	g_5	g_4	g_4g_5	$g_2g_3g_5$	g_2	g_2g_3	g_3g_4	g_2g_5	g_3	g_1	$g_1g_2g_3$	g_1g_2	g_1g_3
χ_{K_D}	9	1	-3	-3	-3	-3	1	1	1	1	1	-1	-1	1

Now from the character table of $G(32, 27)$ (see [Appendix A](#)), we can see that $\chi_{K_C} = \chi_7 + \chi_9 + \chi_{11}$. Similarly, for T_2 , the numbers of fixed points are given by [Table 3](#).

The genus of D is 9, so the character values of χ_{K_D} at the fourteen conjugacy classes ordered as above are as in [Table 4](#).

We find $\chi_{K_D} = \chi_4 + \chi_{10} + \chi_{12} + \chi_{13} + \chi_{14}$ from the character table again.

3.2. Constructing line bundles on C

Consider the normal subgroup $\langle g_5 \rangle \trianglelefteq G(32, 27)$ and the quotient map $\pi: C \rightarrow B$ where B is a quotient curve $B := C/\langle g_5 \rangle$. Then by the unmixed ramification structure and the Riemann–Hurwitz formula, we see that B is an elliptic curve and $\deg R = 8$ where R denotes a ramification divisor of π . Moreover, since $G(32, 27)$ act on C , $\overline{G} := G(32, 27)/\langle g_5 \rangle$ also acts on B . At first, we will show that there exists a non-trivial \overline{G} -invariant theta characteristic of B . To see this, note that there are 4 theta characteristics $\eta_0 := \mathcal{O}_B$, η_1 , η_2 and η_3 on B . Now consider the \overline{G} -orbit $\overline{G}\eta_i$ of η_i . Since \overline{G} acts on the set of all theta characteristics and $\eta_0 = \mathcal{O}_B$ is fixed under the \overline{G} -action, we have 3 possibilities: $(\{i, j, k\} = \{1, 2, 3\})$

- (1) $\overline{G}\eta_i = \{\eta_i\}$: This means η_i is a \overline{G} -invariant, so we are done.
- (2) $\overline{G}\eta_i = \{\eta_i, \eta_j\}$: Take $\eta_k \notin \{\eta_i, \eta_j\}$. Then we have $\overline{G}\eta_k = \{\eta_k\}$.
- (3) $\overline{G}\eta_i = \{\eta_i, \eta_j, \eta_k\}$: By the orbit-stabilizer theorem, we have

$$|\overline{G}\eta_i| |\text{Stab}_{\overline{G}}(\eta_i)| = |\overline{G}|$$

where $\text{Stab}_{\overline{G}}(\eta_i)$ denotes the stabilizer of η_i under the \overline{G} -action. However, since $3 = |\overline{G}\eta_i| \nmid |\overline{G}| = 16$, this is impossible.

Thus, there is a \overline{G} -invariant theta characteristic η of B . We see that $h^0(B, \eta) = 0$ since there is no global section for any non-trivial line bundle of degree 0 of B , and $h^1(B, \eta)$ also vanishes by the Riemann–Roch theorem. With the existence of such η , we can prove the following lemma:

Lemma 3.1. *C has a $G(32, 27)$ -invariant ineffective theta characteristic \mathcal{L} .*

Proof. Let R be a ramification divisor of $\pi: C \rightarrow B$ of degree 8 as above and consider a normal subgroup $H := \langle g_2 g_5, g_4 \rangle \trianglelefteq G(32, 27)$ of order 4. Then by the unmixed ramification structure and the Riemann–Hurwitz formula, $C/H \cong \mathbb{P}^1$ and since g_2 and g_4 freely act on C , the H -action on R is free. Thus, $\mathcal{O}_C(R) = \pi_H^* \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2}$ where $\pi_H: C \rightarrow \mathbb{P}^1$ is the quotient map induced by H . Because $\mathcal{O}_{\mathbb{P}^1}(1)$ is $G(32, 27)/H$ -invariant, $\pi_H^* \mathcal{O}_{\mathbb{P}^1}(1)$ is $G(32, 27)$ -invariant. Therefore, $\mathcal{L} := \pi^* \eta \otimes \pi_H^* \mathcal{O}_{\mathbb{P}^1}(1)$ is a $G(32, 27)$ -invariant line bundle and since

