Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 316 (2007) 79–108
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Uno's invariant conjecture for Steinberg's triality groups in defining characteristic

Jianbei An [a], Frank Himstedt [b,*], Shih-chang Huang [a]

[a] *Department of Mathematics, University of Auckland, Private Bag 92019, Auckland, New Zealand*
[b] *Technische Universität München, Zentrum Mathematik—M11, Boltzmannstr. 3, 85748 Garching, Germany*

Received 10 February 2006

Available online 10 August 2007

Communicated by Meinolf Geck

Abstract

We verify Uno's invariant conjecture for Steinberg's triality groups ${}^3D_4(q)$, q a power of an odd prime p, in the defining characteristic p. Uno's invariant conjecture is a refinement of Dade's invariant conjecture.

Keywords: Modular representation; Uno's conjecture; Dade's conjecture; Steinberg's triality groups; Defining characteristic

1. Introduction

Let G be a finite group and p a prime dividing the order of G. There are several conjectures connecting the representation theory of G with the representation theory of certain p-local subgroups (i.e. the p-subgroups and their normalizers) of G. For example, it seems to be true, that if P is a Sylow p-subgroup of G, then the number of complex irreducible characters of G of degree coprime with p equals the same number for the normalizer $N_G(P)$.

This conjecture, called McKay conjecture [22], and its block-theoretic version due to Alperin [1] were generalized by various authors. In [20], Isaacs and Navarro proposed the following refinement of the McKay conjecture: If k is a residue class modulo p different from zero,

* Corresponding author.
E-mail addresses: an@math.auckland.ac.nz (J. An), himstedt@ma.tum.de (F. Himstedt), shua003@math.auckland.ac.nz (S. Huang).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.07.006

then the two numbers above should still be equal when we count only those characters having a degree in the residue classes k or $-k$.

In a series of papers [8–10], Dade developed several conjectures expressing the number of complex irreducible characters with a fixed defect in a given p-block of G in terms of an alternating sum of related values for p-blocks of certain p-local subgroups of G. In [9], Dade proved that his (projective) conjecture implies the McKay conjecture. Motivated by the Isaacs–Navarro conjecture, Uno [23] suggested a further refinement of Dade's conjecture.

In this paper, we show that Uno's invariant conjecture holds for Steinberg's simple triality groups ${}^3D_4(q)$ with q a power of an odd prime p, in the defining characteristic p. This implies that Dade's invariant conjecture is true for ${}^3D_4(q)$, q odd, in the defining characteristic. Since ${}^3D_4(q)$ has a trivial Schur multiplier and a cyclic outer automorphism group, it follows that Dade's inductive conjecture is also true for ${}^3D_4(q)$ in this case. Together with the results in [2] this completes the proof of Dade's conjecture for ${}^3D_4(q)$, q odd.

The methods we use are similar to those in [18]. By a theorem of Borel and Tits [5], the normalizers in G of radical p-chains are exactly the parabolic subgroups of G. So we count characters of these chain normalizers which are fixed by certain outer automorphisms. Our calculations are based on the character table of ${}^3D_4(q)$ in the character table library of the Maple [7] part of CHEVIE [13] and the character tables of the parabolic subgroups of ${}^3D_4(q)$ which have been computed in [16] (and which are also implemented as generic CHEVIE character tables).

This paper is organized as follows: In Section 2, we fix notation and state Dade's and Uno's invariant conjectures in detail. In Section 3, we state and prove some lemmas from elementary number theory which we use to count fixed points of certain automorphisms of ${}^3D_4(q)$. In Section 4, we compute the fixed points of the outer automorphisms of ${}^3D_4(q)$, q odd, on the irreducible characters of the triality groups and their parabolic subgroups. In Section 5, we verify Uno's invariant conjecture for ${}^3D_4(q)$, $q = p^n$ odd, in the defining characteristic p. Details on irreducible characters and conjugacy classes of the triality groups are summarized in tabular form in Appendix A.

2. Conjectures of Dade and Uno

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) := N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and by $\mathrm{Blk}(G)$ the set of p-blocks. If $H \leqslant G$, $\widetilde{B} \in \mathrm{Blk}(G)$, and d is an integer, we denote by $\mathrm{Irr}(H, \widetilde{B}, d)$ the set of characters $\chi \in \mathrm{Irr}(H)$ satisfying $\mathrm{d}(\chi) = d$ and $b(\chi)^G = \widetilde{B}$ (in the sense of Brauer), where $\mathrm{d}(\chi) = \log_p(|H|_p) - \log_p(\chi(1)_p)$ is the p-defect of χ and $b(\chi)$ is the block of H containing χ.

Given a p-subgroup chain

$$C: \quad P_0 < P_1 < \cdots < P_n$$

of G, define the length $|C| := n$, C_k: $P_0 < P_1 < \cdots < P_k$ and

$$N(C) = N_G(C) := N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n).$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and
(b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Suppose $1 \to G \to E \to \overline{E} \to 1$ is an exact sequence, so that E is an extension of G by $\overline{E}$. Then E acts on $\mathcal{R}$ by conjugation. Given $C \in \mathcal{R}$ and $\psi \in \mathrm{Irr}(N_G(C))$, let $N_E(C, \psi)$ be the stabilizer of (C, ψ) in E, and

$$N_{\overline{E}}(C, \psi) := N_E(C, \psi)/N_G(C).$$

For $\widetilde{B} \in \mathrm{Blk}(G)$, an integer $d \geqslant 0$ and $U \leqslant \overline{E}$, let $k(N_G(C), \widetilde{B}, d, U)$ be the number of characters in the set

$$\mathrm{Irr}\big(N_G(C), \widetilde{B}, d, U\big) := \big\{\psi \in \mathrm{Irr}\big(N_G(C), \widetilde{B}, d\big) \bigm| N_{\overline{E}}(C, \psi) = U\big\}.$$

Dade's invariant conjecture can be stated as follows:

Dade's Invariant Conjecture. *(See [10].) If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k\big(N_G(C), \widetilde{B}, d, U\big) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let H be a subgroup of G, $\varphi \in \mathrm{Irr}(H)$, and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \leqslant (p-1)$ such that the p'-part $(|H|/\varphi(1))_{p'}$ of $|H|/\varphi(1)$ satisfies

$$\left(\frac{|H|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \mod p.$$

Given $1 \leqslant r < (p+1)/2$, let $\mathrm{Irr}(H, [r])$ be the subset of $\mathrm{Irr}(H)$ consisting of those characters φ with $r(\varphi) \equiv \pm r \mod p$. For $\widetilde{B} \in \mathrm{Blk}(G)$, $C \in \mathcal{R}$, an integer $d \geqslant 0$ and $U \leqslant \overline{E}$, we define

$$\mathrm{Irr}\big(N_G(C), \widetilde{B}, d, U, [r]\big) := \mathrm{Irr}\big(N_G(C), \widetilde{B}, d, U\big) \cap \mathrm{Irr}\big(N_G(C), [r]\big)$$

and $k(N_G(C), \widetilde{B}, d, U, [r]) := |\mathrm{Irr}(N_G(C), \widetilde{B}, d, U, [r])|$. The following refinement of Dade's conjecture is due to Uno.

Uno's Invariant Conjecture. *(See [23, Conjecture 3.2].) If $O_p(G) = 1$ and $\widetilde{B} \in \mathrm{Blk}(G)$ with defect group $D(\widetilde{B}) \neq 1$, then for all integers $d \geqslant 0$ and $1 \leqslant r < (p+1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k\big(N_G(C), \widetilde{B}, d, U, [r]\big) = 0.$$

Note that if $p = 2$ or 3, then Uno's conjecture is equivalent to Dade's conjecture.

Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ be the automorphism and outer automorphism groups of G, respectively. We may suppose $\overline{E} = \mathrm{Out}(G)$. If moreover, $\mathrm{Out}(G)$ is cyclic, then we write

$$k\big(N_G(C), \widetilde{B}, d, |U|, [r]\big) := k\big(N_G(C), \widetilde{B}, d, U, [r]\big).$$

For $G = {}^3D_4(q)$, $\mathrm{Out}(G)$ is cyclic and the Schur multiplier of G is trivial. So the invariant conjecture for G is equivalent to its inductive conjecture.

3. Notation and lemmas from elementary number theory

From now on, we always assume that p is an odd prime, n is a positive integer and $q = p^n$. We write ϕ_i for the ith cyclotomic polynomial in q, for example: $\phi_1 = q-1$, $\phi_2 = q+1$, $\phi_3 = q^2+q+1$, $\phi_6 = q^2-q+1$, $\phi_{12} = q^4-q^2+1$. We denote by $\mathbb{N} = \{0,1,2,\ldots\}$ the set of natural numbers including zero. In the next section, we will use the following lemmas, the first of which is folklore:

Lemma 3.1. *Suppose $m,n,a \in \mathbb{Z}$ with $m,n > 0$. Then $\gcd(a^m-1, a^n-1) = |a^d-1|$ where $d := \gcd(m,n)$.*

Proof. Since $a^m - 1 = (a^d-1)\cdot\sum_{i=0}^{\frac{m}{d}-1} a^{di}$ and $a^n-1 = (a^d-1)\cdot\sum_{i=0}^{\frac{n}{d}-1} a^{di}$ we have $a^d-1 \mid a^m-1, a^n-1$. Now let $t \mid a^m-1, a^n-1$, so $a^m \equiv a^n \equiv 1 \bmod t$. Hence the multiplicative order of $a \bmod t$ is a divisor of m,n and then also of d. Thus $a^d \equiv 1 \bmod t$. □

Lemma 3.2. *Let t be a positive integer with $t \mid 3n$. Define $\delta := 1$ if $t \mid n$ and $\delta := \frac{1}{3}$ if $t \nmid n$. Then the following hold.*

(i) $\gcd(p^t-1, q-1) = p^{\delta t}-1$.
(ii) $\gcd(p^t-1, q+1) = 2$.
(iii) $\gcd(p^t-1, q^3-1) = p^t-1$.
(iv) $\gcd(p^t-1, q^3+1) = 2$.

(v) $\gcd(p^t+1, q-1) = \begin{cases} p^{\delta t}+1 & \text{if } 2\delta t \mid n, \\ 2 & \text{if } 2\delta t \nmid n. \end{cases}$

(vi) $\gcd(p^t+1, q+1) = \begin{cases} 2 & \text{if } 2\delta t \mid n, \\ p^{\delta t}+1 & \text{if } 2\delta t \nmid n. \end{cases}$

(vii) $\gcd(p^t+1, q^3-1) = \begin{cases} p^t+1 & \text{if } 2\delta t \mid n, \\ 2 & \text{if } 2\delta t \nmid n. \end{cases}$

(viii) $\gcd(p^t+1, q^3+1) = \begin{cases} 2 & \text{if } 2\delta t \mid n, \\ p^t+1 & \text{if } 2\delta t \nmid n. \end{cases}$

Proof. (i) and (iii) are clear by Lemma 3.1.

(ii) Suppose $d = \gcd(p^t-1, q+1)$. Since $q+1 \mid q^3+1$ and $p^t-1 \mid q^3-1$ by Lemma 3.1, it follows that $d \mid \gcd(q^3-1, q^3+1) = 2$.

(iv) is analogous to (ii).

(vi) Suppose $2\delta t \nmid n$. If $d \mid p^t+1, q+1$, then $d \mid p^{2t}-1$ and so $d \mid p^{3n}-1$ as $2t \mid 3n$. Thus $d \mid q^3-1, q^3+1$ and $d \mid (q^3+1)-(q^3-1) = 2$.

Suppose $2\delta t \nmid n$. Then t,n have "the same 2-part," i.e. there are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $p^t+1 = -((-p^{2^k})^{t_u}-1)$ and $q+1 = -((-p^{2^k})^{n_u}-1)$. So Lemma 3.1 implies $\gcd(p^t+1, q+1) = \gcd((-p^{2^k})^{t_u}-1, (-p^{2^k})^{n_u}-1) = |(-p^{2^k})^{\delta t_u}-1| = p^{\delta t}+1$.

(viii) Suppose $2\delta t \nmid n$. If $d \mid p^t + 1, q^3 + 1$, then $d \mid p^{2t} - 1$ and $p^{2t} - 1 \mid p^{3n} - 1$, so that $d \mid \gcd(q^3 - 1, q^3 + 1) = 2$.

Suppose $2\delta t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with odd $t_u n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. By Lemma 3.1, $p^t + 1 = -((-p^{2^k})^{t_u} - 1) \mid (-p^{2^k})^{3n_u} - 1$. So $p^t + 1 \mid q^3 + 1$.

(v) Suppose $2\delta t \nmid n$. There are $k, t_u, n_u \in \mathbb{N}$ with $2 \nmid t_u$ and $2 \mid n_u$ such that $t = 2^k \cdot t_u$, $n = 2^k \cdot n_u$. Hence $p^t + 1 = -((-p^{2^k})^{t_u} - 1)$ and $q - 1 = (-p^{2^k})^{n_u} - 1$. So Lemma 3.1 implies $\gcd(p^t + 1, q - 1) = \gcd((-p^{2^k})^{t_u} - 1, (-p^{2^k})^{n_u} - 1) = |(-p^{2^k})^{\delta t_u} - 1| = p^{\delta t} + 1$.

Suppose $2\delta t \nmid n$. If $d \mid p^t + 1, q - 1$, then by (viii), $d \mid q^3 + 1, q - 1$ and so $d \mid \gcd(q^3 + 1, q^3 - 1) = 2$.

(vii) Suppose $2\delta t \nmid n$. Then $2t \mid 3n$ and $p^t + 1 \mid p^{2t} - 1$. Hence $p^t + 1 \mid p^{3n} - 1 = q^3 - 1$. Suppose $2\delta t \nmid n$, then the proof is analogous to (v). $\square$

Lemma 3.3. *Let t, m be positive integers and let δ be as in Lemma 3.2. Suppose $t \mid 3n$ and $2\delta t \nmid n$. If $2^m \mid q - 1$, then $2^m \mid p^{\delta t} - 1$.*

Proof. Suppose $q = p^n \equiv 1 \bmod 2^m$ and let φ be the Euler function. Then $\varphi(2^m) = 2^{m-1}$ and $p^{\varphi(2^m)} \equiv 1 \bmod 2^m$. The conditions $t \mid 3n$ and $2\delta t \nmid n$ imply that $\frac{n}{\delta t} \in \mathbb{Z}$ is odd. Thus, $\gcd(\frac{n}{\delta t}, \varphi(2^m)) = 1$ and there are $x, y \in \mathbb{Z}$ such that $x \cdot \frac{n}{\delta t} + y \cdot \varphi(2^m) = 1$. So $p^{\delta t} = p^{\delta t(x \cdot \frac{n}{\delta t} + y \cdot \varphi(2^m))} \equiv q^x \cdot p^{\delta t y \varphi(2^m)} \equiv 1 \bmod 2^m$. $\square$

Lemma 3.4. *Let $a, b, c, d, i, j, m_1, m_2 \in \mathbb{Z}$ with $m_1 \mid b$,*

$$a \cdot i + b \cdot j \equiv 0 \mod (m_1 \cdot m_2) \quad \textit{and} \tag{1}$$

$$c \cdot i + d \cdot j \equiv 0 \mod (m_2). \tag{2}$$

Suppose $D := \gcd(ad - bc, m_1 \cdot m_2)$ is a divisor of m_2. Then

(a) *$m_1 \mid i$.*
(b) *If $m_1 = 1$, then $\frac{m_2}{D} \mid i$ and $\frac{m_2}{D} \mid j$.*

Proof. (a) Since $m_1 \mid b$ there is a $b' \in \mathbb{Z}$ such that $b = b' \cdot m_1$. Multiplying the second congruence with m_1, we get

$$a \cdot i + m_1 b' \cdot j \equiv 0 \mod (m_1 \cdot m_2) \quad \text{and}$$

$$m_1 c \cdot i + m_1 d \cdot j \equiv 0 \mod (m_1 \cdot m_2).$$

Multiplying the first congruence with d, the second with $-b'$ and adding up, we get $(ad - bc)i \equiv 0 \bmod m_1 \cdot m_2$. Since D divides m_2, we can conclude $m_2 \cdot i \equiv 0 \bmod m_1 \cdot m_2$ and hence $m_1 \mid i$.

(b) Multiplying congruence (1) by d and subtracting b times congruence (2), we get $D \cdot i \equiv 0 \bmod m_2$, hence $\frac{m_2}{D} \mid i$. Multiplying congruence (2) by a and subtracting c times congruence (1), we get $D \cdot j \equiv 0 \bmod m_2$, hence $\frac{m_2}{D} \mid j$. $\square$

4. Action of automorphisms on irreducible characters

Let $G = {}^3D_4(q)$ be Steinberg's simple triality group defined over a finite field with $q = p^n$ elements (always assuming that p is odd). Let $O = \mathrm{Out}(G)$ and $A = \mathrm{Aut}(G)$. Then $O = \langle \alpha \rangle$

and $A = G \rtimes \langle\alpha\rangle$, where α is a field automorphism of order $3n$. We fix a Borel subgroup B and maximal parabolic subgroups P and Q of G containing B as in [16]. In particular, α stabilizes B, P and Q.

In this section, we determine the action of $O = \mathrm{Out}(G)$ on the irreducible characters of the chain normalizers. Our notation for the parameter sets of the irreducible characters of G, B, P and Q is similar to the CHEVIE notation and is given in Table A.1 in Appendix A.

