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Covariant functors and asymptotic stability

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ABSTRACT

Let R be a commutative Noetherian ring, I, J ideals of R and M a finitely generated R-module. Let F be a covariant R-linear functor from the category of finitely generated R-modules to itself. We first show that if F is coherent, then the sets $\operatorname{Ass}_R F(M/I^nM)$, $\operatorname{Ass}_R F(I^{n-1}M/I^nM)$ and the values $\operatorname{depth}_J F(M/I^nM)$, $\operatorname{depth}_J F(I^{n-1}M/I^nM)$ become independent of n for large n. Next, we consider several examples in which F is a rather familiar functor, but is not coherent or not even finitely generated in general. In these cases, the sets $\operatorname{Ass}_R F(M/I^nM)$ still become independent of n for large n. We then show one negative result where F is not finitely generated. Finally, we give a positive result where F belongs to a special class of functors which are not finitely generated in general, an example of which is the zeroth local cohomology functor.

1. Introduction

In this paper, we will extend two results on asymptotic stability by M. Brodmann. Let us begin by fixing some terminology. A ring will mean a commutative ring with

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unity, unless specified otherwise. For a ring R, we let $\operatorname{Mod}(R)$ denote the category of R-modules and $\operatorname{mod}(R)$ the category of finitely generated R-modules. A functor will mean a covariant functor. For a nonempty set X and a sequence of elements $\{x_n\}_{n\geqslant k}$ of X, we say that asymptotic stability holds for the elements x_n, or that the elements x_n stabilize, if the sequence $\{x_n\}_{n\geqslant k}$ is eventually constant.

For the rest of this section, we will let R be a Noetherian ring unless specified otherwise, $L, M, N \in \operatorname{mod}(R)$ and I, J be ideals of R. The background of our project can be traced back to one of Ratliff's papers.

Question 1.1. [1, Introduction] Suppose that R is a domain and P is a prime ideal of R. If $P \in \operatorname{Ass}_R(R/I^k)$ for some $k \geqslant 1$, is $P \in \operatorname{Ass}_R(R/I^n)$ for all large n ?

Brodmann [2, (9)] gave a negative answer to the question, but at the same time, he proved a related, by now well-known result. Using the notation established so far, we will state his first result that we are interested in.

Theorem 1.2. *[2, page 16] The sets $\operatorname{Ass}_R(M/I^nM)$ and $\operatorname{Ass}_R(I^{n-1}M/I^nM)$ stabilize.*

The second result that we are interested in is as follows.

Theorem 1.3. *[3, Theorems 2(i) and 12(i)] The values $\operatorname{depth}_J(M/I^nM)$ and $\operatorname{depth}_J(I^{n-1}M/I^nM)$ stabilize.*

Most of this paper will be related to Theorem 1.2. There have been numerous generalizations of the theorem over the years. Here are a few of them.[2]

Theorem 1.4. *[5, Theorem 1] The sets $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, R/I^n)$ and $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, I^{n-1}/I^n)$ stabilize for any $i \geqslant 0$.*

Theorem 1.5. *[4, Proposition 3.4] Let $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ be a complex. Suppose that $L' \subseteq L$, $M' \subseteq M$ and $N' \subseteq N$ are submodules such that $\alpha(L') \subseteq M'$ and $\beta(M') \subseteq N'$. For $n \geqslant 0$, let $\mathrm{H}(n)$ denote the homology of the induced complex*

$$\frac{L}{I^nL'} \xrightarrow{\alpha_n} \frac{M}{I^nM'} \xrightarrow{\beta_n} \frac{N}{I^nN'}$$

Then the sets $\operatorname{Ass}_R \mathrm{H}(n)$ stabilize.

Corollary 1.6. *[4, Corollary 3.5] Let $M' \subseteq M$ be a submodule. Then for any $i \geqslant 0$, the sets $\operatorname{Ass}_R \operatorname{Tor}_i^R(N, M/I^nM')$ and $\operatorname{Ass}_R \operatorname{Ext}_R^i(N, M/I^nM')$ stabilize.*

[2] Although the theorems quoted here are related, the authors of [4] and [5] did not seem to know about the results of each other.

A rather extensive introduction to results related to Theorems 1.2 and 1.3 can be found in [6]. However, we will proceed in a different direction. Our main goal is to relate the theorems to the following notions.

Notation 1.7. Let R be a commutative ring and $M \in \mathrm{Mod}(R)$. Then we let h_M denote the functor $\mathrm{Hom}_R(M, -)$. We let $\mathscr{F}$ denote the category of R-linear covariant functors F from $\mathrm{mod}(R)$ to itself.

Definition 1.8. [7, page 53] Let R be a Noetherian ring and $F \in \mathscr{F}$. We say that:

(1) F is representable if $F \cong h_M$ for some $M \in \mathrm{mod}(R)$;
(2) F is finitely presented if there exist $M, N \in \mathrm{mod}(R)$ and an exact sequence $h_N \to h_M \to F \to 0$;
(3) F is finitely generated if there exist $M \in \mathrm{mod}(R)$ and an exact sequence $h_M \to F \to 0$.

Remark 1.9. By [8, Section 2], finitely presented functors are exactly the coherent objects in $\mathscr{F}$. Hence, following [7], we will use the term "coherent functors" instead of "finitely presented functors."

Remark 1.10. Representable $\Rightarrow$ coherent $\Rightarrow$ finitely generated $\Rightarrow$ R-linear

We can now state our main result, which will be proved in several steps in Section 2.

Theorem 1.11. *Let R be a Noetherian ring, I, J ideals of R, $M \in \mathrm{mod}(R)$ and F be a coherent functor. Then the sets $\mathrm{Ass}_R F(M/I^nM)$, $\mathrm{Ass}_R F(I^{n-1}M/I^nM)$ and the values $\mathrm{depth}_J F(M/I^nM)$, $\mathrm{depth}_J F(I^{n-1}M/I^nM)$ stabilize.*

Remark 1.12. Theorem 1.11 gives an extension of Theorem 1.5 in the following sense. Using the notation in Theorem 1.5, let $L = L'$, $M = M'$ and $N = N'$. Then Theorem 1.5 is an instance of Theorem 1.11 by Lemma 2.3(b) (cf. proof of Theorem 5.6). However, by [7, Example 5.5], not all coherent functors are of the form given by Lemma 2.3(b). A technical generalization of Theorem 1.5 is given by Corollary 2.2.

A summary of the rest of the paper is as follows. In Section 3, we consider two covariant R-linear functors, the zeroth local cohomology functor Γ_I where I is an ideal of R, and the torsion functor τ_S where S is a multiplicatively closed subset of R. We show that in most cases, the functors id/Γ_I and id/τ_S are finitely generated but not coherent, while the functors Γ_I and τ_S are not even finitely generated. However, if $F = \mathrm{id}/\Gamma_I$, id/τ_S, Γ_I or τ_S, then whether or not F is coherent, the sets $\mathrm{Ass}_R F(M/I^nM)$ and $\mathrm{Ass}_R F(I^{n-1}M/I^nM)$ always stabilize. In Section 4, we consider the case where R is a Dedekind domain. We show that if F is a finitely generated functor, then the sets $\mathrm{Ass}_R F(M/I^nM)$ stabilize. We give a family of non-finitely generated functors F such that the sets $\operatorname{Ass}_R F(M/I^nM)$ do not stabilize. In Section 5, we consider a complex $\mathscr{S}\colon A \to B \to C$ of R-modules where $B \in \operatorname{mod}(R)$ and the functor $F(-) = \mathrm{H}(\mathscr{S} \otimes -)$, an example of which is the zeroth local cohomology functor. We show that if R is a one-dimensional Noetherian domain, then the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.