$$\begin{aligned} \mathcal{L}^{\otimes 2} &= (\pi^* \eta \otimes \pi_H^* \mathcal{O}_{\mathbb{P}^1}(1))^{\otimes 2} = \pi^* \eta^{\otimes 2} \otimes \pi_H^* \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2} \\ &= \pi^*(K_B) \otimes \mathcal{O}_C(R) = K_C, \end{aligned}$$

\mathcal{L} is a theta characteristic. Moreover we have $h^0(C, \mathcal{L}) = 2h^0(B, \eta) = 0$ by [Proposition 2.7](#). \square

3.3. Constructing line bundles on D

We begin with the following lemma which will be useful to construct some line bundle on D .

Lemma 3.2. *Let A_1, A_2, A_3, A_4 be the set-theoretic fibers consisting of the ramification points of the map $\pi_D: D \rightarrow D/G(32, 27)$ whose stabilizer group is $\langle g_2 g_3 g_4 \rangle$, $\langle g_2 \rangle$, $\langle g_1 g_2 g_3 g_5 \rangle$ and $\langle g_1 g_2 \rangle$ respectively. Then the following linear equivalence relations hold:*

- (1) $A_1 \sim A_2 \sim 2A_3 \sim 2A_4$.
- (2) $A_3 \approx A_4$.

Proof. Note that the orders of the stabilizer groups $\langle g_2 g_3 g_4 \rangle$, $\langle g_2 \rangle$, $\langle g_1 g_2 g_3 g_5 \rangle$ and $\langle g_1 g_2 \rangle$ are 2, 2, 4 and 4 respectively; see [Remark A.2](#).

- (1) Consider the subgroup $H_1 = \langle g_1 g_2 g_4, g_4, g_5 \rangle$ of $G(32, 27)$ of order 8 and the quotient curve D/H_1 . Since H_1 is disjoint to conjugacy classes of stabilizer groups $\langle g_2 g_3 g_4 \rangle$, $\langle g_2 \rangle$ of A_1 and A_2 , respectively, H_1 acts freely on both A_1 and A_2 . Similarly, since $(g_1 g_2 g_3 g_5)^2 = g_4 g_5 \in H_1$ and same holds for any other representatives of its conjugacy classes, H_1 acts on A_3 with stabilizer group \mathbb{Z}_2 . Finally, since $g_1 g_2$ and other representatives of its conjugacy classes are in H_1 , H_1 acts on A_4 with stabilizer group

\mathbb{Z}_4 . Then, by the Riemann–Hurwitz formula, we have $D/H_1 \cong \mathbb{P}^1$ and hence, we get $A_1 \sim A_2 \sim 2A_3$. Next, consider $H_2 = \langle g_1 g_2 g_3 g_4 g_5, g_4, g_5 \rangle$. By the similar argument for H_2 , we can get $A_1 \sim A_2 \sim 2A_4$.

- (2) Now consider the $H_3 = \langle g_2, g_4 \rangle$ -action on D . Then similar argument as above shows that A_3 contains no ramification point of $\pi_3: D \rightarrow D/H_3$ and A_4 contains some ramification points of π_3 . Moreover, we can see that A_3 is given by a pull-back of some divisor of D/H_3 . However, since A_2 also contains ramification points, A_4 does not contain all ramification points and hence, it cannot be a pull-back of some divisor of D/H_3 by Lemma 2.6. Therefore, $A_3 \not\sim A_4$. \square

Remark 3.3. From the above Lemma, we see that $\mathcal{O}(A_1 - A_3)$, $\mathcal{O}(A_1 - A_4)$, $\mathcal{O}(A_2 - A_3)$, $\mathcal{O}(A_2 - A_4)$ are effective line bundles.

