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On Young modules of general linear groups

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Abstract

We study ℓ-permutation modules of finite general linear groups $\mathrm{GL}_n(q)$ acting on partial flags in the natural module, where the coefficient field of the modules has characteristic ℓ, for $\ell \nmid q$. We call the indecomposable summands of these permutation modules linear Young modules. We determine their vertices and Green correspondents, by methods relying only on the representation theory of $\mathrm{GL}_n(q)$.

Furthermore, we show that when the multiplicative order of q modulo ℓ is strictly greater than 1, the Specht modules for $\mathrm{GL}_n(q)$ in characteristic ℓ form a stratifying system. This implies in particular, that for $\mathrm{GL}_n(q)$-modules with Specht filtration, the filtration multiplicities are independent of the filtration. This is an analogue of a recent theorem by Hemmer and Nakano.

Keywords: Finite general linear groups; Young modules; Specht modules; Filtration multiplicities; Stratifying systems

1. Introduction

In the modular representation theory of finite groups, the connections between modules for a group and modules for its subgroups are the most important tools. One such tool is provided by the basic notion of projectivity relative to a subgroup: A G-module M is relatively H-projective for a subgroup H of the group G if M is a direct summand of some module induced from H. If so, this then tells us that many properties of M will be determined by the representation theory of H. To be of optimal use, H should be as small as possible, and a minimal such H is called a

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vertex of M. Any indecomposable module has a vertex, which is unique up to conjugation in the group. The Green correspondence then provides a 1–1 correspondence between the indecomposable G-modules with vertex H, and the indecomposable N-modules with vertex H, where N is any subgroup containing the normaliser of H in G. Knowing Green correspondents has many benefits, as the subgroups N usually have properties that make them easier to deal with.

Let G be the finite general linear group, $G = \mathrm{GL}_n(q)$, we study representations of G over characteristic ℓ for $\ell \nmid q$. We are interested in the permutation modules of G acting on partial flags in its natural n-dimensional module. We call an indecomposable summand of such permutation module a 'linear Young module'; these are analogous to Young modules of symmetric groups (see Section 2). In Theorem 2.2, the main result, we completely determine the vertices and Green correspondents of the linear Young modules. By doing this we recover the parametrisation of the linear Young modules which was originally given by G. James in [19,20]. The vertices turn out to be the ℓ-Sylow subgroups of the Harish-Chandra vertices of the linear Young modules [8].

Our main motivation is the connection with tilting modules of q-Schur algebras $S_q(n,n)$. The q-Schur algebras are q-analogues of the usual Schur algebras and their representations are equivalent to polynomial representations of quantum general linear groups [5]. Furthermore, it is known that $S_q(n,n)$ can be identified in a natural way with a quotient of the group algebra of $\mathrm{GL}_n(q)$. In particular, this identifies the indecomposable tilting modules with indecomposable summands of permutation modules of $\mathrm{GL}_n(q)$. Through the above identification, the vertices and correspondents described in Theorem 2.2 then give us a representation theoretic invariant of the indecomposable tilting modules of $S_q(n,n)$, see Theorem 3.2. Furthermore, the projective covers we construct for the permutation modules of $\mathrm{GL}_n(q)$ provide us with a tool to construct projective covers for tilting modules of $S_q(n,n)$, Theorem 3.3. Although the arithmetic conditions restrict the q-Schur algebras we are dealing with, all classical Schur algebras are covered.

The Specht modules of $\mathrm{GL}_n(q)$ are identified with the q-Weyl modules for the q-Schur algebra. In Section 4 we show that they share some of the striking homological properties of the q-Weyl modules. More precisely, we prove that for $\mathrm{GL}_n(q)$-modules which have a Specht filtration, the number of times a given Specht module occurs is independent of the filtration provided the multiplicative order of q modulo ℓ is strictly greater than 1. This is an analogue of a recent (surprising) theorem proved by Hemmer and Nakano [16]. Our approach, via stratifying systems, gives rise to a homological characterisation of the linear Young modules, relying only on properties of $\mathrm{GL}_n(q)$-modules, by [21]. Namely, they are the unique indecomposable modules with Specht filtration which play a role analogous to tilting modules (see 4.1, 4.3).

We stress that for our study of vertices and correspondents of the linear Young modules, we work entirely with $\mathrm{GL}_n(q)$-modules, and our tools are basic modular representation theory, similar to [13]. This distinguishes our approach from that of Dipper and Du. In [8] they determine Harish-Chandra vertices and sources of linear Young modules, by relating them to vertices and sources of indecomposable modules for Hecke algebras. These in turn are found in [6] by relating them to q-Schur algebras. Theorem (3.3.4) in [8] asserts a 1–1 correspondence on the level of Hecke algebras, without describing correspondents explicitly.

2. Indecomposable summands of permutation modules

For the convenience of the reader we recall some standard notation.

A composition λ of a natural number n is a sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. A partition λ of n is a composition of n with entries in decreasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$. The degree of the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ is defined to be $\sum_{i=1}^{r} \lambda_i$. An unordered partition is a composition where all entries are non-zero.

Let $\mathrm{GL}_n(q)$ be the finite general linear group with coefficients in the finite field $\mathbb{F}_q$. Throughout we will denote $\mathrm{GL}_n(q)$ by G unless otherwise stated. We study representations of G in non-describing characteristic. That is, let ℓ be a prime not dividing q and let $(K, \mathcal{O}, k)$ be an ℓ-modular system. We study representations over both $\mathcal{O}$ and k. Let e be minimal such that ℓ divides $q^e - 1$.

Fix the following notation throughout. Let T be the subgroup of G given by the invertible diagonal matrices. Let B be the subgroup of invertible upper triangular matrices of G and U the subgroup of upper unitriangular matrices of G such that $B = TU$. We identify $\mathfrak{S}_n$ with the subgroup of permutation matrices of G. For each composition λ of n denote by $\mathfrak{S}_\lambda$ the corresponding Young subgroup of $\mathfrak{S}_n$. The standard parabolic subgroups of G are the groups $P_\lambda = B\mathfrak{S}_\lambda B$ for λ a composition of n. Note that P_λ decomposes into $P_\lambda = L_\lambda \ltimes U_\lambda$, where U_λ is an ℓ'-subgroup of G. We set $e_{U_\lambda} = 1/|U_\lambda| \sum_{u \in U_\lambda} u$. Then $B = P_{(1^n)}$ and every parabolic subgroup P of G is conjugate to a standard parabolic subgroup P_λ and we say that P is of type λ. If $\lambda = (\lambda_1, \ldots, \lambda_h)$ then the standard Levi complement L_λ of P_λ is isomorphic to the product $\mathrm{GL}_{\lambda_1}(q) \times \cdots \times \mathrm{GL}_{\lambda_h}(q)$. We write Q_λ for a diagonally embedded ℓ-Sylow subgroup of L_λ.

For each composition λ of n, the $\mathcal{O}G$-permutation module M_λ is given by permutations of the cosets of a parabolic subgroup of type λ of G. Let μ be a composition of n obtained by rearranging the parts of λ. Then M_μ is isomorphic to M_λ (see for example [19, 14.7]). Therefore it is enough to consider the modules M_λ, where λ is a partition of n. Note that if $\bar{M}_\lambda$ (respectively $\hat{M}_\lambda$) is the permutation kG-module (respectively KG-module) given by the permutations on a coset of P_λ then $k \otimes_{\mathcal{O}} M_\lambda \cong \bar{M}_\lambda$ (respectively $K \otimes_{\mathcal{O}} M_\lambda \cong \hat{M}_\lambda$).