The first column of this table defines a name for the parameter set which parameterizes those characters which are listed in the second column of the table. The characters of G are numbered according to the character table of ${}^3D_4(q)$ in the CHEVIE library, and for the characters of B, P, Q we use the notation from [16]. The list of parameters in the third column of Table A.1 in Appendix A is of the form

$$k = 0, \dots, n_1 - 1 \quad \text{or} \quad \begin{array}{l} k = 0, \dots, n_1 - 1, \\ l = 0, \dots, n_2 - 1 \end{array}$$

where the n_j's are polynomials in q with integer coefficients. In the first case, the parameter k can be substituted by an element of $\mathbb{Z}$, but two parameters which differ by an element of $n_1\mathbb{Z}$ yield the same character. In the second case, the parameter vector (k,l) can be substituted by an element of $\mathbb{Z} \times \mathbb{Z}$, but two parameter vectors which differ by an element of $n_1\mathbb{Z} \times n_2\mathbb{Z}$ yield the same character. In other words, k can be taken to be an element of $\mathbb{Z}_{n_1}$ and (k,l) can be taken to be an element of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The groups $\mathbb{Z}_{n_1}$ and $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ are also called *character parameter groups* (see Section 3.7 of the CHEVIE [13] manual). The next lines of Table A.1 list elements which have to be excluded from the character parameter group. The remaining parameters are called *admissible* in the following. Different values of admissible parameters may give the same character. The fourth column of Table A.1 defines an equivalence relation on the set of admissible parameters. If no equivalence relation is listed we mean the identity relation. The parameter set is defined to be the set of these equivalence classes. Finally, the last column of Table A.1 gives the cardinality of the parameter set.

We consider the example ${}_PI_3$. The character parameter group is $\mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$. The parameter vectors (k,l) and $(-k, k+l)$ yield the same character and the equivalence class of (k,l) is $\{(k,l), (-k,k+l)\}$. Hence, the characters ${}_P\chi_3(k,l)$ are parameterized by the set

$${}_PI_3 = \bigl\{\bigl\{(k,l), (-k,k+l)\bigr\} \bigm| (k,l) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1},\ k \neq 0\bigr\}.$$

If we want to emphasize the dependence of a parameter set, say ${}_PI_3$, from q we write ${}_PI_3(q)$. Table A.1 does not give any detailed information about the parameter sets ${}_GI_{20}$, ${}_GI_{23}$, ${}_GI_{29}$, ${}_GI_{30}$, ${}_GI_{31}$, ${}_GI_{32}$, ${}_GI_{33}$, since we will not need an explicit knowledge of these sets (note that ${}_GI_{20}$, ${}_GI_{23}$, ${}_GI_{29}$, ${}_GI_{30}$, ${}_GI_{31}$, ${}_GI_{32}$, ${}_GI_{33}$ parameterize the regular semisimple irreducible characters of G). The data in Table A.1 is taken from the CHEVIE library and the appendix of [16].

We will also consider the action of $O = \mathrm{Out}(G)$ on the regular semisimple conjugacy classes of G. The parameter sets for these classes are defined analogously to the parameter sets for the irreducible characters (cf. Section 3.8 of the CHEVIE manual) and are listed in Table A.2 in Appendix A. The first column of Table A.2 defines a name for the parameter set which parameterizes those conjugacy classes of G which are listed in the second column of the table. The notation for these classes is taken from Table A.2 in [16]. The third column of Table A.2 in Appendix A describes the *class parameter groups* and the admissible parameters. The fourth column defines an equivalence relation on the set of admissible parameters, and the parameter set is defined to be the set of these equivalence classes. Finally, the last column gives the cardinality of the parameter set.

The information about the parameters in Table A.2 (except for the equivalence relations) is taken from Table A.1 in the appendix of [16]. The equivalence relations were determined as follows: Up to conjugacy, G has exactly 7 maximal tori and these are described by sets $T_0, T_1, \dots, T_6$ (see [11, Table 1.1], and the remarks in [16, Section 3]). The regular semisimple conjugacy classes $c_6(i,j)$, $c_8(i)$, $c_{11}(i)$, $c_{12}(i,j)$, $c_{13}(i,j)$, $c_{14}(i)$, $c_{15}(i,j)$ correspond to T_0, T_1, T_2, T_3, T_4, T_5, T_6 respectively. Let W_j, $j = 0,1,\dots,6$, be the Weyl group of T_j (see p. 42 in [11]). The equivalence classes in Table A.2 in Appendix A correspond to the orbits of W_j on T_j. Using Table 2.1 in [16], the representatives in Table A.1 in [16] and the information about the W_j's in Table A.3 in [15], one can compute the orbits of W_j on T_j. These computations were carried out using computer programs, written by the second author, which are based on the GAP [12] part of CHEVIE.

The action of $O = \mathrm{Out}(G)$ on the conjugacy classes of elements of G, B, P and Q induces an action of O on the sets $\mathrm{Irr}(G)$, $\mathrm{Irr}(B)$, $\mathrm{Irr}(P)$ and $\mathrm{Irr}(Q)$ and then an action on the parameter sets. Using the values of the irreducible characters of G, B, P and Q on the classes listed in the last column of Tables A.3–A.7 we can describe the action of O on the parameter sets.

For an O-set I and each subgroup $H \leqslant O$ let $C_I(H)$ denote the set of fixed points of I under the action of H. In the following proposition we determine $|C_I(H)|$ where I runs through all (disjoint) unions of parameter sets which are listed in Table A.8 except for ${}_GI_{20} \cup {}_GI_{23} \cup {}_GI_{29} \cup {}_GI_{30} \cup {}_GI_{31} \cup {}_GI_{32} \cup {}_GI_{33}$. This last union of parameter sets parameterizes the regular semisimple irreducible characters of G and will be treated separately since it requires different methods.

Proposition 4.1. *Let $t \mid 3n$ and $I \neq {}_GI_{20} \cup {}_GI_{23} \cup {}_GI_{29} \cup {}_GI_{30} \cup {}_GI_{31} \cup {}_GI_{32} \cup {}_GI_{33}$ be one of the (disjoint) unions of parameter sets listed in Table* A.8. *If $H = \langle \alpha^t \rangle$ is a subgroup of O, then the second and third columns of Table* A.8 *show the number of fixed points $|C_I(H)|$ of I under the action of H.*

Proof. We have to consider the following parameter sets I.

First let

$$\begin{aligned} I \in \{ & {}_GI_1 \cup {}_GI_9, {}_GI_2 \cup {}_GI_{10}, {}_GI_3 \cup {}_GI_4 \cup {}_GI_5 \cup {}_GI_6, {}_GI_7, {}_GI_{11}, {}_GI_{12}, \\ & {}_BI_4, {}_BI_8 \cup {}_BI_9 \cup {}_BI_{10} \cup {}_BI_{11}, {}_BI_{18} \cup {}_BI_{19} \cup {}_BI_{20} \cup {}_BI_{21}, \\ & {}_PI_6, {}_PI_9 \cup {}_PI_{10} \cup {}_PI_{12} \cup {}_PI_{13}, {}_PI_{15}, {}_PI_{16}, {}_PI_{17} \cup {}_PI_{18} \cup {}_PI_{19} \cup {}_PI_{20}, \\ & {}_QI_6, {}_QI_7, {}_QI_8, {}_QI_9, {}_QI_{10} \cup {}_QI_{11} \cup {}_QI_{12} \cup {}_QI_{13} \}. \end{aligned}$$

The degrees and character values on the conjugacy classes listed in Tables A.3–A.7 show $C_I(H) = I$ and hence $|C_I(H)| = |I|$. We demonstrate this for the parameter set $I = {}_PI_9 \cup {}_PI_{10} \cup {}_PI_{12} \cup {}_PI_{13}$. The degrees in Table A.5 show that ${}_P\chi_9$ and ${}_P\chi_{10}$ are the only irreducible characters of P of degree $\frac{1}{2}q^3(q^3+1)(q-1)^2$. Furthermore, ${}_P\chi_{12}$ and ${}_P\chi_{13}$ are the only irreducible characters of P of degree $\frac{1}{2}q^3(q^3-1)(q^2-1)$. Hence, ${}_P\chi_9{}^\alpha \in \{{}_P\chi_9, {}_P\chi_{10}\}$ and ${}_P\chi_{12}{}^\alpha \in \{{}_P\chi_{12}, {}_P\chi_{13}\}$. The class representatives in Table A.7 in [16] show that the conjugacy class $c_{3,0}$ is fixed by α and we can see from the character Table A.10 of P in [16] that the values of ${}_P\chi_9$ and ${}_P\chi_{10}$ on $c_{3,0}$ are different. Similarly, the values of ${}_P\chi_{12}$ and ${}_P\chi_{13}$ on $c_{3,0}$ are different. So, ${}_P\chi_i{}^\alpha = {}_P\chi_i$ for $i = 9, 10, 12, 13$ and $|C_I(H)| = |I|$.

In each of the following cases, we have that the action of α on I is given by $x^\alpha = px$ for all $x \in I$ using the character values on the classes listed in the last column of Tables A.4–A.7. We demonstrate this for the parameter set $I = {}_PI_3 \cup {}_PI_4$. The degrees in Table A.7 show that the ${}_P\chi_3(k,l)$'s are the only irreducible characters of P of degree q^3+1, so ${}_P\chi_3(k,l)^\alpha = {}_P\chi_3(k',l')$ for some $\{(k',l'),\dots\} \in {}_PI_3$. We see from the class representatives in Table A.7 in [16] that α acts on the semisimple conjugacy classes of P like the pth power map which implies that the values of ${}_P\chi_3(k',l')$ and ${}_P\chi_3(pk,pl)$ on the semisimple classes coincide. Then, the character values of ${}_P\chi_3(k,l)$ (see the character Table A.10 in [16]) imply that the values of ${}_P\chi_3(k',l')$ and ${}_P\chi_3(pk,pl)$ coincide on all classes, hence ${}_P\chi_3(k',l') = {}_P\chi_3(pk,pl)$ and therefore ${}_P\chi_3(k,l)^\alpha = {}_P\chi_3(pk,pl)$. Similarly, ${}_P\chi_4(k)^\alpha = {}_P\chi_4(pk)$. Hence, $x^\alpha = px$ for all $x \in I$.

Let $I = {}_GI_{13} \cup {}_GI_{21}$. If $x = \{k,-k\} \in I$, then $x \in C_I(H)$ if and only if $(p^t-1)k \equiv 0$ or $(p^t+1)k \equiv 0$. Let

$$C_\pm := \bigl\{\{k,-k\} \in C_I(H) \bigm| \bigl(p^t \pm 1\bigr)k \equiv 0\bigr\},$$

so that $C_I(H) = C_- \cup C_+$ and $C_- \cap C_+ = \emptyset$. We claim

$$C_- = \left\{\{k,-k\} \in {}_GI_{13} \;\middle|\; k \text{ is a multiple of } \frac{q-1}{p^{\delta t}-1}\right\}.$$

The inclusion $\supseteq$ is clear. Let $x = \{k,-k\} \in C_-$. If $x \in {}_GI_{21}$, then $(p^t-1)k \equiv 0 \bmod q+1$ and Lemma 3.2(ii) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in {}_GI_{13}$ and $(p^t-1)k \equiv 0 \bmod q-1$. By Lemma 3.2(i), k is a multiple of $(q-1)/(p^{\delta t}-1)$, proving the claim. Now we consider C_+.

If $2\delta t \nmid n$, we claim $C_+ = \{\{k,-k\} \in {}_GI_{13} \mid k \text{ is a multiple of } (q-1)/(p^{\delta t}+1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k,-k\} \in C_+$. If $x \in {}_GI_{21}$, then $(p^t+1)k \equiv 0 \bmod q+1$ and Lemma 3.2(vi) implies $2 \cdot k \equiv 0$ which is impossible. Hence $x \in {}_GI_{13}$ and $(p^t+1)k \equiv 0 \bmod q-1$. By Lemma 3.2(v), k is a multiple of $(q-1)/(p^{\delta t}+1)$ and the claim holds.

If $2\delta t \nmid n$, we claim $C_+ = \{\{k,-k\} \in {}_GI_{21} \mid k \text{ is a multiple of } (q+1)/(p^{\delta t}+1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k,-k\} \in C_+$. If $x \in {}_GI_{13}$, then $(p^t+1)k \equiv 0 \bmod q-1$ and Lemma 3.2(v) implies $2 \cdot k \equiv 0$ which is impossible. Hence $x \in {}_GI_{21}$ and $(p^t+1)k \equiv 0 \bmod q+1$. By Lemma 3.2(vi), k is a multiple of $(q+1)/(p^{\delta t}+1)$ and the claim holds.

Thus in all cases, $|C_I(H)| = |C_-| + |C_+| = (p^{\delta t}-3)/2 + (p^{\delta t}-1)/2 = p^{\delta t}-2$.

Let $I \in \{{}_GI_{15} \cup {}_GI_{18} \cup {}_GI_{24} \cup {}_GI_{27}, {}_GI_{16} \cup {}_GI_{19} \cup {}_GI_{25} \cup {}_GI_{28}\}$. Then ${}_GI_{15} \cup {}_GI_{18} \cup {}_GI_{24} \cup {}_GI_{27}$ and ${}_GI_{16} \cup {}_GI_{19} \cup {}_GI_{25} \cup {}_GI_{28}$ are isomorphic H-sets, so that we can assume $I = {}_GI_{15} \cup {}_GI_{18} \cup {}_GI_{24} \cup {}_GI_{27}$. Define $J := \{\{k,-k\} \mid k \in \mathbb{Z}_{q^3-1}\} \setminus \{\{0\}, \{(q^3-1)/2\}\}$ and $J' := \{\{k,-k\} \mid k \in \mathbb{Z}_{q^3+1}\} \setminus \{\{0\}, \{(q^3+1)/2\}\}$. The sets J and J' become H-sets by $x^\alpha := px$ for all $x \in J, J'$. By construction and the definition of character parameter groups, ${}_GI_{18} \simeq \{\{k,-k\} \in J \mid q-1 \nmid k\}$ as H-sets. Furthermore, mapping $\{m,-m\} \mapsto \{(q-1)\cdot m, -(q-1)\cdot m\}$ defines an isomorphism of H-sets ${}_GI_{15} \simeq \{\{k,-k\} \in J \mid q-1 \mid k\}$. Hence, $J \simeq {}_GI_{15} \cup {}_GI_{18}$ as H-sets. Similarly, $J' \simeq {}_GI_{24} \cup {}_GI_{27}$ as H-sets, and finally $I \simeq J \cup J'$ (disjoint union) as H-sets, so that we can identify $I = J \cup J'$.

If $x = \{k,-k\} \in I$, then $x \in C_I(H)$ if and only if $(p^t-1)k \equiv 0$ or $(p^t+1)k \equiv 0$. Let

$$C_\pm := \bigl\{\{k,-k\} \in C_I(H) \bigm| \bigl(p^t \pm 1\bigr)k \equiv 0\bigr\},$$

so that $C_I(H) = C_- \cup C_+$ and $C_- \cap C_+ = \emptyset$. We claim

$$C_- = \left\{ \{k, -k\} \in J \;\middle|\; k \text{ is a multiple of } \frac{q^3-1}{p^t-1} \right\}.$$

The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_-$. If $x \in J'$, then $(p^t - 1)k \equiv 0 \bmod q^3 + 1$ and Lemma 3.2(iv) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in J$ and $(p^t - 1)k \equiv 0 \bmod q^3 - 1$. By Lemma 3.2(iii), k is a multiple of $(q^3 - 1)/(p^t - 1)$, proving the claim. Next, we consider C_+.

If $2\delta t \nmid n$, we claim $C_+ = \{\{k, -k\} \in J \mid k \text{ is a multiple of } (q^3 - 1)/(p^t + 1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in J'$, then $(p^t + 1)k \equiv 0 \bmod q^3 + 1$ and Lemma 3.2(viii) implies $2 \cdot k \equiv 0$, which is impossible. Hence $x \in J$ and $(p^t + 1)k \equiv 0 \bmod q^3 - 1$. By Lemma 3.2(vii), k is a multiple of $(q^3 - 1)/(p^t + 1)$ and the claim holds.

If $2\delta t \nmid n$, we claim $C_+ = \{\{k, -k\} \in J' \mid k \text{ is a multiple of } (q^3 + 1)/(p^t + 1)\}$. The inclusion $\supseteq$ is clear. Let $x = \{k, -k\} \in C_+$. If $x \in J$, then $(p^t + 1)k \equiv 0 \bmod q^3 - 1$ and Lemma 3.2(vii) implies $2 \cdot k \equiv 0$ which is impossible. Hence $x \in J'$ and $(p^t + 1)k \equiv 0 \bmod q^3 + 1$. By Lemma 3.2(viii), k is a multiple of $(q^3 + 1)/(p^t + 1)$ and the claim holds.

Thus in all cases, $|C_I(H)| = |C_-| + |C_+| = (p^t - 3)/2 + (p^t - 1)/2 = p^t - 2$.