2. Proof of stability results

In this section, we let R be a Noetherian ring. All R-modules will be finitely generated unless specified otherwise. We will prove our main result, Theorem 1.11, which will follow from Corollaries 2.4, 2.9 and 2.13. First, we need a slightly more general result than Theorem 1.5. We recall that the Theorem follows from an even more general result.

Theorem 2.1. *[4, Proof of Proposition 3.4] Let $I \subseteq R$ be an ideal, $T \in \operatorname{mod}(R)$ and U, V, W submodules of T such that $W \subseteq V$. Then the sets $\operatorname{Ass}_R((U + I^nV)/I^nW)$ stabilize.*

For the reader's convenience, we give here an outline of the proof of Theorem 2.1 in [4]. First, it was shown in [9, Lemma 1.2] using an extended Rees ring that $\cup_{n\geqslant 0} \operatorname{Ass}_R(T/I^nW)$ is finite. Hence the subset $S = \cup_{n\geqslant 0} \operatorname{Ass}_R((U + I^nV)/I^nW)$ is also finite. One then uses the Artin–Rees Lemma at most $|S| + 2$ times to find an m so large such that $\cup_{n\geqslant m} \operatorname{Ass}_R((U + I^nV)/I^nW) \subseteq S$ is an increasing union (cf. [2, (4) and (5)]). Since S is finite, the sets $\operatorname{Ass}_R((U + I^nV)/I^nW)$ stabilize. The reader is invited to consult the references for further details.

Corollary 2.2. *Consider the situation as in Theorem 1.5. Let $c \in \mathbb{N}$ and L_1, L_2 be submodules of L such that $I^cL' \subseteq L_2$. For $n \geqslant c$, let $\mathrm{H}(n)$ denote the homology of the induced complex*

$$\frac{L_1 + I^{n-c}L_2}{I^nL'} \xrightarrow{\alpha_n} \frac{M}{I^nM'} \xrightarrow{\beta_n} \frac{N}{I^nN'}$$

Then the sets $\operatorname{Ass}_R \mathrm{H}(n)$ stabilize.

Proof. We follow [4, Proof of Proposition 3.4]. By the Artin–Rees Lemma, there is $d \geqslant c$ such that for all $n \geqslant d$, $\beta(M) \cap I^nN' = I^{n-d}(\beta(M) \cap I^dN')$. Then for $n \geqslant d$, we have

$$\begin{aligned}\mathrm{H}(n) &= \frac{\ker(\beta_n)}{\operatorname{im}(\alpha_n)} \\ &= \frac{\beta^{-1}(I^nN')/I^nM'}{[\alpha(L_1 + I^{n-c}L_2) + I^nM']/I^nM'} \\ &= \frac{\ker(\beta) + I^{n-d}(\beta^{-1}(I^dN'))}{\alpha(L_1) + I^{n-d}(I^{d-c}\alpha(L_2) + I^dM')}\,.\end{aligned}$$

The result then follows from Theorem 2.1 by letting

$$T = \frac{M}{\alpha(L_1)}\,, \qquad U = \frac{\ker(\beta)}{\alpha(L_1)}\,,$$
$$V = \frac{\beta^{-1}(I^d N') + \alpha(L_1)}{\alpha(L_1)} \quad \text{and} \quad W = \frac{I^{d-c}\alpha(L_2) + I^d M' + \alpha(L_1)}{\alpha(L_1)}\,. \qquad \square$$

Next, we recall some results from [7].

Lemma 2.3. *[7, Lemma 1.2, Examples 2.1–2.5]*

(a) *For any $M \in \operatorname{mod}(R)$ and $F \in \mathscr{F}$, there is a natural isomorphism $\operatorname{Nat}_{\mathscr{F}}(h_M, F) \cong F(M)$ given by $T \mapsto T_M(\mathrm{id}_M)$.*
(b) *Let $P_\bullet$ be a complex of finitely generated R-modules. Then for any $i \in \mathbb{Z}$, the functor $\mathrm{H}_i(P_\bullet \otimes -)$ is coherent.*
(c) *Let $M \in \operatorname{mod}(R)$. Then for any $i \geqslant 0$, the functors $\operatorname{Tor}_i^R(M, -)$ and $\operatorname{Ext}_R^i(M, -)$ are coherent.*

We then obtain the following generalization of the first half of Theorem 1.2. By Lemma 2.3(c), Corollary 2.4 may also be viewed as a generalization of Corollary 1.6.

Corollary 2.4. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, M' be a submodule of M and $I \subseteq R$ an ideal. Then the sets $\operatorname{Ass}_R F(M/I^n M')$ stabilize.*

Proof. Let F be given by $h_L \to h_K \to F \to 0$. By Lemma 2.3(a), the map $h_L \to h_K$ arises from a map $f\colon K \to L$. Choose free resolutions of K and L and a lift of f such that the following diagram commutes.

$$\begin{tikzcd}
R^{\oplus k_1} \arrow[r] \arrow[d, "\beta"] & R^{\oplus \ell_1} \arrow[d, "\gamma"] \\
R^{\oplus k_0} \arrow[r, "\alpha"] \arrow[d] & R^{\oplus \ell_0} \arrow[d] \\
K \arrow[r, "f"] \arrow[d] & L \arrow[d] \\
0 & 0
\end{tikzcd} \tag{1}$$

Apply $\operatorname{Hom}_R(-, M/I^n M')$ to get the commutative diagram

$$\begin{tikzcd}
\dfrac{M^{\oplus \ell_1}}{I^n\left((M')^{\oplus \ell_1}\right)} \arrow[r] & \dfrac{M^{\oplus k_1}}{I^n\left((M')^{\oplus k_1}\right)} & & \\
\dfrac{M^{\oplus \ell_0}}{I^n\left((M')^{\oplus \ell_0}\right)} \arrow[r, "\alpha_n^*"] \arrow[u, "\gamma_n^*"] & \dfrac{M^{\oplus k_0}}{I^n\left((M')^{\oplus k_0}\right)} \arrow[u, "\beta_n^*"] & & \\
h_L\left(\dfrac{M}{I^n M'}\right) \arrow[r, "f_n^*"] \arrow[u] & h_K\left(\dfrac{M}{I^n M'}\right) \arrow[r] \arrow[u] & F\left(\dfrac{M}{I^n M'}\right) \arrow[r] & 0 \\
0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $f_n^*, \alpha_n^*, \beta_n^*, \gamma_n^*$ are induced by f, α, β, γ respectively. Then we have

$$F\left(\frac{M}{I^n M'}\right) \cong \frac{\ker \beta_n^*}{\alpha_n^*\left(\ker \gamma_n^*\right)} .$$