In [18] James parametrised the indecomposable summands of the Young permutation modules for the symmetric groups. We have the analogue of this theorem for general linear groups:

Theorem 2.1. *There is a set of indecomposable $\mathcal{O}\mathrm{GL}_n(q)$-modules Y_μ, one for each partition μ of n, such that the following holds for all partitions λ of n:*

(i) *M_λ is a direct sum of $\mathcal{O}\mathrm{GL}_n(q)$-modules, each of which is isomorphic to some Y_μ with $\mu \geqslant \lambda$, and precisely one summand is isomorphic to Y_λ.*
(ii) *If $Y_\lambda \cong Y_\mu$, then $\lambda = \mu$.*

Here $\geqslant$ denotes the dominance order of partitions of n.

Theorem 2.1 is implicitly contained in the work of James in [19,20]. However, we give a direct proof which only uses properties of general linear groups developed in what follows. In analogy to the symmetric group (see Section 2.4.1 below), we call these indecomposable summands linear Young modules.

We will identify the vertices and the Green correspondents of the linear Young modules. The result is as follows.

Theorem 2.2. *Let λ be a partition of n. The linear Young $\mathcal{O}\mathrm{GL}_n(q)$-module Y_λ is projective if and only if λ is e-restricted.*

Suppose λ is not e-restricted, and write $\lambda = \lambda(-1) + \sum_{m=0}^{s} \lambda(m) e\ell^m$ where $\lambda(-1)$ is an e-restricted partition and where for $m \geqslant 0$, the $\lambda(m)$ are ℓ-restricted partitions. For $m \geqslant -1$, let b_m be the degree of $\lambda(m)$ and let ρ be the partition of n which has b_{-1} parts equal to 1 and b_m parts equal to $e\ell^m$. Then

(i) *Q_ρ is a vertex of Y_λ, and*
(ii) *The Green correspondent of Y_λ in $kN_G(Q_\rho)$ is the inflation of the module*

$$\bar{Y}_{\lambda(-1)} \otimes \bar{Y}^{\lambda(0)} \otimes \bar{Y}^{\lambda(1)} \otimes \cdots \otimes \bar{Y}^{\lambda(s)}$$

for the quotient $k[\mathrm{GL}_{b_{-1}}(q) \times \mathfrak{S}_{b_0} \times \cdots \times \mathfrak{S}_{b_s}]$. Here $\bar{Y}^{\lambda(m)}$ denotes the Young module of $k\mathfrak{S}_{b_m}$ labelled by $\lambda(m)$.

Clearly Theorems 2.1 and 2.2 are still true over k.

Remarks. (1) In [8], Dipper and Du determine a complete set of indecomposable summands of the sum of all permutation kG-modules $\bar{M}_\lambda$ for λ a partition of n. They calculate the Harish-Chandra vertices of these indecomposable summands and show that they correspond exactly to a full set of trivial Harish-Chandra source modules. The linear Young modules $\bar{Y}_\lambda$ with Green vertices Q_ρ are precisely the indecomposable kG-modules with Harish-Chandra vertices L_ρ. Therefore the set of linear Young modules is a complete set of trivial Harish-Chandra source modules.

(2) The above two theorems are analogues of similar theorems for Young modules of symmetric groups, originally due to James [18] and Grabmeier [15]; subsequently a new proof was given in [13]. In fact, the proof we will give here, for the general linear group case, also works for symmetric groups and it improves the approach in [13]. Our counting argument makes it unnecessary to identify Young modules explicitly (which had caused errors in Lemma 3 of [13]). We are grateful to Burkhard Külshammer for suggesting the idea of counting to replace technical computations.

2.1. Projective indecomposable summands

The module $M_{(1^n)}$ has been much studied, for example in [3,17,19,24] and elsewhere. Very important is the classical fact going back to Iwahori [17] that the $\mathcal{O}G$-endomorphism ring of $M_{(1^n)}$ is isomorphic to the Iwahori–Hecke algebra $\mathcal{H}_q(n)$ associated to the symmetric group $\mathfrak{S}_n$. This fact contains sufficient information to parametrise the indecomposable projective summands of M_λ for arbitrary λ.

Lemma 2.3. *The total number of indecomposable projective summands of M_λ up to isomorphism, as λ varies through all partitions of n, is equal to the number of e-restricted partitions of n.*

Proof. Take an indecomposable summand Z of M_λ. There is a surjective $\mathcal{O}G$-module homomorphism from $M_{(1^n)}$ onto M_λ since each standard parabolic subgroup contains B. Composing this homomorphism with the epimorphism from M_λ onto Z gives a surjective module homomorphism from $M_{(1^n)}$ onto Z. This surjection splits if Z is projective.

Assume first that $e > 1$. Then every indecomposable summand of $M_{(1^n)}$ is projective, and the total number of indecomposable summands is equal to the number of simple modules of the endomorphism algebra $\mathrm{End}_{\mathcal{O}G}(M_{(1^n)})$ which is isomorphic to the Hecke algebra $\mathcal{H}_q(n)$. The irreducible representations over k of this algebra are in 1–1 correspondence with the e-restricted partitions of n. (See [9].) Furthermore, the projective indecomposable modules over kG are in bijection with the projective indecomposable modules over $\mathcal{O}G$ and the result follows.

Suppose now that $e = 1$. Then $M_{(1^n)}$ does not have any projective summands. For example, the centre of G acts trivially on $M_{(1^n)}$ and it has a subgroup of order divisible by ℓ. Therefore the restriction of $M_{(1^n)}$ to the centre of G does not have a projective summand and thus $M_{(1^n)}$ does not have a projective summand. On the other hand there are no 1-restricted partitions except for the empty partition, and hence the statement holds as well. □

Lemma 2.3 can be used to parametrise the indecomposable projective summands of M_λ as λ varies through all partitions of n, via the Hecke algebra $\mathcal{H}_q(n)$. We will not go into details of how this works since we will get a consistent parametrisation for all indecomposable summands of the modules M_λ in 2.4.1.

To study the non-projective indecomposable summands, we will use the more general theory of permutation modules, as it was presented by Broué.

2.2. *Broué correspondence*

For an arbitrary finite group G, and $\mathcal{O}$ a complete discrete valuation ring with residue field of characteristic ℓ and maximal ideal $\mathfrak{p}$, an $\mathcal{O}G$-module M is said to be an ℓ-permutation module if for any ℓ-subgroup P of G there is a P-invariant basis $\mathcal{B}$ of M. The indecomposable ℓ-permutation modules are, up to isomorphism, precisely the indecomposable summands of transitive permutation modules.

In [2], a main result is the parametrisation of indecomposable ℓ-permutation modules by using the Brauer construction. This is defined as follows. Let P be a (non-trivial) ℓ-subgroup of G, and let $\bar{N} := N_G(P)/P$. Then we have a functor

$$-(P) : \mathcal{O}G\text{-mod} \to k\bar{N}\text{-mod}$$

which takes an $\mathcal{O}G$-module V to a $k\bar{N}$-module $V(P)$, where

$$V(P) := V^P \Big/ \left(\sum_{Q<P} Tr_Q^P\big(V^Q\big) + \mathfrak{p}V^P \right).$$

Here V^R are the fixed points under the action of R, and Tr_Q^P is the linear map from V^Q to V^P defined by $Tr_Q^P(v) = \sum_i g_i v$, the sum taken over some transversal for the cosets of Q in P.