Let $I = {}_BI_1$. If $(k, l) \in I$, then $(k, l) \in C_I(H)$ if and only if $(p^t - 1)k \equiv 0 \bmod q^3 - 1$ and $(p^t - 1)l \equiv 0 \bmod q - 1$. By Lemma 3.2(i) and (iii), this is equivalent with $(p^t - 1)k \equiv 0 \bmod q^3 - 1$ and $(p^{\delta t} - 1)l \equiv 0 \bmod q - 1$. Hence

$$C_I(H) = \left\{ (k, l) \in I \;\middle|\; k \text{ is a multiple of } \frac{q^3-1}{p^t-1} \text{ and } l \text{ is a multiple of } \frac{q-1}{p^{\delta t}-1} \right\}$$

and $|C_I(H)| = (p^t - 1)(p^{\delta t} - 1)$.

Let $I \in \{{}_BI_2, {}_BI_5, {}_PI_1, {}_PI_7, {}_QI_5\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(p^t - 1)k \equiv 0 \bmod q - 1$. By Lemma 3.2(i), this is equivalent with $(p^{\delta t} - 1)k \equiv 0 \bmod q - 1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q - 1)/(p^{\delta t} - 1)\}$ and $|C_I(H)| = p^{\delta t} - 1$.

Let $I \in \{{}_BI_3, {}_BI_{17}, {}_PI_5, {}_QI_1, {}_QI_2\}$. If $k \in I$, then $k \in C_I(H)$ if and only if $(p^t - 1)k \equiv 0 \bmod q^3 - 1$. So we get $C_I(H) = \{k \in I \mid k \text{ is a multiple of } (q^3 - 1)/(p^t - 1)\}$ and $|C_I(H)| = p^t - 1$.

Let $I = {}_PI_3 \cup {}_PI_4$. First, we compute $|C_{{}_PI_3}(H)|$. Let

$$U_i := \begin{cases} \{\{(k, l), (-k, k + l)\} \in C_{{}_PI_3}(H) \mid p^t k \equiv k,\ p^t l \equiv l\} & \text{if } i = 1, \\ \{\{(k, l), (-k, k + l)\} \in C_{{}_PI_3}(H) \mid p^t k \equiv -k,\ p^t l \equiv k + l\} & \text{if } i = 2. \end{cases}$$

If $x = \{(k, l), (-k, k + l)\} \in {}_PI_3$, then $x \in U_1$ if and only if $(p^t - 1)k \equiv 0 \bmod q^3 - 1$ and $(p^t - 1)l \equiv 0 \bmod q - 1$. By Lemma 3.2(i) and (iii), this is equivalent with $(p^t - 1)k \equiv 0 \bmod q^3 - 1$ and $(p^{\delta t} - 1)l \equiv 0 \bmod q - 1$. Hence

$$U_1 = \left\{ \bigl\{(k, l), (-k, k + l)\bigr\} \in {}_PI_3 \;\middle|\; \frac{q^3-1}{p^t-1} \;\middle|\; k \text{ and } \frac{q-1}{p^{\delta t}-1} \;\middle|\; l \right\}$$

and $|U_1| = (p^t - 2)(p^{\delta t} - 1)/2$.

Suppose $2\delta t \nmid n$. If $x = \{(k, l), (-k, k + l)\} \in {}_PI_3$, then $x \in U_2$ if and only if $(p^t + 1)k \equiv 0 \bmod q^3 - 1$ and $(p^t - 1)l \equiv k \bmod q - 1$. Since $2\delta t \nmid n$ we have $2t \mid 3n$, and hence, by Lemma 3.2(i) and (vii), $(q - 1)/(p^{\delta t} - 1)$, $(q^3 - 1)/(p^t + 1)$, $(q^3 - 1)/(p^{2t} - 1) \in \mathbb{Z}$. We claim

$$U_2 = \left\{ \left\{(k,l), (-k, k+l)\right\} \in {}_P I_3 \;\middle|\; \text{there exists } \; m \in \mathbb{Z} \text{ such that} \right.$$
$$\left. k = \frac{q^3-1}{p^t+1} \cdot m \text{ and } l \equiv \frac{q^3-1}{p^{2t}-1} \cdot m \bmod \frac{q-1}{p^{\delta t}-1} \right\}.$$

The inclusion $\supseteq$ is clear. Suppose $x = \{(k,l), (-k, k+l)\} \in U_2$. Then $(p^t+1)k \equiv 0 \bmod q^3-1$ and Lemma 3.2(vii) imply that there exists $m \in \mathbb{Z}$ such that $k = m \cdot (q^3-1)/(p^t+1)$. Because $(p^t-1)l \equiv k \bmod q-1$ there exists $z \in \mathbb{Z}$ such that $(p^t-1)l = m(q^3-1)/(p^t+1) + z \cdot (q-1)$. Thus

$$l = \frac{q^3-1}{p^{2t}-1} \cdot m + \frac{z}{c} \cdot \frac{q-1}{p^{\delta t}-1}$$

with $c := \frac{p^t-1}{p^{\delta t}-1} \in \mathbb{Z}$. Since $\frac{z}{c} \cdot \frac{q-1}{p^{\delta t}-1} \in \mathbb{Z}$ and $\gcd(c, \frac{q-1}{p^{\delta t}-1}) = 1$ by Lemma 3.2(i), we conclude $(z/c) \in \mathbb{Z}$, proving the claim. Hence $|U_2| = p^t(p^{\delta t}-1)/2$.

Suppose $2\delta t \nmid n$. If $\{(k,l), (-k, k+l)\} \in U_2$, then $(p^t+1)k \equiv 0 \bmod q^3-1$, Lemma 3.2(vii) and the definition of ${}_P I_3$ imply $k \equiv (q^3-1)/2 \bmod q^3-1$. Thus, $(p^t-1)l \equiv k \equiv (q^2+q+1) \times (q-1)/2 \equiv 3(q-1)/2 \equiv (q-1)/2 \bmod q-1$. By Lemma 3.3, each power of 2 dividing $q-1$ also divides p^t-1 and hence also $(q-1)/2$, a contradiction. Hence, $U_2 = \emptyset$. So

$$\left|C_{{}_P I_3}(H)\right| = |U_1| + |U_2| = \begin{cases} (p^t-1)(p^{\delta t}-1) & \text{if } 2\delta t \nmid n, \\ (p^t-2)(p^{\delta t}-1)/2 & \text{if } 2\delta t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_P I_4}(H)|$. If $x = \{k, q^3 k\} \in {}_P I_4$, then $x \in C_{{}_P I_4}(H)$ if and only if $(p^t-1) \times k \equiv 0$ or $(p^t - q^3)k \equiv 0 \bmod (q^3+1)(q-1)$. Suppose $(p^t-1)k \equiv 0$. By Lemma 3.2(i) and (iv), we can find $x, y \in \mathbb{Z}$ and an odd integer u such that $\gcd(x,u) = \gcd(y, \frac{p^t-1}{2}) = 1$ and $p^t - 1 = (p^{\delta t}-1) \cdot u$, $q-1 = (p^{\delta t}-1) \cdot x$ and $q^3+1 = 2y$. Hence $\gcd(p^t-1, (q^3+1)(q-1)) = \gcd((p^{\delta t}-1) \cdot u, (p^{\delta t}-1)2xy) = (p^{\delta t}-1) \cdot \gcd(u, 2xy) = (p^{\delta t}-1)$. Thus $(p^{\delta t}-1)k \equiv 0 \bmod (q^3+1)(q-1)$ and so $\frac{(q^3+1)(q-1)}{p^{\delta t}-1} \mid k$. But then $(q^3+1) \mid k$, a contradiction to the definition of ${}_P I_4$. So we have proved that $x \in C_{{}_P I_4}(H)$ if and only if $(p^t - q^3)k \equiv 0 \bmod (q^3+1)(q-1)$.

Suppose $2\delta t \nmid n$. If $\{k, q^3 k\} \in C_{{}_P I_4}(H)$, then $(p^t - q^3)k \equiv 0 \bmod (q^3+1)(q-1)$. Thus $(p^t+1)k \equiv 0 \bmod (q^3+1)$ and $(p^t-1)k \equiv 0 \bmod q-1$. By Lemma 3.2(i) and (viii), we get $\frac{q^3+1}{2} \mid k$ and $\frac{q-1}{p^{\delta t}-1} \mid k$. Since $\frac{q^3+1}{2} \mid q^3+1$ and $\frac{q-1}{p^{\delta t}-1} \mid q^3-1$ and since $p^{\delta t}-1$ is even, we have $\gcd(\frac{q^3+1}{2}, \frac{q-1}{p^{\delta t}-1}) = 1$ and so $\frac{q^3+1}{2} \cdot \frac{q-1}{p^{\delta t}-1} \mid k$. The condition $2\delta t \nmid n$ implies $(p^{\delta t}-1)(p^{\delta t}+1) \mid p^n - 1 = q-1$, so that $\frac{q-1}{p^{\delta t}-1}$ is even. Thus $q^3+1 \mid k$, contradicting the definition of ${}_P I_4$. Hence in this case $C_{{}_P I_4}(H) = \emptyset$.

Suppose $2\delta t \nmid n$. We claim

$$C_{{}_P I_4}(H) = \left\{ \left\{k, q^3 k\right\} \in {}_P I_4 \;\middle|\; k \text{ is a multiple of } \frac{(q^3+1)(q-1)}{(p^t+1)(p^{\delta t}-1)} \right\}.$$

Let $k = \frac{(q^3+1)(q-1)}{(p^t+1)(p^{\delta t}-1)} \cdot m$ for some $m \in \mathbb{Z}$. Because $t \mid 3n$ and $2\delta t \nmid n$ we have $2t \mid 3n - t$. Since $(p^t+1)(p^{\delta t}-1)$ is a divisor of $(p^t-1)(p^t+1) = p^{2t}-1$ we then get $(p^t+1)(p^{\delta t}-1) \mid$

$p^{3n-t}-1$. Thus $(p^{3n-t}-1)k=\frac{p^{3n-t}-1}{(p^t+1)(p^{\delta t}-1)}(q^3+1)(q-1)\cdot m\equiv 0 \bmod (q^3+1)(q-1)$. So $(p^t-q^3)k\equiv 0 \bmod (q^3+1)(q-1)$ and $\{k,q^3k\}\in C_{{}_PI_4}(H)$.

Conversely, suppose $\{k,q^3k\}\in C_{{}_PI_4}(H)$. Then $(p^t-q^3)k\equiv 0 \bmod (q^3+1)(q-1)$. Hence $(p^t+1)k\equiv 0 \bmod q^3+1$ and $(p^t-1)k\equiv 0 \bmod q-1$. By Lemma 3.2(i) and (viii), $(p^t+1)\times k\equiv 0 \bmod q^3+1$ and $(p^{\delta t}-1)k\equiv 0 \bmod q-1$. So $\frac{q^3+1}{p^t+1}\mid k$ and $\frac{q-1}{p^{\delta t}-1}\mid k$. Since $\frac{q^3+1}{p^t+1}\mid q^3+1$ and $\frac{q-1}{p^{\delta t}-1}\mid q^3-1$ and since $\frac{q-1}{p^{\delta t}-1}$ is odd by Lemma 3.3, we have $\gcd(\frac{q^3+1}{p^t+1},\frac{q-1}{p^{\delta t}-1})=1$. Therefore $\frac{(q^3+1)(q-1)}{(p^t+1)(p^{\delta t}-1)}\mid k$, and the claim holds. So by the definition of ${}_PI_4$, we get $|C_{{}_PI_4}(H)|=p^t(p^{\delta t}-1)/2$.

So in both cases, $|C_I(H)|=|C_{{}_PI_3}(H)|+|C_{{}_PI_4}(H)|=(p^t-1)(p^{\delta t}-1)$.

Let $I={}_PI_{21}\cup{}_PI_{22}$. Then ${}_PI_{21}(q)\simeq{}_GI_{13}(q^3)$ and ${}_PI_{22}(q)\simeq{}_GI_{21}(q^3)$ as H-sets. Thus

$$\left|C_I(H)\right|=\left|C_{{}_PI_{21}(q)\cup{}_PI_{22}(q)}(H)\right|=\left|C_{{}_GI_{13}(q^3)\cup{}_GI_{21}(q^3)}(H)\right|=p^t-2.$$

Let $I={}_QI_3\cup{}_QI_4$. First, we compute $|C_{{}_QI_3}(H)|$. Let

$$U_i:=\begin{cases}\{\{(k,l),(k+\phi_3l,-l)\}\in C_{{}_QI_3}(H)\mid p^tk\equiv k,\ p^tl\equiv l\} & \text{if } i=1,\\ \{\{(k,l),(k+\phi_3l,-l)\}\in C_{{}_QI_3}(H)\mid p^tk\equiv k+\phi_3l,\ p^tl\equiv -l\} & \text{if } i=2.\end{cases}$$

If $x=\{(k,l),(k+\phi_3l,-l)\}\in{}_QI_3$, then $x\in U_1$ if and only if $(p^t-1)k\equiv 0 \bmod q^3-1$ and $(p^t-1)l\equiv 0 \bmod q-1$. By Lemma 3.2(i) and (iii), this is equivalent with $(p^t-1)k\equiv 0 \bmod q^3-1$ and $(p^{\delta t}-1)l\equiv 0 \bmod q-1$. Hence

$$U_1=\left\{\{(k,l),(k+\phi_3l,-l)\}\in{}_QI_3\ \Big|\ \frac{q^3-1}{p^t-1}\ \Big|\ k \text{ and } \frac{q-1}{p^{\delta t}-1}\ \Big|\ l\right\}$$

and $|U_1|=(p^t-1)(p^{\delta t}-2)/2$.

Suppose $2\delta t\nmid n$. If $x=\{(k,l),(k+\phi_3l,-l)\}\in{}_QI_3$, then $x\in U_2$ if and only if $(p^t-1)k\equiv \phi_3l \bmod q^3-1$ and $(p^t+1)l\equiv 0 \bmod q-1$. We have $(p^{\delta t}+1)(p^t-1)\mid (p^t+1)(p^t-1)=p^{2t}-1$ which is a divisor of $p^{3n}-1=q^3-1$ because $2t\mid 3n$. Together with Lemma 3.2(iii) and (v) this implies $\frac{q-1}{p^{\delta t}+1},\frac{q^3-1}{(p^{\delta t}+1)(p^t-1)},\frac{q^3-1}{p^t-1}\in\mathbb{Z}$. We claim

$$\begin{aligned}U_2=\Bigg\{&\{(k,l),(k+\phi_3l,-l)\}\in{}_QI_3\ \Big|\ \exists m\in\mathbb{Z}\text{ such that}\\ &l=\frac{q-1}{p^{\delta t}+1}\cdot m\text{ and } k\equiv\frac{q^3-1}{(p^{\delta t}+1)(p^t-1)}\cdot m \bmod \frac{q^3-1}{p^t-1}\Bigg\}.\end{aligned}$$

The inclusion $\supseteq$ is clear. Suppose $x=\{(k,l),(k+\phi_3l,-l)\}\in U_2$. Then $(p^t+1)l\equiv 0 \bmod q-1$ and Lemma 3.2(v) imply that there exists $m\in\mathbb{Z}$ such that $l=m\cdot(q-1)/(p^{\delta t}+1)$. Because $(p^t-1)k\equiv\phi_3l \bmod q^3-1$ there exists $z\in\mathbb{Z}$ such that $(p^t-1)k=m\phi_3(q-1)/(p^{\delta t}+1)+z\cdot(q^3-1)$. Thus

$$k=\frac{q^3-1}{(p^{\delta t}+1)(p^t-1)}\cdot m+z\cdot\frac{q^3-1}{p^t-1}.$$

This proves the claim and shows $|U_2| = p^{\delta t}(p^t-1)/2$.

Suppose $2\delta t \nmid n$. If $\{(k,l),(k+\phi_3 l,-l)\} \in U_2$, then $(p^t+1)l \equiv 0 \bmod q-1$, Lemma 3.2(v) and the definition of ${}_QI_3$ imply $l \equiv (q-1)/2 \bmod q-1$. Thus $(p^t-1)k \equiv \phi_3 l \equiv (q^3-1)/2 \bmod q^3-1$. By Lemma 3.3, each power of 2 dividing $q^3-1=(q-1)(q^2+q+1)$ also divides p^t-1 and hence also $(q^3-1)/2$, a contradiction. Hence $U_2=\emptyset$. So

$$\left|C_{{}_QI_3}(H)\right| = |U_1|+|U_2| = \begin{cases} (p^t-1)(p^{\delta t}-1) & \text{if } 2\delta t \nmid n, \\ (p^t-1)(p^{\delta t}-2)/2 & \text{if } 2\delta t \nmid n. \end{cases}$$

Next we calculate $|C_{{}_QI_4}(H)|$. If $x=\{k,q^3k\}\in {}_QI_4$, then $x\in C_{{}_QI_4}(H)$ if and only if $(p^t-1)\times k\equiv 0$ or $(p^t-q^3)k\equiv 0 \bmod (q^3-1)(q+1)$. Suppose $(p^t-1)k\equiv\ 0$. By Lemma 3.2(iii), we have $\frac{(q^3-1)(q+1)}{p^t-1} \mid k$ and so $q+1\mid k$, a contradiction to the definition of ${}_QI_4$. So $x\in C_{{}_QI_4}(H)$ if and only if $(p^t-q^3)k\equiv 0 \bmod (q^3-1)(q+1)$.

