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Irreducible decomposition of powers of edge ideals [☆]

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In this paper by using some tools from graph theory, mainly the theorem on ear decomposition of factor-critical graphs given by L. Lovász and the canonical decomposition of a graph given by Edmonds and Gallai, we can describe each irreducible component of powers of edge ideals of a graph. As an application we use our results to persistence of irreducible components, to bound dstab and to count the number of irreducible components of powers of edge ideals of a graph.

Contents

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1. Introduction

Irredundant irreducible decomposition of ideals is an important tool in Commutative Algebra and Algebraic Geometry. It leads to the notions of primary decomposition, associated primes as well as to arithmetic properties. Many recent works (see [2], [5], [6], [11], [12], [13], [17], [20], [21], [22]) concern associated primes of powers of a square free monomial ideal. The most recent work in this topic is the preprint in [13] in which the authors describe the set of associated primes of powers of an edge ideal $I_G \subset R := K[x_1, \ldots, x_d]$ of a graph G. Let recall that the asymptotic stability of $\mathrm{Ass}(R/I^k)$ was proved by M. Brodmann [1]. For an edge ideal I_G, Chen, Morey and Sung [2] give a process to described prime ideals in $\mathrm{Ass}(R/I_G^k)$, in particular they proved that if G is a simple connected non bipartite graph then $\mathfrak{m} \in \mathrm{Ass}(R/I_G^k)$ for k large enough. On the other hand Martinez-Bernal, Morey and Villarreal [17] proved that $\mathrm{Ass}(R/I_G^k) \subset \mathrm{Ass}(R/I_G^{k+1})$ for $k \geq 1$, this result is known as persistence of associated primes for edge ideals of graphs. If G is a simple connected bipartite graph then by Theorem 5.9 of [20] we know that $\mathrm{Ass}(R/I_G^l) = \mathrm{Ass}(R/I_G^{l+1})$ for $l \geq 1$, the smallest k such that $\mathrm{Ass}(R/I_G^l) = \mathrm{Ass}(R/I_G^{l+1})$ (or $\mathrm{depth}(R/I_G^l) = 0$, respectively) for all $l \geq k$ is denoted by $\mathrm{astab}(I_G)$ (or $\mathrm{dstab}(I_G)$, respectively). Note that since $\mathrm{depth}(R/I_G^l) = 0$ if and only if $\mathfrak{m} \in \mathrm{Ass}(R/I_G^l)$, we have that $\mathrm{dstab}(I_G) \leq \mathrm{astab}(I_G)$. Recently, T.N. Trung [22] has improved the upper bound for $\mathrm{dstab}(I_G)$ resulting from the upper bound for $\mathrm{astab}(I_G)$ in [2].

It is well known that for any monomial ideal J, the irredundant irreducible decomposition of J is unique up to order. The set of ideals appearing in the irredundant irreducible decomposition of J is denoted by $\mathrm{Irr}(J)$ and its elements are called irreducible components of J. In this work for the first time we are able to describe explicitly the set $\mathrm{Irr}(I_G^k)$ and so we improve some results contained in the mentioned papers. To be more precise, it is well known that an irreducible component of I_G^k can be written as $\mathfrak{m}^{\mathbf{b}} := (x_i^{b_i} \mid b_i > 0, i = 1, \ldots, d)R$, where $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{N}^d \setminus \{0\}$. We associate to $\mathfrak{m}^{\mathbf{b}}$ the sets $U = \{x_i \mid b_i \geq 1\}, Z = \{x_i \mid b_i = 0\}$ and the vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ defined by $a_i = b_i - 1$ if $b_i \geq 2$ and $a_i = 0$ otherwise. In all this work we study the relations between the sets U, Z and the vector $\mathbf{a}$.

In section 2 we recall some facts about irreducible decomposition of monomial ideals and their translation in terms of corner elements as studied in the book [18]. We also give some definitions and basic properties of edge ideals of a graph.

In section 3 we describe the non embedded irreducible components and especially give a formula for the number of non embedded irreducible components of powers of edge ideals I_G of a graph G (see Theorem 3.1). It coincides with a polynomial of degree the big height of I_G in accord with the main result in [3]. The above result can be applied to the case of square free monomial ideals, it will be published in our forthcoming paper.

In section 4 we give important properties of graphs related to factor-critical. In particular we recall Lovász's Theorem [14] on an ear decomposition of factor-critical graphs and the canonical decomposition of a graph given by Edmonds and Gallai into three sets $A(G), C(G), D(G)$, known as the Gallai-Edmonds Structure Theorem (see [4] and [7]).

In section 5 we prove that embedded irreducible components of powers of edge ideals of graphs are described in terms of factor-critical sets by using Gallai-Edmonds Structure Theorem. Concretely, we prove in Theorem 5.3 that an irreducible component is given by a vector $\mathbf{a}$ such that the replication $S = p_{\mathbf{a}}(G)$ has $C(S) = \emptyset$ (see Definition 2.10). As a consequence we can show that graphs G with $C(G) = \emptyset$ play a crucial role in this subject. We prove in Corollary 5.14 the strong persistence of associated primes: namely if J is an irreducible component of I_G^k then we can describe several irreducible components of I_G^{k+1} that comes directly from J and have the same radical as J, which improves the result in [17]. If G is a simple connected bipartite graph in Corollary 5.12 we get a short proof of Theorem 5.9 of [20].

In section 6, we will apply our main results to study the set $\mathrm{Irr}(I_G^k)$ for $k \gg 0$. From one side we improve the main results of [2] and [22] by giving short and conceptual proofs in Theorem 6.3 and Theorem 6.9. From the other side we can precise the main result of [3] that counted the number of irreducible components of I_G^k for $k \gg 0$ (see Theorem 6.11). Moreover, we also improve the results in [12] and [21] by describing graphs which have $\mathrm{dstab}(I_G) \leq 3$ with a short proof (see Corollary 6.6).

2. Irreducible decomposition and corner elements

Let K be a field, $R := K[x_1, \ldots, x_d]$ a polynomial ring, $\mathfrak{m} := (x_1, \ldots, x_d)$ its unique graded maximal ideal and $J \subset R$ be a monomial ideal. We denote by $[[R]]$ the set of all monomials of R, $V = \{x_1, \ldots, x_d\}$, and $\mu(J)$ the number of minimal generators of J.

Notation 2.1. (i) For a non zero vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$, set $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_d^{a_d}$, $\mathfrak{m}^{\mathbf{a}} := (x_i^{a_i} \mid\ a_i > 0, i = 1, \ldots, d)R$ and $\mathrm{Supp}(\mathbf{a}) = \mathrm{Supp}(\mathbf{x}^{\mathbf{a}}) := \{x_i \mid a_i > 0\}$.

(ii) For every set $S \subset V$, let $\mathbf{1}_S$ be its characteristic vector, i.e. its ith-coordinate is 1 if $x_i \in S$ and 0 otherwise. For instance we have $\mathbf{1}_V = (1, \ldots, 1)$.

(iii) In this article we use the same notation for a subset $F \subset V$ and the induced subgraph $G[F]$ on F, unless is ambiguous.

Now we need some results from [18].

Definition 2.2. A monomial $M \in [[R]]$ is a *J-corner element* if $M \notin J$ but $x_1 M, \ldots,$ $x_d M \in J$. The set of corner elements of J in $[[R]]$ is denoted by $C_R(J)$.

Fact 2.3. (i) It is clear that the J-corner elements are precisely the monomials in $(J :_R \mathfrak{m}) \setminus J$, or in other words, $C_R(J) = [[(J :_R \mathfrak{m})]] \setminus [[J]]$.

(ii) The set $C_R(J)$ is finite.

(iii) If $\mathrm{rad}(J) = \mathfrak{m}$, then it is well known that $t(R/J) = \mathrm{card}(C_R(J))$ is the type of the ring R/J.

The following theorem gives us some methods for computing irreducible decompositions for monomial ideals (see [18], Theorem 6.3.5, Theorem 7.5.3 and Theorem 7.5.5). Set $\mathrm{Irr}(J)$ be the set of irredundant irreducible components of a monomial ideal J. Let recall that every irreducible ideal in the ring R is of the type $\mathfrak{m}^{\mathbf{b}}$ for some non zero vector $\mathbf{b} \in \mathbb{N}^d$.

Theorem 2.4. *Let $J \subset R$ be a monomial ideal.*

(i) Assume that $\mathrm{rad}(J) = \mathfrak{m}$. Let $C_R(J) = \{\mathbf{x}^{\mathbf{c_j}} \mid \mathbf{c_j} \in \mathbb{N}^d, j = 1, \ldots, t(R/J)\}$ be the set of corner elements of J. Then $J = \cap_{j=1}^{t(R/J)} \mathfrak{m}^{\mathbf{c_j}+\mathbf{1}_V}$ is the unique irredundant irreducible decomposition of J.

(ii) Assume that $\mathrm{rad}(J) \neq \mathfrak{m}$ and $J = (\mathbf{x}^{\mathbf{b_j}} \mid \mathbf{b_j} \in \mathbb{N}^d, j = 1, \ldots, \mu(J))R$. Let m be an integer which is equal or bigger than every coordinate of the vectors $\mathbf{b_j}$. Set $J' := J + \mathfrak{m}^{(m+1)\mathbf{1}_V}$ and $C_R(J') = \{\mathbf{x}^{\mathbf{c_j}} \mid \mathbf{c_j} \in \mathbb{N}^d, j = 1, \ldots, t(R/J')\}$ be the set of corner elements of J'. Then $J = \cap_{j=1}^{t(R/J')} \widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}_V}}$ is the unique irredundant irreducible decomposition of J, where $\widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}_V}}$ is obtained from $\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}_V}$ by deleting all monomials of the type $x_1^{m+1}, \ldots, x_d^{m+1}$ from its generators.

Remark 2.5. (i) With the notations of Theorem 2.4, let $\mathbf{x}^{\mathbf{c_j}}$ be a corner element of $J' := J + \mathfrak{m}^{(m+1)\mathbf{1}_V}$. Note that since J' is $\mathfrak{m}$-primary, we have all coordinates of $\mathbf{c_j}$ are non zero. Let $U_j := \{x_i \mid \mathbf{c_j}_i < m\}, Z_j := V \setminus U_j$. We can write $\mathbf{c_j} = \mathbf{a_j} + m\mathbf{1}_{Z_j}$, with $\mathrm{Supp}(\mathbf{a_j}) = U_j$. Then $\mathfrak{m}^{\mathbf{a_j}+\mathbf{1}_{U_j}}$ is an irreducible component of J. Note that U_j is independent of m and in fact for all $i, \mathbf{a_j}_i < \max\{\mathbf{b_j}_i \mid 1 \leq j \leq \mu(J), 1 \leq i \leq d\}$, where $\mathbf{x}^{\mathbf{b_j}}$ is generators of J.

(ii) It follows that any irreducible component of a monomial ideal $J \subset R$ is of the type $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ for some vector $\mathbf{a} \in \mathbb{N}^d$ with $\mathrm{Supp}(\mathbf{a}) \subset U \subset V$.

From now on, let $G = (V, E)$ be a simple connected graph with the vertex set $V = V(G) = \{x_1, \ldots, x_d\}$, the edge set $E = E(G)$ and $I_G := (x_i x_j \mid x_i x_j \in E)R$ its edge ideal. Now we need some definitions.

Definition 2.6. (i) A set $C \subset V$ is a *vertex cover* of G if for every edge $xy \in E$ we have either $x \in C$ or $y \in C$.

(ii) A set $S \subset V$ is called a *clique set* of G if the induced subgraph $G[S]$ is a complete graph and it is called a *coclique (or independent set)* of G if the induced subgraph $G[S]$ has no edges. The family of coclique sets of G, denoted by $\Delta(G)$, is the simplicial complex called *independence complex* of G.

(iii) A *matching* of G is a set of disjoint edges of G. The maximum cardinal of all matchings in G, denoted by $\nu(G)$, is called the *matching number* of G. A *maximum matching* of G is a matching whose cardinal is $\nu(G)$. A *perfect matching* of G is a matching that all the vertices of G are involved.

For a set $S \subset V$ we denote by $N(S)$ the set of vertices that are adjacent to some element in S.

Remark 2.7. (i) A set $C \subset V$ is a vertex cover of G if and only if $V \setminus C$ is a coclique and C is a minimal vertex cover of G if and only if $V \setminus C$ is a maximal coclique.

(ii) A set $Z \subset V$ is a coclique if and only if $N(Z) \cap Z = \emptyset$ and Z is a maximal coclique if and only if $V = N(Z) \cup Z$.