Similarly, we apply $\mathrm{Hom}_R(-, M)$ to (1) to get maps $\alpha^*, \beta^*, \gamma^*$ induced by α, β, γ respectively. Let $A = M^{\oplus \ell_0}$, $A' = (M')^{\oplus \ell_0}$ and $B' = (M')^{\oplus \ell_1}$. As in the proof of Corollary 2.2, there is $c \in \mathbb{N}$ such that $\gamma^*(A) \cap I^n B' = I^{n-c}(\gamma^*(A) \cap I^c B')$ for all $n \geqslant c$, and hence

$$\ker(\gamma_n^*) = \frac{\ker(\gamma^*) + I^{n-c}\left(\left(\gamma^*\right)^{-1}\left(I^c B'\right)\right)}{I^n A'} .$$

The result then follows by applying Corollary 2.2 to the maps $\alpha_n^*|_{\ker(\gamma_n^*)}$ and β_n^*. $\square$

We next generalize the first half of Theorem 1.3 along similar lines.

Notation 2.5. Let T, U, V, W be as in Theorem 2.1. We let $T_n = (T, U, V, W)_n = (U + I^n V)/I^n W$.

Remark 2.6. Let L be an ideal of R. For a submodule S of T, we let $\overline{S}$ be the image of S under the natural projection $T \to T/LU$. Then we have

$$\begin{aligned}
\frac{T_n}{LT_n} &= \frac{U + I^n V}{LU + LI^n V + I^n W} \\
&= \frac{\overline{U} + I^n \overline{V}}{LI^n \overline{V} + I^n \overline{W}} \\
&= (\,\overline{T}, \overline{U}, \overline{V}, \overline{LV + W}\,)_n
\end{aligned}$$

Theorem 2.7. *The values* $\mathrm{depth}_J\, T_n$ *stabilize.*

Proof. First, suppose that $T_n/JT_n = (\overline{T}, \overline{U}, \overline{V}, \overline{JV + W})_n = 0$ for infinitely many n. Then by Theorem 2.1, we see that $\operatorname{Ass}_R \overline{T}_n = \emptyset$ for large n. So for all large n, we have $T_n/JT_n = 0$ and hence $\operatorname{depth}_J T_n = \infty$. Hence we may assume that $T_n \neq JT_n$ for large n.

The rest of the proof is the same as that in [3, Theorem 2(i)]. We let $i_T = \liminf_{n\to\infty} \operatorname{depth}_J(T_n)$, $\ell_T = \lim_{n\to\infty} \operatorname{depth}_J(T_n)$ if such exists, and prove by induction on i_T that $\ell_T = i_T$. Suppose that $i_T = 0$. Then $J \subseteq \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for infinitely many n. By Theorem 2.1, we have $J \subseteq \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for all large n, so $\ell_T = i_T = 0$.

Now suppose that $i_T > 0$. Then by Theorem 2.1, there is $x \in J$ such that $x \notin \{r \in P \mid P \in \operatorname{Ass}_R T_n\}$ for all large n. Writing $T_n/xT_n = (\overline{T}, \overline{U}, \overline{V}, \overline{xV + W})_n$, we have $\operatorname{depth}_J \overline{T}_n = \operatorname{depth}_J T_n - 1$ for all large n. Hence $i_{\overline{T}} = i_T - 1$. By induction, we have $\ell_{\overline{T}} = i_{\overline{T}}$, so $\ell_T = \ell_{\overline{T}} + 1 = i_T$. □

Corollary 2.8. *Let $J \subseteq R$ be an ideal. Consider the situation as in Corollary 2.2 with the complexes*

$$\frac{L_1 + I^{n-c}L_2}{I^n L'} \xrightarrow{\alpha_n} \frac{M}{I^n M'} \xrightarrow{\beta_n} \frac{N}{I^n N'}$$

and $\mathrm{H}(n)$ denoting the homology of the complex. Then the values $\operatorname{depth}_J \mathrm{H}(n)$ stabilize.

Proof. As in the proof of Corollary 2.2, we have $\mathrm{H}(n) = T_{n-d} = (T, U, V, W)_{n-d}$. The result then follows from Theorem 2.7. □

Corollary 2.9. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, M' be a submodule of M and I, J be ideals of R. Then the values $\operatorname{depth}_J F(M/I^n M')$ stabilize.*

Proof. We only need to apply Corollary 2.8 to the maps $\alpha_n^*|_{\ker(\gamma_n^*)}$ and β_n^* in the proof of Corollary 2.4. □

In order to generalize the rest of Theorems 1.2 and 1.3, we let $S = \bigoplus_{n\geqslant 0} S_n$ be a Noetherian R-algebra generated in degree 1 with $S_0 = R$. We will use a result from [5].

Theorem 2.10. *[5, Lemma 2.1] Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded S-module. Then the sets $\operatorname{Ass}_R M_n$ stabilize.*

Corollary 2.11. *Let $L \to M \to N$ be a complex of $\mathbb{Z}$-graded S-modules, where the maps are homogeneous and $M \in \operatorname{mod}(S)$. Let $H = \bigoplus_{n\in\mathbb{Z}} H_n$ be the homology of the complex. Then the sets $\operatorname{Ass}_R H_n$ stabilize.*

Corollary 2.12. *Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded S-module, for example the module H as in Corollary 2.11. Let J be an ideal of R. Then the values $\operatorname{depth}_J M_n$ stabilize.*

Proof. The proof of Theorem 2.7 works by applying Theorem 2.10 to the finitely generated graded S-modules M, $M/JM = \bigoplus_{n\in\mathbb{Z}}(M_n/JM_n)$ and M/xM. □

Corollary 2.13. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, M' be a submodule of M and I, J be ideals of R. Then the sets $\operatorname{Ass}_R F(I^n M/I^n M')$ and the values $\operatorname{depth}_J F(I^n M/I^n M')$ stabilize.*

Proof. As in Corollary 2.4, we apply $\operatorname{Hom}_R(-, I^n M/I^n M')$ to (1) to get

$$\begin{tikzcd}
\dfrac{I^n\left(M^{\oplus \ell_1}\right)}{I^n\left((M')^{\oplus \ell_1}\right)} \arrow[r] & \dfrac{I^n\left(M^{\oplus k_1}\right)}{I^n\left((M')^{\oplus k_1}\right)} & & \\
\dfrac{I^n\left(M^{\oplus \ell_0}\right)}{I^n\left((M')^{\oplus \ell_0}\right)} \arrow[u, "\gamma_n^*"] \arrow[r, "\alpha_n^*"] & \dfrac{I^n\left(M^{\oplus k_0}\right)}{I^n\left((M')^{\oplus k_0}\right)} \arrow[u, "\beta_n^*"'] & & \\
h_L\left(\dfrac{I^n M}{I^n M'}\right) \arrow[u] \arrow[r, "f_n^*"] & h_K\left(\dfrac{I^n M}{I^n M'}\right) \arrow[u] \arrow[r] & F\left(\dfrac{I^n M}{I^n M'}\right) \arrow[r] & 0 \\
0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Again we have $F\left(\dfrac{I^n M}{I^n M'}\right) \cong \dfrac{\ker \beta_n^*}{\alpha_n^*\left(\ker \gamma_n^*\right)}$, where $\alpha_n^*, \beta_n^*, \gamma_n^*$ are the maps induced by α, β, γ in (1) respectively, so the result follows by applying Corollaries 2.11 and 2.12 to $S = \bigoplus_{n \geqslant 0} I^n$ and the maps $\bigoplus_{n\geqslant 0}(\alpha_n^*|_{\ker(\gamma_n^*)})$ and $\bigoplus_{n \geqslant 0} \beta_n^*$. $\square$

A coherent functor F given by $h_L \to h_K \to F \to 0$ can be considered as a functor $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$ since h_L and h_K are (cf. [7, Remark 3.3]). So the proof of Corollary 2.13 gives the next result.