Suppose $2\delta t\nmid n$. If $\{k,q^3k\}\in C_{{}_QI_4}(H)$, then $(p^t-q^3)k\equiv 0 \bmod (q^3-1)(q+1)$. Thus $(p^t-1)k\equiv 0 \bmod (q^3-1)$ and $(p^t+1)k\equiv 0 \bmod q+1$. By Lemma 3.2(iii) and (vi), we get $\frac{q+1}{2}\mid k$ and $\frac{q^3-1}{p^t-1}\mid k$. Since $\frac{q+1}{2}\mid q^3+1$ and $\frac{q^3-1}{p^t-1}\mid q^3-1$ and since p^t-1 is even, we have $\gcd(\frac{q+1}{2},\frac{q^3-1}{p^t-1})=1$ and so $\frac{q+1}{2}\cdot\frac{q^3-1}{p^t-1}\mid k$. The condition $2\delta t\nmid n$ implies $(p^t-1)(p^t+1)\mid p^{3n}-1=q^3-1$, so that $\frac{q^3-1}{p^t-1}$ is even. Thus $q+1\mid k$, contradicting the definition of ${}_QI_4$. Hence in this case, $C_{{}_QI_4}(H)=\emptyset$.

Suppose $2\delta t\nmid n$. We claim

$$C_{{}_QI_4}(H)=\left\{\{k,q^3k\}\in {}_QI_4 \;\middle|\; k \text{ is a multiple of } \frac{(q^3-1)(q+1)}{(p^{\delta t}+1)(p^t-1)}\right\}.$$

Let $k=\frac{(q^3-1)(q+1)}{(p^{\delta t}+1)(p^t-1)}\cdot m$ for some $m\in\mathbb{Z}$. Because $t\mid 3n$ and $2\delta t\nmid n$ we have $2t\mid 3n-t$. Since $(p^{\delta t}+1)(p^t-1)$ is a divisor of $(p^t+1)(p^t-1)=p^{2t}-1$ we then get $(p^{\delta t}+1)(p^t-1)\mid p^{3n-t}-1$. Thus $(p^{3n-t}-1)k=\frac{p^{3n-t}-1}{(p^{\delta t}+1)(p^t-1)}(q^3-1)(q+1)\cdot m\equiv 0 \bmod (q^3-1)(q+1)$. So $(p^t-q^3)k\equiv 0 \bmod (q^3-1)(q+1)$ and $\{k,q^3k\}\in C_{{}_QI_4}(H)$.

Conversely, suppose $\{k,q^3k\}\in C_{{}_QI_4}(H)$. Then $(p^t-q^3)k\equiv 0 \bmod (q^3-1)(q+1)$. Hence $(p^t-1)k\equiv 0 \bmod q^3-1$ and $(p^t+1)k\equiv 0\bmod q+1$. Lemma 3.2(vi) implies $\frac{q^3-1}{p^t-1}\mid k$ and $\frac{q+1}{p^{\delta t}+1}\mid k$. Since $\frac{q^3-1}{p^t-1}\mid q^3-1$ and $\frac{q+1}{p^{\delta t}+1}\mid q^3+1$ and since $\frac{q^3-1}{p^t-1}$ is odd by Lemma 3.3 we have $\gcd(\frac{q^3-1}{p^t-1},\frac{q+1}{p^{\delta t}+1})=1$. Therefore $\frac{(q^3-1)(q+1)}{(p^t-1)(p^{\delta t}+1)}\mid k$, proving the claim. So by the definition of ${}_QI_4$ we get $|C_{{}_QI_4}(H)|=p^{\delta t}(p^t-1)/2$.

So we get in both cases $|C_I(H)|=|C_{{}_PI_3}(H)|+|C_{{}_QI_4}(H)|=(p^t-1)(p^{\delta t}-1)$. $\square$

Now, we deal with the regular semisimple irreducible characters of G.

Proposition 4.2. *Let $t\mid 3n$, $I:={}_GI_{20}\cup{}_GI_{23}\cup{}_GI_{29}\cup{}_GI_{30}\cup{}_GI_{31}\cup{}_GI_{32}\cup{}_GI_{33}$ and $H=\langle\alpha^t\rangle$ a subgroup of O. Then*

(a) *$|C_I(H)|$ is equal to the number of those regular semisimple conjugacy classes of G which are stabilized by α^t.*

(b) *If* $t \nmid n$ (*respectively* $t \mid n$), *then* $|C_I(H)|$ *is equal to the number of regular semisimple conjugacy classes of* ${}^3D_4(p^{t/3})$ (*respectively* $G_2(p^t)$).
(c) $|C_I(H)| = p^t \cdot p^{\delta t} - p^t - p^{\delta t} + 2$ *with* δ *as in Lemma* 3.2.

Proof. The set I parameterizes the regular semisimple irreducible characters of G. We fix some notation. Let $\mathbb{F}_q$ be a finite field with q elements, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_q$ and $\overline{G}$ a simple simply connected algebraic group of Dynkin type D_4 defined over $\mathbb{F}$. In the same way as in [11, Section 1], we choose a graph automorphism $\gamma : \overline{G} \to \overline{G}$ of order 3 arising from the symmetry of the D_4 Dynkin diagram and a field automorphism $\bar{\alpha} : \overline{G} \to \overline{G}$ obtained from the map $\mathbb{F} \to \mathbb{F}$, $x \mapsto x^p$. Setting $F := \bar{\alpha}^n \circ \gamma = \gamma \circ \bar{\alpha}^n$ we get $G = {}^3D_4(q) = \overline{G}^F = \{g \in \overline{G} \mid F(g) = g\}$. Since the restriction $\bar{\alpha}|_G : G \to G$ of $\bar{\alpha}$ to G generates $\mathrm{Out}(G)$ we can assume $\bar{\alpha}|_G = \alpha$.

For $L \in \{G, \overline{G}\}$, let $\mathcal{S}_{\mathrm{reg}}(L)$ be the set of all regular semisimple conjugacy classes of L. If ρ is an endomorphism of L, then let $\mathcal{S}_{\mathrm{reg}}(L)^\rho := \{C \in \mathcal{S}_{\mathrm{reg}}(L) \mid C^\rho = C\}$ be the set of ρ-stable regular semisimple conjugacy classes of L. Finally, let $\mathrm{Irr}^{\mathrm{ss}}_{\mathrm{reg}}(G)$ be the set of regular semisimple irreducible characters of G.

By Corollary 3.10 of Springer–Steinberg [4, p. 197], the map $C \mapsto C \cap \overline{G}^F$ is a bijection from $\mathcal{S}_{\mathrm{reg}}(\overline{G})^F$ onto $\mathcal{S}_{\mathrm{reg}}(G)$ and this bijection induces a bijection between the set of regular semisimple conjugacy classes of G fixed by α^t and the set of F-stable regular semisimple conjugacy classes of $\overline{G}$ fixed by $\bar{\alpha}^t$. It follows that, since $\bar{\alpha}^t$ raises every element of a maximally split torus of $\overline{G}$ to its p^tth power, the automorphism α^t maps each regular semisimple conjugacy class $(g)_G$ of G to the class $(g^{p^t})_G$. In other words, α^t acts on the regular semisimple conjugacy classes of G like the p^tth power map (this does not mean, that α^t maps every regular semisimple element of G to its p^tth power).

(a) Since $G = {}^3D_4(q)$ is selfdual (in the sense of [6, Section 4.4, p. 120]), the number $|\mathcal{S}_{\mathrm{reg}}(G)^{\alpha^t}|$ of fixed points of α^t on $\mathcal{S}_{\mathrm{reg}}(G)$ is equal to the number of fixed points of α^t on $\mathrm{Irr}^{\mathrm{ss}}_{\mathrm{reg}}(G)$. By definition, the latter equals $|C_I(H)|$.

(b) In this part of the proof, we imitate an argument which is used in the proof of Lemma 4.1 in [3]. As we have seen at the beginning of this proof, there is a bijection from the set of regular semisimple conjugacy classes of G fixed by α^t onto $\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\langle F, \bar{\alpha}^t\rangle}$, the set of fixed points of $\mathcal{S}_{\mathrm{reg}}(\overline{G})$ under the action of the group $\langle F, \bar{\alpha}^t\rangle$. So by (a), we have $|C_I(H)| = |\mathcal{S}_{\mathrm{reg}}(G)^{\alpha^t}| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\langle F, \bar{\alpha}^t\rangle}|$.

Case 1. Suppose $t \nmid n$, then $\langle F, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^n \circ \gamma, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^{\pm t/3} \circ \gamma, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^{\pm t/3} \circ \gamma\rangle$. Thus $|C_I(H)| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\bar{\alpha}^{\pm t/3}\circ\gamma}| = |\mathcal{S}_{\mathrm{reg}}(\overline{G}^{\bar{\alpha}^{\pm t/3}\circ\gamma})|$. Since $\overline{G}^{\bar{\alpha}^{\pm t/3}\circ\gamma} \cong {}^3D_4(p^{t/3})$, we get $|C_I(H)| = |\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^{t/3}))|$, proving (b) in this case.

Case 2. Suppose $t \mid n$. By the character table of $G_2(p^t)$ in the CHEVIE library, the number of regular semisimple conjugacy classes of $G_2(p^t)$ is $p^{2t} - 2p^t + 2$. So we have to show, that the number $|C_I(H)|$ of fixed points of α^t on $I = I(q)$ is equal to $p^{2t} - 2p^t + 2$. As a first step, we reduce to the case $t = n$. By assumption $t \mid n$, so $\langle F, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^n \circ \gamma, \bar{\alpha}^t\rangle = \langle \gamma, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^t \circ \gamma, \bar{\alpha}^t\rangle$. Thus $|C_I(H)| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\langle F, \bar{\alpha}^t\rangle}| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\bar{\alpha}^t\circ\gamma} \cap \mathcal{S}_{\mathrm{reg}}(\overline{G})^{\bar{\alpha}^t}| = |\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^{\alpha^t}|$. From part (a), we know $|\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^{\alpha^t}| = |C_{I(p^t)}(H)|$ and thus $|C_I(H)| = |C_{I(p^t)}(H)|$, i.e. $|C_I(H)|$ is equal to the number of fixed points of α^t on $I(p^t)$. Hence, in the following, we can and do assume that $t = n$.

So we have to show that the number of fixed points of α^n on $I(p^n) = I(q) = I$ is equal to $p^{2n} - 2p^n + 2 = q^2 - 2q + 2$. By part (a), we know that the number $|C_I(H)|$ of fixed points of α^n on I is equal to the number of regular semisimple conjugacy classes of G stabilized by α^n. The regular semisimple conjugacy classes of G are parameterized by the sets J_6, J_8, J_{11}, J_{12}, J_{13}, J_{14}, J_{15} in Table A.2 in Appendix A via the representatives given in Table A.1 in [16]. The action of α^n on the regular semisimple conjugacy classes of G induces an action on the set $J := J_6 \cup J_8 \cup J_{11} \cup J_{12} \cup J_{13} \cup J_{14} \cup J_{15}$ (disjoint union). Since α^n acts on the regular semisimple conjugacy classes like the qth power map, we can see from the representatives in Table A.1 in [16], that the action of α^n on J is given by $x^{\alpha^n} = qx$ for all $x \in J$ and that each of the sets J_j ($j = 6, 8, 11, 12, 13, 14, 15$) is invariant under this action. In particular, we have $|C_J(H)| = \sum_{j \in J_G} |C_{J_j}(H)|$ where $J_G := \{6, 8, 11, 12, 13, 14, 15\}$.

Now, we determine the numbers $|C_{J_j}(H)|$ of fixed points of α^n on J_j by a direct calculation using the parameter sets in Table A.2.

Suppose $J' = J_6$. As we can see from Table A.2, each element of $J' = J_6$ is an equivalence class consisting of 12 vectors and we number these vectors according to their order in Table A.2, i.e. the vector (i, j) gets the number 1, the vector $(-i, -j)$ gets the number 2, $(i, i - j)$ gets the number 3, and so on. We will consider the following sets of fixed points:

$$\begin{aligned}
U_1 &:= \big\{\{(i, j), \ldots\} \in C_{J'}(H) \bigm| qi \equiv i,\ qj \equiv j\big\},\\
U_2 &:= \big\{\{(i, j), \ldots\} \in C_{J'}(H) \bigm| qi \equiv -i,\ qj \equiv -j\big\},\\
&\ \ \vdots\\
U_{12} &:= \big\{\{(i, j), \ldots\} \in C_{J'}(H) \bigm| qi \equiv -q\phi_2 i + \phi_3 j,\ qj \equiv -i + 2j\big\},
\end{aligned}$$

where the first congruence is always modulo $q^3 - 1$, the second always modulo $q - 1$. Then $C_{J'}(H) = \bigcup_{m=1}^{12} U_m$. We claim that $U_m = \emptyset$ for $m = 2, 3, \ldots, 12$.

Suppose $\{(i, j), \ldots\} \in U_3 \cup U_{10}$. Then $qj \equiv i - j \bmod q - 1$. Hence $q - 1 \mid i - 2j$, contradicting the definition of J_6. Thus $U_3 = U_{10} = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_8 \cup U_{12}$. Then $qj \equiv -i + 2j \bmod q - 1$. Hence $q - 1 \mid i - j$, contradicting the definition of J_6. Thus $U_8 = U_{12} = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_2$. Then we have $(q + 1)i \equiv 0 \bmod q^3 - 1$ and $(q + 1)j \equiv 0 \bmod q - 1$. By Lemma 3.2(v) and (vii), we have $\gcd(q + 1, q^3 - 1) = \gcd(q + 1, q - 1) = 2$, so $2i \equiv 0 \bmod q^3 - 1$ and $2j \equiv 0 \bmod q - 1$. By definition of $J' = J_6$, we have $i \not\equiv 0 \bmod q^3 - 1$ and $j \not\equiv 0 \bmod q - 1$. It follows that $i \equiv (q^3 - 1)/2 \bmod q^3 - 1$ and $j \equiv (q - 1)/2 \bmod q - 1$. Setting $l := (q - 1)/2$, we get $i \equiv \phi_3 l \bmod q^3 - 1$ and $j \equiv l \bmod q - 1$, contradicting the definition of J_6. Thus $U_2 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_4$. Then $(q + 1)i \equiv 0 \bmod q^3 - 1$ and $i + (q - 1)j \equiv 0 \bmod q - 1$. From Lemma 3.4(a) (with $m_1 = \phi_3$ and $m_2 = \phi_1$), we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. So $(q + 1)\phi_3 l \equiv 0 \bmod \phi_3(q - 1)$ and $\phi_3 l \equiv 0 \bmod q - 1$. Hence $(q + 1)l \equiv 0 \bmod q - 1$ and $\phi_3 l \equiv 0 \bmod q - 1$. Thus $2l \equiv 0 \bmod q - 1$ and $3l \equiv 0 \bmod q - 1$. This implies $l \equiv 0 \bmod q - 1$ and so $i \equiv 0 \bmod q^3 - 1$, contradicting the definition of J_6. Thus $U_4 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_5$. Then $(q - 1)i + \phi_3 j \equiv 0 \bmod q^3 - 1$ and $(q + 1)j \equiv 0 \bmod q - 1$. From Lemma 3.4(a) we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. Thus, $(q - 1)\phi_3 l + \phi_3 j \equiv 0 \bmod q^3 - 1$. Hence, $(q - 1)l + j \equiv 0 \bmod q - 1$, and so $j \equiv 0 \bmod q - 1$, contradicting the definition of J_6. Thus, $U_5 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_6$. Then $(q-1)i + \phi_3 j \equiv 0 \bmod q^3-1$ and $-i + (q+2)j \equiv 0 \bmod q-1$. From Lemma 3.4(a) we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. Thus $(q-1)\phi_3 l + \phi_3 j \equiv 0 \bmod q^3-1$. Hence $(q-1)l + j \equiv 0 \bmod q-1$, and so $j \equiv 0 \bmod q-1$, contradicting the definition of J_6. Thus $U_6 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_7$. Then $(q+1)i - \phi_3 j \equiv 0 \bmod q^3-1$ and $(q-1)j \equiv 0 \bmod q-1$. From Lemma 3.4(a) we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. Thus $(q+1)\phi_3 l - \phi_3 j \equiv 0 \bmod q^3-1$. Hence $(q+1)l - j \equiv 0 \bmod q-1$, and so $j \equiv 2l \bmod q-1$, contradicting the definition of J_6. Thus $U_7 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_9$. Then $q^2 i - \phi_3 j \equiv 0 \bmod q^3-1$ and $-i + (q+2)j \equiv 0 \bmod q-1$. From Lemma 3.4(a) we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. Thus $q^2\phi_3 l - \phi_3 j \equiv 0 \bmod q^3-1$. Hence $q^2 l - j \equiv 0 \bmod q-1$, and so $j \equiv l \bmod q-1$, contradicting the definition of J_6. Thus $U_9 = \emptyset$.

Suppose $\{(i, j), \ldots\} \in U_{11}$. Then we have $(q^2+2q)i - \phi_3 j \equiv 0 \bmod q^3-1$ and $i + (q-1)j \equiv 0 \bmod q-1$. From Lemma 3.4(a) we get that there is $l \in \mathbb{Z}$ with $i = \phi_3 l$. So $(q^2+2q)\phi_3 l - \phi_3 j \equiv 0 \bmod \phi_3(q-1)$ and $\phi_3 l + (q-1)j \equiv 0 \bmod q-1$. Hence $(q^2+2q)l - j \equiv 0 \bmod q-1$ and $\phi_3 l \equiv 0 \bmod q-1$. Thus $3l - j \equiv 0 \bmod q-1$ and $3l \equiv 0 \bmod q-1$. This implies $j \equiv 0 \bmod q-1$, contradicting the definition of J_6. Thus $U_{11} = \emptyset$.