Let $U \cup Z = V$ be a partition of V. With the notations in 2.1, we can resume the above facts for edge ideals in the following corollary.

Corollary 2.8. *Let $k, m \in \mathbb{N}$ such that $m \geq k$ and $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ such that* $\mathrm{Supp}(\mathbf{a}) \subset U$ *(see Remark 2.5). The following conditions are equivalent:*

(i) The ideal $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ *belongs to* $\mathrm{Irr}(I_G^k)$.

(ii) $a_i < k$ for all $i = 1, \ldots, d$ and the ideal $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} + \mathfrak{m}^{(m+1)\mathbf{1}_Z} \in \mathrm{Irr}(I_G^k + \mathfrak{m}^{(m+1)\mathbf{1}_V})$.

(iii) $a_i < k$ for all $i = 1, \ldots, d$ and the monomial $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z}$ is a corner element of $I_G^k + \mathfrak{m}^{(m+1)\mathbf{1}_V}$.

(iv) $a_i < k$ for all $i = 1, \ldots, d$ and we have

(1) $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^k + \mathfrak{m}^{(m+1)\mathbf{1}_V}$.
(2) For every $u \in V$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k + \mathfrak{m}^{(m+1)\mathbf{1}_V}$.

(v) $a_i < k$ for all $i = 1, \ldots, d$ and we have

(1) $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^k$.
(2) For every $u \in U$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k$.

Remark 2.9. Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^k)$, where $\mathbf{a} \in \mathbb{N}^d$ with $\mathrm{Supp}(\mathbf{a}) \subset U$ and $Z = V \setminus U$. We have

(i) The set Z is a coclique. Indeed, it is certainly true if $\sharp Z \leq 1$. We can assume $\sharp Z \geq 2$. Suppose that there exist $u \neq v \in Z$ such that $uv \in I_G$. Then $(uv)^m \in I_G^m \subset I_G^k$ for $m \geq k$, which implies $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k + \mathfrak{m}^{(m+1)\mathbf{1}_V}$, a contradiction.

(ii) For any $u \notin Z$ we have either $u \in N(\mathrm{Supp}\ (\mathbf{a}))$ or $u \in N(Z)$.

Definition 2.10. Let $\mathbf{a} \in \mathbb{N}^d$ be a non zero vector and we set $A_i = \{x_i = x_i^{(1)}, \ldots, x_i^{(a_i)}\}$ for each $a_i > 0$. The graph $S := p_{\mathbf{a}}(G)$ with the vertex set $V(S) = \cup_{a_i>0} A_i$ and the edge set $E(S) = \{x_i^{(l)}x_j^{(m)} \mid x_i \in A_i, x_j \in A_j, x_ix_j \in E\}$ is called *the replication* of G by

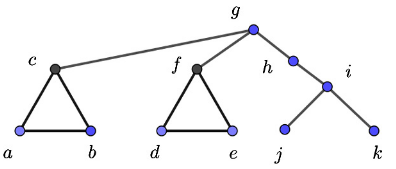

Fig. 1. I_G has 22 irreducible components.

the vector $\mathbf{a}$. The *support* of S is the set $\mathrm{Supp}(S) := V(S) \cap V = \mathrm{Supp}(\mathbf{a})$, we denote $N_G(S) = N(V(S) \cap V)$. For small values of $a_i \leq 3$ sometimes we will write x_i, x_i', x_i'' instead of $x_i^{(1)}, x_i^{(2)}, x_i^{(3)}$.

Example 2.11. As an application of the above result, let us compute the irreducible decomposition of I_G. Since I_G is a square free ideal, any ideal in $\mathrm{Irr}(I_G)$ is of the type $\mathfrak{m}^{\mathbf{1}_U}$ for some $U \subset V$. Let $Z = V \setminus U$, then $\mathbf{x}^{\mathbf{1}_Z}$ is a corner element of $I_G + \mathfrak{m}^{2(\mathbf{1}_V)}$, which implies that Z is a coclique. Moreover, it is a maximal coclique in V, since for every $u \in U$, we have $u\mathbf{x}^{\mathbf{1}_Z} \in I_G + \mathfrak{m}^{2(\mathbf{1}_V)}$, which implies that there exists some $v \in Z$ such that uv is an edge in G.

This proves that the irreducible (prime) ideals in $\mathrm{Irr}(I_G)$ are of the type $\mathfrak{m}^{\mathbf{1}_U}$ for some set $U \subset V$ such that $Z = V \setminus U$ is a maximal coclique in V. This also shows that I_G is the Stanley-Reisner ideal associated to $\Delta(G)$. Note that the set $\mathrm{Irr}(I_G)$ is also the set of minimal associated primes of I_G^k, for any $k \geq 1$.

Example 2.12. Let G be the graph with $\nu(G) = 4$ in Fig. 1. Let consider the 11-variables polynomial ring $K[a, b, \ldots, k]$. Then we have 22 maximal coclique sets

$$\{a,d,h,j,k\}, \{a,d,g,i\}, \{b,d,h,j,k\}, \{b,d,g,i\}, \{c,d,h,j,k\}, \{c,d,i\}, \{a,e,h,j,k\},$$
$$\{a,e,g,i\}, \{b,e,h,j,k\}, \{b,e,g,i\}, \{c,e,h,j,k\}, \{c,e,i\}, \{a,f,h,j,k\}, \{a,f,i\},$$
$$\{b,f,h,j,k\}, \{b,f,i\}, \{c,f,h,j,k\}, \{c,f,i\}, \{a,d,g,j,k\}, \{b,d,g,j,k\}, \{a,e,g,j,k\},$$
$$\{b,e,g,j,k\}.$$

Hence I_G has 22 irreducible components.

3. Non embedded irreducible components of I_G^k

Now we can describe the non embedded components of I_G^k for any k. The proof with minor changes can be extended to hypergraphs and will appear in a forthcoming paper on hypergraphs.

Theorem 3.1. *(i) Let $Z \subset V$ be a maximal coclique, $U := V \setminus Z$ and M a monomial. Then $M\mathbf{x}^{k\mathbf{1}_Z}$ is a corner element of $I_G^k + \mathfrak{m}^{(k+1)\mathbf{1}_V}$ if and only if M is a monomial of degree $k - 1$ with support in U.*

(ii) Every non embedded irreducible component of I_G^k can be written as $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ for some set $U \subset V$ such that $\operatorname{Supp}(\mathbf{a}) \subset U$*, $Z := V \setminus U$ is a maximal coclique inside V and $M := \mathbf{x}^{\mathbf{a}}$ is a monomial of degree $k-1$.*

(iii) Let $Z_1, \dots, Z_\rho$ be the maximal coclique sets inside V and $\mu_i = d - \sharp Z_i$. Then the number of non embedded irreducible components of I_G^k is exactly $\sum_{i=1}^{\rho} \binom{\mu_i - 1 + k - 1}{\mu_i - 1}$, it coincides with a polynomial of degree $\operatorname{bight}(I_G) - 1$*, where* $\operatorname{bight}(I_G)$ *is the biggest height of irreducible components of I_G.*

Proof. (i) Let M be a monomial of degree $k-1$ with support in U. Since Z is a maximal coclique, for each $u \in U$ we have $u\mathbf{x}^{\mathbf{1}_Z} \in I_G$. Hence we have that $M\mathbf{x}^{(k-1)\,\mathbf{1}_Z} \in I_G^{k-1}$ and $uM\mathbf{x}^{k\,\mathbf{1}_Z} \in I_G^k$. Now we prove that $M\mathbf{x}^{k\,\mathbf{1}_Z} \notin I_G^k$. Assume conversely, so there exist generators $M_1, \dots, M_k$ of I_G and a monomial N such that $M\mathbf{x}^{k\,\mathbf{1}_Z} = M_1 \cdots M_k N$. If every monomial M_i contains at least one variable of M then we have a contradiction with the fact that M is a monomial of degree $k-1$. Hence there is a monomial, say M_1, not containing any variable in M. That means $\operatorname{Supp} M_1 \subset Z$. It is a contradiction since Z is an independent set.

Conversely, let M be any monomial such that $M\mathbf{x}^{k\mathbf{1}_Z}$ is a corner element of $I_G^k + \mathfrak{m}^{(k+1)\mathbf{1}_V}$. Note that since $M\mathbf{x}^{k\mathbf{1}_Z} \notin I_G^k + \mathfrak{m}^{(k+1)\mathbf{1}_V}$, we have Supp $(M) \subset U$. Since Z is maximal coclique, for any $u \in U$, we have $u\mathbf{x}^{\mathbf{1}_Z} \in I_G$. Hence if $\deg(M) > k$ then $M\mathbf{x}^{k\mathbf{1}_Z} \in I_G^k$, a contradiction. So $\deg(M) \leq k-1$. Also for $u \in U$ we have $u\mathbf{x}^{k\mathbf{1}_Z} \in I_G^k$, then there exist generators $M_1, \dots, M_k$ of I_G and a monomial N such that $uM\mathbf{x}^{k\mathbf{1}_Z} = M_1 \dots M_k N$. For $i = 1, \dots, k$, every monomial generator M_i of I_G contains at least one variable of uM. This implies $\deg(uM) \geq k$, so $\deg(M) \geq k-1$. Therefore we get that $\deg(M) = k-1$ and the support of M is contained in U as required.

(ii) Since any non embedded irreducible component corresponds to a maximal coclique set, this claim is implied immediately from (i).

(iii) In order to count the number of non embedded irreducible components of I_G^k we have to count the monomials of degree $k-1$ with support in the complement of each maximal coclique set. Assume that $Z_1, \dots, Z_\rho$ are maximal coclique sets inside V. Then for each Z_i we have a complement set $U_i = V \setminus Z_i$ with its cardinal is μ_i, for $i = 1, \dots, \rho$. It is well known that the number of monomials of degree $k-1$ with support in U_i is provided by the Hilbert function $H(k) = \binom{\mu_i - 1 + k - 1}{\mu_i - 1}$ of the polynomials ring with μ_i variables. It coincides with a polynomial of degree $\mu_i - 1$ for $k \geq 0$. Therefore the number of non embedded irreducible components of I_G^k is exactly $\sum_{i=1}^{\rho} \binom{\mu_i - 1 + k - 1}{\mu_i - 1}$. $\square$

Example 3.2. In Example 2.12, the graph G has 22 maximal coclique sets, where 13 with 5 elements, 4 with 4 elements and 5 with 3 elements. Hence there are 22 minimal vertex cover sets, where 13 with 6 elements, 4 with 7 elements and 5 with 8 elements. By Theorem 3.1 there are exactly $13\binom{5+k}{k} + 4\binom{6+k}{k} + 5\binom{7+k}{k}$ non embedded irreducible components of I_G^{k+1} for $k \geq 0$.

4. Factor-critical graphs, Gallai-Edmonds's canonical decomposition

In this section we study factor-critical graphs. The notion factor-critical graph was introduced by Gallai [8]. Factor-critical graphs may be characterized in several different ways, other than their definition by Gallai [8], Edmonds [4], Lovász [14]. For basic definitions please refer to the books of Lovász and Plummer [16], Yu and Lu [23] and of Schrijver [19].

Definition 4.1. A graph G is called *factor-critical* if for any vertex v in G the graph $G-v$ has a perfect matching. A set $F \subset V$ is called *factor-critical* if the induced subgraph on F is factor-critical. A set $H \subset V$ is called *matching-critical* if the induced subgraph on H is a disjoint union of factor-critical graphs.

A *path* P in G is a subgraph, given by a sequence of distinct vertices $v_0, \ldots, v_k$ such that $v_i v_{i+1}$ is an edge in G for all $i = 0, \ldots, k-1$. The vertices v_0, v_k are called the *end points* of P. A *circuit* or *closed path* is a subgraph of G with a vertex set $v_0, \ldots, v_k$ and an edge set all the edges $v_i v_{i+1}$ for $i = 0, \ldots, k$, where $v_{k+1} = v_0$. Note that this definition implies that P has no chords, but like an induced subgraph of G it can have chords.

Remark 4.2. (i) Let $v \in G$, set $F = \{v\}$, then F is factor-critical.

(ii) It is clear that odd circuits are factor-critical.

(iii) If F is factor-critical, then

(1) The number of vertices of F is odd and $\nu(F) = \frac{\sharp(F)-1}{2}$.

(2) Every graph $\widetilde{F}$ such that $V(\widetilde{F}) = V(F)$ and $E(F) \subset E(\widetilde{F})$ is factor-critical. In particular complete odd graphs are factor-critical.

