

Residual intersections in Cohen–Macaulay rings

Christine Cumming

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

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Abstract

Residual intersection generalizes the notion of linkage. The central questions in residual intersection theory are: for a residual intersection J of I , when is R/J Cohen–Macaulay, and what is the canonical module for R/J ? Our main result gives answers to the two central questions in the case where R is merely Cohen–Macaulay.

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1. Introduction

We study when residual intersections are Cohen–Macaulay. Let R be a local Noetherian ring. An ideal J of R is an *s-residual intersection* of I if there exists an s -generated ideal $\mathfrak{a} \subset I$ such that $J = \mathfrak{a} : I$ and $\text{ht } J \geq s$ (see [1]). This is a generalization of the notion of linkage. Two ideals I and J are *linked* when $I = \mathfrak{a} : J$, $J = \mathfrak{a} : I$, and $\mathfrak{a}$ is generated by a regular sequence. Some common examples of residual intersections are ideals of maximal minors [8] and defining ideals of extended Rees rings of ideals of linear type and more generally of minimal reductions [8,13].

Peskine and Szpiro gave conditions for when a link of a given ideal is Cohen–Macaulay [10], but it is not clear when residual intersections of a given ideal are Cohen–Macaulay. Huneke defined strongly Cohen–Macaulay ideals to determine when residual intersections are Cohen–Macaulay [8]. Herzog, Vasconcelos, and Villarreal used ideals that satisfy sliding depth to

E-mail address: ccumming@math.purdue.edu.

URL: <http://www.math.purdue.edu/~ccumming>.

generalize some of Huneke's work [6]. Huneke and Ulrich generalized the statements further using linkage theory and Gorensteinness [9]. In 1994, Ulrich used conditions on the depths of the powers of an ideal to determine when residual intersections are Cohen–Macaulay in a Gorenstein ring [12]. We prove similar results without assuming these strong conditions on I or on R . We show in our main theorem (4.2) that if R is Cohen–Macaulay with canonical module ω , I satisfies the local condition G_s (which is defined later), and $\text{depth } \omega/I^j \omega \geq \dim R/I - j + 1$ for $1 \leq j \leq s - \text{ht } I + 1$, then every s -residual intersection J of I is Cohen–Macaulay.

The next question is if J is a residual intersection of I , what is the canonical module of R/J ? Huneke and Ulrich studied this question when R is Gorenstein [9,12]. Integral to these proofs is the knowledge that the canonical module of R is isomorphic to R . Previously, this question was not studied in the case where R is only Cohen–Macaulay. We show in the main theorem (4.2) that the canonical module of R/J has an “expected form.”

Many proofs involve adaptations of the corresponding earlier ones, but serious difficulties arise when we consider a Cohen–Macaulay ring or, more generally, a finite Cohen–Macaulay module.

We start by reviewing the sections. In the second section, we study the properties of an ideal I such that $M/\mathfrak{a}M :_M I$ satisfies Serre's condition S_2 where $\mathfrak{a} : I$ is a residual intersection of I and M is a finite Cohen–Macaulay module. This builds up to the main theorem in the section. Note an s -residual intersection J of I is a *geometric s -residual intersection* of I if $\text{ht } I + J \geq s + 1$. This is a simplified version of that theorem.

Theorem 2.8. *Let R be a local Cohen–Macaulay ring with canonical module ω , and let I be an R -ideal with height g satisfying G_s where s is an integer such that $g \leq s$. Suppose that $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ and $\text{Ext}_R^{g+j}(\omega/I^j \omega, \omega) = 0$ for $1 \leq j \leq s - g + 1$. If $J = \mathfrak{a} : I$ is a geometric s -residual intersection of I , then R/J and $\omega/\mathfrak{a}\omega :_\omega I$ satisfy Serre's condition S_2 .*

In the third section, we review the properties of M -strongly Cohen–Macaulay ideals. If ω is the canonical module of a local Cohen–Macaulay ring, then a result of Sanders says that I is strongly Cohen–Macaulay if and only if I is ω -strongly Cohen–Macaulay [11]. We use this fact to generalize work by Avramov and Herzog [2]. By assuming only that the ring is Cohen–Macaulay with a canonical module, instead of Gorenstein, that the difference between the minimal number of generators and the height of I is at most two, and $\omega/I\omega$ is Cohen–Macaulay, we show that I is strongly Cohen–Macaulay.

In the fourth section, we use the methods developed earlier to prove the main theorem and its applications. The main theorem generalizes the result of Ulrich outlined above. If the ring is Gorenstein, it recovers this result. The following is a simplified version of the main theorem:

Theorem 4.2. *Let R be a local Cohen–Macaulay ring with canonical module ω , and let I be an R -ideal with height g satisfying G_s where s is an integer such that $g \leq s$. Suppose that $\text{depth } \omega/I^j \omega \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g + 1$ and $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $1 \leq j \leq s - g - 1$. If $J = \mathfrak{a} : I$ is an i -residual intersection of I for $g \leq i \leq s$, then*

- (a) R/J is Cohen–Macaulay,
- (b) $\omega_{R/J} \cong I^{i-g+1} \omega / \mathfrak{a} I^{i-g} \omega$.

2. M - s -residually S_2

Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring of dimension d , M be a finite Cohen–Macaulay R -module, I be an R -ideal of height g , and s be an integer that often satisfies $s \geq g$. Recall the ideal J is an s -residual intersection of I if there exists an s -generated ideal $\mathfrak{a} \subset I$ where $J = \mathfrak{a} : I$ and $\text{ht } \mathfrak{a} : I \geq s$. Similarly, the ideal J is a *geometric s -residual intersection* of I if J is an s -residual intersection of I and $\text{ht } I + J \geq s + 1$. The ideal I satisfies G_s if $\mu(I_p) \leq \dim R_p$ for every $p \in V(I)$ with $\dim R_p \leq s - 1$. Also, I satisfies G_∞ if I satisfies G_s for every s .

An example of a residual intersection is a link of an ideal. Huneke shows that determinantal ideals and defining ideals of $R[Jt, t^{-1}]$ where I is of linear type are a just few examples of geometric residual intersections [8]. Given the assumptions of our main theorem, for any minimal reduction J of I , one can show the ideal $J : I$ is a residual intersection of I which is rarely geometric, and one can show the defining ideal of $R[Jt, t^{-1}]$ is a residual intersection of (I, U) with U a variable (by a similar proof as in [13, 2.1]).

Given an $(s - 1)$ -residual intersection, we can use the following result of Artin and Nagata to systematically define subideals that are also residual intersections and some of which are geometric [12, 1.6(a)]. This allows one to build up residual intersections from smaller ones. We often use this technique to do induction on s .

Remark 2.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring, let s be an integer, let $\mathfrak{a} \subset I$ be R -ideals with $\text{ht } \mathfrak{a} : I \geq s \geq \mu(\mathfrak{a})$, and assume that I satisfies G_s . Then there exists a generating sequence $\underline{a} = a_1, \dots, a_s$ of $\mathfrak{a}$ such that, with $\mathfrak{a}_i = (a_1, \dots, a_i)$ and $J_i = \mathfrak{a}_i : I$,

- (a) $\text{ht } J_i \geq i$ if $0 \leq i \leq s$ and
- (b) $\text{ht } I + J_i \geq i + 1$ if $0 \leq i \leq s - 1$.

Recall if R is a Noetherian local ring and M is a finite R -module, M satisfies Serre's condition S_k if $\text{depth } M_p \geq \min\{\dim M_p, k\}$ for every $p \in \text{Supp}(M)$.

Definition 2.2. The ideal I is (weakly) M - s -residually S_2 if for every i such that $g \leq i \leq s$ and every (geometric) i -residual intersection $\mathfrak{a} : I$ of I , $M/\mathfrak{a}M :_M I$ satisfies Serre's condition S_2 .

Definition 2.3. The ideal I satisfies M - AN_s (or M - AN_s^-) if for every $g \leq i \leq s$ and every (geometric) i -residual intersection $J = \mathfrak{a} : I$ of I , $M/\mathfrak{a}M :_M I$ is Cohen–Macaulay.

For each of these definitions if M is not specified, let $M = R$ and say I is (weakly) s -residually S_2 and I satisfies AN_s (or AN_s^-), respectively. More about these definitions can be found in [3] and [12].

2.1. “Good” properties

Ideals that are weakly M - s -residually S_2 have some nice properties that will be used in later proofs.