So only U_1 contributes to the fixed points, i.e. $|C_{J'}(H)| = |U_1|$. Since $qi \equiv i \bmod q^3-1$ and $qj \equiv j \bmod q-1$ is equivalent with $(q-1)i \equiv 0 \bmod q^3-1$ which is equivalent to $\phi_3 \mid i$, we get $|C_{J'}(H)| = |\{\{(i, j), \ldots\} \in J_6 \mid i \text{ is a multiple of } \phi_3\}|$. So we have to compute the number N of all admissible parameter vectors (i, j) with $\phi_3 \mid i$. By definition of J_6, a vector $(i, j) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$ is admissible if and only if it does not satisfy any of the following conditions (i)–(vi):

(i) $i \equiv 0 \bmod q^3-1$,
(ii) $j \equiv 0 \bmod q-1$,
(iii) $i \equiv \phi_3 l \bmod q^3-1$ and $j \equiv l \bmod q-1$ for some $l \in \mathbb{Z}$,
(iv) $i \equiv \phi_3 l \bmod q^3-1$ and $j \equiv 2l \bmod q-1$ for some $l \in \mathbb{Z}$,
(v) $i \equiv j \bmod q-1$,
(vi) $i \equiv 2j \bmod q-1$.

It is straightforward to calculate the number of all vectors $(i, j) \in \mathbb{Z}_{q^3-1} \times \mathbb{Z}_{q-1}$ with $\phi_3 \nmid i$ satisfying one of the conditions (i), ..., (vi), then to calculate the number of vectors satisfying two of the conditions (i), ..., (vi) and so on. From these numbers, using the include–exclude formula, we get the number N' of all admissible parameter vectors (i, j) with $\phi_3 \nmid i$:

$$N' = \begin{cases} q^4 - 4q^3 + q^2 + 6q - 4 & \text{if } q \equiv 1 \bmod 3, \\ q^4 - 4q^3 + q^2 + 6q & \text{if } q \not\equiv 1 \bmod 3. \end{cases}$$

By Table A.2, the number N_{ad} of all admissible parameter vectors is $q^4 - 4q^3 + 2q^2 - 2q + 15$. From this, we get the number $N = N_{ad} - N'$ of all admissible parameter vectors (i, j) with $\phi_3 \mid i$. Dividing N by 12, the cardinality of the equivalence classes, we get:

$$\left|C_{J'}(H)\right| = \frac{N}{12} = \begin{cases} \frac{q^2-8q+19}{12} & \text{if } q \equiv 1 \bmod 3, \\ \frac{(q-3)(q-5)}{12} & \text{if } q \not\equiv 1 \bmod 3. \end{cases}$$

Suppose $J' = J_{15}$. Analogously to J_6, using Lemma 3.4(a) with $m_1 = \phi_6$ and $m_2 = \phi_2$, we can show, that only the second parameter vector contributes to the fixed points, namely $qi \equiv -i \bmod q^3+1$ and $qj \equiv -j \bmod q+1$ which is equivalent with $(q+1)i \equiv 0 \bmod q^3+1$ which is also equivalent with $\phi_6 \mid i$. So by computing admissible parameter vectors using the definition of $J' = J_{15}$ we get

$$\left|C_{J'}(H)\right| = \begin{cases} \frac{q^2-4q+7}{12} & \text{if } q \equiv -1 \bmod 3, \\ \frac{(q-1)(q-3)}{12} & \text{if } q \not\equiv -1 \bmod 3. \end{cases}$$

Suppose $J' = J_8$. Let

$$U_{\pm 1} := \left\{\left\{i, -i, q^3 i, -q^3 i\right\} \in C_{J'}(H) \;\middle|\; qi \equiv \pm i\right\},$$
$$U_{\pm 2} := \left\{\left\{i, -i, q^3 i, -q^3 i\right\} \in C_{J'}(H) \;\middle|\; qi \equiv \pm q^3 i\right\},$$

where the congruences are mod $(q^3-1)(q+1)$. Then $C_{J'}(H) = U_1 \cup U_{-1} \cup U_2 \cup U_{-2}$. We claim that $U_{\pm 1} = U_{-2} = \emptyset$. Suppose $\{i, -i, q^3 i, -q^3 i\} \in U_1$. Then $(q-1)i \equiv 0 \bmod (q^3-1)(q+1)$. Hence $\phi_3(q+1) \mid i$. In particular, $q+1 \mid i$, contradicting the definition of J_8. So $U_1 = \emptyset$. Suppose $\{i, -i, q^3 i, -q^3 i\} \in U_{-1}$. Then $(q+1)i \equiv 0 \bmod (q^3-1)(q+1)$. Hence $q^3-1 \mid i$, contradicting the definition of J_8. So $U_{-1} = \emptyset$. Suppose $\{i, -i, q^3 i, -q^3 i\} \in U_{-2}$. Then $qi \equiv -q^3 i \bmod (q^3-1)(q+1)$ and hence $q(q^2+1)i \equiv 0 \bmod (q^3-1)(q+1)$. Since $q(q^2+1) \mid q(q^6+1)$ and $(q^3-1)(q+1) \mid q^6-1$, we have $\gcd(q(q^2+1), (q^3-1)(q+1)) = 2$ and so $2i \equiv 0 \bmod (q^3-1)(q+1)$. Thus $(q^3-1)(q+1)/2 \mid i$ and in particular $q+1 \mid i$, contradicting the definition of J_8. So $U_{-2} = \emptyset$. Thus only U_2 contributes to the fixed points, namely $qi \equiv q^3 i \bmod (q^3-1)(q+1)$ which is equivalent with $q(q^2-1)i \equiv 0 \bmod (q^3-1)(q+1)$. Since $\gcd(q(q^2-1), (q^3-1)(q+1)) = q^2-1$, this is equivalent with $(q^2-1)i \equiv 0 \bmod (q^3-1)(q+1)$, which is equivalent with $\phi_3 \mid i$. So by the definition of $J' = J_8$ we get $|C_{J'}(H)| = \frac{(q-1)^2}{4}$.

Suppose $J' = J_{11}$. Analogously to J_8, we can show that only the third parameter vector contributes to the fixed points, namely $qi \equiv q^3 i \bmod (q^3+1)(q-1)$ which is equivalent with $(q^2-1)i \equiv 0 \bmod (q^3+1)(q-1)$, which is equivalent with $\phi_6 \mid i$. So by the definition of $J' = J_{11}$ we get $|C_{J'}(H)| = \frac{(q-1)^2}{4}$.

Suppose $J' = J_{12}$. As we can see from Table A.2 each element of $J' = J_{12}$ is an equivalence class consisting of 24 vectors and we number these vectors according to their order in Table A.2, i.e. the vector (i, j) gets the number 1, the vector $((2q+1)i - qj, \phi_2(2i-j))$ gets the number 2, and so on. Let

$$U_1 := \left\{\left\{(i,j), \ldots\right\} \in C_{J'}(H) \;\middle|\; qi \equiv i,\ qj \equiv j\right\},$$
$$U_2 := \left\{\left\{(i,j), \ldots\right\} \in C_{J'}(H) \;\middle|\; qi \equiv (2q+1)i - qj,\ qj \equiv \phi_2(2i-j)\right\},$$
$$\vdots$$
$$U_{24} := \left\{\left\{(i,j), \ldots\right\} \in C_{J'}(H) \;\middle|\; qi \equiv j - i,\ qj \equiv j - 2i\right\},$$

where the congruences are always modulo q^2+q+1. Then, $C_{J'}(H) = \bigcup_{m=1}^{24} U_m$.

Suppose $\{(i,j),\ldots\} \in U_1$. Then we have $(q-1)i \equiv (q-1)j \equiv 0 \bmod q^2+q+1$. We have

$$\gcd\bigl((q-1)^2, q^2+q+1\bigr) = \begin{cases} 3 & \text{if } q \equiv 1 \bmod 3, \\ 1 & \text{if } q \not\equiv 1 \bmod 3. \end{cases}$$

If $q \equiv 1 \bmod 3$, then Lemma 3.4(b) (with $m_1 = 1$ and $m_2 = q^2+q+1$) implies $\frac{q^2+q+1}{3} \mid i$ and $\frac{q^2+q+1}{3} \mid j$, which implies $j \equiv 0$ or $j \equiv i \equiv -2qi$ or $2i \equiv j$ or $2i \equiv 0 \equiv (1-q^2)j \bmod q^2+q+1$, a contradiction to the definition of $J' = J_{12}$. If $q \not\equiv 1 \bmod 3$, then Lemma 3.4(b) (with $m_1 = 1$ and $m_2 = q^2+q+1$) implies $j \equiv 0 \bmod q^2+q+1$, contradicting the definition of $J' = J_{12}$. Hence $U_1 = \emptyset$. Analogously, using Lemma 3.4(b), it is straightforward to see that $U_m = \emptyset$ for all $m \neq 5, 7, 11, 18$.

Suppose $\{(i,j),\ldots\} \in U_5$. Then $qi \equiv i-j$ and $qj \equiv qj \bmod q^2+q+1$, which is equivalent with $j \equiv (1-q)i \bmod q^2+q+1$. Suppose $\{(i,j),\ldots\} \in U_7$. Then $qi \equiv (2q+1)i - qj$ and $qj \equiv 2(q+1)i - qj \bmod q^2+q+1$, which is equivalent with $(q+1)i \equiv qj \bmod q^2+q+1$. This again is equivalent with $j \equiv -qi \bmod q^2+q+1$. Suppose $\{(i,j),\ldots\} \in U_{11}$. Then $qi \equiv i$ and $qj \equiv qj - 2qi \bmod q^2+q+1$, which is equivalent with $i \equiv 0 \bmod q^2+q+1$. Suppose $\{(i,j),\ldots\} \in U_{18}$. Then $qi \equiv -i + (q+1)j$ and $qj \equiv -2i + (q+2)j \bmod q^2+q+1$, which is equivalent with $i \equiv j \bmod q^2+q+1$. So by the definition of $J' = J_{12}$ we get

$$\bigl|C_{J'}(H)\bigr| = \begin{cases} \frac{4(q+2)(q-1)}{24} & \text{if } q \equiv 1 \bmod 3, \\ \frac{4q(q+1)}{24} & \text{if } q \not\equiv 1 \bmod 3. \end{cases}$$

Suppose $J' = J_{13}$. As we can see from Table A.2, each element of J is an equivalence class consisting of 24 vectors which we number from 1 to 24 according to their order in Table A.2. We define $U_1, \ldots, U_{24}$ analogously to the case $J' = J_{12}$. Analogously to the case $J' = J_{12}$, using Lemma 3.4(b), it is straightforward to see that $U_m = \emptyset$ for all $m \neq 6, 17, 19, 23$.

Suppose $\{(i,j),\ldots\} \in U_6$. Then $qi \equiv i + (q-1)j$ and $qj \equiv 2i + (q-2)j \bmod q^2-q+1$, which is equivalent with $i \equiv j \bmod q^2-q+1$. Suppose $\{(i,j),\ldots\} \in U_{17}$. Then $qi \equiv j-i$ and $qj \equiv qj \bmod q^2-q+1$, which is equivalent with $j \equiv (q+1)i \bmod q^2-q+1$. Suppose $\{(i,j),\ldots\} \in U_{19}$. Then $qi \equiv (2q-1)i - qj$ and $qj \equiv 2(q-1)i - qj \bmod q^2-q+1$, which is equivalent with $j \equiv qi \bmod q^2-q+1$. Suppose $\{(i,j),\ldots\} \in U_{23}$. Then $qi \equiv -i$ and $qj \equiv qj - 2qi \bmod q^2-q+1$, which is equivalent with $i \equiv 0 \bmod q^2-q+1$. So by the definition of $J' = J_{13}$ we get

$$\bigl|C_{J'}(H)\bigr| = \begin{cases} \frac{4(q-2)(q+1)}{24} & \text{if } q \equiv -1 \bmod 3, \\ \frac{4q(q-1)}{24} & \text{if } q \not\equiv -1 \bmod 3. \end{cases}$$

Suppose $J' = J_{14}$. Analogously to J_8, we can define $U_{\pm 1}$ and $U_{\pm 2}$ and show $U_{\pm 1} = U_{\pm 2} = \emptyset$ (using the fact that $q-1, q+1, q(q^2-1), q(q^2+1)$ are relatively prime to q^4-q^2+1). Hence $|C_{J'}(H)| = 0$.

So finally, we have

$$\bigl|C_I(H)\bigr| = \bigl|C_J(H)\bigr| = \sum_{j \in J_G} \bigl|C_{J_j}(H)\bigr| = q^2 - 2q + 2 = p^{2t} - 2p^t + 2,$$

where $J_G := \{6, 8, 11, 12, 13, 14, 15\}$.

(c) From the character tables of ${}^3D_4(p^{t/3})$ and $G_2(p^t)$ in the CHEVIE library we get that if $t \nmid n$ (respectively $t \mid n$), then the number of regular semisimple conjugacy classes of ${}^3D_4(p^{t/3})$ (respectively $G_2(p^t)$) is equal to $p^{4t/3} - p^t - p^{t/3} + 2$ (respectively $p^{2t} - 2p^t + 2$). $\square$

Remark. The field automorphism α^n and the graph automorphism γ are both outer automorphisms of order 3 of ${}^3D_4(q)$. Hence, the number of regular semisimple classes of ${}^3D_4(q)$ fixed by α^n is equal to the number of regular semisimple classes of ${}^3D_4(q)$ fixed by γ. So when counting such conjugacy classes we may replace α^n by γ. By [14, p. 104], the subgroup ${}^3D_4(q)^\gamma$ of fixed points of ${}^3D_4(q)$ under the graph automorphism γ is isomorphic with $G_2(q)$. So after the reduction to $t = n$ the natural way to prove part (b) of Proposition 4.2 in case $t \mid n$ would be to show that every regular semisimple class of ${}^3D_4(q)$ fixed by γ intersects ${}^3D_4(q)^\gamma \cong G_2(q)$ in exactly one regular semisimple conjugacy class and that every regular semisimple class of ${}^3D_4(q)^\gamma$ occurs in this way. We have not been able to find such a proof using only general arguments on endomorphisms of groups of Lie type. In the following, we sketch a proof of Proposition 4.2, part (b), in case $t \mid n$, which uses specific properties of the structure of ${}^3D_4(q)$.

We continue to use the notation of the proof of Proposition 4.2. Suppose $t \mid n$. Then $\langle F, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^n \circ \gamma, \bar{\alpha}^t\rangle = \langle \gamma, \bar{\alpha}^t\rangle = \langle \bar{\alpha}^t \circ \gamma, \bar{\alpha}^t\rangle$. Thus $|C_I(H)| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\langle F, \bar{\alpha}^t\rangle}| = |\mathcal{S}_{\mathrm{reg}}(\overline{G})^{\bar{\alpha}^t \circ \gamma} \cap \mathcal{S}_{\mathrm{reg}}(\overline{G})^{\bar{\alpha}^t}| = |\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^{\alpha^t}|$. It suffices to show that $|\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^{\alpha^t}| = |\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^{\gamma}| = |\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t)^{\gamma})| = |\mathcal{S}_{\mathrm{reg}}(G_2(p^t))|$ and so we may assume that $t = n$.

Let $\widehat{G} = G \rtimes \langle\gamma\rangle$ and $s \in \mathcal{S}_{\mathrm{reg}}(G)$ such that $T = C_G(s)$ is a maximal torus. Since α^n stabilizes the conjugate G-class $(s)_G$, it follows that $[C_{\widehat{G}}(s) : T] = 3$ and so $C_{\widehat{G}}(s) \leqslant N_{\widehat{G}}(T)$. But $N_{\widehat{G}}(T) = N_G(T) \rtimes \mathbb{Z}_3$, so

$$C_{\widehat{G}}(s) = T \rtimes \mathbb{Z}_3.$$

By [14, 9–1(3)] $\widehat{G}$ contains exactly two conjugacy classes C_1 and C_2 of elements of order 3 such that $C_i \not\subseteq G$. As shown in the proof of [14, 9–1(3)] we may suppose $\gamma \in C_1$ and $g\gamma \in C_2$ for some $g \in M := G^\gamma \simeq G_2(q)$ with $|g| = 3$. In addition, $G^{g\gamma} \cong \mathrm{PGL}_3^\eta(q)$ or $[q^5].\mathrm{SL}_2(q)$ according as $q \equiv \eta 1 \bmod 3$ or $3 \mid q$. In the later case, moreover $G^{g\gamma} = C_M(g) \leqslant M$. Replacing s by its G-conjugate, we may suppose $\gamma \in C_{\widehat{G}}(s)$ or $g\gamma \in C_{\widehat{G}}(s)$. We claim that we may suppose

$$\gamma \in C_{\widehat{G}}(s) \quad \text{or equivalently} \quad s \in M.$$

Suppose $\gamma \notin C_{\widehat{G}}(s)$, so that $q \equiv \eta 1 \bmod 3$ and $s \in K := G^{g\gamma} \simeq \mathrm{PGL}_3^\eta(q)$. Since $|g| = 3$, it follows by [17, Appendix B], that $C_M(g) \simeq \mathrm{SL}_3^\eta(q)$ or $\mathrm{GL}_2^\eta(q)$. But $g \in K$ and $C_M(g) \leqslant K$, so $C_M(g) \simeq \mathrm{GL}_2^\eta(q)$.