Definition 4.3. ([9]) *An ear decomposition* $G_0, G_1, \ldots, G_k = G$ of a graph G is a sequence of graphs with the first graph G_0 being a vertex, edge, even cycle, or odd cycle, and each graph G_{i+1} is obtained from G_i by adding an ear.

Adding an ear is done as follows: take two vertices a and b of G_i and add a path P_i from a to b such that all vertices on the path except a and b are new vertices (present in G_{i+1} but not in G_i). An ear with $a \neq b$ is called *open*, otherwise, *closed*. An ear with P_i having an odd (even) number of edges is called *odd (even)*.

The following result in [14] and [16] gives us a nice characterization of a graph G being factor-critical.

Theorem 4.4. *A simple graph* G *has an odd ear decomposition* $G_0, G_1, \ldots, G_r = G$ *if and only if* G *is factor-critical.*

Let G be a factor-critical graph such that $\sharp(V) \geq 3$. We have the following remark.

Remark 4.5. (i) Let $G_0, G_1, \ldots, G_r = G$ be an odd ear decomposition of G. Then

(1) G_0 can be either a vertex or an odd circuit. If G_0 is a vertex then G_1 is an odd circuit. Hence without loss of generality we can assume that G_0 is an odd circuit.

(2) We can assume that the circuit G_0 is chord-less, since otherwise G_0 can be decomposed in an odd circuit and an odd ear.

(ii) For any vertex $u \in V$, there is an odd ear decomposition $G_0, G_1, \ldots, G_r = G$ with $u \in V(G_0)$. In that case the G_0 can have chords.

(iii) For any vertex $u \in V$, the set $N(u)$ contains at least two vertices. In particular G does not have leaves.

(iv) If $G = p_{\mathbf{a}}(H)$ for some graph H with $l = \sharp V(H)$, $\mathbf{a} := (a_1, \ldots, a_l) \in \mathbb{N}^l \setminus \{0\}$ and $G_0, G_1, \ldots, G_r = G$ is an odd ear decomposition, then we can assume that $G_0 \subset V(H)$ and G_0 is chord-less.

(v) A factor-critical graph can have several odd ear decompositions.

Example 4.6. (i) A factor-critical graphs has 3 vertices if and only if it is a triangle.

(ii) A factor-critical graph G has 5 vertices if and only if

(1) It is the union of two triangles with a common vertex.
(2) It is a pentagon with a set (eventually empty) of chords.

(iii) If G has an ear decomposition given by a triangle and an open ear of length 3 then G is a pentagon with chords.

Concerning odd ear decompositions we have some results.

Lemma 4.7. *(i) Let F be factor-critical with $\sharp(F) \geq 3$ and an edge ab such that $a \in F, b \notin F$. Then $p_{\mathbf{1}_{\{a,b\}}+\mathbf{1}_F}(G)$ is factor-critical with $\nu(p_{\mathbf{1}_{\{a,b\}}+\mathbf{1}_F}(G)) = \nu(F) + 1$.*

(ii) Let F be factor-critical and an edge ab in F. Then $p_{\mathbf{1}_{\{a,b\}}+\mathbf{1}_F}(G)$ is factor-critical.

(iii) Let F be factor-critical with $\sharp(F) \geq 3$, $F_0, F_1, \ldots, F_r = F$ an odd ear decomposition, i.e. F_{i+1} is obtained from F_i by adding an odd ear P_i (open or closed) with end points a_i, b_i for $i = 0, \ldots, r$. Then $p_{\mathbf{1}_{\{a_i,b_i\}}+\mathbf{1}_F}(G)$ is factor-critical.

(iv) Let F, F' be factor-critical graphs with $\sharp(F), \sharp(F') \geq 3$ such that $F \cap F' = \{a\}$. Then $F \cup F'$ is factor-critical.

(v) Let F, F' be factor-critical graphs with $\sharp(F), \sharp(F') \geq 3$ such that $F \cap F' = \emptyset$ and assume that there is an edge ab in G such that $a \in F, b \in F'$. Then $p_{\mathbf{1}_{\{a\}}+\mathbf{1}_F+\mathbf{1}_{F'}}(G)$ is factor-critical.

Proof. In order to prove that replicated graphs are factor-critical, by Theorem 4.4 we need only to consider their ear decompositions.

(i) Let $F_0, F_1, \ldots, F_r = F$ be an odd ear decomposition. Let $c \in V(F)$ be a neighbor of a and P the path with edges $ca', a'b, ba$. Then $F_0, F_1, \ldots, F_r, F_r \cup P$ is an odd ear decomposition of $p_{\mathbf{1}_{\{a,b\}}+\mathbf{1}_F}(G)$.

(ii) Do similarly to the claim (i) by choosing the path P with edges $ba', a'b', b'a$.

(iii) Recall that F_{i+1} is obtained from F_i by adding an odd ear P_i. Let P_i : $a_i d_1, d_1 d_2, \ldots, d_l b_i$ be a path, where $d_1, \ldots, d_l \notin V(F_i)$. On the other hand there exist $c, e \in F_i$ such that c is a neighbor of a_i and e is a neighbor of b_i. Let P_i' : $ca_i', a_i' d_1, d_1 d_2, \ldots, d_l b_i', b_i' e$ be a path. Now we change the path P_i by P_i' in the ear decomposition of F, the other ears are unchangeable, we get an ear decomposition of $p_{\mathbf{1}_{\{a_i,b_i\}}+\mathbf{1}_F}(G)$.

(iv) Let $F_0, F_1, \ldots, F_r = F$ and $F_0', F_1', \ldots, F_s' = F'$ be odd ear decompositions of F and F', respectively such that $F \cap F' = \{a\}$. By Remark 4.5 we can assume that $a \in F_0'$. Then $F_0, F_1, \ldots, F_r, F_0' \cup F, F_1' \cup F, \ldots, F_s' \cup F$ is an odd ear decomposition of $F \cup F'$.

(v) By (i) we have that $p_{\mathbf{1}_{\{a,b\}}+\mathbf{1}_F}(G)$ is factor-critical having only a common point with F', so we can apply (iv). $\square$

For any simple graph G, denote by $D(G)$ the set of all vertices in G which are missed by at least one maximum matching of G, and $A(G)$ the set of vertices in $V - D(G)$ adjacent to at least one vertex in $D(G)$. Let $C(G) = V - A(G) - D(G)$ and odd $(D(G))$ be the number of odd connected components of $D(G)$. More generally let $S \subset V$ and $G[S]$ its induced subgraph of G. The induced graphs $A(G[S]), D(G[S]), C(G[S])$ will be denoted by $A(S), D(S), C(S)$.

At first, we recall the Gallai-Edmonds Structure Theorem given independently by J. Edmonds [4] and T. Gallai [7]. We give here a condensed version of [23, Theorem 1.5.3].

Theorem 4.8. *Let G be a graph. Then*

(i) *Every odd component H of $G - A(G)$ is factor-critical and $V(H) \subseteq D(G)$.*
(ii) *Every even component H of $G - A(G)$ has a perfect matching and $V(H) \subseteq C(G)$.*
(iii) *For every non empty set $X \subseteq A(G)$, the set $N_G(X)$ contains vertices in at least $\sharp(X) + 1$ odd components of $G - A(G)$.*
(iv) $\nu(G) = \frac{1}{2}[\sharp(V(G)) - \text{odd } (D(G)) + \sharp(A(G))]$.

Following Lovász [15] we can describe maximum matchings in G as follows: Every maximum matching of G is a union of a perfect matching of $C(G)$, a matching from $A(G)$ to the components of $D(G)$, that is a set of $\sharp(A(G))$ edges, each such edge contains a vertex in $A(G)$ and a vertex in some component of $D(G)$ and a maximum matching of each component of $D(G)$.

The set A(G) is also called Gallai-Edmonds set and is the unique subset of G satisfying the Gallai-Edmonds Structure Theorem as expressed in the following Corollary.

Corollary 4.9. *Let $G = A \cup D \cup C$ be a partition such that D is a matching-critical set, C has a perfect matching and for any $X \subset A$, the set $N_G(X)$ contains vertices in at least $\sharp(X) + 1$ odd components of D. Then $A(G) = A, D(G) = D$ and $C(G) = C$.*

As an application we have the following result.

Lemma 4.10. *Let $A(G), D(G), C(G)$ be the canonical decomposition of the graph G, $e = \{x_i, x_j\}$ an edge in $D(G)$ and $S = p_{\mathbf{1}_e+\mathbf{1}_G}(G)$. Then $A(S) = A(G), D(S) = D(G) \cup \{x_i^{(2)}, x_j^{(2)}\}$ and $C(S) = C(G)$.*

Proof. Let $F_1, \ldots, F_s$ be the connected components of $D(G)$, we can assume that e is an edge of F_1. Then we have by Lemma 4.7, (ii) that $p_{\mathbf{1}_e+\mathbf{1}_{F_1}}(G)$ is factor-critical. So $p_{\mathbf{1}_e+\mathbf{1}_{F_1}}(G) \cup F_2 \cup \ldots \cup F_s = p_{\mathbf{1}_e+\mathbf{1}_{D(G)}}(G)$ is matching-critical. Now we apply Corollary 4.9 to S with $A = A(G), D = p_{\mathbf{1}_e+\mathbf{1}_{D(G)}}(G)$ and $C = C(G)$. Note that vertices of e in $\mathbf{1}_e + \mathbf{1}_{D(G)}$ have coordinate 2 while the other in $D(G)$ have coordinate 1. Hence we have $A(S) = A(G), D(S) = D(G) \cup \{x_i^{(2)}, x_j^{(2)}\}$ and $C(S) = C(G)$ as required. $\square$

We will need the following particular case of Gallai-Edmonds Structure Theorem.

Corollary 4.11. *If $C(G) = \emptyset$ then*

(i) $\nu(G) = \nu(D(G)) + \sharp(A(G))$.

(ii) All elements in $A(G)$ are involved in any maximum matching of G and each edge of this matching contains at most one element of $A(G)$.

(iii) For any odd component F of $D(G)$ and every $u \in F$, each maximum matching $\mathcal{M}$ of $G - u$ provides a perfect matching of $F - u$.

(iv) $\sharp(A(G)) < \text{odd } (D(G))$.

(v) For every subset $X \in A(G)$, the set $N_G(X)$ contains vertices in at least $\sharp X + 1$ components of $D(G)$.

The following corollary is a part of the Stability Lemma [23, Theorem 1.5.5].

Corollary 4.12. *Let A' be a subset of $A(G)$. Then*

$$A(G - A') = A(G) - A', D(G - A') = D(G), C(G - A') = C(G).$$

5. Embedded irreducible components of I_G^k

In this section, we will study embedded irreducible components of I_G^k by using Gallai-Edmonds Structure Theorem 4.8. We know that any embedded irreducible component of I_G^k can be written as $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$, where $\mathbf{a} \in \mathbb{N}^d$ is a non zero vector and $\mathrm{Supp}(\mathbf{a}) \subset U \subset V$. In Theorem 5.3 and Theorem 5.7, we will give necessary and sufficient conditions for ideals of the form $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ belong to $\mathrm{Irr}(I_G^k)$. At first, we need the following lemmas.

Lemma 5.1. *Let M, M' be two monomials of R.*

(i) Let G be a simple graph. Suppose that $\mathrm{Supp}(M') \cap N(\mathrm{Supp} M) = \emptyset$ and $\mathrm{Supp}(M')$ is a coclique set. Then $M \in I_G^k$ if and only if $MM' \in I_G^k$.

(ii) Let I, J be monomial ideals. If $M \in I + J$ and $M \notin J$ then $M \in I$.

From now on, let $G = (V, E)$ be a simple graph, $\mathbf{a} \in \mathbb{N}^d$ a nonzero vector and $S := p_{\mathbf{a}}(G)$ the replication of G by vector $\mathbf{a}$ (see Definition 2.10).

Lemma 5.2. *Let $D(S), A(S)$ and $C(S)$ be the Gallai-Edmonds decomposition of S. Then we have $x_i^{(j)} \in D(S)$ (or in $A(S)$ or in $C(S)$, respectively) if and only if $x_i^{(1)} \in D(S)$ (or in $A(S)$ or in $C(S)$, respectively), for all $i = 1, \dots, d$ and $j = 1, \dots, a_i$.*

Proof. It is enough to prove the necessary condition, for $j \in \{2, \dots, a_i\}$. Firstly, suppose that $x_i^{(1)} \in D(S)$. Let $\mathcal{M}$ be a maximum matching of S avoiding $x_i^{(1)}$. We have to consider two cases:

(a) If $\mathcal{M}$ avoids $x_i^{(j)}$ then we have $\nu(S - x_i^{(1)}) = \nu(S - x_i^{(j)}) = \nu(S)$.