From the above computations we also have the following Lemma.

Lemma 3.4. *As a G -module, $H^0(D, \mathcal{O}(A_3)) \cong \chi_1 \oplus \alpha$ where α is a 2-dimensional irreducible representation of G .*

Proof. First, we claim that χ_1 is the unique 1-dimensional subrepresentation of $H^0(D, \mathcal{O}(A_3))$. If there is another 1-dimensional subrepresentation of $H^0(D, \mathcal{O}(A_3))$, then there should be a G -invariant effective divisor which is linearly equivalent to A_3 . However A_3 is the unique G -invariant effective divisor of degree 8 linearly equivalent to itself. Therefore there is no 1-dimensional subrepresentation of $H^0(D, \mathcal{O}(A_3))$ other than χ_1 . Note that D is not a hyperelliptic curve (cf. [25]). From the Clifford theorem we see that $1 \leq h^0(D, \mathcal{O}(A_3)) \leq 4$. Moreover from the analysis of the previous Lemma, we see that there is an effective divisor of degree 8 not equal to A_3 but linearly equivalent to A_3 . Therefore $h^0(D, \mathcal{O}(A_3)) > 1$ and since there is no n -dimensional irreducible representation of G for $n > 2$ (cf. Remark A.3), we get $H^0(D, \mathcal{O}(A_3)) = \chi_1 \oplus \alpha$ where α is a 2-dimensional irreducible representation. \square

Now we express K_D in terms of A_3 and A_4 .

Lemma 3.5. $K_D \cong \mathcal{O}(A_3 + A_4)$.

Proof. Consider $H_4 = \langle g_4, g_5 \rangle$. From Riemann–Hurwitz formula we see that D/H_4 is an elliptic curve. Because $A_3 + A_4$ are ramification divisor with stabilizer group \mathbb{Z}_2 , we get $K_D \cong \mathcal{O}(A_3 + A_4)$. \square

Let $\mathcal{M} := \mathcal{O}(A_3)$ and $\mathcal{M}' := \mathcal{O}(A_4)$. Because A_3 and A_4 are G -invariant divisors of degree 8 in D , the action on the function field of D induces natural G -linearizations on \mathcal{M} and \mathcal{M}' . Therefore \mathcal{M} and \mathcal{M}' are G -equivariant line bundles on D of degree 8 (see [13,14,23] for more details). We also consider the map $H^0(D, \mathcal{O}(A_3)) \otimes$

$H^0(D, \mathcal{O}(A_4)) \rightarrow H^0(D, \mathcal{O}(A_3 + A_4))$. From Lemma 3.4 and Lemma 3.5, we see that $h^0(D, \mathcal{M}) = h^0(D, \mathcal{M}') = 3$.

Lemma 3.6. *As a G -module, $H^1(D, \mathcal{M}) \cong \chi_4 \oplus \beta$ where β is an irreducible 2-dimensional representation of G .*

Proof. From Serre duality we see that $H^1(D, \mathcal{M}) \cong H^0(D, K_D \otimes \mathcal{M}^{-1})^*$. Consider the natural map $H^0(D, \mathcal{M}) \otimes H^0(D, K_D \otimes \mathcal{M}^{-1}) \rightarrow H^0(D, K_D)$. This map is the same as $H^0(D, \mathcal{O}(A_3)) \otimes H^0(D, \mathcal{O}(A_4)) \otimes \chi \rightarrow H^0(D, \mathcal{O}(A_3 + A_4)) \otimes \chi$ for a character χ . Because constant function belongs to $H^0(D, \mathcal{O}(A_3 + A_4))$, χ_1 is a G -submodule of $H^0(D, \mathcal{O}(A_3 + A_4))$. From Lefschetz fixed point formula we see that $H^0(D, K_D) = \chi_4 + \chi_{10} + \chi_{12} + \chi_{13} + \chi_{14}$. Note that $\chi_{10}, \chi_{12}, \chi_{13}, \chi_{14}$ are 2-dimensional irreducible representations of G . Therefore we get $\chi = \chi_4$ and $H^1(D, \mathcal{M}) \cong H^0(D, K_D \otimes \mathcal{M}^{-1})^* = \chi_4 \oplus \beta$ where β is a 2-dimensional irreducible representation. \square

