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Tight monomials and the monomial basis property ☆

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ABSTRACT

We generalize a criterion for tight monomials of quantum enveloping algebras associated with symmetric generalized Cartan matrices and a monomial basis property of those associated with symmetric (classical) Cartan matrices to their respective symmetrizable case. We then link the two by establishing that a tight monomial is necessarily a monomial defined by a weakly distinguished word. As an application, we develop an algorithm to compute all tight monomials in the rank 2 Dynkin case.

The existence of Hall polynomials for Dynkin or cyclic quivers not only gives rise to a simple realization of the $\pm$-part of the corresponding quantum enveloping algebras, but also results in interesting applications. For example, by specializing q to 0, degenerate quantum enveloping algebras have been investigated in the context of generic extensions [20,8], while through a certain non-triviality property of Hall polynomials, the authors [4,5] have established a monomial basis property for quantum enveloping algebras associated with Dynkin and cyclic quivers. This property describes a systematic construction of many monomial/integral monomial bases some of which have already been studied in the context of elementary algebraic constructions of canonical bases; see, e.g., [15,27,21,5] in the simply-laced Dynkin case and [3,18], [9, Ch. 11] in general. Moreover, in the cyclic quiver case, it has also been used in [10] to obtain an elementary construction of PBW-type bases, and hence, of

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canonical bases for quantum affine $\mathfrak{sl}_n$. In this paper, we will complete this program by proving this property for *all* finite types.

The second purpose of the paper is to establish some relationship between the monomial basis property and tight monomials. Following Lusztig [17], a monomial which is also a canonical basis element is called a tight monomial. See also [19,1] for further work on tight monomials. We first prove that the tight monomial criterion given in [22] for quantum enveloping algebras associated with symmetric generalized Cartan matrices works also for all symmetrizable ones. This criterion is built on Lusztig's criterion for signed bases [16, Ch. 14]. Then we show in the finite type case that a tight monomial is necessarily a monomial associated with a weakly distinguished word. Thus, to test a monomial to be tight, it suffices to test monomials associated with weakly distinguished words. We further conjecture that monomials associated with distinguished words cover all tight monomials.

We organize the paper as follows. Beginning with a quiver Q with automorphism σ, we first briefly review the relationship between representations of the path algebra $\mathcal{A} = kQ$ over the algebraic closure $k = \overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and the fixed point $\mathbb{F}_q$-algebra $\mathcal{A}^F$ of the Frobenius morphism $F = F_{Q,\sigma,q}$. The generalization of the criterion for tight monomials is presented in Section 2. From Section 3 onwards, we assume that Q is a Dynkin quiver. In Sections 3–5, we define the generic extension map $\wp$ for a Dynkin quiver Q with automorphism σ and use it to establish the monomial basis property for quantum enveloping algebras associated with (Q,σ). In the last three sections, we discuss the relationship between tight monomials and the monomial basis property. In Section 6, we prove that a tight monomial is necessarily a monomial associated with a weakly distinguished word, and in Section 7, we develop an algorithm to compute tight monomials of rank 2 and determine explicitly those of type B_2. Finally, in the last section, we verify the conjecture for type B_2 that tight monomials all arise from directed distinguished words and identify them as the canonical basis elements described in [29].

Throughout the paper, $\mathbb{F}_q$ denotes the finite field of q elements and k is the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. For an algebra B over a field, the category of finite-dimensional left B-modules will be denoted by B-**mod**.

1. Preliminaries

Let $Q = (Q_0, Q_1)$ be a finite quiver with vertex set Q_0 and arrow set Q_1, and let σ be an automorphism of Q, that is, σ is a permutation on the vertices of Q and on the arrows of Q such that $\sigma(\mathrm{h}\rho) = \mathrm{h}\sigma(\rho)$ and $\sigma(\mathrm{t}\rho) = \mathrm{t}\sigma(\rho)$ for any $\rho \in Q_1$, where $\mathrm{h}\rho$ and $\mathrm{t}\rho$ denote the head and the tail of ρ, respectively.