Since $T = C_G(s)$ is a maximal torus of G, it follows that $C_K(s) \leqslant T$ is a maximal torus of K, so that $C_K(s) \simeq \mathbb{Z}_{q^2-1}, \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1}$ or $\mathbb{Z}_{q^2+\eta q+1}$. In the first two cases, $C_K(s)$ is conjugate in K to a subgroup of $C_M(g)$, so that we may suppose $s \in C_M(g) \leqslant M$.

Suppose $C_K(s) \simeq \mathbb{Z}_{q^2+\eta q+1}$, so that $T \simeq \mathbb{Z}_{q^2+\eta q+1} \times \mathbb{Z}_{q^2+\eta q+1}$. Let x be an element of M of order 3 such that $L_\eta := C_M(x) \simeq \mathrm{SL}_3^\eta(q)$. By [21, Table II], $L_\eta \leqslant C_G(x) \simeq \mathbb{Z}_{q^2+\eta q+1} \circ \mathrm{SL}_3^\eta(q).\mathbb{Z}_3$. If $Z := Z(C_G(x))$, then $Z \simeq \mathbb{Z}_{q^2+\eta q+1}$ and $Z \cap L_\eta = Z(L_\eta) \simeq \mathbb{Z}_3$. Let $Z' \simeq \mathbb{Z}_{q^2+\eta q+1}$ be a maximal torus of L_η, so that $C_G(ZZ') \simeq \mathbb{Z}_{q^2+\eta q+1} \times \mathbb{Z}_{q^2+\eta q+1}$. Replacing $C_G(ZZ')$ by a G-conjugate, we may suppose $T = C_G(ZZ')$.

As shown in the proof of (2.11) in [2],

$$N_M(Z') = N_{N_M(L_\eta)}(Z') = \langle Z', \rho, \sigma \rangle \quad \text{and} \quad N_{N_G(Z)}(T) = \langle T, \rho, \sigma \rangle$$

where $\rho \in N_M(L_\eta) \setminus L_\eta$ with $|\rho| = 2$ and $\sigma \in L_\eta$ with $|\sigma| = 3$, and $\rho\sigma = \sigma\rho$. Since $C_G(L_\eta) = Z$ and since $\gamma \in C_{\widehat{G}}(L_\eta)$, it follows that γ normalizes Z and so normalizes $T = C_G(ZZ')$. But $g\gamma$ also normalizes T, so $g \in N_G(T) \setminus T$.

Now $N_G(T)/T \simeq \mathrm{SL}_2(3)$ has a cyclic Sylow 3-subgroup, we may suppose $g = \sigma$, so g centralizes ρ. Since $\rho \in M$, it follows that $g\gamma$ centralizes ρ and $\rho \in G^{g\gamma} = K$. In particular, ρ normalizes $T^{g\gamma} = C_K(s)$ and $2 \mid |N_K(C_K(s))/C_K(s)|$. This is impossible, since $N_K(C_K(s))/C_K(s) \simeq \mathbb{Z}_3$.

Thus the claim holds and if γ stabilizes a G-class $(s)_G$, then

$$M \cap (s)_G \neq \emptyset.$$

If $t, t' \in M \cap (s)_G$, then $C_M(t)$ and $C_M(t')$ are two maximal tori of M. Since $C_G(t)$ and $C_G(t')$ are G-conjugate, it follows that $C_M(t)$ and $C_M(t')$ are M-conjugate and so $(t)_M = (t')_M$. Thus $|\mathcal{S}_{\mathrm{reg}}({}^3D_4(p^t))^\gamma| = |\mathcal{S}_{\mathrm{reg}}(G_2(p^t))|$.

Proposition 4.3. *Let $t \mid 3n$, $H = \langle \alpha^t \rangle \leqslant O$ and let $(I, J) \in \{({}_BI_6, {}_PI_8), ({}_BI_7, {}_PI_2), ({}_GI_{14} \cup {}_GI_{22}, {}_QI_{14} \cup {}_QI_{15}), ({}_GI_{17} \cup {}_GI_{26}, {}_PI_{11} \cup {}_PI_{14}), ({}_BI_{12}, {}_QI_{16}), ({}_BI_{13}, {}_QI_{17}), ({}_BI_{14}, {}_QI_{18}), ({}_BI_{15}, {}_QI_{19})({}_BI_{16}, {}_QI_{20})\}$. Then $|C_I(H)| = |C_J(H)|$.*

Proof. By construction, ${}_P\chi_8(k) = {}_B\chi_6(k)^P$ is induced from the α-stable Borel subgroup B for all $k \in {}_BI_6 = {}_PI_8$ (see p. 790 in [16]) and induction of characters induces a bijection from $\{{}_B\chi_6(k) \mid k \in {}_BI_6\}$ onto $\{{}_P\chi_8(k) \mid k \in {}_PI_8\}$ mapping ${}_B\chi_6(k)$ to ${}_P\chi_8(k)$. We have ${}_P\chi_8(k^\alpha) = {}_P\chi_8(k)^\alpha = ({}_B\chi_6(k)^P)^\alpha = ({}_B\chi_6(k)^\alpha)^P = ({}_B\chi_6(k^\alpha))^P$. So the above-mentioned bijection is an isomorphism of H-sets. Hence, ${}_BI_6 \simeq {}_PI_8$ as H-sets and $|C_{{}_BI_6}(H)| = |C_{{}_PI_8}(H)|$. Analogously, $|C_{{}_BI_{14}}(H)| = |C_{{}_QI_{18}}(H)|$, $|C_{{}_BI_{15}}(H)| = |C_{{}_QI_{19}}(H)|$ and $|C_{{}_BI_{16}}(H)| = |C_{{}_QI_{20}}(H)|$ (see the construction of ${}_Q\chi_{18}(k)$, ${}_Q\chi_{19}(k)$ and ${}_Q\chi_{20}(k)$ on p. 795 in [16]).

Let (I, J) be one of the remaining pairs. Then $I = J$ as sets. Using the character values on the classes listed in the last column of Table A.5, we know that the action of α on I, J is given by $x^\alpha = px$ for all $x \in I, J$. Hence $I \simeq J$ as H-sets. □

5. Uno's invariant conjecture for ${}^3D_4(q)$, q odd

In this section, we prove Uno's invariant conjecture for $G = {}^3D_4(q)$ in the defining characteristic p, where $q = p^n$ with an odd prime p. As in the previous section, let $O = \mathrm{Out}(G) = \langle \alpha \rangle$, where α is a field automorphism of order $3n$. We fix a Borel subgroup B and maximal parabolic subgroups P and Q of G containing B as in [16]. In particular, we may assume that α stabilizes B, P and Q.

By the remarks on p. 152 in [19], G has only two p-blocks, the principal block B_0 and one defect-0-block (corresponding to the Steinberg character). Hence we have to verify Uno's conjecture only for the principal block B_0.

According to the Borel–Tits theorem [5], the normalizers of radical p-subgroups are parabolic subgroups. The radical p-chains of G (up to G-conjugacy) are given in Table 1.

Table 1
Radical p-chains of G

C		$N_G(C)$	$N_A(C)$
C_1	$\{1\}$	G	A
C_2	$\{1\} < O_p(P)$	P	$P \rtimes \langle\alpha\rangle$
C_3	$\{1\} < O_p(P) < O_p(B)$	B	$B \rtimes \langle\alpha\rangle$
C_4	$\{1\} < O_p(Q)$	Q	$Q \rtimes \langle\alpha\rangle$
C_5	$\{1\} < O_p(Q) < O_p(B)$	B	$B \rtimes \langle\alpha\rangle$
C_6	$\{1\} < O_p(B)$	B	$B \rtimes \langle\alpha\rangle$

Since C_5 and C_6 have the same normalizers $N_G(C_5) = N_G(C_6)$ and $N_A(C_5) = N_A(C_6)$, it follows that

$$k\big(N_G(C_5), B_0, d, u, [r]\big) = k\big(N_G(C_6), B_0, d, u, [r]\big)$$

for all $d \in \mathbb{N}$ and $u \mid 3n$. Thus the contribution of C_5 and C_6 in the alternating sum of Uno's invariant conjecture is zero. So Uno's invariant conjecture for G is equivalent to

$$k\big(G, B_0, d, u, [r]\big) + k\big(B, B_0, d, u, [r]\big) = k\big(P, B_0, d, u, [r]\big) + k\big(Q, B_0, d, u, [r]\big) \tag{3}$$

for all $d \in \mathbb{N}$, $u \mid 3n$ and $1 \leqslant r < (p+1)/2$.

Theorem 5.1. *Let $p > 2$ be a prime and $\widetilde{B}$ a p-block of $G = {}^3D_4(p^n)$ of positive defect. Then $\widetilde{B}$ satisfies Uno's invariant conjecture.*

Proof. By the proceeding remarks, we can assume $\widetilde{B} = B_0$. Suppose $u \mid 3n$ and set $t := \frac{3n}{u}$ and $H := \langle \alpha^t \rangle$. Let $S \in \{G, B, P, Q\}$. By the character tables in the appendix of [16], we have $k(S, B_0, d, u, [r]) = 0$ when $d \notin \{5n, 8n, 9n, 11n, 12n\}$ and $[r] \notin \{[1], [2]\}$. Moreover, if $[r] = [2]$, then we may suppose $d = 8n$ or $9n$.

(i) If $d = 5n$, then we have $k(G, B_0, d, u, [1]) = |C_{{}_G I_7}(H)| = k(P, B_0, d, u, [1]) = |C_{{}_P I_{16}}(H)| = 1$, and $k(B, B_0, d, u, [1]) = k(Q, B_0, d, u, [1]) = 0$ by Tables A.3 and A.8. Thus (3) holds in this case.

(ii) If $d = 8n$, then we have $k(G, B_0, d, u, [1]) = |C_{{}_G I_{12}}(H)| = k(Q, B_0, d, u, [1]) = |C_{{}_Q I_9}(H)| = 1$. By Tables A.4 and A.8, we have $k(B, B_0, d, u, [1]) = |C_{{}_B I_{17}}(H)| = p^t - 1$ and $k(P, B_0, d, u, [1]) = \sum_{j \in \{15,21,22\}} |C_{{}_P I_j}(H)| = p^t - 1$. In addition, we have $k(B, B_0, d, u, [2]) = \sum_{j \in J_B} |C_{{}_B I_j}(H)| = 4$ with $J_B := \{18, 19, 20, 21\}$ and $k(P, B_0, d, u, [2]) = \sum_{j \in J_P} |C_{{}_P I_j}(H)| = 4$ with $J_P := \{17, 18, 19, 20\}$. Thus (3) holds in this case.

(iii) If $d = 9n$ and $[r] = [1]$, then Table A.5 and Proposition 4.3 imply, that (3) is equivalent to $|C_{{}_G I_{11}}(H)| = |C_{{}_Q I_8}(H)|$, which is true by Table A.8. If $d = 9n$ and $[r] = [2]$, then Table A.5 and Proposition 4.3 imply, that (3) is equivalent to

$$\sum_{j \in J_G} \big|C_{{}_G I_j}(H)\big| + \sum_{j \in J_B} \big|C_{{}_B I_j}(H)\big| = \sum_{j \in J_P} \big|C_{{}_P I_j}(H)\big| + \sum_{j \in J_Q} \big|C_{{}_Q I_j}(H)\big|$$

with the index sets $J_G := \{3, 4, 5, 6\}$, $J_B := \{8, 9, 10, 11\}$, $J_P := \{9, 10, 12, 13\}$ and $J_Q := \{10, 11, 12, 13\}$. By Table A.8, the sums on both sides of the above equation are equal. Thus (3) also holds in this case.

(iv) If $d = 11n$, then Table A.6 and Proposition 4.3 imply, that (3) is equivalent to

$$\sum_{j\in J_G} \left|C_{GI_j}(H)\right| + \left|C_{BI_5}(H)\right| = \left|C_{PI_7}(H)\right| + \left|C_{QI_2}(H)\right| + \left|C_{QI_7}(H)\right|$$

with $J_G := \{2, 10, 16, 19, 25, 28\}$. By Table A.8, we have

$$\sum_{j\in J_G} \left|C_{GI_j}(H)\right| + \left|C_{BI_5}(H)\right| = p^t + p^{\delta t} - 1 = \left|C_{PI_7}(H)\right| + \left|C_{QI_2}(H)\right| + \left|C_{QI_7}(H)\right|$$

with δ as in Lemma 3.2. Thus (3) also holds in this case.

(v) If $d = 12n$, then Table A.7 implies, that (3) is equivalent to

$$\sum_{j\in J_G} \left|C_{GI_j}(H)\right| + \sum_{j\in J_B} \left|C_{BI_j}(H)\right| = \sum_{j\in J_P} \left|C_{PI_j}(H)\right| + \sum_{j\in J_Q} \left|C_{QI_j}(H)\right|$$

with $J_G := \{1, 9, 13, 15, 18, 20, 21, 23, 24, 27, 29, 30, 31, 32, 33\}$, $J_B := \{1, 2, 3, 4\}$, $J_P := \{1, 3, 4, 5, 6\}$ and $J_Q := \{1, 3, 4, 5, 6\}$. By Tables A.7 and A.8, we have

$$k\bigl(G, B_0, d, u, [1]\bigr) + k\bigl(B, B_0, d, u, [1]\bigr) = \sum_{j\in J_G} \left|C_{GI_j}(H)\right| + \sum_{j\in J_B} \left|C_{BI_j}(H)\right| = 2p^t p^{\delta t}$$

and

$$k\bigl(P, B_0, d, u, [1]\bigr) + k\bigl(Q, B_0, d, u, [1]\bigr) = \sum_{j\in J_P} \left|C_{PI_j}(H)\right| + \sum_{j\in J_Q} \left|C_{QI_j}(H)\right| = 2p^t p^{\delta t}$$

with δ as in Lemma 3.2. Thus (3) also holds in this case. $\square$

Acknowledgments

Part of this work was done during a visit of the second author at the Department of Mathematics of the University of Auckland. He wishes to express his sincere thanks to all the persons of the department for their hospitality, and also to the Marsden Fund of New Zealand who supported his visit.

Appendix A

Table A.1
Parameter sets for the irreducible characters of G, B, P, Q

Parameter set	Characters	Parameters	Equivalence-relation	Number of characters
${}_G I_1 = \cdots = {}_G I_{12}$	$\chi_1, \ldots, \chi_{12}$			1
${}_G I_{13} = {}_G I_{14}$	$\chi_{13}(k)$, $\chi_{14}(k)$	$k = 0, \ldots, q-2$, $k \neq 0, \frac{q-1}{2}$	$\{k \equiv -k\}$	$\frac{q-3}{2}$
${}_G I_{15} = {}_G I_{16} = {}_G I_{17}$	$\chi_{15}(k)$, $\chi_{16}(k)$, $\chi_{17}(k)$	$k = 0, \ldots, q^2+q$, $k \neq 0$	$\{k \equiv -k\}$	$\frac{q^2+q}{2}$
${}_G I_{18} = {}_G I_{19}$	$\chi_{18}(k)$, $\chi_{19}(k)$	$k = 0, \ldots, q^3-2$, $q-1 \nmid k$, $k \neq 0, \frac{q^3-1}{2}$	$\{k \equiv -k\}$	$\frac{q^3-q^2-q-3}{2}$
${}_G I_{20}$	$\chi_{20}(k,l)$	see the remarks in Section 4		$\frac{q^4-4q^3+2q^2-2q+15}{12}$
${}_G I_{21} = {}_G I_{22}$	$\chi_{21}(k)$, $\chi_{22}(k)$	$k = 0, \ldots, q$, $k \neq 0, \frac{q+1}{2}$	$\{k \equiv -k\}$	$\frac{q-1}{2}$
${}_G I_{23}$	$\chi_{23}(k)$	see the remarks in Section 4		$\frac{q^4-2q+1}{4}$
${}_G I_{24} = {}_G I_{25} = {}_G I_{26}$	$\chi_{24}(k)$, $\chi_{25}(k)$, $\chi_{26}(k)$	$k = 0, \ldots, q^2-q$, $k \neq 0$	$\{k \equiv -k\}$	$\frac{q^2-q}{2}$
${}_G I_{27} = {}_G I_{28}$	$\chi_{27}(k)$, $\chi_{28}(k)$	$k = 0, \ldots, q^3$, $q+1 \nmid k$, $k \neq 0, \frac{q^3+1}{2}$	$\{k \equiv -k\}$	$\frac{q^3-q^2+q-1}{2}$
${}_G I_{29}$	$\chi_{29}(k)$	see the remarks in Section 4		$\frac{q^4-2q^3+1}{4}$
${}_G I_{30}$	$\chi_{30}(k,l)$	see the remarks in Section 4		$\frac{q^4+2q^3-q^2-2q}{24}$
${}_G I_{31}$	$\chi_{31}(k,l)$	see the remarks in Section 4		$\frac{q^4-2q^3-q^2+2q}{24}$
${}_G I_{32}$	$\chi_{32}(k)$	see the remarks in Section 4		$\frac{q^4-q^2}{4}$
${}_G I_{33}$	$\chi_{33}(k,l)$	see the remarks in Section 4		$\frac{q^4-2q^3+2q^2-4q+3}{12}$
${}_B I_1$	${}_B\chi_1(k,l)$	$k = 0, \ldots, q^3-2$, $l = 0, \ldots, q-2$		$(q^3-1)(q-1)$
${}_B I_2$	${}_B\chi_2(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_B I_3$	${}_B\chi_3(k)$	$k = 0, \ldots, q^3-2$		q^3-1
${}_B I_4$	${}_B\chi_4$			1
${}_B I_5$	${}_B\chi_5(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_B I_6$	${}_B\chi_6(k)$	$k = 1, \ldots, q+1$		$q+1$
${}_B I_7$	${}_B\chi_7(k)$	$k = 0, \ldots, q-2$		$q-1$
${}_B I_8 = \cdots = {}_B I_{11}$	${}_B\chi_8, \ldots, {}_B\chi_{11}$			1
${}_B I_{12}$	${}_B\chi_{12}(k)$	$k = 0, \ldots, q^3-2$		q^3-1
${}_B I_{13}$	${}_B\chi_{13}(k)$	$k = 0, \ldots, q^2+q$		q^2+q+1
${}_B I_{14} = {}_B I_{15}$	${}_B\chi_{14}(k)$, ${}_B\chi_{15}(k)$	$k = 0, 1$		2