3.4. Exceptional sequences of line bundles on S

It is well known that $D_G^b(C \times D) \simeq D^b(S)$ since G acts freely on $C \times D$ (see [26, Appendix] for more details). Therefore it suffices to construct exceptional sequence in $D_G^b(C \times D)$. We consider equivariant line bundles on $C \times D$. By abuse of notation, we denote a G -equivariant line bundle on $C \times D$ and its descent on S by the same symbol. Note that \mathcal{L} is not a G -equivariant line bundle on C . However from the Dolgachev's theorem we can prove that there exists a G -invariant line bundle \mathcal{N} such that $\mathcal{L} \boxtimes \mathcal{N}$ is a G -equivariant line bundle on $C \times D$.

Theorem 3.1. *There is a character $\chi \in \widehat{G}$ such that $\mathcal{L} \boxtimes (\mathcal{M} \otimes \mathcal{N})(\chi)$, $\mathcal{L} \boxtimes \mathcal{N}$, $\mathcal{O}_C \boxtimes \mathcal{M}(\chi)$, $\mathcal{O}_C \boxtimes \mathcal{O}_D$ descend to an exceptional sequence of line bundles on S .*

Proof. Because S is a surface with $p_g = q = 0$, every line bundle on S is an exceptional object. Now from the Künneth formula, we see that, for all i ,

$$H^i(S, \mathcal{L} \boxtimes \mathcal{N}) = \left(\bigoplus_{j+k=i} H^j(C, \mathcal{L}) \otimes H^k(D, \mathcal{N}) \right)^G = 0,$$

$$H^i(S, \mathcal{L} \boxtimes (\mathcal{M} \otimes \mathcal{N})(\chi)) = \left(\bigoplus_{j+k=i} H^j(C, \mathcal{L}) \otimes H^k(D, (\mathcal{M} \otimes \mathcal{N})) \otimes \chi \right)^G = 0,$$

$$H^i(S, \mathcal{L} \boxtimes (\mathcal{M}^{-1} \otimes \mathcal{N})(\chi^{-1})) = \left(\bigoplus_{j+k=i} H^j(C, \mathcal{L}) \otimes H^k(D, (\mathcal{M}^{-1} \otimes \mathcal{N})) \otimes \chi^{-1} \right)^G = 0$$

as \mathcal{L} is an ineffective theta characteristic on C . From Riemann–Roch formula, the Euler–Poincaré characteristic of $\mathcal{O}_C \boxtimes \mathcal{M}(\chi)$ on S is equal to 0. Then by Künneth formula we

see that it is enough to show that the G -invariant parts of the following vector spaces are all zero.

$$H^0(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi)) = (H^0(C, \mathcal{O}_C) \otimes H^0(D, \mathcal{M}) \otimes \chi)^G$$

$$H^2(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi)) = (H^1(C, \mathcal{O}_C) \otimes H^1(D, \mathcal{M}) \otimes \chi)^G$$