The next proposition will be used primarily in the case when I is M -($s - 1$)-residually S_2 . It outlines some of the good residual properties mentioned above and is a generalization of [12, 1.7] (these properties are recovered by setting $M = R$).

Proposition 2.4. Let R be a local Cohen–Macaulay ring of dimension d , I an R -ideal with height g satisfying G_s for some $s \geq g$, and M a finite Cohen–Macaulay R -module with $\text{Supp}(M) = \text{Spec}(R)$. Let $\mathfrak{a} : I$ be an s -residual intersection and $J_i = \mathfrak{a}_i : I$ as in Remark 2.1. Let $K_i = \mathfrak{a}_i M :_M I$ for all i , and let “ $-$ ” denote the images in M/K_i . Assume M/K_i is S_2 for $g \leq i \leq s-1$. Then for $0 \leq i \leq s$,

- (a) $\overline{M}$ is unmixed of dimension $d-i$.
- (b) The associated primes of the R -module $M/\mathfrak{a}_i M$ have height at most i .
- (c) $\mathfrak{a}_i M = IM \cap (\mathfrak{a}_i M :_M I)$ if $0 \leq i \leq s-1$ or if $i = s$ and $\mathfrak{a} : I$ is a geometric s -residual intersection.

Assume M/K_i is S_2 for $g \leq i \leq s-2$. Then for $0 \leq i \leq s-1$,

- (d) $\overline{\mathfrak{a}_{i+1} M :_M I} = \overline{\mathfrak{a}_{i+1} M} :_{\overline{M}} I$ and $\mathfrak{a}_{i+1}$ is $\overline{M}$ -regular.
- (e) $\mathfrak{a}_i M :_M I = \mathfrak{a}_i M :_M (\mathfrak{a}_{i+1})$.

Proof. After adjoining a variable to R , M , and $\mathfrak{a}$ and localizing, we may suppose that $g \geq 1$. We proceed by induction on i , $0 \leq i \leq s$. If $0 \leq i \leq g-1$, the claims are trivial. Assume the claim holds for i such that $g-1 \leq i \leq s-1$. We want to show the assertion for $i+1$.

(a) Part (d) for i implies $\text{Ass}(\frac{M}{K_{i+1}}) \subseteq \text{Ass}(\frac{\overline{M}}{\mathfrak{a}_{i+1} \overline{M}})$ and $\mathfrak{a}_{i+1}$ is $\overline{M}$ -regular. Take $p \in \text{Ass}(\frac{\overline{M}}{\mathfrak{a}_{i+1} \overline{M}})$. Thus $\text{depth } \overline{M}_p = 1$. Since $\overline{M}$ is S_2 , $\dim \overline{M}_p = 1$. By the inductive hypothesis, the height of an associated prime of $\overline{M}$ is i . Therefore, $\text{ht } p = i+1$ since R is Cohen–Macaulay. Therefore, M/K_{i+1} is unmixed. Since all minimal primes of $\text{Supp}(\frac{M}{K_{i+1}})$ have height $i+1$, $\dim M/K_{i+1} = d - (i+1)$.

(b) By (e) for i , we have $K_i = \mathfrak{a}_i M :_M (\mathfrak{a}_{i+1})$. That gives the exact sequence

$$0 \rightarrow \frac{M}{K_i} \xrightarrow{\cdot \mathfrak{a}_{i+1}} \frac{M}{\mathfrak{a}_i M} \rightarrow \frac{M}{\mathfrak{a}_{i+1} M} \rightarrow 0. \quad (2.1)$$

Let $p \in \text{Spec}(R)$ such that $\text{ht } p \geq i+2$. By the inductive hypothesis, $p \notin \text{Ass}(\frac{M}{\mathfrak{a}_i M})$. Thus, $\text{depth}(M/\mathfrak{a}_i M)_p \geq 1$. Since M/K_i is unmixed of dimension $d-i$ by (a) for i , we have $\text{ht } p - \text{ht } \text{Ann}(M/K_i) \geq 2$. Using the fact that R is Cohen–Macaulay, $\dim(M/K_i)_p \geq 2$. Since M/K_i is S_2 , $\text{depth}(M/K_i)_p \geq 2$. By the depth lemma, $\text{depth}(M/\mathfrak{a}_{i+1} M)_p \geq 1$. Therefore, $p \notin \text{Ass}(\frac{M}{\mathfrak{a}_{i+1} M})$.

(c) It suffices to show $\mathfrak{a}_{i+1} M \supseteq IM \cap (\mathfrak{a}_{i+1} M :_M I)$ locally at the associated primes of $M/\mathfrak{a}_{i+1} M$. Take $p \in \text{Ass}(\frac{M}{\mathfrak{a}_{i+1} M})$. By (b) for $i+1$, $\text{ht } p \leq i+1$. Remark 2.1 implies $\text{ht } J_{i+1} \geq i+1$ for $i+1 \leq s$ and $\text{ht}(I + J_{i+1}) \geq i+2$ for $i+1 \leq s-1$. If $i+1 = s$, then by assumption J_{i+1} is a geometric $(i+1)$ -residual intersection of I and $\text{ht}(I + J_{i+1}) \geq i+2$ by definition. In both cases, we have either $J_{i+1} \not\subseteq p$ or $I \not\subseteq p$. If $J_{i+1} \not\subseteq p$, then $(\mathfrak{a}_{i+1})_p = I_p$ and $(K_{i+1})_p = M_p$. Hence $(IM \cap K_{i+1})_p = (\mathfrak{a}_{i+1} M)_p \cap M_p = (\mathfrak{a}_{i+1} M)_p$. If $I \not\subseteq p$, then $I_p = R_p$ and $(K_{i+1})_p = (\mathfrak{a}_{i+1} M)_p$. Thus $(IM \cap K_{i+1})_p = M_p \cap (\mathfrak{a}_{i+1} M)_p = (\mathfrak{a}_{i+1} M)_p$.

(d) Note $K_{i+2} = (\mathfrak{a}_{i+2} + \mathfrak{a}_{i+1})M :_M I = (\mathfrak{a}_{i+2}M + \mathfrak{a}_{i+1}M) :_M I$. By part (c) for $i+1$, we have $K_{i+2} = [\mathfrak{a}_{i+2}M + (IM \cap K_{i+1})] :_M I = (\mathfrak{a}_{i+2}M + K_{i+1}) :_M I$. Hence $\frac{K_{i+2}}{K_{i+1}} = \frac{\mathfrak{a}_{i+2}M + K_{i+1}}{K_{i+1}} : \frac{M}{K_{i+1}} I$.

To show a_{i+2} is $\frac{M}{K_{i+1}}$ -regular, it is enough to show $a_{i+2} \notin p$ for all $p \in \text{Ass}(\frac{M}{K_{i+1}})$. Let $p \in \text{Ass}(\frac{M}{K_{i+1}})$. By (a) for $i+1$, $\text{ht } p = i+1$. Since $i+1 \leq s-1$, Remark 2.1 gives us that $\text{ht } J_{i+1} \geq i+1$, $\text{ht } J_{i+2} \geq i+2$, and $\text{ht}(I + J_{i+1}) \geq i+2$. Since $\text{Ass}(\frac{M}{\mathfrak{a}_{i+1}M:M I}) \subset \text{Supp}(\frac{M}{\mathfrak{a}_{i+1}M:M I}) = V(\text{Ann}(\frac{M}{\mathfrak{a}_{i+1}M:M I}))$, we have $\mathfrak{a}_{i+1} : I \subseteq p$. So, we have $J_{i+2} \not\subseteq p$ and $I \not\subseteq p$. Since $\mathfrak{a}_{i+1} \subseteq p$, this means that $a_{i+2} \notin p$.

(e) It is enough to prove $\mathfrak{a}_{i+1}M :_M I \supseteq \mathfrak{a}_{i+1}M :_M (a_{i+2})$. We may show this at the associated primes of M/K_{i+1} . Take $p \in \text{Ass}(\frac{M}{K_{i+1}})$. By (a) for $i+1$, we have $\text{ht } p = i+1$. As in the proof of part (d), we can show $J_{i+1} \subseteq p$, $I \not\subseteq p$, and $a_{i+2} \notin p$. Thus $(\mathfrak{a}_{i+1}M :_M (a_{i+2}))_p = (\mathfrak{a}_{i+1}M)_p :_{M_p} R_p = (K_{i+1})_p$. Therefore, $K_{i+1} = \mathfrak{a}_{i+1}M :_M (a_{i+2})$. $\square$

Recall the definition of a d -sequence. A sequence of elements $x_1, \dots, x_n$ in R is a d -sequence with respect to M if (1) $x_1, \dots, x_n$ is a minimal generating set of the ideal $I = (x)$ and (2) $(x_1, \dots, x_i)M :_M x_{i+1}x_k = (x_1, \dots, x_i)M :_M x_k$, for $0 \leq i \leq n-1$ and $k \geq i+1$. These sequences are critical in the proof of the following lemma which is similar to [12, 2.7]. This lemma is primarily needed for the proof of Theorem 4.2. It appears here, because concepts from this proof will be used in the proof of Theorem 2.8.