(b) If $\mathcal{M}$ does not avoid $x_i^{(j)}$ then by interchanging $x_i^{(1)}$ with $x_i^{(j)}$, we get a maximum matching $\mathcal{M}_j$ of S containing $x_i^{(1)}$, but avoiding $x_i^{(j)}$. Then $\nu(S - x_i^{(j)}) = \nu(S)$.

Thus both cases imply that $x_i^{(j)} \in D(S)$ by definition.

Secondly, suppose that $x_i^{(1)} \in A(S) = N_S(D(S)) - D(S)$. Then there exists $u \in D(S)$ such that u is a neighbor of $x_i^{(1)}$. It implies that $x_i^{(j)}$ is a neighbor of u, but $x_i^{(j)} \notin D(S)$, otherwise by the first claim $x_i^{(1)} \in D(S)$, a contradiction. Hence $x_i^{(j)} \in A(S)$.

Finally, the claim for the set $C(S)$ follows by the claims for $D(S)$ and $A(S)$. $\square$

Theorem 5.3. *Let $k \geqslant 2$ be an integer, $\mathbf{a} \in \mathbb{N}^d$ be a nonzero vector, $U \subset V$ such that* $\operatorname{Supp}(\mathbf{a}) \subset U$*. Let denote $Z := V \setminus U$ and $S := p_{\mathbf{a}}(G)$ the replication of G by* $\mathbf{a}$*. Assume that* $\operatorname{Supp}(\mathbf{a}) \cap N(Z) = \emptyset$*. Then the following statements are equivalent:*

(i) $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ is an embedded irreducible component of I_G^k.

(ii) The sets S and Z satisfy the following properties

(1) Z is a coclique set, $\nu(S) = k - 1$ and $V = N_G(D(S)) \cup Z \cup N(Z)$.

(2) $C(S) = \emptyset$, i.e. $S = D(S) \cup A(S)$ in the Gallai-Edmonds's canonical decomposition.

Proof. Firstly, if we choose $m = k$ in Corollary 2.8(v) then we have that $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ belongs to $\operatorname{Irr}(I_G^k)$ if and only if $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \notin I_G^k$ and for any $u \in U$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \in I_G^k$.

(i) $\Rightarrow$ (ii). Since $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ is an embedded irreducible component of I_G^k, by Remark 2.9 and Example 2.11, Z is a non maximal coclique set and $V \setminus Z \cup N(Z) \neq \emptyset$. Let $u \in V \setminus Z \cup N(Z)$, we will prove that $u \in N(D(S))$. Since $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \notin I_G^k$ we have $\nu(S) < k$. Moreover since $u \in U$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \in I_G^k$. Since Z is a coclique set and $\operatorname{Supp} \mathbf{a} \cap N(Z) = \emptyset$, by Lemma 5.1, (i) we have $u\mathbf{x}^{\mathbf{a}} \in I_G^k$. Then $u\mathbf{x}^{\mathbf{a}} = M_{e_1} \dots M_{e_k} M'$, for some $e_1, e_2, \dots, e_k \in E(G)$, but since $\mathbf{x}^{\mathbf{a}} \notin I_G^k$, the vertex u must belong to some of the edges $e_1, e_2, \dots, e_k$. Hence there exists $1 \leq i \leq k, v \in \operatorname{Supp}(\mathbf{a})$ such that $e_i = uv$. It follows that $k > \nu(S) \geq \nu(S - v) \geq k - 1$, which implies $\nu(S) = \nu(S - v) = k - 1$. Therefore $v \in D(S)$ by definition of $D(S)$, so $u \in N(D(S))$ as required and claim (1) is over.

In order to prove claim (2), note that by Theorem 4.8, $D(S)$ is matching-critical. By hypothesis $\operatorname{Supp}(\mathbf{a}) \cap (Z \cup N(Z)) = \emptyset$, so by (1) we have $\operatorname{Supp}(\mathbf{a}) \subset N_G(D(S))$. Since $(S \setminus D(S)) \cap V \subset \operatorname{Supp}(\mathbf{a}) \subset N_G(D(S))$, we have $(S \setminus D(S)) \cap V \subset A(S) \subset S \setminus D(S)$ by

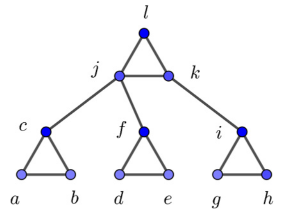

Fig. 2. $(a^2, \ldots, l^2)$ is not an irreducible component of I_G^6.

definition. We can see from Lemma 5.2 that $S \setminus D(S) \neq \emptyset$ if and only if $(S \setminus D(S)) \cap V \neq \emptyset$ and $S \setminus D(S) \subset A(S)$, hence $S \setminus D(S) = A(S)$ and we have $C(S) = \emptyset$ by definition.

(ii) $\Rightarrow$ (i). Since $\nu(S) = k - 1$, we have $\mathbf{x}^{\mathbf{a}} \in I_G^{k-1} \setminus I_G^k$. On the other hand Z is a coclique set and $N(\mathrm{Supp}(\mathbf{a})) \cap Z = \emptyset$. Hence $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \notin I_G^k$ by Lemma 5.1 (i). In order to complete our claim we have to prove that for any $u \in U$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} \in I_G^k$. We have two cases:

• If $u \in U \cap N(Z)$ then there exists $v \in Z$ such that $uv \in I_G$. So $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{k\mathbf{1}_Z} = (uv)\mathbf{x}^{\mathbf{a}}(\frac{\mathbf{x}^{k\mathbf{1}_Z}}{v}) \in I_G^k$.

• If $u \in U \setminus N(Z)$ then by hypothesis $u \in N_G(D(S))$. Hence there exists $w \in D(S)$ such that $uw \in I_G$. By the definition of $D(S)$ we have $\nu(S) = \nu(S - w)$, which implies that $\frac{\mathbf{x}^{\mathbf{a}}}{w} \in I_G^{k-1}$, hence $u\mathbf{x}^{\mathbf{a}} = (uw)\frac{\mathbf{x}^{\mathbf{a}}}{w} \in I_G^k$. $\square$

With the notations of the above Theorem we will show in Lemma 5.5 that every connected component of $D(S)$ contains an odd circuit. In the next example we illustrate this condition.

Example 5.4. In Fig. 2 we have $A(G) = \{j, k\}$, $D(G)$ has four connected components $F_1 = \{a, b, c\}, F_2 = \{d, e, f\}, F_3 = \{g, h, i\}, F_4 = \{l\}$ and $C(G) = \emptyset$. Note that F_4 is an isolated point in $D(G)$. We have $a \cdots l \in I_G^5 \setminus I_G^6$ but is not a corner element of I_G^6, since $la \cdots l \notin I_G^6$. This shows that $(a^2, \ldots, l^2)$ is not an irreducible component of I_G^6.

Note that for any sets $F, Z \subset V$, $F \cap N(Z) = \emptyset$ is equivalent to $N(F) \cap Z = \emptyset$.

Lemma 5.5. *Let $F, Z \subset V$ be two disjoint sets and $D \subset F$. Suppose that Z is a coclique set and $N(F) \cap Z = \emptyset$.*

(i) The following statements are equivalent:

(1) $V = N(D) \cup Z \cup N(Z)$

(2) $N(F) \setminus N(Z) \subset N(D)$ and Z is maximal such that $N(F) \cap Z = \emptyset$.

(ii) If one of the above conditions (1) or (2) is satisfied then D has no isolated vertices. In addition, if D is matching-critical then every connected component of the induced graph on D contains an odd circuit.

Proof. (i) (1) $\Rightarrow$ (2). Let $u \in N(F) \setminus N(Z)$. Since $N(F) \cap Z = \emptyset$ we have $u \notin Z$. Hence the equality $V = N(D) \cup Z \cup N(Z)$ implies $u \in N(D)$. Now suppose that Z is not maximal such that $N(F) \cap Z = \emptyset$. Then there exists $Z' \supsetneq Z$ a coclique set such that $N(F) \cap Z' = \emptyset$. Let $v \in Z' \setminus Z$. Then we have $v \notin N(F)$ and therefore $v \notin N(D)$. Hence the equality $V = N(D) \cup Z \cup N(Z)$ implies $v \in N(Z)$, a contradiction since Z' is coclique.

(2) $\Rightarrow$ (1). Let $u \in V \setminus Z \cup N(Z)$. Then the set $Z' = Z \cup \{u\}$ is coclique. If $u \notin N(F)$ then we have $N(F) \cap Z' = \emptyset$, it is a contradiction to the maximality of Z. Hence $u \in N(F) \setminus N(Z) \subset N(D)$. Our claim is done.

(ii) We have $D \cap (N(Z) \cup Z) = \emptyset$ by the hypothesis $F \cap (N(Z) \cup Z) = \emptyset$. By (1), we have $D \subset N(D)$, i.e. for every vertex $u \in D$, there is at least an edge uv with $v \in D$. Moreover, if D is matching-critical, every connected component of the induced graph on D contains an odd circuit by Theorem 4.4 and Remark 4.5. $\square$

Lemma 5.6. *Let G be a simple connected graph. $Z := \{z_1, \dots, z_\lambda\} \subset V$ be any non empty coclique set and $R' = R[z_1^{-1}, \dots, z_\lambda^{-1}]$. Let $\mathbf{b} \in \mathbb{N}^d$ such that $\operatorname{Supp}(\mathbf{x}^{\mathbf{b}}) \subset N(Z)$ and M a monomial with $\operatorname{Supp}(M) \subset V \setminus (Z \cup N(Z))$. Then for $l \in \mathbb{N}$, we have $M\mathbf{x}^{\mathbf{b}} \in I_G^l R' \setminus I_G^{l+1} R'$ if and only if $M \in I_G^{l-|\mathbf{b}|} \setminus I_G^{l+1-|\mathbf{b}|}$.*

Proof. Since Z is coclique $I_G R'$ is a proper ideal. Note that for every $u \in N(Z)$, there exists some j such that $uz_j \in I_G$ and $u = uz_j z_j^{-1}$. Hence we have $u \in I_G R'$. This implies that $I_G R' = I_{G \setminus (N(Z) \cup Z)} R' + \mathfrak{N} R'$, where $\mathfrak{N}$ is the ideal generated by $N(Z)$.

Suppose that $M\mathbf{x}^{\mathbf{b}} \in I_G^l R' \setminus I_G^{l+1} R'$. Since $I_G^l R'$ is a proper monomial ideal, there are monomial generators $f_1, \dots, f_l$ of $I_G R'$ and a monomial $N \in R'$ such that $M\mathbf{x}^{\mathbf{b}} = f_1 \dots f_l N$. It is an equality in R', so we can assume that no unit appears in the right member. Since $M\mathbf{x}^{\mathbf{b}} \notin I_G^{l+1} R'$ we have $\operatorname{Supp}(N) \subset V \setminus (N(Z) \cup Z)$. Suppose that $f_1, \dots, f_i \in I_{G \setminus (N(Z) \cup Z)} R'$ and $f_j \in N(Z)$ for every $j > i$. Because of the equality of monomials we have $i = l - |\mathbf{b}|$. Hence $M = f_1 \dots f_{l-|\mathbf{b}|} N$ with $f_1, \dots, f_{l-|\mathbf{b}|} \in I_G$, which means that $M \in I_G^{l-|\mathbf{b}|}$. On the other hand if $M \in I_G^{l+1-|\mathbf{b}|}$ then $M \in I_G^{l+1-|\mathbf{b}|} R'$ since $\operatorname{Supp}(M) \subset V \setminus (Z \cup N(Z))$. So we have $M\mathbf{x}^{\mathbf{b}} \in I_G^{l+1} R'$, a contradiction. Hence $M \in I_G^{l-|\mathbf{b}|} \setminus I_G^{l+1-|\mathbf{b}|}$ as required.