From the equivariant Serre duality we see that $H^1(C, \mathcal{O}_C) \cong H^0(C, K_C)^*$ as G -modules and we know the representation of $H^1(C, \mathcal{O}_C) = \chi_7 + \chi_9 + \chi_{11}$. Moreover $H^0(D, \mathcal{M}) = \chi_1 \oplus A$ and $H^1(D, \mathcal{M}) = \chi_4 \oplus B$ where A and B are 2-dimensional irreducible representations of G (cf. [Lemma 3.4](#), [Lemma 3.6](#)). Finally we can check that for any possible representation $H^0(D, \mathcal{M})$, $H^1(D, \mathcal{M})$ there exists $\chi \in \widehat{G}$ such that

$$h^0(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi)) = h^1(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi)) = h^2(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi)) = 0$$

using Magma (cf. [Appendix B](#)). In other word, we can always find $\chi \in \widehat{G}$ such that a sequence of 4 equivariant line bundles $\mathcal{L} \boxtimes (\mathcal{M} \otimes \mathcal{N})(\chi)$, $\mathcal{L} \boxtimes \mathcal{N}$, $\mathcal{O}_C \boxtimes \mathcal{M}(\chi)$, $\mathcal{O}_C \boxtimes \mathcal{O}_D$ on $C \times D$ descent to an exceptional sequence of line bundles on S . \square

3.5. Quasiphantom categories

From the exceptional collections of maximal length 4, we obtain examples of quasiphantom categories. We recall the definitions of quasiphantom and phantom category.

Definition 3.7. [[18](#), [Definition 1.8](#)] Let S be a smooth projective variety. Let \mathcal{A} be an admissible triangulated subcategory of $D^b(S)$. Then \mathcal{A} is called a quasiphantom category if the Hochschild homology of \mathcal{A} vanishes, and the Grothendieck group of \mathcal{A} is finite. If the Grothendieck group of \mathcal{A} also vanishes, then \mathcal{A} is called a phantom category.

Now we obtain new examples of quasiphantom categories as follows.

Proposition 3.8. *The orthogonal complements of the exceptional sequence of [Theorem 3.1](#) are quasiphantom categories.*

Proof. There is a semiorthogonal decomposition $D^b(S) = \langle \mathcal{A}, \mathcal{B} \rangle$ where \mathcal{B} is the full triangulated subcategory of $D^b(S)$ generated by exceptional collection of length 4 constructed above. From [[21](#)], we see that the Hochschild homology of \mathcal{A} vanishes and from [[2](#)], we see that the Grothendieck group of \mathcal{A} is $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$. Therefore \mathcal{A} is a quasiphantom category. \square

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Appendix A. $G(32, 27)$

For the convenience of readers, we list basic properties of $G(32, 27)$. The following data were obtained using GAP [17] and Magma [8]. $SmallGroup(a, b)$ denotes a finite group of order a having number b in the list in Magma library.

Definition A.1. We write $G(32, 27)$ using finite polycyclic presentation.

$$\begin{aligned} G(32, 27) &:= SmallGroup(32, 27) \\ &= \langle g_1, g_2, g_3, g_4, g_5 \mid g_1^{-1}g_2g_1 = g_2g_4, g_1^{-1}g_3g_1 = g_3g_5 \rangle. \end{aligned}$$

Remark A.2. We see that $G(32, 27)$ is a semidirect product of $N = \mathbb{Z}_2^4$ and $Q = \mathbb{Z}_2$ via the following isomorphism (cf. [4]).

$$\begin{aligned} g_1 &\mapsto (0, 1), & g_2 &\mapsto ((1, 0, 0, 0), 0), & g_3 &\mapsto ((0, 1, 0, 0), 0), \\ g_4 &\mapsto ((0, 0, 1, 0), 0), & g_5 &\mapsto ((0, 0, 0, 1), 0). \end{aligned}$$

Here we write $N \rtimes_{\Phi} Q$ to denote the semidirect product where $\Phi: Q \rightarrow \text{Aut}(N) \cong GL_4(\mathbb{F}_2)$ can be represented by the following matrix.

$$\Phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The multiplication of $G(32, 27) = N \rtimes_{\Phi} Q$ can be defined as follows:

$$(n_1, q_1) \cdot (n_2, q_2) = (n_1 + \Phi_{q_1}(n_2), q_1 + q_2).$$

The list of conjugacy classes of $G(32, 27)$ is as follows.

Table 5
Character table of $G(32, 27)$.