Now suppose that V is an ℓ-permutation $\mathcal{O}G$-module. Then $V(P)$ can be identified with the k-span of the fixed points of P in $\mathcal{B}$ (see for example [25, Proposition 27.6]). We will apply the following, which was suggested by Puig and published in [2].

Theorem 2.4. [2] *Let P be an ℓ-subgroup of G. The functor $-(P)$ induces a* 1-1 *correspondence between the isomorphism classes of*

(i) *indecomposable ℓ-permutation $\mathcal{O}G$-modules with vertex P and*
(ii) *indecomposable projective $k\bar{N}$-modules.*

In particular, if M is any ℓ-permutation module and if we have a direct sum decomposition $M = \bigoplus Y_i$ with all Y_i indecomposable, then Y_i has vertex P if and only if $Y_i(P)$ is indecomposable projective as a module for $k\bar{N}$. Suppose Y is any indecomposable ℓ-permutation $\mathcal{O}G$-module with vertex P, then $Y \cong Y'$ if and only if $Y(P) \cong Y'(P)$. We remark that $Y(P)$ (as a module for $kN_G(P)$) is isomorphic to the Green correspondent in $kN_G(P)$ of Y.

In [13] it is shown that given a permutation module M, not all ℓ-subgroups can occur as vertices of summands of M:

Lemma 2.5. [13] *Suppose M is any ℓ-permutation $\mathcal{O}G$-module. If R is an ℓ-subgroup of G and Q is a proper subgroup of R such that $M(R) = M(Q)$, then M does not have an indecomposable summand with vertex Q.*

2.3. The structure of $M_\lambda(Q)$ for some non-trivial vertex Q

Now let $G = \mathrm{GL}_n(q)$, and let ℓ and e be as before. For a fixed partition λ of n, we consider the Brauer quotient $M_\lambda(Q)$ of the ℓ-permutation $\mathcal{O}G$-module M_λ, where Q is some non-trivial ℓ-subgroup of G. First we show that if Q is a vertex of some indecomposable summand of M_λ then Q must be an ℓ-Sylow subgroup of some Levi subgroup L_ρ of G where all parts of the partition ρ are equal to 1 or of the form $e\ell^m$ for some $m \geqslant 0$. Furthermore, we determine a direct sum decomposition of $M_\lambda(Q)$ as a module for $N_G(Q)/Q$.

To do so, it is convenient to view M_λ as the permutation module on all flags of type λ. A flag of type λ is a filtration of the natural module for G, that is a filtration of the vector space $V = \mathbb{F}_q^n$ by vector subspaces W_i of dimension $\lambda_1 + \cdots + \lambda_i$. Such a flag is fixed by Q if and only if each term is an $\mathbb{F}_q Q$-submodule of V.

Since ℓ does not divide q we know that V is semi-simple as a module for Q over $\mathbb{F}_q$. So we write

$$V = \bigoplus_{m=1}^{s} V_m$$

where the V_m are simple $\mathbb{F}_q Q$-modules. Suppose V_m has dimension n_m, without loss of generality, we can suppose $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s \geqslant 1$.

With respect to a basis consisting of bases of the V_m we can write every $g \in Q$ as a block diagonal matrix. If we replace Q by a conjugate then we can assume that the basis we chose is the standard basis of $\mathbb{F}_q^n$ and then Q is a product of block diagonal ℓ-subgroups.

We have a Levi subgroup associated to Q. Let $\rho = (n_1, n_2, \ldots, n_s)$, then Q is contained in the Levi subgroup $L_\rho = \prod_{m=1}^{s} \mathrm{GL}_{n_m}(q)$. Call L_ρ the associated Levi subgroup. The modules V_m are the natural $\mathbb{F}_q$-modules for $\mathrm{GL}_{n_m}(q)$, and hence they are also irreducible as modules for L_ρ. Note that if V_l and V_m have dimension > 1 then they are not isomorphic as $\mathbb{F}_q Q$-modules or as $\mathbb{F}_q L_\rho$-modules, since, for example, their kernels are different with respect to the action of Q or L_ρ. Furthermore, as submodules of V the V_m are unique, for $m \geqslant 1$.

Following Weir's notation in [26] we denote by G_0 an ℓ-Sylow subgroup of $\mathrm{GL}_e(q)$. Define recursively the groups $G_m \cong G_{m-1} \wr C_\ell$, where C_ℓ denotes a cyclic group of order ℓ. An ℓ-Sylow subgroup of $\mathrm{GL}_{e\ell^m}(q)$ is then isomorphic to G_m.

Lemma 2.6. *Suppose M_λ has a summand with non-trivial vertex Q. Then Q is an ℓ-Sylow subgroup of L_ρ. In particular, all parts of ρ are equal to* 1 *or of the form $e\ell^m$ for some $m \geqslant 0$.*

Proof. There is some ℓ-Sylow subgroup R of L_ρ containing Q. Assume for a contradiction that $Q \neq R$. We claim that $M_\lambda(Q) = M_\lambda(R)$ and this is a contradiction by Lemma 2.5.

Clearly $M_\lambda(R) \subseteq M_\lambda(Q)$. To see the converse, take $\mathcal{F}$ to be a flag of type λ fixed by Q. Each term of $\mathcal{F}$ is an $\mathbb{F}_q Q$-module and hence is a direct sum of some the V_m. But each of these is also an $\mathbb{F}_q L_\rho$-module and therefore an $\mathbb{F}_q R$-module. So the terms of the flag are $\mathbb{F}_q R$-modules, that is they are fixed by R and thus $\mathcal{F}$ belongs to $M_\lambda(R)$.

This implies now that Q is a direct product of groups isomorphic to G_m, where G_m is a Sylow subgroup of $\mathrm{GL}_{e\ell^m}(q)$ as defined above. In particular, the simple summands of V as $\mathbb{F}_q Q$-modules have dimensions $e\ell^m$. Therefore the partition ρ is as stated. □

2.3.1. The group N_ρ

Suppose ρ has b_{-1} parts equal to 1 and b_m parts equal to $e\ell^m$, for $0 \leqslant m \leqslant s$ and let Q be an ℓ-Sylow subgroup of L_ρ. We must understand the structure of $M_\lambda(Q)$ as a module for the factor group $N_G(Q)/Q$.

Lemma 2.7. *There is a normal subgroup N_1 of $N_G(Q)$ containing Q which acts trivially on $M_\lambda(Q)$, and such that the quotient $N_\rho := N_G(Q)/N_1$ is the direct product of $\mathrm{GL}_{b_{-1}}(q)$ with the Young subgroup $\mathfrak{S}_{b_0} \times \cdots \times \mathfrak{S}_{b_s}$.*

Proof. Take an element $g \in N_G(Q)$, then for each $\mathbb{F}_q Q$-submodule W of V, the shifted module gW is also an $\mathbb{F}_q Q$-submodule. Furthermore, W is irreducible if and only if gW is irreducible. That is, $N_G(Q)$ acts on the direct sum decomposition of V into irreducible $\mathbb{F}_q Q$-modules. Any two irreducible summands of dimension strictly greater than 1 are non-isomorphic as modules of Q. Hence $N_G(Q)$ permutes the summands of a fixed dimension strictly greater than 1 amongst each other. By writing down matrices one sees that all permutations can occur.