Conversely, suppose that $M \in I_G^{l-|\mathbf{b}|} \setminus I_G^{l+1-|\mathbf{b}|}$. Since $\operatorname{Supp}(M) \subset V \setminus (Z \cup N(Z))$ and $\operatorname{Supp}(\mathbf{x}^{\mathbf{b}}) \subset N(Z)$ we have $M\mathbf{x}^{\mathbf{b}} \in I_G^l R'$. Assume that $M\mathbf{x}^{\mathbf{b}} \in I_G^{l+1} R'$. Since $I_G^{l+1} R'$ is a proper monomial ideal, there are monomial generators $f_1, \dots, f_{l+1}$ of $I_G R'$ and a monomial $N \in R'$ such that $M\mathbf{x}^{\mathbf{b}} = f_1 \dots f_{l+1} N$. It is an equality in R', so we can assume that no unit appears in the right member. If there exists $u \in \operatorname{Supp}(\mathbf{x}^{\mathbf{b}})$ that divides N then we can cancel it in both sides of the equality. Hence we can suppose that $M\mathbf{x}^{\mathbf{b}'} = f_1 \dots f_{l+1} N'$ for some monomial $\mathbf{x}^{\mathbf{b}'}$ dividing $\mathbf{x}^{\mathbf{b}}$ and $\operatorname{Supp}(N') \cap \operatorname{Supp}(\mathbf{b}') = \emptyset$. Every variable appearing in $\mathbf{x}^{\mathbf{b}'}$ should appear in $f_1 \dots f_{l+1}$. By canceling in both sides the variables in $\mathbf{x}^{\mathbf{b}'}$ we get $M \in I_G^{l+1-|\mathbf{b}'|} \subset I_G^{l+1-|\mathbf{b}|}$, a contradiction. $\square$

Theorem 5.7. *Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ be an embedded irreducible component of I_G^k and $Z = V \setminus U$. Assume that $\operatorname{Supp}(\mathbf{a}) \cap N(Z) \neq \emptyset$. Let $\mathbf{a} = \mathbf{b} + \mathbf{c}$ with $\mathbf{b}, \mathbf{c} \in \mathbb{N}^d$ the unique decomposition* *such that* $\mathrm{Supp}(\mathbf{b}) = \mathrm{Supp}(\mathbf{a}) \cap N(Z), \mathrm{Supp}(\mathbf{c}) = \mathrm{Supp}(\mathbf{a}) \setminus N(Z)$ *and* $\delta =\mid \mathbf{b} \mid$. *Then* $\mathrm{Supp}(\mathbf{c}) \cap N(Z) = \emptyset$ *and* $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_U}$ *is an embedded irreducible component of* $I_G^{k-\delta}$.

Proof. Since $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ is an embedded irreducible component of I_G^k, by Remark 2.9, Z is a coclique set and by Example 2.11, Z is not maximal. Hence we need only to prove that $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^{k-\delta})$. For $m \geq k$, we have by Corollary 2.8(v) that $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^k)$ if and only if

$$\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^k \text{ (1) and for every } u \notin Z \; u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k \text{ (2)}.$$

By definition of $\mathbf{c}$, we have $N(\mathrm{Supp}(\mathbf{c})) \cap Z = \emptyset$. In order to prove $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^{k-\delta})$, again by Corollary 2.8(v) we need to prove

$$\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^{k-\delta} \text{ (1}'\text{) and for every } u \notin Z \; u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k-\delta} \text{ (2}'\text{)}.$$

Suppose conversely $\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k-\delta}$. Then there are $f_1, \ldots, f_{k-\delta}$ monomial generators of I_G and a monomial N such that $\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} = f_1 \ldots f_{k-\delta}N$. If there exist $z \in Z$ and an index i such that $f_i = zu$ then $u \in N(Z)$, it is impossible since $(\mathrm{Supp}(\mathbf{c}) \cup Z) \cap N(Z) = \emptyset$. It follows that $\mathbf{x}^{m\mathbf{1}_Z}$ divides N. Hence $\mathbf{x}^{\mathbf{c}} = f_1 \ldots f_{k-\delta}N'$. On the other hand, by the definition of $\mathbf{b}$ and note that $\mathrm{Supp}(\mathbf{c}) = \mathrm{Supp}(\mathbf{a}) \setminus N(Z)$, we have $\mathrm{Supp}(\mathbf{x}^{\mathbf{b}}) \subset N(Z)$. So for $m \geq \delta$ we have $\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{\delta}$. Finally, we get $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} = \mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k$, a contradiction to (1). Therefore (1$'$) is proved.

Now we prove (2$'$). Let $u \notin Z$ and $R' = R[z_1^{-1}, \ldots, z_\lambda^{-1}]$ be a localization of R by Z. We have from (2) that $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k R$, which implies $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}} \in I_G^k R'$. By definition of localization, the condition $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}} \in I_G^{k+1}R'$ implies that $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k+1}R$ for $m \gg 0$, writing $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z}$ as a product of $k+1$ generators of $I_G R$ gives us a contradiction to (1). Hence $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{\mathbf{b}} \notin I_G^{k+1}R'$. Now by applying Lemma 5.6 for two cases, we have: if $u \in N(Z)$ then $\mathbf{x}^{\mathbf{c}} \in I_G^{k-1-\delta}$, which implies $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k-\delta}$; if $u \in V \setminus (N(Z) \cup Z)$, then we have $u\mathbf{x}^{\mathbf{c}} \in I_G^{k-\delta}$, hence $u\mathbf{x}^{\mathbf{c}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k-\delta}$ as required. $\square$

Theorem 5.8. *Let* $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ *be an embedded irreducible component of* I_G^k *and* $\mathbf{x}^{\mathbf{b}}$ *be a monomial with support in* $N(Z)$. *Suppose that* $N(\mathrm{Supp}(\mathbf{a})) \cap Z = \emptyset$. *Then* $\mathfrak{m}^{\mathbf{a}+\mathbf{b}+\mathbf{1}_U}$ *is an embedded irreducible component of* $I_G^{k+|\mathbf{b}|}$.

Proof. Similar to Theorem 5.7. Since $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ is an embedded irreducible component of I_G^k, the coclique set Z is not maximal, hence we need only to prove that $\mathfrak{m}^{\mathbf{a}+\mathbf{b}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^{k-\delta})$. By Theorem 5.3 and Corollary 2.8(v), for $m \geq k$ we have that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^k$ and $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k$ for every $u \notin Z$. By Corollary 2.8(v) we have to prove that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \notin I_G^{k+|\mathbf{b}|}$ for every $m \geq k + |\mathbf{b}|$ and $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k+|\mathbf{b}|}$ for every $u \notin Z$.

Suppose that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k+|\mathbf{b}|}$. Then there are $f_1, \ldots, f_{k+|\mathbf{b}|}$ monomial generators of I_G and a monomial N such that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{m\mathbf{1}_Z} = f_1 \ldots f_{k+|\mathbf{b}|}N$. Let $v \in \mathrm{Supp}(\mathbf{x}^{\mathbf{b}})$, then either v divides N or f_i for some $1 \leq i \leq k + |\mathbf{b}|$. In both cases we have that

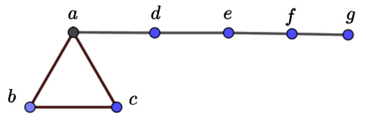

Fig. 3. $(a^2, b^2, c^2, d^2, f), (a^2, b^2, c^2, d, f^2), (a^2, b^2, c^2, d, e, g^2), (a^2, b^2, c^2, d, e^2, g) \in \mathrm{Irr}(I_G^k)$.

$\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}-\mathbf{1}_v}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^{k+|\mathbf{b}|-1}$. Repeating this argument we will have $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k$, a contradiction.

Let $u \notin Z$ and $x^{\mathbf{b}} = x_1^{b_1} \dots x_d^{b_d}$. For each $b_i > 0$, there exist $z_i \in Z$ such that $x_i z_i \in E$ which implies $(x_i z_i)^{b_i} \in I_G^{b_i}$. Hence $\prod_i (x_i z_i)^{b_i} \in I_G^{|\mathbf{b}|}$. Note that even if an element z_i can appear several times in this product then its power is at most $|\mathbf{b}|$. So $\prod_i (x_i z_i)^{b_i}$ divides $\mathbf{x}^{\mathbf{b}}\mathbf{x}^{|\mathbf{b}|\mathbf{1}_Z}$ which implies $\mathbf{x}^{\mathbf{b}}\mathbf{x}^{|\mathbf{b}|\mathbf{1}_Z} \in I_G^{|\mathbf{b}|}$. On the other hand since for any $u \notin Z$ and $m \geq k$ we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{m\mathbf{1}_Z} \in I_G^k$, so we have $u\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}\mathbf{x}^{(|\mathbf{b}|+m)\mathbf{1}_Z} \in I_G^{k+|\mathbf{b}|}$. Our claim is done. □

Remark 5.9. By the Theorem 5.7, in order to describe the embedded irreducible components of I_G^k, it is necessary and sufficient to describe the pair $(S := p_{\mathbf{a}}(G), Z)$ as in Theorem 5.3, where $S = A(S) \cup D(S)$ is in the Gallai-Edmonds decomposition such that $\nu(S) = k-1$, and Z is a coclique set such that $N_G(S) \cap Z = \emptyset, V = N_G(D(S)) \cup N(Z) \cup Z$. In this situation, let $\mathbf{b}$ be any vector with support in $N(Z)$. Then the irreducible component associated to $p_{\mathbf{a}+\mathbf{b}}(G)$ is an embedded irreducible component of $I_G^{k+|\mathbf{b}|}$.

Example 5.10. Let G be the graph as in Fig. 3, with $V = \{a, b, c, d, e, f, g\}$ and $E = \{ab, ac, bc, ad, de, ef, fg\}$. Consider the ring $R = K[a, b, c, d, e, f, g]$ and $I_G = (ab, ac, bc, ad, de, ef, fg) \subset R$. We have only one matching-critical subgraph $Y \subset G$ which is the triangle with vertices a, b, c. There are two coclique sets $Z_1 = \{e, g\}$ and $Z_2 = \{f\}$ such that $N(Y) \cap Z_1 = \emptyset$, $N(Y) \cap Z_2 = \emptyset$ and maximal for this property. Hence $(a^2, b^2, c^2, d, f), (a^2, b^2, c^2, d, e, g)$ are irreducible components of I_G^2. On the other hand we have $N(Z_1) = \{d, f\}, N(Z_2) = \{e, g\}$. Hence by applying Theorem 5.7, Theorem 5.8 and Remark 5.9 we have $(a^2, b^2, c^2, d^2, f), (a^2, b^2, c^2, d, f^2)$, and $(a^2, b^2, c^2, d, e, g^2), (a^2, b^2, c^2, d, e^2, g)$ are irreducible components of I_G^3.

Corollary 5.11. *There is at least an embedded component in* $\mathrm{Irr}(I_G^k)$ *if and only if there is an odd circuit* C *in* G *with* $\nu(C) \leq k-1$.

Proof. Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \mathrm{Irr}(I_G^k)$ be an embedded component and $Z := V \setminus U$. Let $\mathbf{a} = \mathbf{b} + \mathbf{c}$ be the decomposition given in Theorem 5.7, in particular we have $\mathrm{Supp}(\mathbf{b}) \subset \mathrm{Supp}(\mathbf{a}) \cap N(Z)$ and $\mathrm{Supp}(\mathbf{c}) \cap N(Z) = \emptyset$. So $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_U}$ is an embedded irreducible component of $I_G^{k-|\mathbf{b}|}$ by Theorem 5.7. Set $S := p_{\mathbf{c}}(G)$, then we have by Theorem 5.3 that $S = A(S) \cup D(S)$ in the Gallai-Edmonds decomposition, and $\nu(S) = k - \mid \mathbf{b} \mid -1$. Let $F_1 \subset D(S)$ be a factor-critical connected component of the matching-critical set $D(S)$. We have from Theorem 4.4 that F_1 has an odd ear decomposition $P_0 \subset P_1 \subset \dots \subset P_r = F_1$ such that

P_0 is an odd circuit. We can assume by Remark 4.5 that P_0 is a subgraph of G. Hence we have a circuit P_0 in G with $\nu(P_0) \leq \nu(F_1) \leq \nu(D(S)) \leq \nu(S) = k- \mid \mathbf{b} \mid -1$.

Conversely, let C be an odd circuit with $\nu(C) \leq k-1$. Then C is factor-critical. Let Z be any coclique set such that $N(C) \cap Z = \emptyset$ maximal for this property and $U = V \setminus Z$. It implies by Theorem 5.3 that $\mathfrak{m}^{\mathbf{1}_C + \mathbf{1}_U}$ is an embedded irreducible component of $I_G^{k-|\mathbf{b}|} I_G^{\nu(C)+1}$. Let e be any edge of C. Then by Lemma 4.7 $p_{\mathbf{1}_C + (k-\nu(C)-1)\mathbf{1}_e}$ is factor-critical and hence we have $\mathfrak{m}^{\mathbf{1}_C + (k-\nu(C)-1)\mathbf{1}_e + \mathbf{1}_U}$ is an embedded irreducible component of I_G^k by Theorem 5.3. $\square$

It is well known that a graph G has no odd cycles if and only if is bipartite. As an immediate consequence of Corollary 5.11 we can recover and precise Theorem 5.9 of [20].