Lemma 2.5. *Let R be a local Cohen–Macaulay ring of dimension d , M a Cohen–Macaulay R -module with $\text{Supp}(M) = \text{Spec}(R)$, I an R -ideal with height $g > 0$, and $s \geq g$. Assume I satisfies G_s , is weakly $(s-2)$ -residually S_2 , and is weakly M -($s-2$)-residually S_2 . Assume $\text{depth } M/I^j M \geq d-g-j+1$ for $1 \leq j \leq s-g+1$. Let $\mathfrak{a} : I$ be an s -residual intersection with $\mathfrak{a} = (a_1, \dots, a_s)$. For $1 \leq i \leq s$, let $J_i = \mathfrak{a}_i : I$ with $\mathfrak{a}_i = (a_1, \dots, a_i)$ as defined in Remark 2.1. For $1 \leq i \leq s$, consider the complexes*

$$C_{ij}: 0 \rightarrow I^{j-1}M/\mathfrak{a}_{i-1}I^{j-2}M \xrightarrow{\varphi} I^jM/\mathfrak{a}_{i-1}I^{j-1}M \rightarrow I^jM/\mathfrak{a}_iI^{j-1}M \rightarrow 0,$$

where φ is induced by multiplication with a_i . Then:

- (a) The complexes C_{ij} are exact whenever $1 \leq i \leq s$ and $i-g+2 \leq j \leq s-g+2$.
- (b) $\text{depth } I^jM/\mathfrak{a}_iI^{j-1}M \geq \min\{d-i, d-g-j+2\}$ whenever $0 \leq i \leq s$ and $i-g+1 \leq j \leq s-g+1$.
- (c) $I^jM/\mathfrak{a}_iI^{j-1}M \cong [I^jM + (\mathfrak{a}_iM :_M I)]/(\mathfrak{a}_iM :_M I)$ whenever $0 \leq i \leq s-1$ and $i-g+1 \leq j \leq s-g+1$, or whenever $i = s$, $j = s-g+1$, and J_s is a geometric s -residual intersection.

Proof. Let $K_i = \mathfrak{a}_iM :_M I$ for all i . We proceed by induction on i . Let $i = 0$. Then (a) is vacuous. For (b), recall $\text{depth } M = d$. That takes care of the case when $j \leq 0$. So we may assume $j \geq 1$. Then we have the exact sequence

$$0 \rightarrow I^jM \rightarrow M \rightarrow M/I^jM \rightarrow 0.$$

By the depth lemma, $\text{depth } I^jM \geq \min\{d, (d-g-j+1)+1\}$. For (c), we only need to consider $g = 0$ and $s > 0$. By Proposition 2.4(c), it suffices to show that $I^jM \cap K_0 = 0$ for $j \geq 1$. We may show this locally at the associated primes of M since $I^jM \cap K_0 \subseteq M$. Since $\text{Ass}(M) \subseteq \text{Min}(R)$, we may localize at minimal primes $\mathfrak{p}$ of R . Since $s > 0$ and the ideal I satisfies G_1 , $I_{\mathfrak{p}}$ is either the zero ideal or the unit ideal. If $I_{\mathfrak{p}} = 0$, then $(I^jM)_{\mathfrak{p}} = 0$ since $j \geq 1$. If $I_{\mathfrak{p}} = R_{\mathfrak{p}}$, then $(K_0)_{\mathfrak{p}} = 0$.

Now let us assume the claims are true for i . If $j \leq 0$, then $i + 1 < g$. Thus $\mathfrak{a}_{i+1}$ is minimally generated by a regular sequence on R and also therefore on M . This implies $\mathfrak{a}_{i+1}M :_M I = \mathfrak{a}_{i+1}M$. Then (a), (b), and (c) are trivially true. So we may assume $j \geq 1$.

For (a), it suffices to show that φ is injective. There is a commutative diagram where φ and ψ are induced by multiplication by a_{i+1} :

$$\begin{array}{ccc} \frac{I^{j-1}M}{\mathfrak{a}_i I^{j-2}M} & \xrightarrow{\varphi} & \frac{I^j M}{\mathfrak{a}_i I^{j-1}M} \\ \downarrow \pi & & \downarrow \\ \frac{(I^{j-1}M + K_i)}{K_i} & \xrightarrow{\psi} & \frac{(I^j M + K_i)}{K_i}. \end{array}$$

By the inductive hypothesis for (c), π is an isomorphism. Proposition 2.4(d) gives that a_{i+1} is M/K_i -regular. Therefore, ψ is injective, and then φ is injective. Thus $C_{i+1,j}$ is exact.

For (b), we have the following exact sequence by part (a) for $i + 1$,

$$0 \rightarrow I^{j-1}M/\mathfrak{a}_i I^{j-2}M \xrightarrow{\varphi} I^j M/\mathfrak{a}_i I^{j-1}M \rightarrow I^j M/\mathfrak{a}_{i+1} I^{j-1}M \rightarrow 0.$$

Hence the inductive hypothesis for (b) gives

$$\text{depth } I^j M/\mathfrak{a}_{i+1} I^{j-1}M \geq \min\{d - (i + 1), d - g - j + 2\}.$$

For (c), we first prove that for every $j \geq 1$ and $0 \leq k \leq s - 1$

$$\mathfrak{a}_k \mathfrak{a}^{j-1}M = \mathfrak{a}^j M \cap (\mathfrak{a}_k M :_M I). \quad (2.2)$$

Let $S = R \ltimes M = \frac{S(M)}{(S_+(M))^2}$ where $S(M)$ is the symmetric algebra of M . We know that

$$\frac{S}{(\mathfrak{a}_k S :_S a_{k+1})} = \frac{R \oplus M}{\mathfrak{a}_k : (a_{k+1}) \oplus \mathfrak{a}_k M :_M (a_{k+1})} = \frac{R \oplus M}{J_k \oplus K_k}$$

by Proposition 2.4(e) for the modules M and R . Therefore, $\frac{S}{(\mathfrak{a}_k S :_S a_{k+1})}$ is S_2 . Proposition 2.4(d), (c), and (e) (applied with $M = S$) implies $\underline{a} = a_1, \dots, a_s$ is a d -sequence on S . Hence the images $a_{k+1}, \dots, a_s$ form a d -sequence in $S/\mathfrak{a}_k S$ [7, p. 252]. Let $X = (a_{k+1}, \dots, a_s)$. So, $\mathfrak{a} = \mathfrak{a}_k + X$. This implies

$$\mathfrak{a}^j M \cap \mathfrak{a}_k M = (\mathfrak{a}_k + X)^j M \cap \mathfrak{a}_k M = \mathfrak{a}_k \mathfrak{a}^{j-1}M + (X^j M \cap \mathfrak{a}_k M).$$

By [7, Theorem 2.1], $X^j S \cap \mathfrak{a}_k S \subseteq \mathfrak{a}_k X^{j-1}S$. This gives

$$X^j M \cap \mathfrak{a}_k M = [X^j S \cap \mathfrak{a}_k S]_1 \subseteq [\mathfrak{a}_k X^{j-1}S]_1 = \mathfrak{a}_k X^{j-1}M.$$

Therefore,

$$\mathfrak{a}^j M \cap \mathfrak{a}_k M = \mathfrak{a}_k \mathfrak{a}^{j-1}M + (X^j M \cap \mathfrak{a}_k M) \subseteq \mathfrak{a}_k \mathfrak{a}^{j-1}M + \mathfrak{a}_k X^{j-1}M.$$

Since $X \subset \mathfrak{a}$, this gives

$$\mathfrak{a}^j M \cap \mathfrak{a}_k M \subseteq \mathfrak{a}_k \mathfrak{a}^{j-1} M \subseteq \mathfrak{a}^j M \cap \mathfrak{a}_k M,$$

where the second inclusion is trivial. Therefore,

$$\mathfrak{a}_k \mathfrak{a}^{j-1} M = \mathfrak{a}^j M \cap \mathfrak{a}_k M. \quad (2.3)$$

Using Proposition 2.4(c),

$$\mathfrak{a}^j M \cap \mathfrak{a}_k M = \mathfrak{a}^j M \cap IM \cap (\mathfrak{a}_k M :_M I) = \mathfrak{a}^j M \cap (\mathfrak{a}_k M :_M I)$$

since $j > 0$. Therefore, (2.2) holds.