Recall from [6] that there is a Frobenius morphism $F_{Q,\sigma;q}$ on the path algebra $\mathcal{A} := kQ$ of Q over $k = \overline{\mathbb{F}}_q$ defined by

$$F = F_{Q,\sigma;q} : \mathcal{A} \to \mathcal{A}, \quad \sum_s x_s p_s \mapsto \sum_s x_s^q \sigma(p_s),$$

where $\sum_s x_s p_s$ is a k-linear combination of paths p_s. This gives an $\mathbb{F}_q$-algebra

$$\mathcal{A}^F = \{a \in \mathcal{A} \mid F(a) = a\}.$$

If Q contains no oriented cycles, then $\mathcal{A}^F$ is a finite-dimensional hereditary $\mathbb{F}_q$-algebra. Conversely, every finite-dimensional hereditary basic $\mathbb{F}_q$-algebra is isomorphic to $\mathcal{A}^F$ for some quiver Q with automorphism σ (see [6, Th. 6.5] or [7, Th. 9.3]).

By [6, Prop. 4.2], there is a Frobenius twist functor

$$(\ )^{[1]} : \mathcal{A}\text{-}\mathbf{mod} \to \mathcal{A}\text{-}\mathbf{mod}, \quad M \mapsto M^{[1]}$$

which is an equivalence of categories. Alternatively, let M be an $\mathcal{A}$-module and $F_M : M \to M$ be an arbitrary Frobenius map on the k-vector space M. We define the F_M-twist of M to be the $\mathcal{A}$-module $M^{[F_M]}$ such that $M^{[F_M]} = M$ as vector spaces and the $\mathcal{A}$-module structure is given by

$$a * m := F_M\big(F_{Q,\sigma;q}^{-1}(a) F_M^{-1}(m)\big) \quad \text{for all } a \in \mathcal{A},\ m \in M.$$

By [7, Lem. 2.5], $M^{[F_M]}$ is isomorphic to $M^{[1]}$.

A module $M \in \mathcal{A}$-**mod** is called F-stable if $M \cong M^{[1]}$. Moreover, for an F-stable $\mathcal{A}$-module M, there is a Frobenius map F_M on M satisfying

$$F_M(am) = F_{Q,\sigma;q}(a) F_M(m) \quad \text{for all } a \in \mathcal{A},\ m \in M,$$

that is, $M^{[F_M]} = M$ as $\mathcal{A}$-modules. Consequently, we obtain an $\mathcal{A}^F$-module

$$M^F = M^{F_M} = \big\{ m \in M \bigm| F_M(m) = m \big\}.$$

By [6, Th. 3.2], the correspondence $M \mapsto M^F$ induces a bijection between the isoclasses of F-stable $\mathcal{A}$-modules and those of $\mathcal{A}^F$-modules.

Suppose that Q contains no oriented cycles. Let I be the set of isoclasses of simple modules in $\mathcal{A}^F$-**mod**. (Note that I identifies with the set of σ-orbits in Q_0.) The Grothendieck group $\mathscr{K}_0(\mathcal{A}^F)$ of $\mathcal{A}^F$-**mod** is then identified with the free abelian group $\mathbb{Z}I$ with basis I. Given a module M in $\mathcal{A}^F$-**mod**, we denote by $\mathbf{dim}\, M$ the image of M in $\mathscr{K}_0(\mathcal{A}^F)$, called *dimension vector* of M. Then, for each $i \in I$, we have a simple $\mathcal{A}^F$-module S_i with dimension vector i. Hence, if $\mathbf{dim}\, M = \sum_{i \in I} x_i i$, then x_i is the number of composition factors isomorphic to S_i in a composition series of M.