The action of $N_G(Q)$ also leaves the direct sum of all one-dimensional summands invariant, and on these it induces a base change, which is an element in $\mathrm{GL}_{b_{-1}}(q)$. All possible base changes occur. We have therefore a surjective group homomorphism $\pi : N_G(Q) \to \mathrm{GL}_{b_{-1}}(q) \times \prod_{m=0}^{s} \mathfrak{S}_{b_m}$. Let N_1 be the kernel of this homomorphism, then N_1 is a normal subgroup, and N_1 is given by all $h \in N_G(Q)$ such that $hV_m = V_m$ for all m with $\dim_{\mathbb{F}_q} V_m > 1$ and such that h acts trivially on any trivial summand of V. That is, the elements of N_1 act trivially on each flag fixed by Q. □

Lemma 2.8. *In order to be able to apply the Broué correspondence, it is sufficient to identify and label the indecomposable direct summands of $M_\lambda(Q)$ as a module for kN_ρ.*

Proof. We have the following well-known more general fact. Let H be a normal subgroup of a finite group R whose order is not divisible by ℓ, the characteristic of k. If X is an indecomposable kR-module on which H acts trivially then X is projective as kR-module if and only if it is projective as kR/H-module. We apply this with $H = N_1/Q$ and $R = N_G(Q)/Q$, and then $R/H = N_\rho$. □

Let $Q = Q_\rho$ be a Sylow subgroup of L_ρ, as before, where ρ is arbitrary but with all parts of the form 1 or $e\ell^m$ for some $m \geqslant 0$. Recall that V is completely reducible as an $\mathbb{F}_q Q$-module. We wish to specify a labelling of the direct sum decomposition of V as a module for Q. If $Q = \prod_{m=0}^{s} G_m^{b_m} \times 1^{b_{-1}}$ then V is a direct sum of irreducible $\mathbb{F}_q Q_\rho$-submodules $L(m)_j$ such that for all $0 \leqslant m \leqslant s$ and $1 \leqslant j, j' \leqslant b_m$ the following holds:

(i) the modules $L(m)_j$ are isomorphic to $\mathbb{F}_q^{e\ell^m}$ as $\mathbb{F}_q$-vector spaces and
(ii) the modules $L(m)_j$ and $L(m)_{j'}$ are isomorphic as $\mathbb{F}_q Q$-modules if and only if $j = j'$.

However, for all $1 \leqslant j, j' \leqslant b_{-1}$, the modules $L(-1)_j$ and $L(-1)_{j'}$ are isomorphic as $\mathbb{F}_q Q$-modules.

We will now view $M_\lambda(Q)$ as a module for the normaliser quotient N_ρ. Say $N_\rho = \mathrm{GL}_{b_{-1}}(q) \times \mathfrak{S}_{b_0} \times \cdots \times \mathfrak{S}_{b_s}$.

For each unordered partition $\alpha(m)$ of b_m we have the Young permutation module $\bar{M}^{\alpha(m)}$ for $k\mathfrak{S}_{b_m}$, and the outer tensor product $\bar{M}_{\alpha(-1)} \otimes \bar{M}^{\alpha(0)} \otimes \bar{M}^{\alpha(1)} \otimes \cdots \otimes \bar{M}^{\alpha(s)}$ is then a module for kN_ρ. The kN_ρ-module $M_\lambda(Q)$ is isomorphic to a direct sum of such tensor products. More precisely

Proposition 2.9. *As a kN_ρ-module, $M_\lambda(Q)$ is isomorphic to the direct sum*

$$\bigoplus_{\tilde{\alpha}} \big(\bar{M}_{\alpha(-1)} \otimes \bar{M}^{\alpha(0)} \otimes \bar{M}^{\alpha(1)} \otimes \cdots \otimes \bar{M}^{\alpha(s)}\big),$$

where the sum is taken over all $\tilde{\alpha} := (\alpha(-1), \alpha(0), \alpha(1), \ldots, \alpha(s))$ such that $\lambda = \alpha(-1) + \sum_{m=0}^{s} \alpha(m)e\ell^m$ and $\alpha(m)$ is an unordered partition of b_m.

Proof. Let $\mathcal{F} = (0 \subset W_1 \subset \cdots \subset W_k = V)$ be a flag which is fixed by Q, so that each term in the flag is a direct sum of $\mathbb{F}_q Q$-modules as described above. Define $\alpha(m)$ by setting $\alpha(m)_i :=$ the number of $e\ell^m$-dimensional summands of W_i/W_{i-1}; and let $\alpha(-1)_i$ be the number of trivial summands of W_i/W_{i-1}. Say that $\mathcal{F}$ has type $\tilde{\alpha}$ where $\tilde{\alpha} = (\alpha(-1), \alpha(0), \ldots, \alpha(m))$. So we have $\lambda_i = \alpha(-1)_i + \sum_m \alpha(m)_i e\ell^m$ and $\lambda = \alpha(-1) + \sum_m \alpha(m)e\ell^m$; and $\alpha(m)$ is an unordered partition of b_m.

For each flag $\mathcal{F}$ as above, we write the ith term as

$$W_i = W_i(-1) \oplus \left(\bigoplus_{m \geqslant 0} W_i(m)\right)$$

where $W_i(-1)$ is the direct sum of all trivial Q-summands of W_i, and $W_i(m)$ is the direct sum of all Q-summands of dimension $e\ell^m$ of W_i. Then for each $m \geqslant -1$ we have a flag

$$\mathcal{F}(m) = \big(0 \subseteq W_1(m) \subseteq \cdots \subseteq W_s(m)\big).$$

If $\mathcal{F}$ has type $\tilde{\alpha}$ then for $m = -1$, the flag $\mathcal{F}(-1)$ is a basis vector in the permutation basis of $\bar{M}_{\alpha(-1)}$. Moreover, all basis vectors for $\bar{M}_{\alpha(-1)}$ occur. For $m \geqslant 0$, the flag $\mathcal{F}(m)$ can be thought of as a row equivalence class of an $\alpha(m)$-tableau for the symmetric group $\mathfrak{S}_{b_m}$ with the ith row corresponding to $W_i(m)/W_{i-1}(m)$. Therefore it can be considered as an element of $\bar{M}^{\alpha(m)}$. All elements in the permutation basis of $\bar{M}^{\alpha(m)}$ occur in this way.

We define a linear map $\psi : M_\lambda(Q) \to \bigoplus_{\tilde{\alpha}} \bar{M}_{\alpha(-1)} \otimes \bar{M}^{\alpha(0)} \otimes \bar{M}^{\alpha(1)} \otimes \cdots \otimes \bar{M}^{\alpha(s)}$ by

$$\psi(\mathcal{F}) := \mathcal{F}(-1) \otimes \mathcal{F}(0) \otimes \cdots \otimes \mathcal{F}(s).$$

Here $\psi(\mathcal{F})$ lies in the summand corresponding to its type. The map ψ takes a basis onto a basis and is therefore a vector space isomorphism.

We claim that it is a homomorphism of kN_ρ-modules. First, consider the action of the factor $\mathrm{GL}_{b_{-1}}(q)$ of N_ρ. Its action on flags $\mathcal{F}$ fixes all Q-summands of dimension > 1, so it fixes all $\mathcal{F}(m)$ for $m \neq -1$; and its action on the parts of flags $\mathcal{F}$ coming from trivial summands is precisely the action on flags $\mathcal{F}(-1)$, which is the natural permutation action on $\bar{M}_{\alpha(-1)}$. Similarly, the action of the factor $\mathfrak{S}_{b_m}$ of N_ρ on flags $\mathcal{F}$ fixes all Q-summands of dimension $\neq e\ell^m$; and its action on the parts of the flags $\mathcal{F}$ coming from summands $L(m)_i$ is precisely the action on flags $\mathcal{F}(m)$, and this is the natural permutation action on $\bar{M}^{\alpha(m)}$. $\square$

2.4. *Proofs of Theorems 2.2 and 2.1*

2.4.1. *On linear Young modules*

In [19], James proved several important results, which show that the kG-modules $\bar{M}_\lambda$ have properties very similar to those of Young permutation modules of symmetric groups.