Consider the natural epimorphism $\pi : I^j M / \mathfrak{a}_{i+1} I^{j-1} M \rightarrow (I^j M + K_{i+1}) / K_{i+1}$. Let $A = I^j M / \mathfrak{a}_{i+1} I^{j-1} M$ and $B = (I^j M + K_{i+1}) / K_{i+1}$. It suffices to show π is an isomorphism locally at the associated primes of A . Let p be an associated prime of A . By (b) for $i+1$, $\text{ht } p \leq \max\{i+1, g+j-2\}$. If $i+1 = s$ and $J_{i+1} = J_s$ is a geometric residual intersection, then $\text{ht } p \leq s$. Otherwise when $i+1 < s$, then $\text{ht } p \leq s-1$.

If $I \not\subseteq p$, $I_p = R_p$ and π_p is an isomorphism. If $I \subseteq p$, then either $\text{ht } p \leq s-1$ or $\text{ht } p \leq s$ and J_s is a geometric s -residual intersection of I . Both cases imply $J_{i+1} \not\subseteq p$, $I_p = \mathfrak{a}_p$, and $(K_{i+1})_p = M_p$. We have $A_p = (\frac{\mathfrak{a}^j M}{\mathfrak{a}_{i+1} \mathfrak{a}^{j-1} M})_p$. If $i+1 \leq s-1$, then (2.2) implies

$$A_p = \left(\frac{\mathfrak{a}^j M}{\mathfrak{a}^j M \cap K_{i+1}} \right)_p \cong \left(\frac{\mathfrak{a}^j M + K_{i+1}}{K_{i+1}} \right)_p = B_p.$$

If $i+1 = s$, then

$$A_p = \left(\frac{\mathfrak{a}^j M}{\mathfrak{a}^j M} \right)_p = 0 = \left(\frac{\mathfrak{a}^j M + K_s}{K_s} \right)_p = B_p.$$

Therefore, π_p is an isomorphism. $\square$

2.2. How to obtain this property

The next lemma is helpful in showing some of the isomorphisms in Theorem 2.8 below.

Lemma 2.6. *Let R be a Noetherian ring, M a finite R -module satisfying S_2 , N a finite R -module such that $\dim M_p \leq 1$ for every $p \in \text{Ass}(N)$, and $\varphi : M \rightarrow N$. Then φ is an isomorphism if and only if φ is an isomorphism locally in codimension one in $\text{Supp}(M)$.*

This proof is left to the reader using the S_1 and S_2 properties.

Note 2.7. If $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring of dimension d with canonical module ω and I is an R -ideal with height g , then by local duality

- (a) $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0 \Leftrightarrow H_{\mathfrak{m}}^{d-g-j}(R/I^j) = 0$,
- (b) $\text{Ext}_R^{g+j}(\omega/I^j \omega, \omega) = 0 \Leftrightarrow H_{\mathfrak{m}}^{d-g-j}(\omega/I^j \omega) = 0$.

In the next theorem, we prove a sufficient condition for the properties weakly s -residually S_2 and weakly ω - s -residually S_2 to hold. This is the main theorem of this section; it generalizes [3, 4.1] where the case of Gorenstein rings was treated. Parts of this proof parallel that of [3, 4.1], but for the sake of readability, we discuss some of these arguments along with the new ones.

Theorem 2.8. *Let R be a local Cohen–Macaulay ring with canonical module ω , and let I be an R -ideal with height g satisfying G_s for some $s \geq g$ and $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ and $\text{Ext}_R^{g+j}(\omega/I^j\omega, \omega) = 0$ both for $1 \leq j \leq s - g$.*

- (i) *If $\text{Ext}_R^{g+j}(\omega/I^j\omega, \omega) = 0$ for $j = s - g + 1$, then I is weakly s -residually S_2 and weakly ω -($s - 1$)-residually S_2 .*
- (ii) *If $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $j = s - g + 1$, then I is weakly ($s - 1$)-residually S_2 and weakly ω - s -residually S_2 .*

Proof. Let $\alpha : I$ be any geometric s -residual intersection of I . As in Remark 2.1, let $J_i = \alpha_i : I$, and write $R_i = R/J_i$. For all i , let $K_i = \alpha_i \omega :_\omega I$ and $\omega_i = \omega/K_i = \frac{\omega}{\alpha_i \omega :_\omega I}$.

The theorem is a consequence of the following assertions, which we shall prove by induction on i . For the proof of part (i), let $r = s$ and $t = s - 1$. For part (ii), let $r = s - 1$ and $t = s$. After making these assumptions, the proof of each part is the same.

- (a) R_i satisfies S_2 for $0 \leq i \leq r$;
- (a') ω_i satisfies S_2 for $0 \leq i \leq t$;
- (b) $\text{Ext}_R^{i+1}(I^{i-g+2}R_i, \omega) = 0$ for $0 \leq i \leq t - 1$;
- (b') $\text{Ext}_R^{i+1}(I^{i-g+2}\omega_i, \omega) = 0$ for $0 \leq i \leq r - 1$;
- (c) $\omega_{R_i} \cong (I^{i-g+1}\omega_i)^{\vee\vee}$ for $0 \leq i \leq s - 1$, where $-\vee = \text{Hom}(-, \omega_{R_i})$. Note that this notation implicitly uses the value of i .

If $s > 0$, then

$$I^j\omega_0 \cong I^j\omega \quad \text{for all } j \geq 1. \quad (2.4)$$

This is equivalent to $I^j\omega \cap (0 :_\omega I) = 0$ for all $j \geq 1$. Since ω is unmixed, we need to prove this after localizing at minimal primes of R . Since I satisfies G_1 , the equality holds. Similarly, if $s > 0$, then

$$I^jR_0 \cong I^j \quad \text{for all } j \geq 1. \quad (2.5)$$

(See [3, p. 208] for more details.)

Next, we prove the above five assertions by induction on i . Let $i = 0$. In this case, our assertions are only nontrivial if $g = 0$ (i.e. $R \neq R_0$). Let $g = 0$. Notice that we need only check (a) if we are proving part (ii) and that it suffices to check (a') in the proof of part (i). Write $-' = \text{Hom}(-, \omega)$. Note $0 : I \cong \text{Hom}(R/I, R) \cong \text{Hom}(R/I, \text{Hom}_R(\omega, \omega)) = \text{Hom}(\omega/I\omega, \omega) = (\omega/I\omega)'$. Consider the exact sequence

$$0 \rightarrow I\omega \rightarrow \omega \rightarrow \omega/I\omega \rightarrow 0.$$

Applying $-'$ and using the assumption $\text{Ext}_R^1(\omega/I\omega, \omega) = 0$ yields the exact sequence

$$0 \rightarrow (\omega/I\omega)' \rightarrow (\omega)' \rightarrow (I\omega)' \rightarrow 0.$$

Rewriting this sequence we obtain

$$0 \rightarrow 0: I \rightarrow R \rightarrow (I\omega)' \rightarrow 0.$$

Therefore, $(I\omega)' \cong R/(0:I) = R_0$. Since $(I\omega)'$ is always S_2 , R_0 is S_2 .

Similarly, note $0:_{\omega} I \cong \text{Hom}(R/I, \omega)$. Since $\text{Ext}_R^1(R/I, \omega) = 0$, apply $-'$ to a free R -resolution of R/I which yields an exact sequence. This implies $\omega_0 = \frac{\omega}{0:_{\omega} I}$ satisfies S_2 . (See [3, p. 208] for more details.)