Table A.1 (*continued*)

Parameter set	Characters	Parameters	Equivalence-relation	Number of characters
${}_BI_{16}$	${}_B\chi_{16}(k)$	$k=1,\dots,q-1$		$q-1$
${}_BI_{17}$	${}_B\chi_{17}(k)$	$k=0,\dots,q^3-2$		q^3-1
${}_BI_{18}=\cdots$ $={}_BI_{21}$	${}_B\chi_{18},\dots,{}_B\chi_{21}$			1
${}_PI_1={}_PI_2$	${}_P\chi_1(k)$, ${}_P\chi_2(k)$	$k=0,\dots,q-2$		$q-1$
${}_PI_3$	${}_P\chi_3(k,l)$	$k=0,\dots,q^3-2$ $l=0,\dots,q-2;\ k\neq 0$	$\{(k,l)$ $\equiv(-k,k+l)\}$	$\frac{(q^3-2)(q-1)}{2}$
${}_PI_4$	${}_P\chi_4(k)$	$k=0,\dots,$ q^4-q^3+q-2 $q^3+1\nmid k$	$\{k\equiv q^3k\}$	$\frac{q^3(q-1)}{2}$
${}_PI_5$	${}_P\chi_5(k)$	$k=0,\dots,q^3-2$		q^3-1
${}_PI_6$	${}_P\chi_6$			1
${}_PI_7$	${}_P\chi_7(k)$	$k=0,\dots,q-2$		$q-1$
${}_PI_8$	${}_P\chi_8(k)$	$k=1,\dots,q+1$		$q+1$
${}_PI_9={}_PI_{10}$	${}_P\chi_9$, ${}_P\chi_{10}$			1
${}_PI_{11}$	${}_P\chi_{11}(k)$	$k=0,\dots,q^2+q$ $k\neq 0$	$\{k\equiv -k\}$	$\frac{q^2+q}{2}$
${}_PI_{12}={}_PI_{13}$	${}_P\chi_{12}$, ${}_P\chi_{13}$			1
${}_PI_{14}$	${}_P\chi_{14}(k)$	$k=0,\dots,q^2-q$ $k\neq 0$	$\{k\equiv -k\}$	$\frac{q^2-q}{2}$
${}_PI_{15}=\cdots$ $={}_PI_{20}$	${}_P\chi_{15},\dots,{}_P\chi_{20}$			1
${}_PI_{21}$	${}_P\chi_{21}(k)$	$k=0,\dots,q^3-2$ $k\neq 0,\ \frac{q^3-1}{2}$	$\{k\equiv -k\}$	$\frac{q^3-3}{2}$
${}_PI_{22}$	${}_P\chi_{22}(k)$	$k=0,\dots,q^3;$ $k\neq 0,\ \frac{q^3+1}{2}$	$\{k\equiv -k\}$	$\frac{q^3-1}{2}$
${}_QI_1={}_QI_2$	${}_Q\chi_1(k)$, ${}_Q\chi_2(k)$	$k=0,\dots,q^3-2$		q^3-1
${}_QI_3$	${}_Q\chi_3(k,l)$	$k=0,\dots,q^3-2$ $l=0,\dots,q-2;\ l\neq 0$	$\{(k,l)$ $\equiv(k+\phi_3 l,-l)\}$	$\frac{(q^3-1)(q-2)}{2}$
${}_QI_4$	${}_Q\chi_4(k)$	$k=0,\dots,q^4+q^3-q-2$ $q+1\nmid k$	$\{k\equiv q^3k\}$	$\frac{q(q^3-1)}{2}$
${}_QI_5$	${}_Q\chi_5(k)$	$k=0,\dots,q-2$		$q-1$
${}_QI_6=\cdots$ $={}_QI_{13}$	${}_Q\chi_6,\dots,{}_Q\chi_{13}$			1
${}_QI_{14}$	${}_Q\chi_{14}(k)$	$k=0,\dots,q-2$ $k\neq 0,\ \frac{q-1}{2}$	$\{k\equiv -k\}$	$\frac{q-3}{2}$
${}_QI_{15}$	${}_Q\chi_{15}(k)$	$k=0,\dots,q;\ k\neq 0,\ \frac{q+1}{2}$	$\{k\equiv -k\}$	$\frac{q-1}{2}$
${}_QI_{16}$	${}_Q\chi_{16}(k)$	$k=0,\dots,q^3-2$		q^3-1
${}_QI_{17}$	${}_Q\chi_{17}(k)$	$k=0,\dots,q^2+q$		q^2+q+1
${}_QI_{18}={}_QI_{19}$	${}_Q\chi_{18}(k)$, ${}_Q\chi_{19}(k)$	$k=0,1$		2
${}_QI_{20}$	${}_Q\chi_{20}(k)$	$k=1,\dots,q-1$		$q-1$

For the parameter sets ${}_GI_{20},{}_GI_{23},{}_GI_{29},\dots,{}_GI_{33}$ see the remarks at the beginning of Section 4.

Table A.2
Parameter sets for the regular semisimple conjugacy classes of G

Param. set	Classes	Parameters	Equivalence relation	Number of classes
J_6	$c_{6,0}(i,j)$	$i=0,\dots,q^3-2$ $j=0,\dots,q-2$ $i,j\neq 0$ $i\neq\phi_3 l$ or $j\neq l$, $l=0,\dots,q-2$ $i\neq\phi_3 l$ or $j\neq 2l$, $l=0,\dots,q-2$ $q-1\nmid i-j$, $q-1\nmid i-2j$	$\{(i,j)$ $\equiv(-i,-j)$ $\equiv(i,i-j)$ $\equiv(-i,-i+j)$ $\equiv(i-\phi_3 j,-j)$ $\equiv(i-\phi_3 j,i-2j)$ $\equiv(-i+\phi_3 j,j)$ $\equiv(-i+\phi_3 j,-i+2j)$ $\equiv(q\phi_2 i-\phi_3 j,i-2j)$ $\equiv(q\phi_2 i-\phi_3 j,i-j)$ $\equiv(-q\phi_2 i+\phi_3 j,-i+j)$ $\equiv(-q\phi_2 i+\phi_3 j,-i+2j)\}$	$\frac{1}{12}(q^4-4q^3$ $+2q^2-2q$ $+15)$
J_8	$c_{8,0}(i)$	$i=0,\dots,$ q^4+q^3-q-2 $q+1,q^3-1\nmid i$,	$\{i\equiv -i\equiv q^3 i\equiv -q^3 i\}$	$\frac{q^4-2q+1}{4}$
J_{11}	$c_{11,0}(i)$	$i=0,\dots,$ q^4-q^3+q-2 $q-1,q^3+1\mid i$,	$\{i\equiv -i\equiv q^3 i\equiv -q^3 i\}$	$\frac{q^4-2q^3+1}{4}$
J_{12}	$c_{12,0}(i,j)$	$i=0,\dots,q^2+q$ $j=0,\dots,q^2+q$ $j\neq 0,-2qi$ $2i\neq j,(1-q^2)j$	$\{(i,j)$ $\equiv((2q+1)i-qj,\phi_2(2i-j))$ $\equiv(qj-(2q+1)i,(2q+1)j-2\phi_2 i)$ $\equiv(i-\phi_2 j,-2qi-j)$ $\equiv(i-j,qj)$ $\equiv(i-\phi_2 j,2i-(q+2)j)$ $\equiv((2q+1)i-qj,2\phi_2 i-qj)$ $\equiv(i,2i-\phi_2 j)$ $\equiv(i-j,\phi_1 j-2qi)$ $\equiv(\phi_2 j-i,\phi_2 j)$ $\equiv(i,qj-2qi)$ $\equiv(i-j,2i-j)$ $\equiv(-i,-j)$ $\equiv(qj-(2q+1)i,\phi_2(j-2i))$ $\equiv((2q+1)i-qj,2\phi_2 i-(2q+1)j)$ $\equiv(\phi_2 j-i,2qi+j)$ $\equiv(j-i,-qj)$ $\equiv(\phi_2 j-i,(q+2)j-2i)$ $\equiv(qj-(2q+1)i,qj-2\phi_2 i)$ $\equiv(-i,\phi_2 j-2i)$ $\equiv(j-i,2qi-\phi_1 j)$ $\equiv(i-\phi_2 j,-\phi_2 j)$ $\equiv(-i,2qi-qj)$ $\equiv(j-i,j-2i)\}$	$\frac{1}{24}(q^4+2q^3$ $-q^2-2q)$

Table A.2 (*continued*)

Param. set	Classes	Parameters	Equivalence relation	Number of classes
J_{13}	$c_{13,0}(i,j)$	$i=0,\dots,q^2-q$ $j=0,\dots,q^2-q$ $j\neq 0,2qi$ $2i\neq j,(1-q^2)j$	$\{(i,j)$ $\equiv(qj-(2q-1)i,\phi_1(j-2i))$ $\equiv((2q-1)i-qj,2\phi_1 i-(2q-1)j)$ $\equiv(i+\phi_1 j,2qi-j)$ $\equiv(i-j,-qj)$ $\equiv(i+\phi_1 j,2i+(q-2)j)$ $\equiv(qj-(2q-1)i,qj-2\phi_1 i)$ $\equiv(i,2i+\phi_1 j)$ $\equiv(i-j,2qi-\phi_2 j)$ $\equiv(-i-\phi_1 j,-\phi_1 j)$ $\equiv(i,2qi-qj)$ $\equiv(i-j,2i-j)$ $\equiv(-i,-j)$ $\equiv((2q-1)i-qj,\phi_1(2i-j))$ $\equiv(qj-(2q-1)i,(2q-1)j-2\phi_1 i)$ $\equiv(-i-\phi_1 j,j-2qi)$ $\equiv(j-i,qj)$ $\equiv(-i-\phi_1 j,-2i-(q-2)j)$ $\equiv((2q-1)i-qj,2\phi_1 i-qj)$ $\equiv(-i,-2i-\phi_1 j)$ $\equiv(j-i,\phi_2 j-2qi)$ $\equiv(i+\phi_1 j,\phi_1 j)$ $\equiv(-i,qj-2qi)$ $\equiv(j-i,j-2i)\}$	$\frac{1}{24}(q^4-2q^3$ $-q^2+2q)$
J_{14}	$c_{14,0}(i)$	$i=0,\dots,q^4-q^2$ $i\neq 0$	$\{i\equiv -i\equiv q^3 i\equiv -q^3 i\}$	$\frac{q^4-q^2}{4}$
J_{15}	$c_{15,0}(i,j)$	$i=0,\dots,q^3$ $j=0,\dots,q$ $i,j\neq 0$ $i\neq\phi_6 l$ or $j\neq l$, $l=0,\dots,q$ $i\neq\phi_6 l$ or $j\neq 2l$, $l=0,\dots,q$ $q+1\nmid i-j$, $q+1\nmid i-2j$	$\{(i,j)\equiv(-i,-j)$ $\equiv(i,i-j)$ $\equiv(-i,-i+j)$ $\equiv(i-\phi_6 j,-j)$ $\equiv(i-\phi_6 j,i-2j)$ $\equiv(-i+\phi_6 j,j)$ $\equiv(-i+\phi_6 j,-i+2j)$ $\equiv(q\phi_1 i-\phi_6 j,i-2j)$ $\equiv(q\phi_1 i-\phi_6 j,i-j)$ $\equiv(-q\phi_1 i+\phi_6 j,-i+j)$ $\equiv(-q\phi_1 i+\phi_6 j,-i+2j)\}$	$\frac{1}{12}(q^4-2q^3$ $+2q^2-4q$ $+3)$

For the definition of the ϕ_i's see the beginning of Section 3.

Table A.3
The irreducible characters of the chain normalizers of defect $5n$

Group	Character	Degree	Parameter	Number	Class
G	χ_7	$q^7\phi_{12}$	${}_G I_7$	1	
P	${}_P\chi_{16}$	$q^7\phi_1$	${}_P I_{16}$	1	

Table A.4
The irreducible characters of the chain normalizers of defect $8n$

Group	Character	Degree	Parameter	Number	Class
G	χ_{12}	$q^4\phi_3\phi_6\phi_{12}$	${}_G I_{12}$	1	
B	${}_B\chi_{17}(k)$	$q^4\phi_1$	${}_B I_{17}$	q^3-1	$c_{11,0}(i)$
	${}_B\chi_{18}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_B I_{18}$	1	$c_{1,11}, c_{2,0}$
	${}_B\chi_{19}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_B I_{19}$	1	$c_{1,11}, c_{2,0}$
	${}_B\chi_{20}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_B I_{20}$	1	$c_{1,11}, c_{2,0}$
	${}_B\chi_{21}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_B I_{21}$	1	$c_{1,11}, c_{2,0}$
P	${}_P\chi_{15}$	$q^4\phi_1$	${}_P I_{15}$	1	
	${}_P\chi_{17}$	$\frac{1}{2}q^4\phi_1\phi_2\phi_6$	${}_P I_{17}$	1	$c_{1,4}$
	${}_P\chi_{18}$	$\frac{1}{2}q^4\phi_1\phi_2\phi_6$	${}_P I_{18}$	1	$c_{1,4}$
	${}_P\chi_{19}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_P I_{19}$	1	$c_{1,4}$
	${}_P\chi_{20}$	$\frac{1}{2}q^4\phi_1^2\phi_3$	${}_P I_{20}$	1	$c_{1,4}$
	${}_P\chi_{21}(k)$	$q^4\phi_1\phi_2\phi_6$	${}_P I_{21}$	$\frac{1}{2}(q^3-3)$	$c_{8,0}(i)$
	${}_P\chi_{22}(k)$	$q^4\phi_1^2\phi_3$	${}_P I_{22}$	$\frac{1}{2}(q^3-1)$	$c_{11,0}(i)$
Q	${}_Q\chi_9$	$q^4\phi_1\phi_3$	${}_Q I_9$	1	

Table A.5
The irreducible characters of the chain normalizers of defect $9n$

Group	Character	Degree	Parameter	Number	Class
G	χ_3	$\frac{1}{2}q^3\phi_2^2\phi_{12}$	${}_G I_3$	1	
	χ_4	$\frac{1}{2}q^3\phi_2^2\phi_6^2$	${}_G I_4$	1	
	χ_5	$\frac{1}{2}q^3\phi_1^2\phi_3^2$	${}_G I_5$	1	
	χ_6	$\frac{1}{2}q^3\phi_1^2\phi_{12}$	${}_G I_6$	1	
	χ_{11}	$q^3\phi_3\phi_6\phi_{12}$	${}_G I_{11}$	1	
	$\chi_{14}(k)$	$q^3\phi_2\phi_3\phi_6\phi_{12}$	${}_G I_{14}$	$\frac{1}{2}(q-3)$	$c_{11,0}(i)$
	$\chi_{17}(k)$	$q^3\phi_2\phi_6^2\phi_{12}$	${}_G I_{17}$	$\frac{1}{2}q(q+1)$	$c_{5,0}(i)$
	$\chi_{22}(k)$	$q^3\phi_1\phi_3\phi_6\phi_{12}$	${}_G I_{22}$	$\frac{1}{2}(q-1)$	$c_{10,0}(i)$
	$\chi_{26}(k)$	$q^3\phi_1\phi_3^2\phi_{12}$	${}_G I_{26}$	$\frac{1}{2}q(q-1)$	$c_{11,0}(i)$
B	${}_B\chi_7(k)$	$q^3\phi_1\phi_3$	${}_B I_7$	$q-1$	$c_{8,0}(i)$
	${}_B\chi_8$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_8$	1	$c_{1,8}, c_{4,0}$
	${}_B\chi_9$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_9$	1	$c_{1,8}, c_{4,0}$
	${}_B\chi_{10}$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_{10}$	1	$c_{1,8}, c_{4,0}$
	${}_B\chi_{11}$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_{11}$	1	$c_{1,8}, c_{4,0}$
	${}_B\chi_{12}(k)$	$q^3\phi_1$	${}_B I_{12}$	q^3-1	$c_{10,0}(i)$
	${}_B\chi_{13}(k)$	$q^3\phi_1^2$	${}_B I_{13}$	q^2+q+1	$c_{5,0}(i)$
	${}_B\chi_{14}(k)$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_{14}$	2	$c_{1,10}, c_{3,0}$
	${}_B\chi_{15}(k)$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_B I_{15}$	2	$c_{1,10}, c_{3,0}$
	${}_B\chi_{16}(k)$	$q^3\phi_1^2\phi_3$	${}_B I_{16}$	$q-1$	$c_{1,6}, c_{1,12}, c_{1,16}$