	1	g_5	g_4	g_4g_5	$g_2g_3g_5$	g_2	g_2g_3	g_3g_4	g_2g_5	g_3	g_1	$g_1g_2g_3$	g_1g_2	g_1g_3
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	1	−1	−1	1	−1	1	−1	1	−1
χ_3	1	1	1	1	1	−1	1	−1	−1	−1	1	1	−1	−1
χ_4	1	1	1	1	−1	−1	−1	1	−1	1	1	−1	−1	1
χ_5	1	1	1	1	−1	1	−1	−1	1	−1	−1	1	−1	1
χ_6	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
χ_7	1	1	1	1	−1	−1	−1	1	−1	1	−1	1	1	−1
χ_8	1	1	1	1	1	−1	1	−1	−1	−1	−1	−1	1	1
χ_9	2	−2	2	−2	0	2	0	0	−2	0	0	0	0	0
χ_{10}	2	2	−2	−2	0	0	0	−2	0	2	0	0	0	0
χ_{11}	2	−2	−2	2	2	0	−2	0	0	0	0	0	0	0
χ_{12}	2	2	−2	−2	0	0	0	2	0	−2	0	0	0	0
χ_{13}	2	−2	2	−2	0	−2	0	0	2	0	0	0	0	0
χ_{14}	2	−2	−2	2	−2	0	2	0	0	0	0	0	0	0

$$[1], [g_5], [g_4], [g_4g_5], [g_2g_3g_4, g_2g_3g_5], [g_2, g_2g_4], [g_2g_3, g_2g_3g_4g_5], \\ [g_3g_4, g_3g_4g_5], [g_2g_5, g_2g_4g_5], [g_3, g_3g_5], [g_1, g_1g_4, g_1g_5, g_1g_4g_5], \\ [g_1g_2g_3, g_1g_2g_3g_4, g_1g_2g_3g_5, g_1g_2g_3g_4g_5], \\ [g_1g_2, g_1g_2g_4, g_1g_2g_5, g_1g_2g_4g_5], [g_1g_3, g_1g_3g_4, g_1g_3g_5, g_1g_3g_4g_5].$$

The list of normal subgroups of $G(32, 27)$ is as follows.

$$G(32, 27), \langle g_1, g_3, g_4, g_5 \rangle, \langle g_1, g_2g_3, g_4, g_5 \rangle, \langle g_1, g_2, g_4, g_5 \rangle, \\ \langle g_2, g_3, g_4, g_5 \rangle, \langle g_1g_2g_4, g_3, g_4, g_5 \rangle, \langle g_1g_3g_5, g_2, g_4, g_5 \rangle, \\ \langle g_1g_2g_4, g_2g_3g_4g_5, g_4, g_5 \rangle, \langle g_3, g_4, g_5 \rangle, \langle g_1g_3g_5, g_4, g_5 \rangle, \langle g_2g_3, g_4, g_5 \rangle, \\ \langle g_1g_2g_3g_4g_5, g_4, g_5 \rangle, \langle g_2, g_4, g_5 \rangle, \langle g_1g_2g_4, g_4, g_5 \rangle, \langle g_1, g_4, g_5 \rangle, \\ \langle g_2, g_4 \rangle, \langle g_2g_5, g_4 \rangle, \langle g_4, g_5 \rangle, \langle g_3, g_5 \rangle, \langle g_3g_4, g_5 \rangle, \\ \langle g_2g_3, g_4g_5 \rangle, \langle g_2g_3g_4, g_4g_5 \rangle, \langle g_5 \rangle, \langle g_4g_5 \rangle, \langle g_4 \rangle, \langle 1 \rangle$$

Table 5 is the character table of $G(32, 27)$.

Remark A.3. Note that all characters of $G(32, 27)$ are real valued and there is no n -dimensional irreducible representation for $n > 2$.