Corollary 2.14. *Let F be a coherent functor, $M \in \operatorname{mod}(R)$, $M' \subseteq M$ be a submodule, I be an ideal of R, $S = \mathscr{R}(I) = \bigoplus_{n\geqslant 0} I^n$ and $\operatorname{gr}(I) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$. Then:*

(a) *$F\left(\bigoplus_{n\geqslant 0} I^n M/I^n M'\right) = \bigoplus_{n\geqslant 0} F(I^n M/I^n M')$ is a finitely generated graded S-module.*
(b) *When $M' = IM$, $F\left(\bigoplus_{n\geqslant 0} I^n M/I^{n+1} M\right) = \bigoplus_{n\geqslant 0} F(I^n M/I^{n+1} M)$ is a finitely generated graded $\operatorname{gr}(I)$-module.*
(c) *The module structures over S and $\operatorname{gr}(I)$ in (a) and (b) respectively correspond to the multiplication maps given by applying F to $I^n M/I^n M' \xrightarrow{x} I^{n+m} M/I^{n+m} M'$, where $x \in I^m$.*

Remark 2.15. Instead of studying asymptotic stability properties of covariant coherent functors, one may want to consider contravariant coherent functors as well. Unfortunately, as stated in [4, Remark 3.6], the sets $\operatorname{Ass}_R \operatorname{Ext}^i_R(R/I^n, R)$ do not stabilize in general, so our main focus will be on covariant functors. See [10, Introduction] and [11, Proposition 2.1] for further details.

3. Examples of non-coherent functors with asymptotic stability

In view of the results in Section 2, one may be interested in knowing whether or not an R-linear covariant functor is coherent. Some important examples of coherent functors are given in Lemma 2.3. In this section, we will study the zeroth local cohomology functor $\Gamma_I = \mathrm{H}^0_I$ where I is an ideal of R, and the torsion functor τ_S where S is a multiplicatively closed subset of R. It turns out that if $F = \Gamma_I$, τ_S, id/Γ_I or id/τ_S, then the functor F is usually not coherent. However, we will see in Corollaries 3.6 and 3.17 that whether or not F is coherent, the sets $\operatorname{Ass}_R F(M/I^nM)$ and $\operatorname{Ass}_R F(I^{n-1}M/I^nM)$ always stabilize.

First, let us consider a Yoneda type result.

Lemma 3.1. *Let R be a Noetherian ring and F be a finitely generated functor given by $h_M \xrightarrow{T} F \to 0$. Then for any $N \in \operatorname{mod}(R)$ and $x \in F(N)$, there is $f \in \operatorname{Hom}_R(M,N)$ such that $x = (F(f) \circ T_M)(\mathrm{id}_M)$. In particular, $x \in \operatorname{im} F(f)$.*

Proof. If $x \in F(N)$, then we let $f \in \operatorname{Hom}_R(M,N)$ be such that $T_N(f) = x$. The result follows from the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_R(M,M) \arrow[r, "T_M"] \arrow[d, "h_M(f)"'] & F(M) \arrow[r] \arrow[d, "F(f)"] & 0 \\ \operatorname{Hom}_R(M,N) \arrow[r, "T_N"] & F(N) \arrow[r] & 0 \end{tikzcd} \qquad \square$$

Corollary 3.2. *Let R be a Noetherian ring and $\{F_\lambda\}_{\lambda\in\Lambda}$ be a direct system of functors in $\mathscr{F}$. Let $F = \varinjlim_{\lambda\in\Lambda} F_\lambda$ be given by $\{T_\lambda\colon F_\lambda \to F\}_{\lambda\in\Lambda}$. If $F \in \mathscr{F}$ and is finitely generated, then $F = \operatorname{im} T_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. In particular, if T_λ is injective for all $\lambda \in \Lambda$, then $F = F_\lambda$ for all $\lambda \geqslant \lambda_0$.*

Proof. Let F be given by $h_M \to F \to 0$. Since $F(M) \in \operatorname{mod}(R)$, there is $\lambda_0 \in \Lambda$ such that $F(M) = \operatorname{im}(T_{\lambda_0})_M$. Let $N \in \operatorname{mod}(R)$ and $x \in F(N)$. By Lemma 3.1, there is $f \in \operatorname{Hom}_R(M,N)$ such that $x \in \operatorname{im} F(f) \subseteq \operatorname{im}(T_{\lambda_0})_N$.

$$\begin{tikzcd} F_{\lambda_0}(M) \arrow[r, two heads, "(T_{\lambda_0})_M"] \arrow[d, "F_{\lambda_0}(f)"'] & F(M) \arrow[d, "F(f)"] \\ F_{\lambda_0}(N) \arrow[r, "(T_{\lambda_0})_N"] & F(N) \end{tikzcd}$$

Therefore $F = T_{\lambda_0}(F_{\lambda_0})$. $\square$

In the following, we will consider two applications of Corollary 3.2.

Corollary 3.3. *Let I be an ideal of a Noetherian ring R. The following are equivalent:*

(a) Γ_I *is representable.*
(b) Γ_I *is finitely generated.*
(c) $I^n = I^{n+1}$ *for some* $n \geqslant 0$.

Proof. For all $M \in \mathrm{Mod}(R)$, we have $\Gamma_I(M) = \varinjlim_n \mathrm{Hom}_R(R/I^n, M) = \varinjlim_n (0 :_M I^n)$. So by Corollary 3.2, Γ_I is finitely generated iff there exists $n \geqslant 0$ such that $\Gamma_I(M) = \mathrm{Hom}_R(R/I^n, M)$ for all $M \in \mathrm{mod}(R)$ iff $I^n = I^{n+1}$ for some $n \geqslant 0$ by considering $M = R/I^{n+1}$ for "only if". $\square$

The relationship between our result and Section 2 is as follows.