For the remaining parts, note that $s > 0$. Statement (2.5) which says $I^2 R_0 \cong I^2$ gives (b). Statement (2.4) implies $I^2 \omega_0 \cong I^2 \omega$ which gives (b'). We know $\omega_{R_0} \cong \text{Hom}_R(R_0, \omega) \cong \text{Hom}((I\omega)', \omega)$ as $R_0 \cong (I\omega)'$. By (2.4), $I\omega \cong I\omega_0$. Using Hom-tensor adjointness, $\omega_{R_0} \cong \text{Hom}([\text{Hom}(I\omega_0 \otimes_{R_0} R_0, \omega)] \otimes_{R_0} R_0, \omega) \cong \text{Hom}(\text{Hom}(I\omega_0, \text{Hom}(R_0, \omega)), \text{Hom}(R_0, \omega)) \cong \text{Hom}(\text{Hom}(I\omega_0, \omega_{R_0}), \omega_{R_0}) = (I\omega_0)^{\vee\vee}$; therefore, we have (c).

Now, we check the inductive step. Assume the assertions are true for $i \geq 0$. We show they hold for $i + 1$.

For (a) and (a'), we may suppose $i \leq s - 2$. Notice that in the proof of part (i), we may also assume $i \leq s - 1$ for (a) and that in the proof of part (ii), we may suppose additionally $i \leq s - 1$ for (a'). By the inductive hypothesis, I is weakly i -residually S_2 and weakly ω - i -residually S_2 . Thus, R_i and ω_i are S_2 . By Proposition 2.4(a) and (d) applied with $M = R$ and $M = \omega$, respectively, R_i and ω_i are unmixed of codimension i in R , a_{i+1} is regular on R_i and on ω_i , $\text{depth}_I(R_i) > 0$, $\text{depth}_I(\omega_i) > 0$, $J_{i+1}R_i = a_{i+1}R_i :_{R_i} IR_i \cong \text{Hom}_{R_i}(IR_i, a_{i+1}R_i) \cong \text{Hom}_{R_i}(IR_i, R_i)$, and $K_{i+1}/K_i = a_{i+1}\omega_i :_{\omega_i} I R_i \cong \text{Hom}_{R_i}(IR_i, a_{i+1}\omega_i)$ which yields

$$K_{i+1}/K_i \cong \text{Hom}_{R_i}(IR_i, \omega_i). \quad (2.6)$$

We start by proving (a). Since R_i is S_2 , $J_{i+1}R_i \cong \text{Hom}_{R_i}(IR_i, \text{Hom}_{R_i}(\omega_{R_i}, \omega_{R_i}))$. By Hom-tensor adjointness and the fact that $\text{depth}_I(\omega_{R_i}) > 0$, we have that $J_{i+1}R_i \cong \text{Hom}_{R_i}(I\omega_{R_i}, \omega_{R_i})$. Hence by (c) for i , $J_{i+1}R_i \cong \text{Hom}_{R_i}(I(I^{i-g+1}\omega_i)^{\vee\vee}, \omega_{R_i})$. By Lemma 2.6, it suffices to show the natural map $(I(I^{i-g+1}\omega_i)^{\vee\vee})^{\vee} \rightarrow (I^{i-g+2}\omega_i)^{\vee}$ is an isomorphism locally in codimension one in R_i . But this is clear since ω_i is torsionfree over R_i by (a') and Proposition 2.4(a), and, therefore, $I^{i-g+1}\omega_i \xrightarrow{\sim} (I^{i-g+1}\omega_i)^{\vee\vee}$ in codimension one. This yields $J_{i+1}R_i \cong \text{Hom}_{R_i}(I^{i-g+2}\omega_i, \omega_{R_i})$. By [3, 4.8], $J_{i+1}R_i \cong \text{Ext}_R^i(I^{i-g+2}\omega_i, \omega)$. By (b') for i , we have $\text{Ext}_R^{i+1}(I^{i-g+2}\omega_i, \omega) = 0$. Since ω_i has codimension i by Proposition 2.4(a), $\text{Ext}_R^{\ell}(I^{i-g+2}\omega_i, \omega) = 0$ whenever $\ell \leq i - 1$. Dualizing a free R -resolution of $I^{i-g+2}\omega_i$ into ω , we see that the R_i -module $\text{Ext}_R^i(I^{i-g+2}\omega_i, \omega)$ satisfies S_3 . Thus $J_{i+1}R_i$ satisfies S_3 . Then R_{i+1} is S_2 . This proves (a) for $i + 1$.

Now we investigate the following claim that we will use to prove (a'),

$$\omega_i \cong \text{Hom}_{R_i}(I^{i-g+1}R_i, (I^{i-g+1}\omega_i)^{\vee\vee}). \quad (2.7)$$

To prove (2.7), let $\psi: \omega_i \rightarrow \text{Hom}_{R_i}(I^{i-g+1}R_i, (I^{i-g+1}\omega_i)^{\vee\vee})$ be the map which to each element $w \in \omega_i$ associates multiplication by w followed by the natural map $M \rightarrow M^{\vee\vee}$. By the inductive hypothesis, ω_i is S_2 . By Lemma 2.6, it is enough to check that ψ is an isomorphism locally

in codimension one in R_i . Take any $p \in \text{Spec}(R)$ such that $\text{ht } p = i + 1$ and $\mathfrak{a}_i : I \subset p$. By Remark 2.1, $\text{ht}(J_{i+1} + I) \geq i + 2$. So either $I \not\subset p$ or $J_{i+1} \not\subset p$. Let $q_i = pR_i$. If $I \not\subset p$, then $I_p = R_p$ and $(IR_i)_{q_i} = (R_i)_{q_i}$. Therefore, ψ_{q_i} is an isomorphism since $(\omega_i^{\vee\vee})_{q_i} \cong (\omega_i)_{q_i}$ naturally. If $J_{i+1} \not\subset p$, then $I_p = (\mathfrak{a}_{i+1})_p$ and $(IR_i)_{q_i} = (\mathfrak{a}_{i+1}R_i)_{q_i} = (y)$ where y is a nonzerodivisor on $(R_i)_{q_i}$. Then $\text{Hom}_{R_i}(I^{i-g+1}R_i, (I^{i-g+1}\omega_i)^{\vee\vee})_{q_i} = \text{Hom}_{(R_i)_{q_i}}((y)^{i-g+1}, (y)^{i-g+1}(\omega_i)_{q_i}^{\vee\vee}) \cong \text{Hom}_{(R_i)_{q_i}}((R_i)_{q_i}, (\omega_i)_{q_i}^{\vee\vee})$. Hence again ψ_{q_i} is an isomorphism. Thus (2.7) holds.

By (2.6) and (2.7), $K_{i+1}/K_i \cong \text{Hom}_{R_i}(IR_i, \text{Hom}_{R_i}(I^{i-g+1}R_i, (I^{i-g+1}\omega_i)^{\vee\vee}))$. By applying Hom-tensor adjointness,

$$K_{i+1}/K_i \cong \text{Hom}_{R_i}(IR_i \otimes_{R_i} I^{i-g+1}R_i, (I^{i-g+1}\omega_i)^{\vee\vee}).$$

Since $\text{depth}_I(R_i) > 0$, the kernel of the map $IR_i \otimes_{R_i} I^{i-g+1}R_i \rightarrow I^{i-g+2}R_i$ is annihilated by a nonzerodivisor on $(I^{i-g+1}\omega_i)^{\vee\vee}$. So, applying $\text{Hom}(-, (I^{i-g+1}\omega_i)^{\vee\vee})$ takes care of the torsion, or more concretely,

$$K_{i+1}/K_i \cong \text{Hom}_{R_i}(I^{i-g+2}R_i, (I^{i-g+1}\omega_i)^{\vee\vee}).$$

We have $K_{i+1}/K_i \cong \text{Hom}_{R_i}(I^{i-g+2}R_i, \omega_{R_i})$ by (c) for i . By [3, 4.8], $K_{i+1}/K_i \cong \text{Ext}_R^i(I^{i-g+2}R_i, \omega)$. By (b) for i , $\text{Ext}_R^{i+1}(I^{i-g+2}R_i, \omega) = 0$. Since R_i has codimension i , $\text{Ext}_R^\ell(I^{i-g+2}R_i, \omega) = 0$ whenever $\ell \leq i - 1$. Dualizing a free R -resolution of $I^{i-g+2}R_i$ into ω , we see that the R_i -module $\text{Ext}_R^i(I^{i-g+2}R_i, \omega)$ satisfies S_3 . Thus K_{i+1}/K_i satisfies S_3 . Since ω_i is S_2 , ω_{i+1} is S_2 . This proves (a') for $i + 1$.