Appendix B. Magma code

A downloadable code which readers can just copy and paste into Magma is uploaded at <https://sites.google.com/site/kklmagnacodes/>. Those who do not have a Magma license can use the online Magma calculator at <http://magma.maths.usyd.edu.au/calc/>.

The following is the basic setup, to ensure that the conjugacy classes and the character table are arranged in the way we presented in our paper. The commands following the symbol $>$ are to be entered into Magma.

```
> G := SmallGroup(32,27);
> G;
GrpPC : G of order 32 = 2^5
PC-Relations:
    G.2^G.1 = G.2 * G.4,
    G.3^G.1 = G.3 * G.5
> cls := Classes(G);
> cls_map := ClassMap(G);
> ct := CharacterTable(G);
> R := ClassFunctionSpace(G);
> cls_reps := [ Identity(G), G.5, G.4, G.4*G.5, G.2*G.3*G.5, G.2, G.2*G.3,
>              G.3*G.4, G.2*G.5, G.3, G.1, G.1*G.2*G.3, G.1*G.2, G.1*G.3 ];
> cls_reps_ind := [ cls_map(cls_reps[i]) : i in [1..#cls] ];
> function Character_for_our_cls_reps(values_at_cls_reps)
>     return R[ values_at_cls_reps[Index([1..#cls], cls_reps_ind[i])] ] :
>             i in [1..#cls] ];
> end function;
```

For example, we check whether the first and the fourth irreducible characters of ct coincide respectively with χ_1 and χ_4 in the table we presented in [Appendix A](#); in principle, one could do such a check for every χ_i in the table in [Appendix A](#).

```
> Character_for_our_cls_reps([1,1,1,1,-1,-1,-1,1,-1,1,1,-1,-1,1]) eq ct[4];
true
> Character_for_our_cls_reps([1,1,1,1,1,1,1,1,1,1,1,1,1,1]) eq ct[1];
true
```

The following code is to get the results of §3.1. We first compute the number of fixed points of elements of $G(32,27)$ for the ramification structure T_1 , and will compute χ_{K_C} accordingly.

```
> H_1 := sub<G|G.4*G.5*G.1>;
> H_2 := sub<G|G.2*G.3*G.4*G.5>;
> H_3 := sub<G|G.2*G.4*G.5>;
> H_4 := sub<G|G.3*G.4*G.5*G.1>;
> standard_stabs_T1 := [H_1,H_2,H_3,H_4];
> subs_T1 := &cat[[H^tr : tr in Transversal(G,H)] : H in standard_stabs_T1];
```

For $i = 1, 2, 3, 4$, $H_i = H_i$ is the stabilizer group of one of the pre-images of the i -th ramification point; call this pre-image x_i . Let's assume that $G(32,27)$ is acting from right. If it was acting from left, then we can turn it into a right action by letting each element of $G(32,27)$ act by its inverse; when doing so, the stabilizer groups H_i do not change. Let g_{i1}, g_{i2}, \dots be the right coset representatives of H_i in $G(32,27) = G$; so $H_i g_{i1}, H_i g_{i2}, \dots$ are the cosets. Then the list of all distinct pre-images of the i -th ramification point

is $x_i g_{i1}, x_i g_{i2}, \dots$, whose stabilizers are $g_{i1}^{-1} H_i g_{i1}, g_{i2}^{-1} H_i g_{i2}, \dots$. So `subs_T1` is the collection of all these stabilizer groups for all $i = 1, 2, 3, 4$, counted with multiplicity; the number of fixed points of an element g is the number of all stabilizer groups in `subs_T1` that contains g .

```
> fixedpts_T1 := [ #[sub : sub in subs_T1 | cls_reps[i] in sub ]
>                 : i in [2..#cls] ];
> fixedpts_T1;
[ 8, 0, 0, 0, 0, 8, 0, 8, 0, 4, 0, 0, 4 ]
```