Corollary 5.12. *A graph G is a bipartite graph if and only if I_G^k has no embedded irreducible components for every $k \geq 1$. For bipartite graphs Theorem 3.1 describes all irreducible components of I_G^k for any $k \geq 1$. In particular the number of irreducible components of I_G^k coincides with a polynomial of degree* $\operatorname{bight}(I_G) - 1$.

The next result improves Theorem 3.1 of [12] and Theorem 2.8 of [21].

Corollary 5.13. *Every embedded component in $\operatorname{Irr}(I_G^2)$ equals $\mathfrak{m}^{\mathbf{1}_F + \mathbf{1}_U}$, where $U \cup Z$ is a partition of V, Z is a coclique set and $F \subset U$ is a triangle such that $N(F) \cap Z = \emptyset, V = N(F) \cup N(Z) \cup Z$.*

Proof. Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \operatorname{Irr}(I_G^2)$ be an embedded component. From the proof of Corollary 5.11 we get $1 \leq \nu(P_0) \leq \nu(F_1) \leq \nu(D(S)) \leq \nu(S) = 2- \mid \mathbf{b} \mid -1$. This implies that the vector $\mathbf{b}$ is null and $P_0 = S$ such that $\nu(P_0) = 1$. Therefore S is a triangle in G by Example 4.6. Now take $F := S$, then we have $\mathbf{a} = \mathbf{1}_F$.

Reciprocally let F be a triangle, $\nu(F) = 1$, we know that F is factor-critical. Let Z be any coclique set such that $N(F) \cap Z = \emptyset$, maximal for this property and $U = V \setminus Z$. Hence $\mathfrak{m}^{\mathbf{1}_F + \mathbf{1}_U}$ is an embedded component of I_G^2 by Lemma 5.5 and Theorem 5.3. $\square$

As a consequence of our results in this section we get the following **strong persistence** which improves [17, Theorem 2.15].

Corollary 5.14. *Let $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \operatorname{Irr}(I_G^k)$. Then we have at least one irreducible component $\mathfrak{m}^{\mathbf{a}'+\mathbf{1}_U} \in \operatorname{Irr}(I_G^{k+1})$ such that $\operatorname{Supp}(\mathbf{a}) = \operatorname{Supp}(\mathbf{a}')$ and $a_i \leq a'_i$ for all $i = 1, \ldots, d$. In particular if $\mathfrak{m}^{\mathbf{1}_U}$ is an associated prime of I_G^k then $\mathfrak{m}^{\mathbf{1}_U}$ is an associated prime of I_G^l for all $l \geq k$.*

Proof. If $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \operatorname{Irr}(I_G^k)$ is a non embedded component then the conclusion follows from Theorem 3.1. If $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U} \in \operatorname{Irr}(I_G^k)$ is an embedded component, let $Z := V \setminus U$ and $\mathbf{a} = \mathbf{b} + \mathbf{c}$ be the decomposition given in Theorem 5.7. Let $S := p_{\mathbf{c}}(G)$. From the proof of Corollary 5.11 we get $\nu(S) = k- \mid \mathbf{b} \mid -1$ and $S = A(S) \cup D(S)$ in the Gallai-Edmonds decomposition, with $D(S)$ a matching-critical set. Let e be any edge in $D(S)$ and $S' = S := p_{\mathbf{c}+\mathbf{1}_e}(G)$, by Lemma 4.10 we have $A(S') = A(S), C(S') = C(S) = \emptyset$ and $D(S')$ is obtained from $D(S)$ by replication of the edge e. So $\nu(D(S')) = \nu(D(S))+1$ which implies $\nu(S') = \nu(S)+1 = k- \mid \mathbf{b} \mid$. Moreover $D(S)\cap V = D(S')\cap V$, so $N_G(D(S')) = N_G(D(S))$. Hence S' satisfies the hypothesis of Theorem 5.3, and $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_e+\mathbf{1}_U} \in \mathrm{Irr}(I_G^{k-|\mathbf{b}|+1})$. By applying Theorem 5.8 we have $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_e+\mathbf{1}_U} \in \mathrm{Irr}(I_G^{k+1})$ Taking $\mathbf{a}' = \mathbf{a} + \mathbf{1}_e$ we get the result. $\square$

6. Behavior of $\mathrm{Irr}(I_G^k)$ for k large enough

The edge subring $K[G] := K[e \mid e \in E] \subset R$ of G is the subalgebra of R generated by the edges of G by considering each edge as a monomial in R. In other words, to any edge $e \in E$ of G we associate a variable Y_e and have a morphism $\varphi : K[Y_e \mid e \in E] \to R$ from a polynomial ring to R defined by $\varphi(Y_e) = e$. Let $I(G)$ be the kernel of φ, then we have $K[Y_e \mid e \in E]/I(G) = K[G]$ and $I(G)$ called *toric edge ideal* is a toric ideal generated by binomials. Note that the edge subring of G is a graded algebra generated in degree 2, thus it can be regarded as a standard graded algebra by assigning degree 1 to its generators and there is a natural homogeneous isomorphism between the edge subring $K[G]$ and the special fiber ring of the edge ideal I_G of G. Therefore, the Krull dimension of $K[G]$ equals the Krull dimension of the special fiber ring of the edge ideal I_G, which is called the analytic spread and denoted by $l(I_G)$. It follows from [10, Theorem 3.3] that $l(I_G) = \dim K[G] = \sharp V$ if G contains an odd circuit and $l(I_G) = \dim K[G] - 1 = \sharp V - 1$ if G is bipartite.

In this section we will apply our main results to study the set $\mathrm{Irr}(I_G^k)$ for $k \gg 0$. From one side we improve the main results of [2] and [22] by giving short and conceptual proofs in Theorem 6.3 and Theorem 6.9. From the other side we can precise the main result of [3] that counted the number of irreducible components of I_G^k for $k \gg 0$.

First we give a direct corollary of Theorem 5.3 for irreducible components of I_G^k such that its radical is the maximal ideal.

Corollary 6.1. *Let $\mathbf{a} \in \mathbb{N}^d$ and $S = p_{\mathbf{a}}(G)$. Then $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_V}$ is an embedded irreducible component of I_G^k if and only if*

(i) $\nu(S) = k - 1$, $V = N_G(D(S))$.

(ii) $S = A(S) \cup D(S)$ in the Gallai-Edmonds decomposition, that is $C(S) = \emptyset$.

Proof. The claim follows by applying Theorem 5.3 with $Z = \emptyset$. $\square$

We have by virtue of Lemma 5.5 that every connected component of the matching-critical set $D(S)$ in 6.1 contains an odd cycle.

Let $S = p_{\mathbf{a}}(G)$ be a matching-critical set. Note that $\nu(S) =\mid \mathbf{a} \mid -\beta_0(\mathrm{Supp}(\mathbf{a}))$, where $\beta_0(\mathrm{Supp}(\mathbf{a})) = \beta_0(S)$ is the number of connected components of $\mathrm{Supp}(\mathbf{a})$.

Lemma 6.2. *Let G be a simple connected non bipartite graph and F a factor-critical subset of G. Then*

(i) If $F \subsetneq G$ then there exists a vector $\mathbf{a} \in \mathbb{N}^d$ with $F \subset \operatorname{Supp}(\mathbf{a})$ such that $p_{\mathbf{a}}(G)$ is factor-critical, $\operatorname{Supp}(\mathbf{a}) \subsetneq N_G(p_{\mathbf{a}}(G)) = V$*, $E_0 \subset V \setminus \operatorname{Supp}(\mathbf{a})$ and $\nu(p_{\mathbf{a}}(G)) = \sharp\operatorname{Supp}(\mathbf{a}) - \nu(F) - 1$, where E_0 is the set of leaves in G.*

(ii) There exists a vector $\mathbf{b} \in \mathbb{N}^d$ with $\operatorname{Supp}(\mathbf{b}) = V$ such that $p_{\mathbf{b}}(G)$ is factor-critical and $\nu(p_{\mathbf{b}}(G)) = \sharp V - \nu(F) - 1$.

Proof. (i) We consider two cases. If $N(F) = V$ then we take $\mathbf{a} = \mathbf{1}_F$, so $p_{\mathbf{a}}(G) = F$ satisfies all conditions.

If $N(F) \neq V$ then there exists an independent set Z which is maximal such that $N(F) \cap Z = \emptyset$. We have by Lemma 5.5 that $V = N(F) \cup Z \cup N(Z)$. Now set $F_1 := F, \mathbf{a}_1 = \mathbf{1}_F$. Since G is connected we have $F_1 \subsetneq N(F_1) \setminus E_0$. Let $uv \in E(G)$ such that $u \in N(F_1) \setminus (E_0 \cup \operatorname{Supp}(\mathbf{a}_1))$ and $v \in F_1$, by applying Lemma 4.7 we have that $p_{\mathbf{a}_1 + \mathbf{1}_{\{u,v\}}}(G)$ is factor-critical. Proceeding similarly for all vertices in $N(F_1) \setminus (E_0 \cup \operatorname{Supp}(\mathbf{a}_1))$ we can construct a factor-critical graph $F_2 = p_{\mathbf{a}_2}(G)$ such that $\operatorname{Supp}(\mathbf{a}_2) = N(F_1) \setminus E_0$ and $\nu(F_2) = \nu(F_1) + \sharp\operatorname{Supp}(\mathbf{a}_2) \setminus \operatorname{Supp}(\mathbf{a}_1)$. Similarly, by successive applications of Lemma 4.7, we can construct factor-critical graphs $F_1, \ldots, F_\tau$ such that for $i = 2, \ldots, \tau$ $F_i = p_{\mathbf{a}_i}(G), \operatorname{Supp}(\mathbf{a}_i) = N_G(F_{i-1}) \setminus E_0$, $\nu(F_i) = \nu(F_{i-1}) + \sharp\operatorname{Supp}(\mathbf{a}_i) \setminus \operatorname{Supp}(\mathbf{a}_{i-1})$ and $F_\tau \subsetneq N(F_\tau) = V$. On the other hand

$$
\begin{aligned}
\nu(F_2) &= \nu(F_1) + \sharp\operatorname{Supp}(\mathbf{a}_2) \setminus \operatorname{Supp}(\mathbf{a}_1) \\
\nu(F_3) &= \nu(F_2) + \sharp\operatorname{Supp}(\mathbf{a}_3) \setminus \operatorname{Supp}(\mathbf{a}_2) \\
&\cdots \\
\nu(F_\tau) &= \nu(F_{\tau-1}) + \sharp\operatorname{Supp}(\mathbf{a}_\tau) \setminus \operatorname{Supp}(\mathbf{a}_{\tau-1})
\end{aligned}
$$

which implies $\nu(F_\tau) = \nu(F_1) + \sharp\operatorname{Supp}(\mathbf{a}_\tau) \setminus \operatorname{Supp}(\mathbf{a}_1)$, but $\sharp\operatorname{Supp}(\mathbf{a}_1) = \sharp F_1 = 2\nu(F_1) + 1$. Hence $\nu(F_\tau) = \nu(F_1) + \sharp\operatorname{Supp}(\mathbf{a}_\tau) - \sharp\operatorname{Supp}(\mathbf{a}_1) = \sharp\operatorname{Supp}(\mathbf{a}_\tau) - \nu(F_1) - 1$. Our claim is proved by taking $\mathbf{a} = \mathbf{a}_\tau$.

(ii) By the claim (i), there exists a vector $\mathbf{a}$ such that $p_{\mathbf{a}}(G)$ is factor-critical and $\operatorname{Supp}(\mathbf{a}) \subsetneq N_G(p_{\mathbf{a}}(G)) = V$. Let $V \setminus \operatorname{Supp}(\mathbf{a}) := \{x_{i_1}, \ldots, x_{i_\sigma}\}$ and e_j an edge which one vertex is x_{i_j} and the second vertex is in $\operatorname{Supp}(\mathbf{a})$, for $j = 1, \ldots, \sigma$. Let $\mathbf{b} = \mathbf{a} + \sum_{j=1}^{\sigma} \mathbf{1}_{e_j}$. Then we have by Lemma 4.7 that $p_{\mathbf{b}}(G)$ is factor-critical with $\operatorname{Supp}(\mathbf{b}) = V$ and $\nu(p_{\mathbf{b}}(G)) = \sharp V - \nu(F) - 1$. □

Now we can improve the main result of [22].