For (b), (b'), and (c), we may suppose that $i + 1 \leq s - 1$. The proof of part (i) only requires $i + 1 \leq s - 2$ for (b), and we merely need $i + 1 \leq s - 2$ for (b') in the proof of part (ii). Using Proposition 2.4 applied with $M = R$ as below, we can show for $1 \leq k \leq i + 1$ and $j \leq s - g + 1$, there are complexes

$$C_{kj}: 0 \rightarrow I^{j-1}R_{k-1} \xrightarrow{a_k} I^jR_{k-1} \rightarrow I^jR_k \rightarrow 0 \quad (2.8)$$

having nontrivial homology at most in the middle. Call this homology H , and note

$$H = \frac{J_k \cap I^j}{(J_{k-1} \cap I^j) + a_k I^{j-1}}.$$

We can show that

$$\text{Ext}_R^\ell(H, \omega) = 0 \quad \text{whenever } \ell \leq \min\{g + j - 1, i + 1\} \quad (2.9)$$

or equivalently $H_p = 0$ whenever $\dim R_p \leq \min\{g + j - 1, i + 1\}$. Since this proof is parallel to the one below, we will only show the latter. For this proof, use [7, Theorem 2.1] instead of (2.3). For more details, see [3, p. 210].

By Proposition 2.4(d), (c), and (e) applied with $M = \omega$, a_k is regular on ω_{k-1} whenever $1 \leq k \leq i + 1$ and $a_1, \dots, a_{i+1}$ is a d -sequence with respect to ω . Thus for $1 \leq k \leq i + 1$ and $j \leq s - g + 1$, there are complexes

$$D_{kj}: 0 \rightarrow I^{j-1}\omega_{k-1} \xrightarrow{a_k} I^j\omega_{k-1} \rightarrow I^j\omega_k \rightarrow 0 \quad (2.10)$$

having nontrivial homology at most in the middle. Call the homology $H^\dagger$, and note

$$H^\dagger = \frac{K_k \cap I^j \omega}{(K_{k-1} \cap I^j \omega) + a_k I^{j-1} \omega}.$$

We will show that

$$\text{Ext}_R^\ell(H^\dagger, \omega) = 0 \quad \text{whenever } \ell \leq \min\{g + j - 1, i + 1\} \quad (2.11)$$

or equivalently $(H^\dagger)_\mathfrak{p} = 0$ whenever $\dim R_\mathfrak{p} \leq \min\{g + j - 1, i + 1\}$.

To see this, let $\mathfrak{p} \in \text{Spec}(R)$ with $\dim R_\mathfrak{p} \leq \min\{g + j - 1, i + 1\}$. If $I \not\subseteq \mathfrak{p}$, then $I^{j-1} \not\subseteq \mathfrak{p}$, $I^j \not\subseteq \mathfrak{p}$, and $(K_k)_\mathfrak{p} = (\alpha_k \omega)_\mathfrak{p}$. Hence $(H^\dagger)_\mathfrak{p} = 0$. If $I \subset \mathfrak{p}$, then $j \geq 1$ since $g \leq \dim R_\mathfrak{p} \leq g + j - 1$. Also $I_\mathfrak{p} = (\alpha_{i+1})_\mathfrak{p}$ as $\dim R_\mathfrak{p} \leq i + 1 \leq s - 1$. Since $j \geq 1$ and I is weakly ω - i -residually S_2 , Proposition 2.4(c) implies $K_{k-1} \cap I^j \omega = K_{k-1} \cap I \omega \cap I^j \omega = \alpha_{k-1} \omega \cap I^j \omega$. This gives us $(K_{k-1} \cap I^j \omega)_\mathfrak{p} = (\alpha_{k-1} \omega \cap I^j \omega)_\mathfrak{p}$. Recall, $a_1, \dots, a_k$ form a d -sequence with respect to ω . Using the fact that I is weakly ω -($i - 1$)-residually S_2 and weakly $(i - 1)$ -residually S_2 , $(K_k \cap I^j \omega)_\mathfrak{p} = (\alpha_k I^{j-1} \omega)_\mathfrak{p}$ by (2.3). Then we have $(K_k \cap I^j \omega)_\mathfrak{p} = (\alpha_{k-1} I^{j-1} \omega + a_k I^{j-1} \omega)_\mathfrak{p} \subseteq (\alpha_{k-1} \omega \cap I^j \omega + a_k I^{j-1} \omega)_\mathfrak{p} \subseteq ((K_{k-1} \cap I^j \omega) + a_k I^{j-1} \omega)_\mathfrak{p}$. Hence $(H^\dagger)_\mathfrak{p} = 0$. Thus (2.11) holds.

To prove (b), we can show by induction on k , $0 \leq k \leq i + 1$, that

$$\text{Ext}_R^{g+j-1}(I^j R_k, \omega) = 0 \quad \text{whenever } k - g + 2 \leq j \leq i - g + 3 \quad (2.12)$$

as we show (2.13). Statement (b) follows when we set $k = i + 1$. The proof parallels that of (b'), so we omit it. See [3, p. 210], for the full details.

Let us prove (b'). We show, by induction on k , $0 \leq k \leq i + 1$,

$$\text{Ext}_R^{g+j-1}(I^j \omega_k, \omega) = 0 \quad \text{whenever } k - g + 2 \leq j \leq i - g + 3. \quad (2.13)$$

Statement (b') follows when we set $k = i + 1$.

First, suppose $k = 0$. Suppose $j \leq 0$. Then $g \geq 2$. Hence $\omega_0 = \omega$ and $I^j \omega_0 = R \omega_0 = \omega$. As above, we have that $g + j - 1 \geq 1$, and $\text{Ext}_R^{g+j-1}(\omega, \omega) = 0$. Assume $j \geq 1$. We have $I^j \omega_0 \cong I^j \omega$ by (2.4). Now the assertion follows because $\text{Ext}_R^{g+j-1}(I^j \omega, \omega) = 0$ for $1 \leq j \leq s - g + 1$.

Next, suppose $1 \leq k \leq i + 1$. Assuming $k - g + 2 \leq j \leq i - g + 3$, as in the desired formula, we have $\text{Ext}_R^{g+j-2}(H^\dagger, \omega) = 0$ by (2.11), and $\text{Ext}_R^{g+j-2}(I^{j-1} \omega_{k-1}, \omega) = \text{Ext}_R^{g+j-1}(I^j \omega_{k-1}, \omega) = 0$ by the induction hypothesis. Now using the complex D_{kj} , we derive $\text{Ext}_R^{g+j-1}(I^j \omega_k, \omega) = 0$, proving (2.13) and part (b').

We prove (c) for $i + 1 \leq s - 1$. By part (a') for i and $i + 1$ and Proposition 2.4(a), we know that ω_i and ω_{i+1} satisfy S_2 and are unmixed of codimension i and $i + 1$ in R . Also by Proposition 2.4(d), we have $K_{i+1}/K_i = a_{i+1} \omega_i :_{\omega_i} I R_i$ and a_{i+1} is ω_i -regular. Write $- ' = \text{Hom}_{R_i}(-, \omega_{R_i})$ and $-^\vee = \text{Hom}_{R_{i+1}}(-, \omega_{R_{i+1}})$.

Recall the complex

$$D_{i+1, i-g+2}: \quad 0 \rightarrow I^{i-g+1} \omega_i \xrightarrow{a_{i+1}} I^{i-g+2} \omega_i \rightarrow I^{i-g+2} \omega_{i+1} \rightarrow 0$$

introduced above which has nontrivial homology $H^\dagger$ at most in the middle. By (2.11), $\text{Ext}^\ell(H^\dagger, \omega) = 0$ for $\ell \leq i + 1$, and by part (b') for i , $\text{Ext}_R^{i+1}(I^{i-g+2}\omega_i, \omega) = 0$. Also, $\text{Ext}_R^i(I^{i-g+2}\omega_{i+1}, \omega) = 0$. Thus the above complex $D_{i+1, i-g+2}$ induces an exact sequence

$$0 \rightarrow \text{Ext}_R^i(I^{i-g+2}\omega_i, \omega) \xrightarrow{\cdot a_{i+1}} \text{Ext}_R^i(I^{i-g+1}\omega_i, \omega) \rightarrow \text{Ext}_R^{i+1}(I^{i-g+2}\omega_{i+1}, \omega) \rightarrow 0$$

which can be rewritten using [3, 4.8],

$$0 \rightarrow (I^{i-g+2}\omega_i)' \xrightarrow{\cdot a_{i+1}} (I^{i-g+1}\omega_i)' \rightarrow (I^{i-g+2}\omega_{i+1})^\vee \rightarrow 0.$$