The module $\bar{M}_\lambda$ has a unique submodule $\bar{S}_\lambda$ which can be defined characteristic-free and which is irreducible in characteristic zero. A version of James' Submodule Theorem holds, namely there is a non-degenerate bilinear form on $\bar{M}_\lambda$ with the property that for any submodule A of $\bar{M}_\lambda$, either $\bar{S}_\lambda \subseteq A$, or $A \subseteq (\bar{S}_\lambda)^\perp$.

This leads naturally to the definition of Young modules in this context. To begin with, write $\bar{M}_\lambda$ as a direct sum of indecomposable kG-modules. By the Submodule Theorem, there is a unique indecomposable summand of $\bar{M}_\lambda$ which contains $\bar{S}_\lambda$. We denote this summand by $\bar{Y}_\lambda$. As a direct summand of a permutation module $\bar{Y}_\lambda$ is liftable [23], and its unique lift to $\mathcal{O}G$ is the summand Y_λ of M_λ, which we called the linear Young module associated to λ in the remark following Theorem 2.1.

Furthermore, write χ_λ for the character of $\hat{S}_\lambda$. It follows from [19] that the character of the $\mathcal{O}G$-module Y_λ is a sum of χ_μ, where χ_λ occurs once and if χ_μ occurs then $\lambda \geqslant \mu$. This implies directly that the Y_λ for different λ are not isomorphic.

2.4.2. Before starting the proof of Theorem 2.1 we recall some basic facts of partitions which we will need now.

Every partition λ of n has a unique $e-\ell$-adic expansion of the form

$$\lambda = \lambda(-1) + \sum_{m=0}^{k} \lambda(m) e\ell^m$$

with $\lambda(-1)$ being an e-restricted partition and $\lambda(m)$ an ℓ-restricted partition. Hence we have a bijection between the set of all partitions of n and the set Γ of all tuples

$$\big(\lambda(-1), \lambda(0), \dots, \lambda(k)\big)$$

where $\lambda(-1)$ is an e-restricted partition and where the $\lambda(m)$ are ℓ-restricted partitions for $m \geqslant 0$ such that the sum $\lambda(-1) + \sum_{m\geqslant 0} \lambda(m) e\ell^m$ is a partition of n.

2.4.3. *Proof of Theorem 2.1*

The linear Young modules Y_μ give a set of pairwise non-isomorphic direct summands of the M_λ. To prove the theorem it suffices to show that the total number of indecomposable direct summands up to isomorphism of all M_λ is equal to the number of partitions of n.

We have already proved that the total number of projective indecomposable $\mathcal{O}G$-summands of all M_λ is equal to the number of e-restricted partitions of n.

We now count the number of non-projective indecomposable $\mathcal{O}G$-summands of all M_λ. By the Broué correspondence, this is equal to the total number of indecomposable summands of all $M_\lambda(Q_\rho)$ which are projective for kN_ρ where ρ runs through all partitions of n with all parts equal to 1 or $e\ell^m$ for non-trivial Q_ρ.

By Proposition 2.9, as a kN_ρ-module we can write $M_\lambda(Q_\rho)$ as a direct sum of modules of the form

$$\bar{M}_{\alpha(-1)} \otimes \bar{M}^{\alpha(0)} \otimes \cdots \otimes \bar{M}^{\alpha(k)}.$$

The indecomposable projective summands of this module are of the form

$$\bar{P}_{-1} \otimes \bar{Y}^{\lambda(0)} \otimes \cdots \otimes \bar{Y}^{\lambda(k)}$$

where $\bar{P}_{-1}$ is an indecomposable projective summand of $\bar{M}_{\alpha(-1)}$ and where the $\bar{Y}^{\lambda(m)}$ are Young modules for the symmetric groups appearing as factors, and where the $\lambda(m)$ are ℓ-restricted (we know this from the theory of Young modules for symmetric groups). The number of possible $\bar{P}_{-1}$ is equal to the number of e-restricted partitions of degree $|\alpha(-1)|$. The number of possible $\lambda(m)$ is the number of ℓ-restricted partitions of degree $|\alpha(m)|$ for each $0 \leqslant m \leqslant k$.

Hence the total number of non-projective indecomposable $\mathcal{O}G$-summands of all M_λ is equal to the number of elements of Γ excluding the tuples with only one term equal to $\lambda(-1)$.

But the number of excluded partitions is precisely the number of projective indecomposable summands of all M_λ. Hence the total number of indecomposable summands is equal to the number of partitions. □

Proof of Theorem 2.2. We proceed by induction on n, and for each n by induction on the dominance order of partitions.

Suppose first $1 \leqslant n \leqslant e$. The module $M_{(n)}$ is equal to the trivial module $\mathcal{O}$ whose vertex always is an ℓ-Sylow subgroup of G. Thus the vertex is equal to the trivial group when $n < e$ and it is equal to G_0 for $n = e$. And if $n = e$ then the Brauer quotient is k, and is equal to $\bar{Y}_{(1)}$ as $k\mathrm{GL}_1(q)$-module. If $\lambda < (n)$ then λ is e-restricted and Y_λ is projective as $\mathcal{O}G$-module, so there is nothing to do.

Now let $n > e$ and assume that the theorem is true for all $k < n$.

Consider first $\lambda = (n)$. The $\mathcal{O}G$-module $M_{(n)}$ is trivial and its vertex is an ℓ-Sylow subgroup of G. We can take it to be Q_ρ where the partition ρ has b_m parts equal to $e\ell^m$ for $m \geqslant 0$ and b_{-1} parts equal to 1, where $n = b_{-1} + \sum_{m \geqslant 0} b_m e\ell^m$ is the $e - \ell$-adic expansion of n (see the explanation following Lemma 2.8). This is the vertex as stated.

Now consider the Brauer quotient, this is the trivial module. Viewed as a module for kN_ρ it is equal to the tensor product of trivial modules, that is

$$\bar{Y}_{(b_{-1})} \otimes \bar{Y}^{(b_0)} \otimes \cdots \otimes \bar{Y}^{(b_s)}.$$

This proves the claim for the partition (n).

For the inductive step, take $\lambda < (n)$ which is not e-restricted, and assume the statement holds for all Y_μ where $\lambda < \mu$.

Suppose Q_ρ is as in the statement. Consider the direct sum decomposition of $M_\lambda(Q_\rho)$ as in Proposition 2.9. One of the expansions $\tilde{\alpha}$ for which there is a summand, is the $e-\ell$-adic expansion of λ. That is, with the notation as in the statement, $M_\lambda(Q_\rho)$ has a direct summand

$$\bar{M}_{\lambda(-1)} \otimes \bar{M}^{\lambda(0)} \otimes \cdots \otimes \bar{M}^{\lambda(s)}$$

as a module for kN_ρ. The degree of $\lambda(-1)$ is strictly less than n and $\lambda(-1)$ is e-restricted. Hence by induction, $\bar{M}_{\lambda(-1)}$ has a direct summand $\bar{Y}_{\lambda(-1)}$ which is projective for $k\mathrm{GL}_{b_{-1}}(q)$. Furthermore, for $m \geqslant 0$ we know that only one copy of $\bar{Y}^{\lambda(m)}$ occurs as an indecomposable direct summand of $\bar{M}^{\lambda(m)}$. Furthermore, since $\lambda(m)$ is ℓ-restricted $\bar{Y}^{\lambda(m)}$ is projective for the symmetric group $\mathfrak{S}_{b_m}$.