Theorem 6.3. *Let G be a simple connected non bipartite graph, I_G its edge ideal and F a factor-critical subset of G with the biggest matching number. Then $\mathfrak{m} \in \operatorname{Ass}(I_G^k)$ for $k = \sharp V - \varepsilon_0(G) - \nu(F)$, where $\varepsilon_0(G)$ is the number of leaves in G. In particular* $\operatorname{dstab}(I_G) \leq \sharp V - \varepsilon_0(G) - \nu(F)$.

Proof. If G is factor-critical then G has the biggest matching number among all factor-critical subgraphs of G. Factor-critical graphs do not have leaves and $\nu(G) = \sharp V - \nu(G) - 1$. We can apply Corollary 6.1 to $S = p_{\mathbf{1}_V}(G) = G$ and $k = \nu(G) + 1$ to get that $\mathfrak{m}^{2(\mathbf{1}_V)} \in \operatorname{Irr}(I_G^k)$. Since $\operatorname{rad}(\mathfrak{m}^{2(\mathbf{1}_V)}) = \mathfrak{m}$ we have $\operatorname{dstab}(I_G) \leq k = \sharp V - \nu(G)$. So we can assume that F is a proper factor-critical subset of G with the biggest matching number. By Lemma 6.2 there exists a vector $\mathbf{a} \in \mathbb{N}^d$ with $F \subset \operatorname{Supp}(\mathbf{a})$ such that $p_{\mathbf{a}}(G)$ is factor-critical, $\operatorname{Supp}(\mathbf{a}) \subsetneqq N_G(p_{\mathbf{a}}(G)) = V$, $E_0 \subset V \setminus \operatorname{Supp}(\mathbf{a})$ and $\nu(p_{\mathbf{a}}(G)) = \sharp\operatorname{Supp}(\mathbf{a}) - \nu(F) - 1$, where E_0 is the set of leaves in G. We can apply Corollary 6.1 to $S = p_{\mathbf{a}}(G)$ and $k = \nu(p_{\mathbf{a}}(G)) + 1$ to get that $\mathfrak{m}^{\mathbf{1}_S + \mathbf{1}_V} \in \operatorname{Irr}(I_G^k)$. Since $\operatorname{rad}(\mathfrak{m}^{\mathbf{1}_S + \mathbf{1}_V}) = \mathfrak{m}$, we have $\operatorname{dstab}(I_G) \leq \nu(p_{\mathbf{a}}(G)) + 1 = \sharp\operatorname{Supp}(\mathbf{a}) - \nu(F) \leq \sharp V - \varepsilon_0(G) - \nu(F)$. $\square$

If G is not factor-critical and has no leaves then we can get better bounds for $\operatorname{dstab}(I_G)$ and $\operatorname{astab}(I_G)$.

Corollary 6.4. *Let G be a simple connected non bipartite graph without leaves. If G is not factor-critical then* $\operatorname{dstab}(I_G) < \sharp V - \nu(F)$ *for any proper factor-critical subgraph F of G.*

Proof. Suppose that $\operatorname{dstab}(I_G) = \sharp V - \nu(F)$ for some proper factor-critical subgraph F of G. By Lemma 6.2 there exist a vector $\mathbf{a} \in \mathbb{N}^d$ with $F \subset \operatorname{Supp}(\mathbf{a})$ such that $p_{\mathbf{a}}(G)$ is factor-critical, $\operatorname{Supp}(\mathbf{a}) \subsetneqq N_G(p_{\mathbf{a}}(G)) = V$ and $\nu(p_{\mathbf{a}}(G)) = \sharp\operatorname{Supp}(\mathbf{a}) - \nu(F) - 1$. Therefore it implies

$$\operatorname{dstab}(I_G) = \sharp V - \nu(F) \leq \nu(p_{\mathbf{a}}(G)) + 1 = \sharp\operatorname{Supp}(\mathbf{a}) - \nu(F).$$

So $\sharp V \leq \sharp\operatorname{Supp}(\mathbf{a})$, but $\operatorname{Supp}(\mathbf{a}) \subseteq V$, hence $\operatorname{Supp}(\mathbf{a}) = V$. It is a contradiction to the fact that $\operatorname{Supp}(\mathbf{a}) \subsetneqq N_G(p_{\mathbf{a}}(G)) = V$. $\square$

In practice we can find the best bound by working on matching-critical sets, but the process is more difficult to control. A set $D \subset V$ is called *dominant* in G if $V = N(D)$. We also say that $p_{\mathbf{a}}(G)$ is dominant in G if $V = N_G(p_{\mathbf{a}}(G))$.

Corollary 6.5. *Let G be a simple connected non bipartite graph. Let $\mathbf{a} \in \mathbb{N}^d$ such that $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_V} \in \operatorname{Irr}(I_G^k)$. Then there exists $\mathbf{c} \in \mathbb{N}^d$ such that $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_V} \in \operatorname{Irr}(I_G^{k'})$, with $k' \leq k$, $c_i \leq a_i$ for all $i = 1, \ldots, d$ and $p_{\mathbf{c}}(G)$ is matching-critical with dominant in G. As a consequence we have*

$$\operatorname{dstab}(I_G) = \min\{\nu(p_{\mathbf{a}}(G)) + 1 \mid \mathbf{a} \in \mathbb{N}^d, p_{\mathbf{a}}(G) \textit{ is matching-critical, dominant in } G\}.$$

Proof. Let $S = p_{\mathbf{a}}(G)$. Then we have by Corollary 6.1 that $S = D(S) \cup A(S)$. Now let $\mathbf{c} \in \mathbb{N}^d$ such that $D(S) = p_{\mathbf{c}}(G)$, we can see $D(S)$ satisfies the conditions of Corollary 6.1. Hence $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_V} \in \operatorname{Irr}(I_G^{k'})$ with $k' = \nu(D(S)) + 1 \leq \nu(S) + 1$. Moreover, we have by Corollary 5.14 that $\operatorname{dstab}(I_G)$ is the smallest k such that $\mathfrak{m} \in \operatorname{Ass}(I_G^k)$. So by computing the minimum in the formula for $\mathrm{dstab}(I_G)$, it is enough to consider vectors $\mathbf{a}$ such that $p_{\mathbf{a}}(G)$ is matching-critical and dominant in G. Note that by Lemma 5.5 every connected component of $p_{\mathbf{a}}(G)$ contains an odd cycle. □

The next result recovers and extends Theorem 3.1 of [12] and Theorem 2.8 of [21].

Corollary 6.6. *Let G be a simple connected non bipartite graph then:*

(i) $\mathrm{dstab}(I_G) = 2$ *if and only if there is a dominant triangle in G.*

(ii) $\mathrm{dstab}(I_G) = 3$ *if and only if no triangle is dominant in G and $p_{\mathbf{a}}(G)$ is dominant in G, where $\mathbf{a}$ is one of the following vectors:*

(1) $\mathbf{1}_\Delta + \mathbf{1}_{\{u,v\}}$, *where Δ is a triangle, $u \in \Delta, v \notin \Delta, uv \in E$.*

(2) $\mathbf{1}_{\Delta_1} + \mathbf{1}_{\Delta_2}$, *where Δ_1, Δ_2 are triangles with at most one common vertex.*

(3) $\mathbf{1}_\Gamma$, *where Γ is a pentagon.*

Proof. (i) By Corollary 6.5, $\mathrm{dstab}(I_G) = 2$ if and only if there is a vector $\mathbf{a}$ such that $p_{\mathbf{a}}(G)$ is matching-critical dominant in G with $\nu(p_{\mathbf{a}}(G)) = 1$. Since $p_{\mathbf{a}}(G)$ contains an odd cycle, we have $p_{\mathbf{a}}(G)$ is a triangle.

(ii) We have by Corollary 6.5 that $\mathrm{dstab}(I_G) = 3$ if and only if there is a vector $\mathbf{a}$ such that $p_{\mathbf{a}}(G)$ is matching-critical dominant in G with $\nu(p_{\mathbf{a}}(G)) = 2$. If $p_{\mathbf{a}}(G)$ is not connected then $p_{\mathbf{a}}(G)$ is the union of two disjoint triangles. If $p_{\mathbf{a}}(G)$ is connected, then $p_{\mathbf{a}}(G)$ is factor-critical and contains an odd cycle, so we have $3 \leq \sharp\mathrm{Supp}(\mathbf{a}) \leq 5$. If $\sharp\mathrm{Supp}(\mathbf{a}) = 3$ then the triangle defined by $\mathrm{Supp}(\mathbf{a})$ is matching-critical dominant in G, hence $\mathrm{dstab}(I_G) = 2$, a contradiction so $4 \leq \sharp\mathrm{Supp}(\mathbf{a}) \leq 5$. We have two cases:

If $\sharp\mathrm{Supp}(\mathbf{a}) = 4$, then let $\mathrm{Supp}(\mathbf{a}) = \{u_1, u_2, u_3, u_4\}$. So we can assume that $V(p_{\mathbf{a}}(G)) := \{u_1, u_1', u_2, u_3, u_4\}$. Since $p_{\mathbf{a}}(G)$ is factor-critical we have that $p_{\mathbf{a}}(G) - u_4$ has a perfect matching, which is necessarily $u_1u_2, u_1'u_3$. On the other hand $p_{\mathbf{a}}(G) - u_3$ has a perfect matching, which is necessarily $u_1u_2, u_1'u_4$. Similarly $p_{\mathbf{a}}(G) - u_1$ has a perfect matching, which is necessarily either (a) u_1u_2, u_3u_4 or (b) u_1u_3, u_2u_4 or (c) u_1u_4, u_2u_3. In case the (a) we have $\mathbf{a} = \mathbf{1}_{\{u_1,u_3,u_4\}} + \mathbf{1}_{\{u_1,u_2\}}$. In case the (b) we have $\mathbf{a} = \mathbf{1}_{\{u_1,u_2,u_4\}} + \mathbf{1}_{\{u_1,u_3\}}$ and in the case (c) we have $\mathbf{a} = \mathbf{1}_{\{u_1,u_2,u_3\}} + \mathbf{1}_{\{u_1,u_4\}}$. So in all the cases $\mathbf{a} = \mathbf{1}_\Delta + \mathbf{1}_{\{u,v\}}$, where Δ is a triangle, $u \in \Delta, v \notin \Delta, uv \in E$.

If $\sharp\mathrm{Supp}(\mathbf{a}) = 5$ then since $p_{\mathbf{a}}(G)$ is factor-critical, all coordinates of the vector $\mathbf{a}$ which are in $\mathrm{Supp}(\mathbf{a})$ equal to 1. Hence $p_{\mathbf{a}}(G)$ is a factor-critical subgraph of G, so is either a pentagon with eventually some chords or a union of two triangles with only a common vertex and eventually some chords. □

In the following example we compute the first irreducible component which radical is the maximal ideal.

Example 6.7. Let $V = \{a, \ldots, l\}$ (see Fig. 4) and choose $S = \{a, \ldots, j\}$. Hence we can check that $C(S) = \emptyset, D(S) = \{a, \ldots, i\}$ and $A(S) = \{j\}$. Therefore S satisfies the second condition of Corollary 6.1 but not the first. By applying Lemma 4.7, (i) we have that the

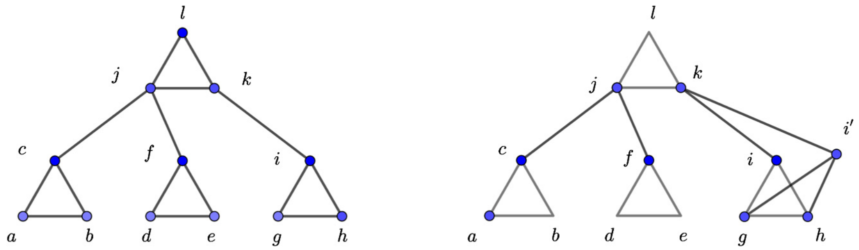

Fig. 4. $\mathrm{dstab}(I_G) = 5$.

graph $S' := p_{\mathbf{1}_S + 1_{\{i,k\}}}(G)$ is matching-critical with $C(S') = \emptyset, D(S') = \{a, \dots, i, i', k\}$ and $A(S') = \emptyset$. Hence S' satisfies the conditions of Corollary 6.1 with $\nu(S') = 4$. Thus $(a^2, \dots, h^2, i^3, j, k^2, l) \in \mathrm{Irr}(I_G^5)$. In this example, the bound given by [2] and [22] is $\mathrm{dstab}(I_G) \leq 11$, but we have seen that $\mathrm{dstab}(I_G) \leq 5$. In fact by using Corollary 6.5 we can prove that $\mathrm{dstab}(I_G) = 5$.