Dualizing and rewriting again, we obtain an exact sequence

$$0 \rightarrow (I^{i-g+1}\omega_i)'' \xrightarrow{\cdot a_{i+1}} (I^{i-g+2}\omega_i)'' \rightarrow (I^{i-g+2}\omega_{i+1})^{\vee\vee} \rightarrow \text{Ext}_R^{i+1}((I^{i-g+1}\omega_i)', \omega).$$

This shows that $(I^{i-g+2}\omega_i)''/a_{i+1}(I^{i-g+1}\omega_i)''$ is isomorphic to an R_{i+1} -submodule of $(I^{i-g+2}\omega_{i+1})^{\vee\vee}$ and that both modules coincide locally in codimension one in R_{i+1} , as R_i satisfies S_2 . Thus, since ω_{i+1} is S_2 ,

$$(I^{i-g+2}\omega_{i+1})^{\vee\vee} \cong ((I^{i-g+2}\omega_i)''/a_{i+1}(I^{i-g+1}\omega_i)'')^{\vee\vee}.$$

Now we must show that the latter module is isomorphic to $\omega_{R_{i+1}}$.

By (c) for i , we have $(I^{i-g+2}\omega_i)''/a_{i+1}(I^{i-g+1}\omega_i)'' \cong (I\omega_{R_i})''/a_{i+1}\omega_{R_i}$. But [3, 4.9] shows that the last module is isomorphic to $\text{Ext}_{R_i}^1(R_{i+1}, \omega_{R_i})$. Thus it remains to prove that $\text{Ext}_{R_i}^1(R_{i+1}, \omega_{R_i})^{\vee\vee} \cong \omega_{R_{i+1}}$. Now, [3, 4.8] yields a natural map $\text{Ext}_{R_i}^1(R_{i+1}, \omega_{R_i}) \rightarrow \omega_{R_{i+1}}$, which is an isomorphism locally in codimension one in R_{i+1} , because R_i satisfies S_2 . Thus, we get $\text{Ext}_{R_i}^1(R_{i+1}, \omega_{R_i})^{\vee\vee} \cong (\omega_{R_{i+1}})^{\vee\vee} \cong \omega_{R_{i+1}}$, proving (c). $\square$

Under the strong assumption that $\text{Ext}_R^{g+j}(R/I^j, \omega)$ and $\text{Ext}_R^{g+j}(\omega/I^j\omega, \omega)$ both vanish for $1 \leq j \leq s - g + 1$, then Theorem 2.8 would follow directly from [3, 4.1] by passing to the Gorenstein ring $R \ltimes \omega$.

3. M -strongly Cohen–Macaulay ideals

Assume R is a Noetherian local ring, $I = (a_1, \dots, a_n) = (\underline{a})$ is an R -ideal, and M is a finite R -module. Let $K_\bullet(\underline{a}; M)$ be the Koszul complex of the sequence $\underline{a}$ with coefficients in M . Let $B_\bullet(\underline{a}; M)$, $Z_\bullet(\underline{a}; M)$, and $H_\bullet(\underline{a}; M)$ be the boundaries, the cycles, and the homologies of $K_\bullet(\underline{a}; M)$, respectively.

Definition 3.1. The ideal $I = (\underline{a})$ is M -strongly Cohen–Macaulay if $H_i(\underline{a}; M)$ are Cohen–Macaulay for all i . When $M = R$, one says that I is *strongly Cohen–Macaulay*.

Note that the sequence $\underline{a}$ need not be a minimal generating set. If the M -strongly Cohen–Macaulay property holds for one generating set of I , it follows for all generating sets of I . Flenner, O’Carroll, and Vogel also introduce this concept [4, 7.2.9]. They show slightly different versions of Proposition 2.4(a), (b), and (c) assuming M is Cohen–Macaulay and I satisfies M -strongly Cohen–Macaulay.

This brings us to an important theorem of Sanders that will be used in the next theorem. If R is a local Cohen–Macaulay ring with canonical module ω and I is a R -ideal, then I is ω -strongly Cohen–Macaulay if and only if it is strongly Cohen–Macaulay [11, 1.6]. The next theorem was proved by Avramov and Herzog for Gorenstein rings [2, p. 259]. We show it holds for rings that are not necessarily Gorenstein.

Theorem 3.2. *Let R be a local Cohen–Macaulay ring with canonical module ω and I an R -ideal with $\mu(I) - \text{ht } I \leq 2$ such that $\omega/I\omega$ is Cohen–Macaulay. Then I is strongly Cohen–Macaulay.*

Proof. By Sanders’ result [11, 1.6], it is enough to show that I is ω -strongly Cohen–Macaulay. Let $\underline{a} = a_1, \dots, a_n$ be a generating set of I such that $n = \text{ht } I + 2$. Let $K_\bullet = K_\bullet(\underline{a}; \omega)$ and $H_\bullet = H_\bullet(\underline{a}; \omega)$. Then H_2 is isomorphic to the canonical module of R/I which is Cohen–Macaulay. Since $H_0 \cong \omega/I\omega$, this module is Cohen–Macaulay by assumption. So it suffices to prove H_1 is Cohen–Macaulay. Let $a \in R$ be regular on R and on R/I . From the short exact sequence

$$0 \rightarrow K_\bullet \xrightarrow{\cdot a} K_\bullet \rightarrow \overline{K}_\bullet \rightarrow 0$$

we obtain the long exact sequence

$$0 \rightarrow H_2 \xrightarrow{\cdot a} H_2 \rightarrow H_2(\overline{K}_\bullet) \rightarrow H_1 \xrightarrow{\cdot a} H_1 \rightarrow H_1(\overline{K}_\bullet) \rightarrow H_0 \xrightarrow{\cdot a} H_0 \rightarrow H_0(\overline{K}_\bullet) \rightarrow 0.$$

Since a is regular on R/I , $H_2/aH_2 \cong \omega_{(R/I)}/a\omega_{(R/I)} \cong \omega_{R/(I,a)} \cong H_2(\overline{K}_\bullet)$. Clearly, a is regular on $\omega/I\omega$ which tells us that the mapping $H_0 \xrightarrow{\cdot a} H_0$ is injective. This implies a is H_1 -regular, and $H_1/aH_1 \cong H_1(\overline{K}_\bullet)$. By induction on the dimension of R/I , the assertion follows. $\square$

4. The main theorem

Remark 4.1. If $(R, \mathfrak{m})$ is a local Cohen–Macaulay ring with canonical module ω and I is an R -ideal with height g , then $\text{depth } \omega/I^j\omega \geq \dim R/I - j + 1$ implies $\text{Ext}_R^{g+j}(\omega/I^j\omega, \omega) = 0$ for any integer j .

Now, we prove our main theorem. Under suitable assumptions, we show that an ideal has G_s and AN_s , and we give a formula for the canonical module of R/J where J is a residual intersection of I . This is a generalization of [12, 2.9] where Ulrich assumed that R was a local Gorenstein ring.

Theorem 4.2. *Let R be a local Cohen–Macaulay ring with canonical module ω ; let I be an R -ideal with height g ; let s be an integer; and assume that for every $p \in \mathbb{V}(I)$ with $\dim R_p \leq s - 1$, either $\mu(I_p) \leq \dim R_p$ or $\ell(I_p) \leq \dim R_p - 1$. Further suppose that $\text{depth } \omega/I^j\omega \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g + 1$ and $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $1 \leq j \leq s - g - 1$. Then*

- (a) I satisfies G_s and AN_s ;
- (b) for every $g \leq i \leq s$ and every i -residual intersection $J = \mathfrak{a} : I$ of I , $\omega_{R/J} \cong I^{i-g+1}\omega / \mathfrak{a}I^{i-g}\omega$, where $\omega_{R/J} \cong (I^{i-g+1}\omega + \mathfrak{a}\omega :_\omega I) / (\mathfrak{a}\omega :_\omega I)$ in case J is a geometric i -residual intersection.

Proof. After adjoining a variable and localizing, we may assume $g \geq 1$ and R has an infinite residue class field. Let $J = \mathfrak{a} : I$ be an s -residual intersection. Let $J_i = \mathfrak{a}_i : I$ be as in Remark 2.1. Let $K = \mathfrak{a}\omega :_\omega I$ and $K_i = \mathfrak{a}_i\omega :_\omega I$ for all i . We proceed by induction on s . Let $s = 0$. Hence I satisfies G_0 trivially. Since $g \geq 1$, I satisfies AN_0 , and (b) trivially holds.