Let $\bar{Z} = \bar{Y}_{\lambda(-1)} \otimes \bar{Y}^{\lambda(0)} \otimes \cdots \otimes \bar{Y}^{\lambda(s)}$. Then $\bar{Z}$ is an indecomposable direct summand of $M_\lambda(Q_\rho)$ and it is projective as a module for kN_ρ.

By the Broué correspondence, this module $\bar{Z}$ corresponds to a unique indecomposable direct summand of M_λ with vertex Q_ρ. By Theorem 2.1 we know that all indecomposable direct summands of M_λ other than Y_λ are of the form Y_μ for $\mu > \lambda$. For those which are not projective we know inductively the vertex, and the correspondent, and none of these has correspondent isomorphic to $\bar{Z}$. The correspondents described in (ii) are pairwise non-isomorphic and hence the correspondent of $\bar{Z}$ must be Y_λ. This proves the statement for λ. $\square$

3. Tilting modules

3.1. Background

The q-Schur algebra $S_q(n,n)$ is quasi-hereditary, with respect to the dominance order of partitions (see [12, Appendix]). For each partition λ there is a standard module $\Delta(\lambda)$, and a costandard module $\nabla(\lambda)$ which, for the q-Schur algebra, are dual to each other through contravariant duality, therefore they are also often called q-Weyl and dual q-Weyl modules. The standard module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$ and otherwise only composition factors $L(\mu)$ satisfying $\mu < \lambda$.

Let $\mathcal{F}(\Delta)$ be the category of modules for $S_q(n,n)$ with Δ-filtration. That is, M belongs to this category if it has submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that M_i/M_{i-1} is isomorphic to some standard module, for all i. Similarly let $\mathcal{F}(\nabla)$ be the category of modules with costandard filtration.

It was proved, in [22] for arbitrary quasi-hereditary algebras, and adapted by [11], that there are only finitely many indecomposable modules which belong to both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ and that they are parametrised by highest weights. To give the labelling, $T(\lambda)$ has highest weight λ (so that λ is a partition of n), it contains $\Delta(\lambda)$ as a submodule, and its other Δ-quotients have labels $\mu < \lambda$. Any module which is a direct sum of such $T(\lambda)$'s is called a tilting module.

3.2. *Identification with* $\mathrm{GL}_n(q)$*-modules*

For the convenience of the reader we recall the identification of certain $\mathrm{GL}_n(q)$-modules with certain $S_q(n,n)$-modules from [3] and [24]. In particular, the work of [3] implicitly gives a parametrisation of the linear Young modules, see Lemma 3.1 below.

Let $M = M_{(1^n)}$. Recall that the quotient $C_{1,n} = \mathcal{O}G/\mathrm{ann}_{\mathcal{O}G}(M)$, called the cuspidal algebra in [3], is Morita equivalent to the q-Schur algebra $S_q(n,n)$. This was first proved in [24] over a field of characteristic ℓ and then in more generality and over the coefficient ring $\mathcal{O}$ in [3], see also [4]. Via the Morita equivalence, the standard module $\Delta(\lambda)$ corresponds to the Specht module $S_{\lambda'}$ labelled by the conjugate partition of λ.

The tilting module $T(\lambda)$ is identified with the linear Young module $Y_{\lambda'}$ in the following way: By [24, 4.4], $\Delta(\lambda)$ corresponds to $\bar{S}_{\lambda'}$ (note that there $\Delta(\lambda)$ is denoted W_λ). The identification of $Y_{\lambda'}$ with the tilting module $T(\lambda)$ can now be deduced from [3]:

Let ν be a composition of n and let $x_\nu \in \mathcal{H}_q(n)$ be the standard element generating the trivial module for the parabolic subalgebra $\mathcal{H}_q(\nu)$. Then in [3, 3.3.6], a module

$$\dot{\bigwedge}^{\nu}(1) = Mx_\nu$$

is defined. Furthermore, $Mx_\nu \cong M_\nu$ (this follows from [24]). By [3, p. 84], the tilting module $T(\lambda')$ of $S_q(n,n)$ corresponds via the Morita equivalence to the $C_{1,n}$-module $T(1,\lambda)$. Then 4.5b in [3] states:

Lemma 3.1. [3] *The indecomposable tilting modules of $C_{1,n}$ are precisely the indecomposable summands of $\dot{\bigwedge}^{\nu}(1)$ where ν runs through the compositions of n. Moreover $T(1,\lambda)$ occurs once in $\dot{\bigwedge}^{\lambda}(1)$ and if $T(1,\mu)$ occurs then $\mu \geqslant \lambda$.*

We have $\Delta(\lambda') \subset T(\lambda')$ and $\Delta(\lambda')$ is identified with S_λ. By definition, Y_λ is the unique indecomposable summand of M_λ that contains S_λ. Therefore, Y_λ is identified with $T(1,\lambda)$ and hence corresponds to $T(\lambda')$.

Therefore our results show that we can define new representation theoretic invariants for the tilting modules for any q-Schur algebra which occurs as a quotient of some group algebra $\mathcal{O}\mathrm{GL}_n(q)$, namely its vertices and its correspondents as described in Theorem 2.2. Thus with the above identifications, we have the following.

Theorem 3.2. *Suppose $S_q(n,n)$ is a q-Schur algebra over the complete discrete valuation ring $\mathcal{O}$ with residue field of characteristic ℓ. Suppose $q \in \mathbb{N}$ is a prime power and has multiplicative order e modulo ℓ and e divides $\ell - 1$.*

Then any indecomposable tilting module of $S_q(n,n)$ has a vertex which is an ℓ-subgroup of the general linear group $\mathrm{GL}_n(q)$ and a Green correspondent which is an indecomposable projective module for a group which is a product of a general linear group with symmetric groups.

Explicitly, the tilting module $T(\lambda')$ is identified with the linear Young module Y_λ and the vertex and the Green correspondent are as described in Theorem 2.2.

3.2.1. Arithmetic conditions

We recall for the convenience of the reader under which arithmetic conditions our results hold. Let ξ be a primitive eth root of 1 in $\mathcal{O}$. Then there is a ξ-Schur algebra $S_\xi(n,n)$ and a Hecke algebra $\mathcal{H}_\xi(n)$.

We would like to know when there is an associated finite general linear group $\mathrm{GL}_n(q)$ such that q has order e modulo ℓ.

If $\mathcal{H}_\xi(n)$ is Morita equivalent to $\mathcal{H}_q(n)$, then the algebra $S_\xi(n,n)$ is Morita equivalent to $S_q(n,n)$.

This is the situation when our results apply to tilting modules of $S_q(n,n)$.

Suppose we have such q, then q viewed as an element in the integers modulo ℓ has multiplicative order e. But the multiplicative group of the non-zero integers modulo ℓ is cyclic of order $\ell - 1$. Hence it follows that we must have e divides $\ell - 1$.

Conversely, assume e divides $\ell - 1$. Then the multiplicative group of the field with ℓ elements contains elements of order e. Let x be an integer between 1 and $\ell - 1$, such that x modulo ℓ has order e. Consider all elements of the form

$$x + \ell y, \quad y \in \mathbb{N}.$$

By Dirichlet's Theorem, there are infinitely many primes of this form. Take any prime $p = x + \ell y$, then p has multiplicative order e since $p \equiv x \pmod{\ell}$. So we can take $q = p$, and the group algebra of $\mathrm{GL}_n(q)$ has the q-Schur algebra $S_q(n,n)$ as a quotient.