Theorem 3.4. *[7, Theorem 1.1(a)] Let F, G be coherent functors and $T \colon F \to G$ be a natural transformation. Then $\ker(T)$, $\mathrm{coker}(T)$ and $\mathrm{im}(T)$ are also coherent.*

Lemma 3.5. *Let R be a Noetherian ring, $I \subseteq R$ be an ideal and $M \in \mathrm{Mod}(R)$. Then $\mathrm{Ass}_R \Gamma_I(M) = \mathrm{Ass}_R(M) \cap V(I)$ and $\mathrm{Ass}_R(M/\Gamma_I(M)) = \mathrm{Ass}_R(M) \setminus V(I)$, where $V(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$.*

Corollary 3.6. *Let R be a Noetherian ring and $\{M_n\}_{n \geqslant 0}$ be a sequence of modules in $\mathrm{mod}(R)$ such that the sets $\mathrm{Ass}_R(M_n)$ stabilize. Let $I \subseteq R$ be an ideal. If $I^n \neq I^{n+1}$ for any n, then the functor id/Γ_I is finitely generated but not coherent, and Γ_I is not finitely generated. However, whether or not $I^n = I^{n+1}$ for any n, the sets $\mathrm{Ass}_R(M_n/\Gamma_I(M_n))$ and $\mathrm{Ass}_R \Gamma_I(M_n)$ always stabilize.*

Now we consider our second example.

Lemma 3.7. *Let R be a ring, possibly noncommutative, with 1. Let $S \subseteq R$ and $f \colon S \times S \to R$ be a function. The following are equivalent:*

(a) *For every $r, s \in S$, left R-module M and $m \in M$, if $rm = 0$, then $f(r,s)m = f(s,r)m = 0$.*
(b) *For every $r, s \in S$ we have $f(r,s) \in Rr \cap Rs$.*

Proof. (a) $\Rightarrow$ (b): Let $r, s \in S$ and $M = R/Rr$. Then $r\overline{1} = \overline{0}$. By assumption, we have $f(r,s)\overline{1} = \overline{0}$, so $f(r,s) \in Rr$. Similarly, with $M = R/Rs$ we have $f(r,s) \in Rs$, so that $f(r,s) \in Rr \cap Rs$.

(b) $\Rightarrow$ (a): Let $r, s \in S$ and $m \in M$. By assumption, $f(r,s), f(s,r) \in Rr$. So if $rm = 0$, then $f(r,s)m, f(s,r)m \in Rrm = 0$. $\square$

Example 3.8. Let R be a UFD, $S = R$ and $f: R \times R \to R$. Then f satisfies the conditions in Lemma 3.7 iff for all $r, s \in R$ we have $f(r, s) \in (\mathrm{lcm}(r, s))$.

Definition 3.9. Let R be a commutative ring with 1.

(1) We say that a subset $S \subseteq R$ is common multiplicatively closed if $S \neq \emptyset$ and there is a function $f: S \times S \to S$ satisfying any condition in Lemma 3.7, or equivalently, for any $r, s \in S$ there is $f(r, s) \in S$ that satisfies any condition in Lemma 3.7.
(2) We say that a (nonempty) subset $S \subseteq R$ is coprincipal if there is $s \in S$ such that $s \in \bigcap_{r \in S} Rr$. Such an s is called a cogenerator of S.
(3) For any $S \subseteq R$ and $M \in \mathrm{Mod}(R)$, we let $\tau_S(M) = \{m \in M \mid rm = 0 \text{ for some } r \in S\}$. If S is common multiplicatively closed, then $\tau_S(M)$ is a submodule of M.

Example 3.10.

(1) Any singleton subset of R is common multiplicatively closed.
(2) In general, any coprincipal subset $S \subseteq R$ is common multiplicatively closed, since if $s \in S$ is a cogenerator, then we can let $f(r, t) = s$ for all $r, t \in S$.
(3) Conversely, if $S = \{s_1, \dots, s_n\} \subseteq R$ is common multiplicatively closed, then S has a cogenerator $f(\cdots f(f(s_1, s_2), s_3), \dots, s_n)$.
(4) Any multiplicatively closed subset of R is common multiplicatively closed.
(5) If $r, s \in \mathbb{Z}$ and $(0) \neq (s) \subsetneq (r)$, then the subset $\{r, s\}$ of $\mathbb{Z}$ is common multiplicatively closed and coprincipal but not multiplicatively closed.
(6) Let $a \in \mathbb{Z}$ such that $a \neq 0, \pm 1$. Let $S = \{a^2\} \cup \{a^{8+12n} \mid n \geqslant 0\}$. Then S is a common multiplicatively closed subset of $\mathbb{Z}$ by the function $f(s, t) = (st)^2$, and S is neither multiplicatively closed nor coprincipal.
(7) Let $a \in \mathbb{Z}$ such that $a \neq 0, \pm 1$. Then the infinite multiplicatively closed subset $S = \{a^{-n} \mid n \geqslant 0\}$ of $\mathbb{Z}_a$ is coprincipal with 1 as a cogenerator; the subset $\{a^n \mid n \geqslant 0\}$ of $\mathbb{Z}$ is not. If $i \geqslant 0$ and $i \neq 1$, then $S \backslash \{a^{-i}\} \subseteq \mathbb{Z}_a$ is coprincipal but not multiplicatively closed.
(8) Let R_1, R_2 be rings and u be a unit in R_1. Let $S \subseteq R_1 \times R_2$ be the subset $\{(u^n, r) \mid n \geqslant 1\} \cup \{(1, 1)\}$. If $u^n \neq 1$ for any $n \geqslant 1$, or if R_2 is infinite, then S is infinite, multiplicatively closed and coprincipal with cogenerator $(u, 0)$.

Remark 3.11. We have now seen that:

- Coprincipal $\Rightarrow$ common multiplicatively closed
- If S is finite, then S is coprincipal $\Leftrightarrow$ S is common multiplicatively closed
- Multiplicatively closed $\Rightarrow$ common multiplicatively closed
- Coprincipal and multiplicatively closed do not imply or refute each other
- Common multiplicatively closed $\not\Rightarrow$ coprincipal
- Common multiplicatively closed $\not\Rightarrow$ multiplicatively closed

Corollary 3.12. *Let R be a Noetherian ring and S be a common multiplicatively closed subset of R. The following are equivalent:*

(a) *τ_S is representable.*
(b) *τ_S is finitely generated.*
(c) *S is coprincipal.*

Proof. First, we note that for all $M \in \mathrm{Mod}(R)$, $\tau_S(M) = \bigcup_{s\in S}(0 :_M s) = \varinjlim_{Rs}(0 :_M s) = \varinjlim_{Rs} \mathrm{Hom}_R(R/(s), M)$, where $Rs = (s) \geqslant (t) = Rt$ iff $(s) \subseteq (t)$ for $s, t \in S$. So by Corollary 3.2, τ_S is finitely generated iff there exists $s \in S$ such that $\tau_S(M) = \mathrm{Hom}_R(R/(s), M)$ for all $M \in \mathrm{mod}(R)$ iff there exists $s \in S$ such that $(s) \subseteq (r)$ for all $r \in S$ by considering $M = R/(r)$ for "only if". $\square$

Notation 3.13. We let $R^\times$ denote the set of units of a ring R.