Lemma 6.8. *Let L be the set of leaves in G. Then for any set $A \subset V$ such that every connected component of the induced subgraph $G[A]$ contains an odd circuit, we have $N(A) = N(A \setminus L)$.*

Proof. It is clear that $N(A \setminus L) \subset N(A)$. We have to prove that $N(A) \subset N(A \setminus L)$. Let $u \in N(A)$, so we have $N(u) \cap A \neq \emptyset$. We have two cases:

If $N(u) \cap A \not\subset L$ then there exists $v \notin A \setminus L$ such that $uv \in E$, so $u \in N(A \setminus L)$.

If $N(u) \cap A \subset L$, let $l \in L$ such that $l \in N(u) \cap A$, since A has no isolated vertices there exists $v \in A$ such that $lv \in E$, but l is a leaf in G so we have $u = v \in A$. Let A' be the connected component of $G[A]$ containing u and w be a vertex in a circuit contained in A'. Since A' is connected, there is a path $v_0 := u, v_1, \dots, v_l = w$ in A'. If $w = v_1$ then $v_1 \notin L$ and if $w \neq v_1$ then v_1 has at least u, v_2 as neighbors. So we have $v_1 \notin L$ and $v_1 \in N(u)$, a contradiction or the second case can not happen. $\square$

Now we can improve the main result of [2]. Note that parts (i) and (ii) are a weak version of [13, Theorem 4.3].

Theorem 6.9. *Let G be a simple connected non bipartite graph, C a smallest odd circuit in G and L the set of leaves in G.*

(i) If $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^k and $Z := V \setminus U$ then there exist a set $H \subset U \setminus L$ such that every connected component of the induced graph on H contains an odd circuit and $U = N(H) \cup N(Z)$.

(ii) Let $U \subset V$ such that $Z := V \setminus U$ is a coclique set and there exist a set $H \subset U \setminus L$ such that every connected component of the induced graph on H contains an odd circuit and $U = N(H) \cup N(Z)$. Then $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^k, for some $k \leq \sharp H - \nu(C)$.

(iii) $\mathrm{astab}(I_G) \leq \sharp(V \setminus L) - \nu(C)$.

(iv) Suppose that G is not factor-critical and has no leaves. Then $\operatorname{astab}(I_G) < \sharp V - \nu(C)$.

Proof. We know that $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^k if an only if there exists some vector $\mathbf{a} \in \mathbb{N}^d$ such that $\operatorname{Supp}(\mathbf{a}) \subset U$ and $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_U}$ is an embedded irreducible component of I_G^k. We have by Remark 2.9 that $Z = V \setminus U$ is a coclique set.

(i) We can suppose that k is the smallest possible number by Theorem 5.14. Let $\mathbf{a} = \mathbf{b} + \mathbf{c}$ be the decomposition given in Theorem 5.7, in particular we have $\operatorname{Supp}(\mathbf{b}) \subset \operatorname{Supp}(\mathbf{a}) \cap N(Z)$ and $\operatorname{Supp}(\mathbf{c}) \cap N(Z) = \emptyset$. So $\mathfrak{m}^{\mathbf{c}+\mathbf{1}_U}$ is an embedded irreducible component of $I_G^{k-|\mathbf{b}|}$, which implies that $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of $I_G^{k-|\mathbf{b}|}$. Since k is the smallest possible number such that $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^k, we have $\mathbf{b}$ is null. Set $S := p_{\mathbf{a}}(G)$, we have by Theorem 5.3 that $S = A(S) \cup D(S)$ in the Gallai-Edmonds decomposition, and $\nu(S) = k - 1$. From Corollary 4.11 we have $\nu(S) = \nu(D(S)) + \sharp(A(S))$. Let $\mathbf{a}'$ be the vector defined by $a'_i = 0$ if $x_i \in A(S) \cap V$ and $a'_i = a_i$ if $x_i \in D(S) \cap V$. Then we have by Theorem 5.3 that $\mathfrak{m}^{\mathbf{a}'+\mathbf{1}_U}$ is an embedded irreducible component of $I_G^{k-\sharp(A(S))}$ and hence $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of $I_G^{k-\sharp(A(S))}$. Therefore $A(S) = \emptyset$ and $S = D(S)$ by the smallest property of k. Thus S is matching-critical such that $N_G(S) \cap Z = \emptyset$ and $U = N_G(S) \cup N(Z)$. Now we need to show that there exists some set $H \subset U \setminus L$ such that $N(H) = N_G(S)$. Indeed, let $S_1, \ldots, S_s$ be the connected components of S. Since S_i is factor-critical for each $i = 1, \ldots, s$, we can assume by Remark 4.5 that there is an odd circuit C_i such that $C_i \subset V(S_i) \cap V$. Moreover, since any vertex in C_i is not a leaf we have $C_i \subset (V(S_i) \cap V) \setminus L$. Therefore, applying Lemma 6.8 to $A := V(S) \cap V$ we have $N(V(S) \cap V) = N((V(S) \cap V) \setminus L)$. Now choose the set $H := (V(S) \cap V) \setminus L$ and taking notice that $N_G(S) = N(V(S) \cap V)$, we get the result.

(ii) Let $H_1, \ldots, H_s$ be the connected components of the induced subgraph on H. Since H_i is connected and contains an odd circuit C_i, by applying Lemma 6.2, there exists a vector $\mathbf{b}_i \in \mathbb{N}^d$ with $\operatorname{Supp}(\mathbf{b}_i) = H_i$ such that $p_{\mathbf{b}_i}(G)$ is factor-critical and $\nu(p_{\mathbf{b}_i}(G)) = \sharp H_i - \nu(C_i) - 1$. Let $\mathbf{b} = \mathbf{b}_1 + \ldots + \mathbf{b}_s$. Then $p_{\mathbf{b}}(G)$ is matching-critical and since $N(H) = N_G(p_{\mathbf{b}}(G))$ we have $U = N_G(p_{\mathbf{b}}(G)) \cup N(Z)$. It implies by Theorem 5.3 that $\mathfrak{m}^{\mathbf{b}+\mathbf{1}_U}$ is an embedded irreducible component of $I_G^{\nu(p_{\mathbf{b}}(G))+1}$. So $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^k for $k = \nu(p_{\mathbf{b}}(G)) + 1$. Our claim is over.

(iii) Let $\mathfrak{m}^{\mathbf{1}_U}$ be an embedded associated prime of I_G^k for some k and $H \subset U \setminus L$ a set defined as in (i). Thanks to (ii), we can choose $k = \nu(p_{\mathbf{b}}(G)) + 1$ and hence we have

$$\nu(p_{\mathbf{b}}(G)) = \sum_{i=1}^{s} (\sharp H_i - \nu(C_i) - 1) \leq \sharp H - \nu(C) - 1 \leq \sharp(V \setminus L) - \nu(C) - 1,$$

which implies by Corollary 5.14 that $\mathfrak{m}^{\mathbf{1}_U}$ is an embedded associated prime of I_G^l for every $l \geq \sharp(V \setminus L) - \nu(C)$. This shows that $\operatorname{astab}(I_G) \leq \sharp(V \setminus L) - \nu(C)$.

(iv) Let $\mathfrak{m}^{\mathbf{1}_U}$ be an embedded associate prime of I_G^k. We assume that k is the smallest possible number. We consider two cases:

(1) If $U = V$ then by Corollary 6.4 we have that $k < \sharp V - \nu(C)$.

(2) If $U \neq V$ then there exists $H \subset U$ as in (i) such that $k \leq \sharp H - \nu(C) \leq \sharp U - \nu(C) < \sharp V - \nu(C)$. It then follows that $\operatorname{astab}(I_G) < \sharp V - \nu(C)$, as required. □

Let $I \subset R$ be an ideal, $\operatorname{bight}(I)$ the biggest height of the associated primes of R/I and $l(I)$ the analytic spread of I. Denote $\operatorname{ir}_I(k)$ be the number of irreducible components of I^k. Now we recall the main result of [3].

Theorem 6.10. *Let I be an ideal of R. Then there exists a polynomial $\operatorname{Ir}_I(k)$ with rational coefficients such that $\operatorname{ir}_I(k) = \operatorname{Ir}_I(k)$ for sufficiently large k. Moreover we have*

$$\operatorname{bight}(I) - 1 \leq \deg(\operatorname{Ir}_I(k)) \leq l(I) - 1.$$

For edge ideals we will prove in the next theorem that $\deg(\operatorname{Ir}_{I_G}(k))$ characterize if a graph is bipartite or not and $\deg(\operatorname{Ir}_{I_G}(k))$ can take only one of the two extreme values.

Theorem 6.11. *Let G be a simple connected graph. Then*

(i) If G is bipartite then for $k \geq 1$, the function $\operatorname{ir}_{I_G}(k)$ coincides with a polynomial with rational coefficients $\operatorname{Ir}_{I_G}(k)$ of degree $\operatorname{bight}(I_G) - 1$.

(ii) If G is non bipartite then the function $\operatorname{ir}_{I_G}(k)$ is bounded below by the Hilbert function of $K[G](-l)$, where $l = \nu(p_{\mathbf{a}}(G)) + 1$ is the smallest number such that there is a factor-critical graph $p_{\mathbf{a}}(G)$ with support V. In particular for $k \gg 0$ the function $\operatorname{ir}_{I_G}(k)$ coincides with a polynomial with rational coefficients $\operatorname{Ir}_{I_G}(k)$ of degree $\sharp V - 1$.

Proof. (i) If G is bipartite then our claim follows immediately from Corollary 5.12.

(ii) Now we assume that G contains an odd circuit. By Lemma 6.2, there exists a replicated graph $S = p_{\mathbf{a}}(G)$ which is factor-critical and with support V, hence $\mathfrak{m}^{\mathbf{a}+\mathbf{1}_V}$ is an irreducible component of $I_G^{\nu(S)+1}$. Let $\Lambda = (\lambda_e) \in \mathbb{N}^E$, $k_\Lambda = \sum_{e\in E} \lambda_e$ and set $P_\Lambda = \prod_{e\in E, \lambda_e \neq 0} e^{\lambda_e} \in [[R]]$. It follows from Lemma 4.7 that $p_{\mathbf{a}+\sum_{e\in E}\lambda_e \mathbf{1}_e}(G)$ is factor-critical with $\nu(p_{\mathbf{a}+\sum_{e\in E}\lambda_e \mathbf{1}_e}(G)) = \nu(S) + k_\Lambda$. Hence $\mathfrak{m}^{\mathbf{a}+(\sum_{e\in E}\lambda_e \mathbf{1}_e)+\mathbf{1}_V}$ is an irreducible component of $I_G^{\nu(S)+k_\Lambda+1}$. In terms of corner elements it is equivalent to say that $P_\Lambda \mathbf{x}^{\mathbf{a}}$ is a corner element of $I_G^{\nu(S)+k_\Lambda+1} + \mathfrak{m}^{(m+1)\mathbf{1}_V}$ where m is an integer bigger or equal than $\nu(S) + k_\Lambda + 1$. Two vectors $\Lambda, \Gamma \in \mathbb{N}^E$ give the same irreducible component of $I_G^{\nu(S)+k_\Lambda+1}$ if and only if $P_\Lambda \mathbf{x}^{\mathbf{a}}, P_\Gamma \mathbf{x}^{\mathbf{a}}$ give the same corner element of $I_G^{\nu(S)+k_\Lambda+1} + \mathfrak{m}^{(m+1)\mathbf{1}_V}$, that is $P_\Lambda \mathbf{x}^{\mathbf{a}} = P_\Gamma \mathbf{x}^{\mathbf{a}}$ which is equivalent to $P_\Lambda = P_\Gamma$. It is equivalent to say that $\prod_e Y_e^{\lambda_e} - \prod_e Y_e^{\gamma_e} \in I(G)$ where $I(G)$ is the toric edge ideal. It follows that the number of irreducible components of the type $\mathfrak{m}^{\mathbf{a}+(\sum_{e\in E}\lambda_e \mathbf{1}_e)+\mathbf{1}_V}$ in $I_G^{\nu(S)+k_\Lambda+1}$ is given by $H_{K[G]}(k_\Lambda)$, where $H_{K[G]}$ is the Hilbert function of the ring $K[G]$. Hence $\operatorname{ir}_{I_G}(\nu(S) + k_\Lambda + 1)$ is bounded below by $H_{K[G]}(k_\Lambda)$. From one side we know that for k big enough $H_{K[G]}(k)$ coincides with a polynomial of degree $\dim K[G] - 1 = \sharp V - 1$. From another side by Theorem 6.10 for k big enough we have $\operatorname{ir}_{I_G}(k)$ coincides with a polynomial of degree at most $\sharp V - 1$. Our claim is over. □

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