Table A.5 (*continued*)

Group	Character	Degree	Parameter	Number	Class
P	${}_P\chi_2(k)$	q^3	${}_PI_2$	$q-1$	$c_{5,0}(i)$
	${}_P\chi_9$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_6$	${}_PI_9$	1	$c_{3,0}$
	${}_P\chi_{10}$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_6$	${}_PI_{10}$	1	$c_{3,0}$
	${}_P\chi_{11}(k)$	$q^3\phi_1^2\phi_2\phi_6$	${}_PI_{11}$	$\frac{1}{2}(q^2+q)$	$c_{6,0}(i)$
	${}_P\chi_{12}$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_3$	${}_PI_{12}$	1	$c_{3,0}$
	${}_P\chi_{13}$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_3$	${}_PI_{13}$	1	$c_{3,0}$
	${}_P\chi_{14}(k)$	$q^3\phi_1^2\phi_2\phi_3$	${}_PI_{14}$	$\frac{1}{2}(q^2-q)$	$c_{10,0}(i)$
Q	${}_Q\chi_8$	$q^3\phi_1\phi_3$	${}_QI_8$	1	
	${}_Q\chi_{10}$	$\frac{1}{2}q^3\phi_1\phi_2\phi_3$	${}_QI_{10}$	1	$c_{3,0}, c_{3,3}$
	${}_Q\chi_{11}$	$\frac{1}{2}q^3\phi_1\phi_2\phi_3$	${}_QI_{11}$	1	$c_{3,0}$, $c_{3,3}$
	${}_Q\chi_{12}$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_QI_{12}$	1	$c_{3,0}, c_{3,3}$
	${}_Q\chi_{13}$	$\frac{1}{2}q^3\phi_1^2\phi_3$	${}_QI_{13}$	1	$c_{3,0}, c_{3,3}$
	${}_Q\chi_{14}(k)$	$q^3\phi_1\phi_2\phi_3$	${}_QI_{14}$	$\frac{1}{2}(q-3)$	$c_{5,0}(i)$
	${}_Q\chi_{15}(k)$	$q^3\phi_1^2\phi_3$	${}_QI_{15}$	$\frac{1}{2}(q-1)$	$c_{10,0}(i)$
	${}_Q\chi_{16}(k)$	$q^3\phi_1\phi_2$	${}_QI_{16}$	q^3-1	$c_{8,0}(i)$
	${}_Q\chi_{17}(k)$	$q^3\phi_1^2\phi_2$	${}_QI_{17}$	q^2+q+1	$c_{6,0}(i)$
	${}_Q\chi_{18}(k)$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_3$	${}_QI_{18}$	2	$c_{1,8}, c_{2,0}$
	${}_Q\chi_{19}(k)$	$\frac{1}{2}q^3\phi_1^2\phi_2\phi_3$	${}_QI_{19}$	2	$c_{1,8}, c_{2,0}$
	${}_Q\chi_{20}(k)$	$q^3\phi_1^2\phi_2\phi_3$	${}_QI_{20}$	$q-1$	$c_{1,5}, c_{1,9}, c_{1,12}$

Table A.6
The irreducible characters of the chain normalizers of defect $11n$

Group	Character	Degree	Parameter	Number	Class
G	χ_2	$q\phi_{12}$	${}_GI_2$	1	
	χ_{10}	$q\phi_3\phi_6\phi_{12}$	${}_GI_{10}$	1	
	$\chi_{16}(k)$	$q\phi_2^2\phi_6^2\phi_{12}$	${}_GI_{16}$	$\frac{1}{2}(q+1)q$	$c_{5,0}(i)$
	$\chi_{19}(k)$	$q\phi_2\phi_3\phi_6^2\phi_{12}$	${}_GI_{19}$	$\frac{1}{2}(q^3-q^2-q-3)$	$c_{8,0}(i)$
	$\chi_{25}(k)$	$q\phi_1^2\phi_3^2\phi_{12}$	${}_GI_{25}$	$\frac{1}{2}q(q-1)$	$c_{10,0}(i)$
	$\chi_{28}(k)$	$q\phi_1\phi_3^2\phi_6\phi_{12}$	${}_GI_{28}$	$\frac{1}{2}(q-1)(q^2+1)$	$c_{11,0}(i)$
B	${}_B\chi_5(k)$	$q\phi_1\phi_3$	${}_BI_5$	$q-1$	$c_{7,0}(i)$
	${}_B\chi_6(k)$	$q\phi_1^2\phi_3$	${}_BI_6$	$q+1$	$c_{1,17}(a')$
P	${}_P\chi_7(k)$	$q\phi_1\phi_2\phi_3\phi_6$	${}_PI_7$	$q-1$	$c_{5,0}(i)$
	${}_P\chi_8(k)$	$q\phi_1^2\phi_2\phi_3\phi_6$	${}_PI_8$	$q+1$	$c_{1,9}(a')$
Q	${}_Q\chi_2(k)$	q	${}_QI_2$	q^3-1	$c_{8,0}(i)$
	${}_Q\chi_7$	$q\phi_1^2\phi_2\phi_3$	${}_QI_7$	1	

Table A.7
The irreducible characters of the chain normalizers of defect $12n$

Group	Character	Degree	Parameter	Number	Class
G	χ_1	1	${}_GI_1$	1	
	χ_9	$\phi_3\phi_6\phi_{12}$	${}_GI_9$	1	
	$\chi_{13}(k)$	$\phi_2\phi_3\phi_6\phi_{12}$	${}_GI_{13}$	$\frac{1}{2}(q-3)$	$c_{5,0}(i)$
	$\chi_{15}(k)$	$\phi_2\phi_6^2\phi_{12}$	${}_GI_{15}$	$\frac{1}{2}q(q+1)$	$c_{8,0}(i)$
	$\chi_{18}(k)$	$\phi_2\phi_3\phi_6^2\phi_{12}$	${}_GI_{18}$	$\frac{1}{2}(q^3-q^2-q-3)$	$c_{8,0}(i)$
	$\chi_{20}(k,l)$	$\phi_2^2\phi_3\phi_6^2\phi_{12}$	${}_GI_{20}$	$\frac{1}{12}(q-3)(q^3-q^2-q-5)$	$c_{6,0}(i,j)$
	$\chi_{21}(k)$	$\phi_1\phi_3\phi_6\phi_{12}$	${}_GI_{21}$	$\frac{1}{2}(q-1)$	$c_{8,0}(i)$
	$\chi_{23}(k)$	$\phi_1\phi_2\phi_3\phi_6^2\phi_{12}$	${}_GI_{23}$	$\frac{1}{4}(q-1)(q^3+q^2+q-1)$	$c_{8,0}(i)$
	$\chi_{24}(k)$	$\phi_1\phi_3^2\phi_{12}$	${}_GI_{24}$	$\frac{1}{2}q(q-1)$	$c_{10,0}(i)$
	$\chi_{27}(k)$	$\phi_1\phi_3^2\phi_6\phi_{12}$	${}_GI_{27}$	$\frac{1}{2}(q-1)(1+q^2)$	$c_{11,0}(i)$
	$\chi_{29}(k)$	$\phi_1\phi_2\phi_3^2\phi_6\phi_{12}$	${}_GI_{29}$	$\frac{1}{4}(q-1)(q^3-q^2-q-1)$	$c_{11,0}(i)$
	$\chi_{30}(k,l)$	$\phi_1^2\phi_2^2\phi_6^2\phi_{12}$	${}_GI_{30}$	$\frac{1}{24}q(q+2)(q+1)(q-1)$	$c_{12,0}(i,j)$
	$\chi_{31}(k,l)$	$\phi_1^2\phi_2^2\phi_3^2\phi_{12}$	${}_GI_{31}$	$\frac{1}{24}q(q+1)(q-1)(q-2)$	$c_{13,0}(i,j)$
	$\chi_{32}(k)$	$\phi_1^2\phi_2^2\phi_3^2\phi_6^2$	${}_GI_{32}$	$\frac{1}{4}q^2(q+1)(q-1)$	$c_{14,0}(i)$
	$\chi_{33}(k,l)$	$\phi_1^2\phi_3^2\phi_6\phi_{12}$	${}_GI_{33}$	$\frac{1}{12}(q-1)(q^3-q^2+q-3)$	$c_{15,0}(i,j)$
B	${}_B\chi_1(k,l)$	1	${}_BI_1$	$(q^3-1)(q-1)$	$c_{12,0}(i,j)$
	${}_B\chi_2(k)$	$\phi_1\phi_3$	${}_BI_2$	$q-1$	$c_{6,0}(i)$
	${}_B\chi_3(k)$	ϕ_1	${}_BI_3$	q^3-1	$c_{9,0}(i)$
	${}_B\chi_4$	$\phi_1^2\phi_3$	${}_BI_4$	1	
P	${}_P\chi_1(k)$	1	${}_PI_1$	$q-1$	$c_{5,0}(i)$
	${}_P\chi_3(k,l)$	$\phi_2\phi_6$	${}_PI_3$	$\frac{1}{2}(q-1)(q^3-2)$	$c_{9,0}(i,j)$
	${}_P\chi_4(k)$	$\phi_1\phi_3$	${}_PI_4$	$\frac{1}{2}q^3(q-1)$	$c_{12,0}(i)$
	${}_P\chi_5(k)$	$\phi_1\phi_2\phi_6$	${}_PI_5$	q^3-1	$c_{7,0}(i)$
	${}_P\chi_6$	$\phi_1^2\phi_2\phi_3\phi_6$	${}_PI_6$	1	
Q	${}_Q\chi_1(k)$	1	${}_QI_1$	q^3-1	$c_{8,0}(i)$
	${}_Q\chi_3(k,l)$	ϕ_2	${}_QI_3$	$\frac{1}{2}(q^3-1)(q-2)$	$c_{9,0}(i,j)$
	${}_Q\chi_4(k)$	ϕ_1	${}_QI_4$	$\frac{1}{2}q(q^3-1)$	$c_{11,0}(i)$
	${}_Q\chi_5(k)$	$\phi_1\phi_2\phi_3$	${}_QI_5$	$q-1$	$c_{4,0}(i)$
	${}_Q\chi_6$	$\phi_1^2\phi_2\phi_3$	${}_QI_6$	1	

Table A.8
Number of fixed points of $H = \langle \alpha^t \rangle$ on parameter sets of the irreducible characters

Parameter set I	Number of fixed points $\lvert C_I(H)\rvert$	
	if $t \mid n$	if $t \nmid n$
${}_GI_1 \cup {}_GI_9$	2	2
${}_GI_2 \cup {}_GI_{10}$	2	2
${}_GI_3 \cup {}_GI_4 \cup {}_GI_5 \cup {}_GI_6$	4	4
${}_GI_7$	1	1
${}_GI_{11}$	1	1
${}_GI_{12}$	1	1
${}_GI_{13} \cup {}_GI_{21}$	$p^t - 2$	$p^{t/3} - 2$
${}_GI_{15} \cup {}_GI_{18} \cup {}_GI_{24} \cup {}_GI_{27}$	$p^t - 2$	$p^t - 2$
${}_GI_{16} \cup {}_GI_{19} \cup {}_GI_{25} \cup {}_GI_{28}$	$p^t - 2$	$p^t - 2$
${}_GI_{20} \cup {}_GI_{23} \cup {}_GI_{29} \cup {}_GI_{30} \cup {}_GI_{31} \cup {}_GI_{32} \cup {}_GI_{33}$	$p^{2t} - 2p^t + 2$	$p^{4t/3} - p^t - p^{t/3} + 2$
${}_BI_1$	$(p^t - 1)^2$	$(p^t - 1)(p^{t/3} - 1)$
${}_BI_2$	$p^t - 1$	$p^{t/3} - 1$
${}_BI_3$	$p^t - 1$	$p^t - 1$
${}_BI_4$	1	1
${}_BI_5$	$p^t - 1$	$p^{t/3} - 1$
${}_BI_8 \cup {}_BI_9 \cup {}_BI_{10} \cup {}_BI_{11}$	4	4
${}_BI_{17}$	$p^t - 1$	$p^t - 1$
${}_BI_{18} \cup {}_BI_{19} \cup {}_BI_{20} \cup {}_BI_{21}$	4	4
${}_PI_1$	$p^t - 1$	$p^{t/3} - 1$
${}_PI_3 \cup {}_PI_4$	$(p^t - 1)^2$	$(p^t - 1)(p^{t/3} - 1)$
${}_PI_5$	$p^t - 1$	$p^t - 1$
${}_PI_6$	1	1
${}_PI_7$	$p^t - 1$	$p^{t/3} - 1$
${}_PI_9 \cup {}_PI_{10} \cup {}_PI_{12} \cup {}_PI_{13}$	4	4
${}_PI_{15}$	1	1
${}_PI_{16}$	1	1
${}_PI_{17} \cup {}_PI_{18} \cup {}_PI_{19} \cup {}_PI_{20}$	4	4
${}_PI_{21} \cup {}_PI_{22}$	$p^t - 2$	$p^t - 2$
${}_QI_1$	$p^t - 1$	$p^t - 1$
${}_QI_2$	$p^t - 1$	$p^t - 1$
${}_QI_3 \cup {}_QI_4$	$(p^t - 1)^2$	$(p^t - 1)(p^{t/3} - 1)$
${}_QI_5$	$p^t - 1$	$p^{t/3} - 1$
${}_QI_6$	1	1
${}_QI_7$	1	1
${}_QI_8$	1	1
${}_QI_9$	1	1
${}_QI_{10} \cup {}_QI_{11} \cup {}_QI_{12} \cup {}_QI_{13}$	4	4

The unions of parameter sets in this table are disjoint unions.

References

[1] J.L. Alperin, The main problem of block theory, in: Proceedings of the Conference on Finite Groups, Univ. Utah, Park City, UT, Academic Press, New York, 1975, pp. 341–356.
[2] J. An, Dade's conjecture for Steinberg triality groups ${}^3D_4(q)$ in non-defining characteristics, Math. Z. 241 (2002) 445–469.
[3] H.I. Blau, G.O. Michler, Modular representation theory of finite groups with T.I. Sylow p-subgroups, Trans. Amer. Math. Soc. 319 (1990) 417–468.
[4] A. Borel, et al., Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math., vol. 131, Springer, Heidelberg, 1970.
[5] N. Burgoyne, C. Williamson, On a theorem of Borel and Tits for finite Chevalley groups, Arch. Math. (Basel) 27 (1976) 489–491.
[6] R.W. Carter, Finite Groups of Lie Type—Conjugacy Classes and Complex Characters, Wiley–Interscience Publication, Chichester, 1985.
[7] B. Char, K. Geddes, G. Gonnet, B. Leong, M. Monagan, S. Watt, Maple V, Language Reference Manual, Springer, 1991.
[8] E.C. Dade, Counting characters in blocks I, Invent. Math. 109 (1992) 187–210.
[9] E.C. Dade, Counting characters in blocks II, J. Reine Angew. Math. 448 (1994) 97–190.
[10] E.C. Dade, Counting characters in blocks 2.9, in: R. Solomon (Ed.), Representation Theory of Finite Groups, de Gruyter, 1997, pp. 45–59.
[11] D.I. Deriziotis, G.O. Michler, Character tables and blocks of finite simple triality groups ${}^3D_4(q)$, Trans. Amer. Math. Soc. 303 (1987) 39–70.
[12] The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.4, http://www.gap-system.org, 2005.
[13] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer, CHEVIE—A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, Appl. Algebra Engrg. Comm. Comput. 7 (1996) 175–210.
[14] D. Gorenstein, R. Lyons, The local structure of finite groups of characteristic 2 type, Mem. Amer. Math. Soc. 276 (1983).
[15] F. Himstedt, Charaktertafeln parabolischer Untergruppen der Steinbergschen Trialitätsgruppen und Anwendungen auf deren Darstellungstheorie, PhD thesis, RWTH Aachen, 2003.
[16] F. Himstedt, Character tables of parabolic subgroups of Steinberg's triality groups, J. Algebra 281 (2004) 774–822.
[17] G. Hiss, On the decomposition numbers of $G_2(q)$, J. Algebra 120 (1989) 339–360.
[18] S. Huang, Dade's invariant conjecture for the Chevalley groups of type G_2 in the defining characteristic, J. Algebra 292 (2005) 110–121.
[19] J. Humphreys, Defect groups for finite groups of Lie type, Math. Z. 119 (1971) 149–152.
[20] I.M. Isaacs, G. Navarro, New refinements of the McKay conjecture for arbitrary finite groups, Ann. of Math. 156 (2002) 333–344.
[21] P. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and their automorphism groups, J. Algebra 115 (1988) 182–199.
[22] J. McKay, A new invariant for simple groups, Notices Amer. Math. Soc. 18 (1971) 397.
[23] K. Uno, Conjectures on character degrees for the simple Thompson group, Osaka J. Math. 41 (2004) 11–36.