Lemma 3.14. *Let R be a ring and S be a subset of R. Consider the following statements.*

(a) *S is coprincipal.*
(b) *There are rings R_1, R_2 such that $R = R_1 \times R_2$, $S \cap (R_1)^\times \neq \emptyset$ and for all $s \in S$ we have $s(1, 0) \in (R_1)^\times$.*

Then (b) $\Rightarrow$ (a). If S is furthermore multiplicatively closed, then (a) $\Rightarrow$ (b).

Proof. (b) $\Rightarrow$ (a): Let $(u, 0) \in S \cap (R_1)^\times$ and $s \in S$. Since $s(1, 0) \in (R_1)^\times$, $(u, 0) \in Rs$. Therefore $(u, 0)$ is a cogenerator of S.

Now suppose that S is multiplicatively closed and coprincipal with cogenerator e. Since S is multiplicatively closed, $e^2 \in S$. Since e is a cogenerator of S, $e = re^2$ for some $r \in R$. Then $(re)^2 = r(re^2) = re$, so re is idempotent. Let $R_1 = R(re)$ and $R_2 = R(1 - re)$, so that $R = R_1 \times R_2$. Then $e(re) = re^2 = e$, so $e \in R_1$, and $e(r^2e) = (re)^2 = re$, so $e \in S \cap (R_1)^\times$. Finally, let $s \in S$. Then $e = r's$ for some $r' \in R$, and $(r'r^2e)(sre) = (re)^3 = re$, so $sre \in (R_1)^\times$. $\square$

Lemma 3.15. *Let R be a ring, $S \subseteq R$ and $M \in \mathrm{Mod}(R)$. If $\tau_S(M)$ is a submodule of M, then $\mathrm{Ass}_R(\tau_S(M)) = \{P \in \mathrm{Ass}_R(M) \mid P \cap S \neq \emptyset\}$. If R is Noetherian and S is a multiplicatively closed subset of R, then $\mathrm{Ass}_R(M/\tau_S(M)) = \{P \in \mathrm{Ass}_R(M) \mid P \cap S = \emptyset\}$.*

Remark 3.16. The second half of Lemma 3.15 is false if S is not multiplicatively closed. For example, let $R = \mathbb{Z}$, $S = \{p\}$ where p is prime, and $M = \mathbb{Z}/(p^2)$. Then $\mathrm{Ass}_R(M/\tau_S(M)) = \{(p)\}$, but $(p) \cap S \neq \emptyset$.

Corollary 3.17. *Let R be a Noetherian ring, S be a multiplicatively closed subset of R and $\{M_n\}_{n\geqslant 0}$ be a sequence of modules in $\mathrm{mod}(R)$ such that the sets $\mathrm{Ass}_R(M_n)$ stabilize. If S is not coprincipal, then the functor id/τ_S is finitely generated but not coherent,* *and τ_S is not finitely generated. However, whether or not S is coprincipal, the sets $\mathrm{Ass}_R(M_n/\tau_S(M_n))$ and $\mathrm{Ass}_R(\tau_S(M_n))$ always stabilize.*

4. Covariant functors over a Dedekind domain

In Section 2, we saw that the sets $\mathrm{Ass}_R F(M/I^nM)$ stabilize whenever F is a coherent functor. One may ask whether such asymptotic stability still holds when F is not coherent. In this section, we consider the case where R is a Dedekind domain. We will see that if F is a finitely generated functor over R, then the sets $\mathrm{Ass}_R F(M/I^nM)$ stabilize. We then construct a family of examples of R-linear covariant functors F such that the sets $\mathrm{Ass}_R F(R/I^n)$ do not stabilize.

Lemma 4.1. *Let R be a ring, F be an R-linear functor from $\mathrm{Mod}(R)$ to itself and $M \in \mathrm{Mod}(R)$. Then $\mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(F(M))$.*

Theorem 4.2. *Let R be a Dedekind domain, I be an ideal of R, $M \in \mathrm{mod}(R)$ and F be a finitely generated functor. Then the sets $\mathrm{Ass}_R F(M/I^nM)$ stabilize.*

Proof. The proof will proceed in several steps.

Step 1. First, we will make some reductions. Since F is additive, it preserves finite direct sums. By the structure theorem for finitely generated modules over a Dedekind domain, we may assume that $M = J$ is an ideal of R or $M = R/P^i$ for some maximal ideal P of R and $i \geqslant 1$. If $M = R/P^i$, then either $M/I^nM = 0$ for all n or $M/I^nM = M$ for all $n \geqslant i$. If $0 \neq M = J \subseteq R$ and $I \neq 0$, then $M/I^nM \cong R/I^n$ for all $n \geqslant 1$. But R/I^n is again a direct sum of modules of the form R/P^{ni}. So it suffices to show that asymptotic stability holds for $\mathrm{Ass}_R F(R/P^n)$, where P is a maximal ideal of R. Furthermore, by Lemma 4.1, $\mathrm{Ass}_R F(R/P^n) = \{P\}$ or $\emptyset$ for all $n \geqslant 1$. So we only need to show that $F(R/P^n)$ is either always 0 or always nonzero for all large n.

Step 2. Let F be given by the surjection $h_L \to F$, where $L \in \mathrm{mod}(R)$. First we consider the case where $L = J$ is an ideal of R. Suppose that $F(R/P^n) = 0$ for infinitely many n. We will show that in fact $F(R/P^n) = 0$ for all n, which will conclude this case. So fix $n \geqslant 1$. Let $N \geqslant n$ be such that $F(R/P^N) = 0$. Let $\pi\colon R/P^N \to R/P^n$ be the natural projection map. Since J is a projective R-module, the map $h_J(\pi)\colon h_J(R/P^N) \to h_J(R/P^n)$ is surjective. From the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_R\left(J, \frac{R}{P^N}\right) \arrow[r, two heads] \arrow[d, "h_J(\pi)"] & F\left(\frac{R}{P^N}\right) = 0 \arrow[d, "F(\pi)"] \\ \mathrm{Hom}_R\left(J, \frac{R}{P^n}\right) \arrow[r, two heads] & F\left(\frac{R}{P^n}\right) \end{tikzcd}$$

we see that $F(\pi)$ is surjective and therefore $F(R/P^n) = 0$.

Step 3. Next, we consider the case where $L = R/Q^i$ such that Q is a maximal ideal of R and $i \geqslant 1$. We may assume that $Q = P$. Suppose that $F(R/P^N) = 0$ for some $N \geqslant i$. We will show that in fact $F(R/P^n) = 0$ for all $n \geqslant N$, concluding this case. We recall the following facts. For any $n_1 \geqslant 1$, R/P^{n_1} is a principal ideal ring. Choose an element $p \in P \setminus P^2$. Then P^{n_2}/P^{n_1} is generated by p^{n_2} for all $0 \leqslant n_2 \leqslant n_1$. Now fix $n \geqslant N$. Let $p^{n-N}\colon R/P^N \to R/P^n$ denote multiplication by p^{n-N}. Again from the commutative diagram

$$\begin{tikzcd}
\frac{R}{P^i} \arrow[r, "\cong", "p^{N-i}"'] \arrow[d, "\mathrm{id}"] & \frac{P^{N-i}}{P^N} = \mathrm{Hom}_R\left(\frac{R}{P^i}, \frac{R}{P^N}\right) \arrow[r, two heads] \arrow[d, "\cong"', "p^{n-N} = h_J\left(p^{n-N}\right)"] & F\left(\frac{R}{P^N}\right) = 0 \arrow[d, "F\left(p^{n-N}\right)"] \\
\frac{R}{P^i} \arrow[r, "\cong", "p^{n-i}"'] & \frac{P^{n-i}}{P^n} = \mathrm{Hom}_R\left(\frac{R}{P^i}, \frac{R}{P^n}\right) \arrow[r, two heads] & F\left(\frac{R}{P^n}\right)
\end{tikzcd}$$

we see that $F(p^{n-N})$ is surjective and therefore $F(R/P^n) = 0$.