If $s \geq 1$, we may assume (a) and (b) hold for $s - 1$. In particular, I satisfies AN_{s-1} . By [12, 1.11], I satisfies G_s .

By Remark 2.1, $J_{s-1} = \mathfrak{a}_{s-1} : I$ is a geometric $(s - 1)$ -residual intersection of I . Let “ $'$ ” denote images in $R' = R/J_{s-1}$. The inductive hypothesis implies that R' is Cohen–Macaulay with $\omega_{R'} \cong \frac{I^{s-g}\omega + K_{s-1}}{K_{s-1}}$. Note by Theorem 2.8(i), I is weakly ω -($s - 2$)-residually S_2 . Given this fact and the fact that I satisfies AN_{s-1} , we may apply Lemma 2.5. By Lemma 2.5(b) and (c), we have $\text{depth} \frac{I^{s-g}\omega + K_{s-1}}{K_{s-1}} \geq d - s + 1$ and $\text{depth} \frac{I^{s-g+1}\omega + K_{s-1}}{K_{s-1}} \geq d - s + 1$. Since $\omega_{R'}/I'\omega_{R'} \cong (\frac{I^{s-g}\omega + K_{s-1}}{K_{s-1}})/(\frac{I^{s-g+1}\omega + K_{s-1}}{K_{s-1}})$, it follows that $\text{depth} \omega_{R'}/I'\omega_{R'} \geq d - s$. As J_{s-1} is a geometric $(s - 1)$ -residual intersection of I , $\text{ht}(J_{s-1} + I) \geq s$, and hence $\dim R'/I' \leq d - s$. Now, the fact that $\text{depth} \omega_{R'}/I'\omega_{R'} \geq d - s$ forces $\omega_{R'}/I'\omega_{R'}$ to be a Cohen–Macaulay module over R'/I' . Applying Proposition 2.4(d) yields $J'_s = (a'_s) : I'$ where a'_s is R' -regular. By [12, 2.3], $R/J \cong R'/J'$ is Cohen–Macaulay. Thus I satisfies AN_s .

We also have from [12, 2.3] that

$$\omega_{R/J} \cong I'\omega_{R'}/(a'_s)\omega_{R'} = \left(\frac{I^{s-g+1}\omega + K_{s-1}}{K_{s-1}} \right) / \left(\frac{a'_s I^{s-g}\omega + K_{s-1}}{K_{s-1}} \right).$$

On the other hand, Lemma 2.5(a) yields the exact sequence

$$\mathcal{C}_{s,s-g+1}: 0 \rightarrow \frac{I^{s-g}\omega}{\mathfrak{a}_{s-1}I^{s-g-1}\omega} \xrightarrow{\cdot a'_s} \frac{I^{s-g+1}\omega}{\mathfrak{a}_{s-1}I^{s-g}\omega} \rightarrow \frac{I^{s-g+1}\omega}{\mathfrak{a}_s I^{s-g}\omega} \rightarrow 0,$$

which by Lemma 2.5(c) is

$$0 \rightarrow \frac{I^{s-g}\omega + K_{s-1}}{K_{s-1}} \xrightarrow{\cdot a'_s} \frac{I^{s-g+1}\omega + K_{s-1}}{K_{s-1}} \rightarrow \frac{I^{s-g+1}\omega}{\mathfrak{a}I^{s-g}\omega} \rightarrow 0.$$

This implies that

$$\left(\frac{I^{s-g+1}\omega + K_{s-1}}{K_{s-1}} \right) / \left(\frac{a'_s I^{s-g}\omega + K_{s-1}}{K_{s-1}} \right) \cong \frac{I^{s-g+1}\omega}{\mathfrak{a}I^{s-g}\omega}.$$

Therefore,

$$\omega_{R/J} \cong \frac{I^{s-g+1}\omega}{\mathfrak{a}I^{s-g}\omega}.$$

If J is a geometric i -residual intersection, Lemma 2.5(c) gives

$$\omega_{R/J} \cong \frac{I^{s-g+1}\omega + K}{K}. \quad \square$$

This theorem leads to two quick corollaries which give some insight into its power. By setting $s = g$, if $\omega/I\omega$ is Cohen–Macaulay, then for every link $J = \mathfrak{a} : I$ of I R/J is Cohen–Macaulay and $\omega_{R/J} \cong I\omega/\mathfrak{a}\omega$. Now setting $s = g + 1$, we can review the first nontrivial case.

Corollary 4.3. *Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring with canonical module ω and let I be a generically complete intersection R -ideal of grade g . Also suppose $\omega/I\omega$ is Cohen–Macaulay, $\text{depth } \omega/I^2\omega \geq \dim R/I - 1$, and $H_{\mathfrak{m}}^{d-g-1}(R/I) = 0$. Then for every $(g+1)$ -residual intersection $J = \mathfrak{a} : I$ of I*

- (a) R/J is Cohen–Macaulay,
- (b) $\omega_{R/J} \cong I^2\omega/\mathfrak{a}I\omega$.

Some of the assumptions of the main theorem can be replaced by other “nicer” assumptions. Note that part (b) of this remark is a generalization of [12, 2.10].

Remark 4.4. Let R be a local Cohen–Macaulay ring with canonical module ω ; let $I = (a_1, \dots, a_n)$ be an R -ideal with height g ; and let s be an integer.

- (a) Assume $\text{depth } \omega/I^j\omega \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g + 1$ and $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $1 \leq j \leq s - g - 1$. If I satisfies G_s , then the conclusions of Theorem 4.2 hold.
- (b) Assume for every $p \in V(I)$ with $\dim R_p \leq s - 1$, either $\mu(I_p) \leq \dim R_p$ or $\ell(I_p) \leq \dim R_p - 1$, and $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $1 \leq j \leq s - g - 1$. If the Koszul homology $H_j(a_1, \dots, a_n; \omega)$ are Cohen–Macaulay for $0 \leq j \leq s - g$, then the conclusions of Theorem 4.2 hold.
- (c) Assume for every $p \in V(I)$ with $\dim R_p \leq s - 1$, either $\mu(I_p) \leq \dim R_p$ or $\ell(I_p) \leq \dim R_p - 1$. If I is strongly Cohen–Macaulay, then the conclusions of Theorem 4.2 hold.

Proof. (a) See the proof of Theorem 4.2.

(b) In light of Theorem 4.2, it suffices to show $\text{depth } \omega/I^{j+1}\omega \geq d - g - j$ for $0 \leq j \leq s - g$. We may assume that R has infinite residue class field. As above, we proceed by induction on s . The case when $s = 0$ is vacuous unless $g = 0$. If $g = s = 0$, the condition is trivially satisfied.

So let $s \geq 1$ and suppose our assertion holds for $s - 1$. Then by Theorem 4.2, I satisfies AN_{s-1} . Hence by [12, 1.11], I is G_s . By [11, 1.3], the Cohen–Macaulay assumption on $H_j(a_1, \dots, a_n; \omega)$ implies for $0 \leq j \leq s - g$ that $\text{depth } H_j(a_1, \dots, a_n; \omega) \geq d - g$. Since I satisfies G_s , we can use the acyclicity lemma as like what is done in the proof of [5, 9.1] to conclude that $\mathcal{M}_j = \mathcal{M}_j(\underline{a}; \omega)$ is acyclic for $0 \leq j \leq s - g$ (see [5, p. 83] for more on the $\mathcal{M}$ -complex). By [5, 4.5, 4.6, and 4.9], $H_0(\mathcal{M}_j) \cong I^j\omega/I^{j+1}\omega$ for $0 \leq j \leq s - g$. Using the fact that $\text{depth } H_j(a_1, \dots, a_n; \omega) \geq d - g$ for $0 \leq j \leq s - g$, one can show that $\text{depth } \omega/I^{j+1}\omega \geq d - g - j$ for $0 \leq j \leq s - g$.

(c) Similarly as in the proof of part (b), one shows that the strongly Cohen–Macaulay property of I implies that $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g - 1$. Hence $\text{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for $1 \leq j \leq s - g - 1$. By Sanders’ result [11, 1.6], this implies $H_i(\underline{x}; \omega)$ is Cohen–Macaulay for all $i \geq 0$. Now use part (b). $\square$

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