Remark. In particular, our results always hold for the classical Schur algebra, that is when $e = 1$. Similarly, since e and ℓ must be coprime, ℓ must be odd for $e = 2$ and the above conditions always hold.

However, if for example $e = 3$ and $\ell = 2$, then no general linear group can realise the q-Schur algebra over characteristic ℓ where q is a primitive cube root of unity.

3.3. Projective covers of tilting modules

We will show how to construct a not necessarily minimal projective cover of the tilting modules $T(\lambda')$. Recall that $G = \mathrm{GL}_n(q)$. Define a central idempotent of KG

$$f = \sum_{\chi} e(\chi)$$

here χ runs over the set of irreducible characters of $\hat{M}_{(1^n)}$ and $e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1})g$. Let I be the annihilator of $M_{(1^n)}$ in $\mathcal{O}G$. Then the algebras $\mathcal{O}Gf$ and $\mathcal{O}G/I = C_{1,n}$ are isomorphic and it follows from [3] that $\mathcal{O}Gf$ and $S_q(n,n)$ are Morita equivalent. Define $P(\lambda')$ as the projective $S_q(n,n)$-module corresponding to projective $\mathcal{O}Gf$-module $\mathcal{O}Gfe_{U_\lambda}$ through this Morita equivalence.

Theorem 3.3. *The $S_q(n,n)$-module $P(\lambda')$ is a projective cover of the indecomposable tilting module $T(\lambda')$.*

Proof. Let

$$0 \to W \to \mathcal{O}L_\lambda \to \mathcal{O} \to 0$$

be the short exact sequence of $\mathcal{O}L_\lambda$-modules given by the projection of $\mathcal{O}L_\lambda$ onto the trivial module. Then since $R_{L_\lambda}^G = \mathcal{O}Ge_{U_\lambda} \otimes_{\mathcal{O}L_\lambda} (-)$ is an exact functor, the sequence

$$0 \to R_{L_\lambda}^G(W) \to \mathcal{O}Ge_{U_\lambda} \to R_{L_\lambda}^G(\mathcal{O}) \to 0$$

is exact. Note that $R^G_{L_\lambda}(\mathcal{O}) = M_\lambda$. All irreducible constituents of $K \otimes_{\mathcal{O}} M_\lambda$ are unipotent. Therefore $fM_\lambda = M_\lambda$ and we obtain an exact sequence

$$0 \to fR^G_{L_\lambda}(W) \to \mathcal{O}Gfe_{U_\lambda} \to M_\lambda \to 0.$$

The idempotent f acts trivially on the modules Y_λ. Thus Y_λ is an indecomposable $\mathcal{O}Gf$-module and $\mathcal{O}Gfe_{U_\lambda}$ is its projective cover. The result then follows from the Morita equivalence between $\mathcal{O}Gf$ and $S_q(n,n)$ and the identifications in 3.2. □

3.4. Further connections

3.4.1. Dipper and Du have introduced vertices of modules for Iwahori–Hecke algebras of symmetric groups [6]. A vertex of an indecomposable module of such a Hecke algebra is defined to be a minimal parabolic subgroup of the symmetric group such that the module is relatively projective with respect to the associated parabolic subalgebra.

They show that the vertices of indecomposable summands of modules $\mathcal{H}x_\lambda$ (the 'trivial source modules,' also known as q-permutation modules), are always $e-\ell$-parabolic; that is, the partition labelling the parabolic subalgebra has all parts of the form 1 or $e\ell^m$ for some $m \geqslant 0$. If Y^λ is the summand of $\mathcal{H}x_\lambda$ containing the Specht module S^λ (note that our Y^λ is denoted by X^λ in their paper) then Theorem 5.8 of [6] gives explicitly its vertex. The paper also mentions alternating source modules $\mathcal{H}x_\lambda$.

Let $S = S_q(n,n)$ be a q-Schur algebra; here q is arbitrary non-zero. Let ξ be the idempotent in S such that $\xi S \xi$ is isomorphic to the Hecke algebra $\mathcal{H}_q(n)$ (see [10]).

If M is an indecomposable S-module such that ξM is an indecomposable module for the Hecke algebra then one can associate a vertex to M, by taking the Dipper–Du vertex of ξM.

When M is indecomposable injective, or is an indecomposable tilting module then ξM is indecomposable. In the first case it is a Young module (it is Y^λ when M is the injective hull of the simple module with highest weight λ); and in the second case it is a summand of $\mathcal{H}y_\lambda$. So in the first case Theorem 5.8 of [6] describes the vertices.

3.4.2. Harish-Chandra induction and restriction for general linear groups can be related to an induction and restriction functor for the q-Schur algebra. More precisely, for a composition $\lambda = (\lambda_1, \ldots, \lambda_h)$ of n, define $S_q(\lambda,\lambda)$ to be equal to $S_q(\lambda_1,\lambda_1) \otimes \cdots \otimes S_q(\lambda_h,\lambda_h)$. Then $S_q(\lambda,\lambda)$ is Morita equivalent to the cuspidal algebra $C_{1,\lambda}$ associated to L_λ. In [3, Section 4.2] it is proved that the algebra $S_q(\lambda,\lambda)$ embeds naturally into $e_\lambda S_q(n,n)e_\lambda$ where the e_λ are idempotents which are explicitly defined. This gives rise to functors:

$$S_q(n,n)e_\lambda \otimes_{S_q(\lambda,\lambda)} - : S_q(\lambda,\lambda)\text{-mod} \to S_q(n,n)\text{-mod},$$

$$e_\lambda.- : S_q(n,n)\text{-mod} \to S_q(\lambda,\lambda)\text{-mod}.$$

Furthermore, there is a commutative diagram (this is a consequence of [3, 4.2.1] and [1, Section 5]):

$$\begin{array}{ccc}
C_{1,n}\text{-mod} & \xrightarrow{P\otimes_{C_{1,n}}-} & S_q(n,n)\text{-mod} \\
{\scriptstyle {}^*R^G_{L_\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle e_\lambda} \\
C_{1,\lambda}\text{-mod} & \xrightarrow{P_\lambda\otimes_{C_{1,\lambda}}-} & S_q(\lambda,\lambda)\text{-mod} \\
{\scriptstyle R^G_{L_\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle S_q(n,n)e_\lambda\otimes_{S_q(\lambda,\lambda)}-} \\
C_{1,n}\text{-mod} & \xrightarrow{P\otimes_{C_{1,n}}-} & S_q(n,n)\text{-mod.}
\end{array}$$

Here the horizontal arrows are Morita equivalences, given by tensoring with bimodules, as discussed in Section 3.2.

One could now define vertices and sources of modules for the q-Schur algebra using the above induction and restriction functors of the q-Schur algebra. That is, vertices should be groups corresponding to minimal q-Schur algebras such that the module is relatively projective with respect to the q-Schur subalgebras. These vertices would then correspond exactly to the vertices and sources obtained as the image through the above diagram of the Harish-Chandra vertices and sources for the general linear group defined by Dipper and Du in [7]. However, as working directly with the q-Schur algebras appears to be difficult, it is not clear that this approach would be more advantageous.