Step 4. Finally, we consider the general case where $L = J_1 \oplus \cdots \oplus J_k \oplus R/Q_1^{i_1} \oplus \cdots \oplus R/Q_\ell^{i_\ell}$ such that $J_1, \ldots, J_k \subseteq R$ are ideals, $Q_1, \ldots, Q_\ell$ are maximal ideals of R and $i_1, \ldots, i_\ell \geqslant 1$. Again we may assume that $Q_1 = \cdots = Q_\ell = P$. Suppose that $F(R/P^n) = 0$ for infinitely many n. Fix $N \geqslant \max\{1, i_1, \ldots, i_\ell\}$ such that $F(R/P^N) = 0$. Then repeating Steps 2 and 3, we see that for all $n \geqslant N$, each direct summand of $h_L(R/P^n) = h_{J_1}(R/P^n) \oplus \cdots \oplus h_{J_k}(R/P^n) \oplus h_{R/P^{i_1}}(R/P^n) \oplus \cdots \oplus h_{R/P^{i_\ell}}(R/P^n)$ is mapped to 0 in $F(R/P^n)$. Therefore $F(R/P^n) = 0$ for all $n \geqslant N$. $\square$

Lemma 4.3. *Let R be a Dedekind domain, I be an ideal of R and $M \in \mathrm{mod}(R)$. Then the modules $I^nM/I^{n+1}M$ are all isomorphic for large n. In particular, let F be any functor from $\mathrm{Mod}(R)$ to itself. Then the sets $\mathrm{Ass}_R F(I^nM/I^{n+1}M)$ stabilize.*

Proof. As in Step 1 of Theorem 4.2, we may assume that $M = J$ is an ideal of R or $M = R/P^i$ for some maximal ideal P of R and $i \geqslant 1$. If $M = J \neq 0$ and $I \neq 0$, then $I^nM/I^{n+1}M \cong R/I$ for all $n \geqslant 0$. If $M = R/P^i$, then $I^nM/I^{n+1}M = 0$ for all $n \geqslant i$. $\square$

Theorem 4.4. *Let R be a Dedekind domain and $I \neq 0$ be an ideal of R. Then there exists $F \in \mathscr{F}$ such that the sets $\mathrm{Ass}_R F(R/I^n)$ do not stabilize. In fact, we may construct F such that $\mathrm{Ass}_R F(R/I^n)$ is given by any sequence of subsets of $\mathrm{Ass}_R(R/I) = V(I)$.*

Proof. First, let $\mathscr{T} \subseteq \mathrm{mod}(R)$ be the full subcategory of finitely generated torsion R-modules. Then the torsion functor $\tau\colon \mathrm{mod}(R) \to \mathscr{T}$ is R-linear. Next, we recall from category theory that any category is naturally equivalent to any skeleton of itself. In particular, given a skeleton $\mathscr{T}_0$ of $\mathscr{T}$, there is an R-linear functor $\pi\colon \mathscr{T} \to \mathscr{T}_0$. Therefore it suffices to construct $F\colon \mathscr{T}_0 \to \mathscr{T}_0$ as in our Theorem.

We will define $\mathscr{T}_0$ as follows. Fix a linear ordering $\preceq$ of the nonzero prime ideals R, and let the objects of $\mathscr{T}_0$ be modules of the form $R/P_1^{e_1} \oplus \cdots \oplus R/P_j^{e_j}$, where $P_1 \preceq \cdots \preceq P_j$

and $e_i \leqslant e_{i+1}$ whenever $P_i = P_{i+1}$. For each maximal ideal P we choose a subset S_P of $\mathbb{N}_{>0}$. Then we define $F(R/P^e) = R/P$ if $e \in S_P$, and 0 otherwise. We let $F(R/P_1^{e_1} \oplus \cdots \oplus R/P_j^{e_j}) = \oplus_{\{i | e_i \in S_{P_i}\}} R/P_i$. Next we define $F(f)$ for $f\colon M \to N$, where $M, N \in \mathscr{T}_0$. It suffices to consider the case where M, N are both P-torsion for some maximal ideal P of R. Fix an element $p \in P \setminus P^2$. Then $\mathrm{Hom}_R(R/P^{n_1}, R/P^{n_2}) = P^{n_2 - n_1}/P^{n_2}$ is generated by $p^{n_2-n_1}$ if $n_2 \geqslant n_1 \geqslant 1$, and $\mathrm{Hom}_R(R/P^{n_1}, R/P^{n_2}) = R/P^{n_2}$ if $n_1 \geqslant n_2 \geqslant 1$. So we can identify f with a square matrix with entries in R (more precisely, in R/P^{e_i} for suitable e_i) viewed as multiplication maps, adding rows or columns of zeroes if necessary. If M, N are both direct sums of copies of $R/P^{e_1}, \ldots, R/P^{e_j}$ with $e_1 < \cdots < e_j$, then we define

$$F(f) = F\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & p* & \ddots & \\ & & & A_j \end{pmatrix} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & 0 & \ddots & \\ & & & A_j \end{pmatrix},$$

where the entries in the lower diagonal of the matrix on the left are multiples of p, and $A_1, A_2, \ldots, A_j$ are the square blocks that correspond to $R/P^{e_1}, \ldots, R/P^{e_j}$ respectively. Since $F(R/P^e) =$ either R/P or 0, the definition of $F(f)$ does not depend on the choice of coset representatives in the entries of f. It is then immediate that F preserves identity maps and is R-linear. Finally, if $f\colon M \to N$ and $g\colon N \to L$ where M, N, L are P-torsion, then

$$\begin{aligned} F(g \circ f) &= F\left(\begin{pmatrix} B_1 & & & * \\ & B_2 & & \\ & p* & \ddots & \\ & & & B_j \end{pmatrix}\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & p* & \ddots & \\ & & & A_j \end{pmatrix}\right) \\ &= F\begin{pmatrix} B_1A_1 + p* & & & * \\ & B_2A_2 + p* & & \\ & p* & \ddots & \\ & & & B_jA_j + p* \end{pmatrix} \\ &= \begin{pmatrix} B_1A_1 + p* & & & 0 \\ & B_2A_2 + p* & & \\ & 0 & \ddots & \\ & & & B_jA_j + p* \end{pmatrix} \\ &= \begin{pmatrix} B_1A_1 & & & 0 \\ & B_2A_2 & & \\ & 0 & \ddots & \\ & & & B_jA_j \end{pmatrix} = F(g)F(f) \end{aligned}$$

Therefore F respects composition. $\square$

Corollary 4.5. *The functors constructed in Theorem 4.4 are not finitely generated.*

Question 4.6. Is there a finitely generated non-coherent functor F such that the sets $\operatorname{Ass}_R F(R/I^n)$ do not stabilize?