Then the character value for χ_{K_C} and its decomposition into irreducible characters are obtained as follows.

```
> values_C := [ (2-fixedpts_T1[i])/2 : i in [1..#fixedpts_T1]];
> values_C;
[ -3, 1, 1, 1, 1, -3, 1, -3, 1, -1, 1, 1, -1 ]
> chi_K_C_at_cls_reps := [ 5 ] cat values_C;
> chi_K_C := Character_for_our_cls_reps(chi_K_C_at_cls_reps);
> [InnerProduct(chi_K_C, ct[i]) : i in [1..#ct]];
[ 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0 ]
> chi_K_C eq ct[7]+ct[9]+ct[11];
true
```

We do likewise for T_2 , to get χ_{K_D} :

```
> H_5 := sub<G|G.2*G.3*G.4>;
> H_6 := sub<G|G.2>;
> H_7 := sub<G|G.2*G.3*G.4*G.1>;
> H_8 := sub<G|G.2*G.4*G.1>;
> standard_stabs_T2 := [H_5,H_6,H_7,H_8];
> subs_T2 := &cat[[H^tr : tr in Transversal(G,H)] : H in standard_stabs_T2];
> fixedpts_T2:=#[sub:sub in subs_T2|cls_reps[i] in sub]:i in [2..#cls]];
> fixedpts_T2;
[ 0, 8, 8, 8, 8, 0, 0, 0, 0, 4, 4, 0 ]
> values_D := [ (2-fixedpts_T2[i])/2 : i in [1..#fixedpts_T2]];
> values_D;
[ 1, -3, -3, -3, -3, 1, 1, 1, 1, -1, -1, 1 ]
> chi_K_D_at_cls_reps := [ 9 ] cat values_D;
> chi_K_D := Character_for_our_cls_reps(chi_K_D_at_cls_reps);
> [InnerProduct(chi_K_D, ct[i]) : i in [1..#ct]];
[ 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1 ]
> chi_K_D eq ct[4]+ct[10]+ct[12]+ct[13]+ct[14];
true
```

The following Magma code enables us to check that for each 2-dimensional irreducible representations A, B , there exists $\chi \in \widehat{G}$ such that

$$\begin{aligned} H^0(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi))^G &= (H^0(C, \mathcal{O}_C) \otimes H^0(D, \mathcal{M}) \otimes \chi)^G = 0 \\ H^2(S, \mathcal{O}_C \boxtimes \mathcal{M}(\chi))^G &= (H^1(C, \mathcal{O}_C) \otimes H^1(D, \mathcal{M}) \otimes \chi)^G = 0 \end{aligned}$$

hold where $H^0(C, \mathcal{O}_C) \cong \chi_1$ and $H^1(C, \mathcal{O}_C) = \chi_7 + \chi_9 + \chi_{11}$ in the character table, $H^0(D, \mathcal{M}) = \chi_1 \oplus A$, $H^1(D, \mathcal{M}) = \chi_4 \oplus B$ (cf. proof of [Theorem 3.1](#)).

```
> H0_C_OC := ct[1];
> H1_C_OC := chi_K_C;
> for i in [9..14] do           // irreducible characters of dimension 2
> for j in [9..14] do           // irreducible characters of dimension 2
>   H0_D_M := ct[1] + ct[i];     // A is ct[i]
>   H1_D_M := ct[4] + ct[j];     // B is ct[j]
>   good_chis := [k:k in [1..8]] // irreducible characters of dimension 1
>   InnerProduct(H0_C_OC * H0_D_M * ct[k], ct[1]) eq 0 and
>   InnerProduct(H1_C_OC * H1_D_M * ct[k], ct[1]) eq 0];
>   if #good_chis eq 0 then
>     printf "\n (i,j)=(%o,%o) is trouble.", i,j;
>   // else
>     printf "\n A=ct[%o], B=ct[%o] -> chi=ct%o works", i,j,good_chis;
>   end if;
> end for;
> end for;
```

We believe that anyone could easily interpret what the above code is doing. If the code does not print any message, it means that the statement is true. Removing `//` from the two lines lets the reader see what χ 's work for each A and B .

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