4. Stratifying systems

In [14] stratifying systems were introduced, as a generalisation of standard modules for quasi-hereditary algebras. This concept proved sufficiently strong to guarantee uniqueness of filtration multiplicities. We will show that for $e > 1$, the Specht modules for $\mathrm{GL}_n(q)$ form a stratifying system. This implies then that for $k\mathrm{GL}_n(q)$-modules with Specht filtration, filtration multiplicities are well-defined, that is they do not depend on the filtration. We will also show that when $e = 1$ and ℓ is a prime then there is always some $q \equiv 1 \pmod{\ell}$ such that the Specht modules do not form a stratifying system.

We recall the main definitions and results from [14]. Following the conventions there, we should work over an algebraically closed field (but this is not strictly necessary). Thus from now on we only consider modules over the field k of characteristic ℓ and we suppose k algebraically closed.

Definition 4.1. Suppose R is a finite-dimensional k-algebra. Given a set of R-modules $\{\Theta(1), \Theta(2), \ldots, \Theta(n)\}$ and a set of indecomposable R-modules $\{Y(1), Y(2), \ldots, Y(n)\}$, we say that $(\Theta(i), Y(i))_i$ is a *stratifying system* provided the following hold:

(1) $\mathrm{Hom}_R(\Theta(i+s), \Theta(i)) = 0$ for $s \geqslant 1$ and all i.
(2) For each i, there is a short exact sequence

$$0 \to \Theta(i) \to Y(i) \to Z(i) \to 0;$$

and $Z(i)$ is filtered by $\Theta(j)$ with $j < i$.
(3) $\mathrm{Ext}^1_R(\mathcal{F}(\Theta), Y) = 0$ where $Y = \bigoplus_{i=1}^n Y(i)$.

Here $\mathcal{F}(\Theta)$ is the category of modules which have a filtration with quotients isomorphic to some $\Theta(j)$. Then [14, Lemma 1.4], states:

Lemma 4.2. [14] *For N in $\mathcal{F}(\Theta)$, the filtration multiplicities are independent of the filtration.*

In [14] it was proved that given a set of indecomposable modules $\{\Theta(i)\}$ satisfying condition (1) and in addition

(4) $\mathrm{Ext}^1_R(\Theta(j), \Theta(i)) = 0$ for $j \geqslant i$;

then there is always a stratifying system $(\Theta(i), Y(i))$. In fact, the modules $Y(i)$ are uniquely determined by the $\Theta(i)$, up to isomorphism, see [21].

Proposition 4.3. *Suppose $e > 1$. Then the Specht modules S_λ for λ a partition of n form a stratifying system for $\mathrm{GL}_n(q)$, with respect to the opposite lexicographic order on partitions. The corresponding $Y(\lambda)$ are precisely the linear Young modules Y_λ.*

Corollary 4.4. *Suppose $e > 1$. Then for $k\mathrm{GL}_n(q)$-modules N with Specht filtration, the filtration multiplicities are well-defined.*

Proof of Proposition 4.3. Let $G = \mathrm{GL}_n(q)$. (1) For all partitions μ of n, S_μ is a submodule of M_μ. The module M_μ has a Specht filtration, and if S_λ occurs then $\lambda \geqslant \mu$, and S_μ occurs exactly once (see [19, 15.16]). We get from [19, 11.7] that if $\mathrm{Hom}_{kG}(S_\lambda, M_\mu) \neq 0$ then $\lambda \geqslant \mu$. So the Hom-condition for a stratifying system is (always) satisfied.

(2) We want to prove condition (4), that is the Ext-vanishing-condition. We have an exact sequence

$$0 \to M_\lambda \to M_{(1^n)} \to W \to 0.$$

Apply $\mathrm{Hom}_{kG}(S_\mu, -)$ and also $\mathrm{Hom}_{kG/I}(S_\mu, -)$ where I is the kernel of the action on $M_{(1^n)}$ as before. Since $e > 1$, the module $M_{(1^n)}$ is projective and gives an exact sequence

$$0 \to \mathrm{Hom}_{kG}(S_\mu, M_\lambda) \to \mathrm{Hom}_{kG}(S_\mu, M_{(1^n)}) \xrightarrow{\delta} \mathrm{Hom}_{kG}(S_\mu, W) \to \mathrm{Ext}^1_{kG}(S_\mu, M_\lambda) \to 0,$$

and there is also an exact sequence

$$0 \to \mathrm{Hom}_{kG/I}(S_\mu, M_\lambda) \to \mathrm{Hom}_{kG/I}(S_\mu, M_{(1^n)}) \xrightarrow{\bar{\delta}} \mathrm{Hom}_{kG/I}(S_\mu, W).$$

If we identify kG/I with the q-Schur algebra, then M_λ corresponds to a tilting module and S_μ corresponds to a standard module (see 3.2). Therefore $\mathrm{Ext}^1_{kG/I}(S_\mu, M_\lambda)$ vanishes and $\bar{\delta}$ is surjective.

The homomorphism spaces for kG and for kG/I are equal, so we get commutative squares where the vertical maps are identity maps. It follows then that the map δ is onto and therefore $\mathrm{Ext}^1_{kG}(S_\mu, M_\lambda) = 0$.

We now proceed by induction on the opposite lexicographic order on partitions. Let $\lambda = (n)$, then $S_\lambda = M_\lambda$ and $\mathrm{Ext}^1_{kG}(S_\lambda, S_\lambda) = 0$. Suppose $\lambda \leqslant \mu$ for some partition μ of n. The short exact sequence

$$o \to S_\lambda \to M_\lambda \to M_\lambda/S_\lambda \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Hom}_{kG}(S_\mu, S_\lambda) \to \mathrm{Hom}_{kG}(S_\mu, M_\lambda) \to \mathrm{Hom}_{kG}(S_\mu, M_\lambda/S_\lambda) \to \mathrm{Ext}^1_{kG}(S_\mu, S_\lambda) \to 0.$$

As M_λ/S_λ has a filtration by S_ν where $\lambda < \nu$, $\mathrm{Hom}_{kG}(S_\mu, M_\lambda/S_\lambda) = 0$ and thus $\mathrm{Ext}^1_{kG}(S_\mu, S_\lambda) = 0$.

The modules Y_λ have a Specht filtration and satisfy properties (2) and (3) of 4.1, this follows from the identifications described in Section 3.2. The uniqueness property established in [21] implies then the last statement of the proposition. □

The case $e = 1$. Assume now that $q \equiv 1$ modulo ℓ. We consider $G = \mathrm{GL}_2(q)$. Then the Specht modules do not form a stratifying system for kG-mod (although, of course, they do as modules for the q-Schur algebra).

(1) Suppose $\ell = 2$, and take $q = 3$. The group $G = \mathrm{GL}_2(q)$ has the subgroup $\mathrm{SL}_2(q)$ of index 2. So there is an indecomposable kG-module of dimension two with trivial composition factors. The trivial module is $S_{(2)}$ and therefore $\mathrm{Ext}^1_{kG}(S_{(2)}, S_{(2)}) \neq 0$.

(2) Now suppose $\ell > 2$. Then there is always a minimal 2-power q such that ℓ divides $q - 1$. Since $q - 1$ is odd, we have $G = \mathrm{GL}_2(q)$ is the direct product of $\mathrm{SL}_2(q)$ with the centre Z of G which is cyclic of order $q - 1$. The ℓ-Sylow subgroup P of Z is cyclic of order > 1. Then $G = P \times H$ for a subgroup H and since the trivial module of P has self-extensions, so does the trivial module of G. That is, as before $\mathrm{Ext}^1_{kG}(S_{(2)}, S_{(2)}) \neq 0$.

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