5. Functors arising from middle finite complexes

In this section, we will study a class of R-linear covariant functors F which arise naturally and are non-finitely generated in general. An example of such kind of functor is the zeroth local cohomology functor. We will obtain results that are related to all the previous sections. Our main result is that over a one-dimensional Noetherian domain R, the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.

Definition 5.1. Let R be a ring and $\mathscr{S}\colon A \to B \to C$ be a complex of R-modules.

(1) We say that an R-linear functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ arises from $\mathscr{S}$ if $F(-) = \mathrm{H}(\mathscr{S} \otimes -)$.
(2) We say that $\mathscr{S}$ is middle finite if $B \in \operatorname{mod}(R)$.

Example 5.2. Let R be a ring and $I = (x_1, \dots, x_n)$ be an ideal of R. Then the functor Γ_I arises from the middle finite complex

$$0 \to R \to R_{x_1} \oplus \cdots \oplus R_{x_n}$$

Remark 5.3. Let R be a Noetherian ring. By Corollary 3.3, a functor that arises from a middle finite complex of R-modules is not finitely generated in general.

Lemma 5.4. *Let R be a Noetherian ring. Let F be a functor that arises from the middle finite complex $A \xrightarrow{\partial_A} B \xrightarrow{\partial_B} C$. Then F is coherent iff it is finitely generated.*

Proof. Suppose that F is finitely generated and is given by the surjection $h_M \to F$. Let K, I denote the functors given by $K(-) = \ker(\partial_B \otimes -)$ and $I(-) = \operatorname{im}(\partial_A \otimes -)$. Let $N \in \operatorname{mod}(R)$ and $n \in K(N)$, so that $n + I(N) \in F(N)$. By Lemma 3.1, there is $f \in \operatorname{Hom}_R(M, N)$ such that $n + I(N) \in \operatorname{im} F(f)$. That is, there are $m \in K(M)$ and $x \in A \otimes N$ such that $n = (\mathrm{id}_B \otimes f)(m) + (\partial_A \otimes \mathrm{id}_N)(x)$. Now $C \otimes M = \varinjlim_D (D \otimes M)$, where D ranges over all finitely generated submodules of C. Since $B \otimes M \in \operatorname{mod}(R)$, there is a finitely generated submodule C_0 of C that contains $\operatorname{im} \partial_B$ such that $\ker(B \otimes M \to C \otimes M) = \ker(B \otimes M \to C_0 \otimes M)$. From the commutative diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\partial_A \otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_A \otimes f"] & B \otimes M \arrow[r, "\partial_B \otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_B \otimes f"] & C_0 \otimes M \arrow[d, "\mathrm{id}_{C_0} \otimes f"] \\
A \otimes N \arrow[r, "\partial_A \otimes \mathrm{id}_N"] & B \otimes N \arrow[r, "\partial_B \otimes \mathrm{id}_N"] & C_0 \otimes N
\end{tikzcd}$$

we see that in fact $n \in \ker(B \otimes N \to C_0 \otimes N)$. Finally, let A_0 be a finitely generated submodule of A such that $\partial_A(A_0) = \partial_A(A)$. Then F arises from the complex $A_0 \to B \to C_0$. Therefore F is coherent by Lemma 2.3. $\square$

Lemma 5.5. *Let R be a Noetherian ring, I, J be ideals of R, $M \in \operatorname{mod}(R)$, M' be a submodule of M and F be a functor that arises from the middle finite complex $A \to B \to C$. Then the sets $\operatorname{Ass}_R F(I^nM/I^nM')$ and the values $\operatorname{depth}_J F(I^nM/I^nM')$ stabilize.*

Proof. The module $\bigoplus_{n\geqslant 0} B \otimes (I^nM/I^nM')$ is finitely generated and graded over $S = \bigoplus_{n\geqslant 0} I^n$, and the maps in the induced complex

$$\bigoplus_{n\geqslant 0} A \otimes \frac{I^nM}{I^nM'} \to \bigoplus_{n\geqslant 0} B \otimes \frac{I^nM}{I^nM'} \to \bigoplus_{n\geqslant 0} C \otimes \frac{I^nM}{I^nM'}$$

are homogeneous of degree 0. The result then follows from Corollaries 2.11 and 2.12. $\square$

Theorem 5.6. *Let R be a one-dimensional Noetherian domain, I be an ideal of R, $M \in \operatorname{mod}(R)$ and F be a functor that arises from the middle finite complex $\mathscr{S}\colon A \xrightarrow{\alpha} B \xrightarrow{\beta} C$. Then the sets $\operatorname{Ass}_R F(M/I^nM)$ stabilize.*

Proof. First, since $\mathscr{S} \otimes (M/I^nM) = (\mathscr{S} \otimes M) \otimes (R/I^n)$, it suffices to show that the sets $\operatorname{Ass}_R F(R/I^n)$ stabilize. We have $\mathscr{S} \otimes (R/I^n)\colon A/I^nA \xrightarrow{\alpha_{n-1}} B/I^nB \xrightarrow{\beta_{n-1}} C/I^nC$, so

$$F(R/I^n) = \frac{\ker \beta_{n-1}}{\operatorname{im} \alpha_{n-1}} = \frac{\beta^{-1}(I^nC)}{\alpha(A) + I^nB} = F'(R/I^n),$$

where F' arises from the complex $0 \to B/\alpha(A) \to C$. So we may assume that $A = 0$. Furthermore, since localization is flat, we may assume that R is local of dimension one. So it remains to show that $F(R/I^n)$ is either always 0 or always nonzero for all large n.

Now let $S = \bigoplus_{n\geqslant 0} I^n$ and $\gamma\colon \bigoplus_{n\geqslant 0}(I^nB/I^{n+1}B) \to \bigoplus_{n\geqslant 0}(I^nC/I^{n+1}C)$ be the map induced by β with graded components γ_n. By Corollary 2.11, there is N so large such that the sets $\operatorname{Ass}_R(\ker \gamma_n)$ are equal for all $n > N$. Again we have $\beta_n\colon B/I^{n+1}B \to C/I^{n+1}C$, so that $F(R/I^n) = \ker \beta_{n-1}$. Suppose that there is $m > N$ such that $\ker \beta_{m-1} = 0$ but $\ker \beta_m \neq 0$. Then $I^{m+1}B \subsetneq \beta^{-1}(I^{m+1}C) \subseteq \beta^{-1}(I^mC) = I^mB$, so that $0 \neq \ker \beta_m \subseteq \ker \gamma_m$, and hence $\ker \gamma_n \neq 0$ for all $n > N$. But $\ker \beta_n \supseteq \ker \gamma_n$ always holds. Therefore we have $\ker \beta_n \neq 0$ for all $n > N$. $\square$

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