The Euler from $\langle -,- \rangle : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ associated with (Q,σ) is defined by

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \dim_{\mathbb{F}_q} \mathrm{Hom}_{\mathcal{A}^F}(M,N) - \dim_{\mathbb{F}_q} \mathrm{Ext}^1_{\mathcal{A}^F}(M,N),$$

for $M, N \in \mathcal{A}^F$-**mod**. For $i \in I$, let $d_i = \dim_{\mathbb{F}_q} \mathrm{End}_{\mathcal{A}^F}(S_i) = \langle i, i \rangle$, and for $i, j \in I$, define

$$c_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ \frac{1}{d_i}(\langle \mathbf{dim}\, S_i, \mathbf{dim}\, S_j \rangle + \langle \mathbf{dim}\, S_j, \mathbf{dim}\, S_i \rangle), & \text{if } i \neq j. \end{cases}$$

The matrix $C_{Q,\sigma} = (c_{i,j})_{i,j \in I}$ is a symmetrizable generalized Cartan matrix with symmetrization $D = \mathrm{diag}(d_i)_{i \in I}$.

Let $\mathfrak{g} = \mathfrak{g}(C_{Q,\sigma})$ be the Kac–Moody algebra associated with $C_{Q,\sigma}$ and let $\mathbf{U} = \mathbf{U}_v(\mathfrak{g})$ be the corresponding quantized enveloping algebra over the fraction field $\mathbb{Q}(v)$ in indeterminate v. We are interested in the positive part $\mathbf{U}^+$ of $\mathbf{U}$, which is by definition the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}$ generated by E_i, $i \in I$. Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over $\mathbb{Z}$. Although the definition of $\mathbf{U}$ depends on a realization of $C_{Q,\sigma}$, $\mathbf{U}^+$ depends only on $C_{Q,\sigma}$ and is called the *quantum algebra* associated with $C_{Q,\sigma}$ (or (Q,σ)) in the sequel. Let U^+ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}^+$ generated by divided powers $E_i^{(m)} := \frac{E_i^m}{[m]_i^!}$ for all $i \in I$, $m \in \mathbb{N}$, where $[m]_i^! = [1]_i [2]_i \cdots [m]_i$ with $[a]_i = \frac{v_i^a - v_i^{-a}}{v_i - v_i^{-1}}$ $(v_i = v^{d_i})$.

Let Ω be the set of all words in the alphabet I. For each word $w = i_1 i_2 \cdots i_m \in \Omega$, $\boldsymbol{i} = (i_1, \ldots, i_t) \in I^t$, and $\boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{N}^t$, define monomials

$$\begin{aligned} E_w &:= E_{i_1} E_{i_2} \cdots E_{i_m} \in \mathbf{U}^+, \\ E_{\boldsymbol{i}}^{(\boldsymbol{a})} &:= E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_t}^{(a_t)} \in U^+. \end{aligned} \tag{1.0.1}$$

Then $\mathbf{U}^+$ is spanned by $\{E_w\}_{w\in\Omega}$, while U^+ is spanned by $\{E_{\boldsymbol{i}}^{(\boldsymbol{a})} \mid \boldsymbol{i}\in I^t,\ \boldsymbol{a}\in\mathbb{N}^t,\ t\in\mathbb{N}\}$. Thus, it would be interesting to ask how to extract monomial bases from these spanning sets. For quantum algebras associated with a Dynkin or cyclic quiver, the answer is given in [4,5]. We will generalize this result in Sections 3–5 to include the nonsimply-laced Dynkin cases.

2. A criterion for tight monomials: the general case

In [22], Reineke gave a criterion for a monomial to be tight (i.e., to be a canonical basis element) in a quantum enveloping algebra associated with a symmetric generalized Cartan matrix. We will see in this section that this criterion can be easily extended to all quantum enveloping algebras.

As in Section 1, let Q be a quiver with automorphism σ. Suppose that Q contains no oriented cycles. Then $C_{Q,\sigma}$ is a symmetrizable generalized Cartan matrix. Moreover, the Euler form $\langle -,-\rangle$ associated with (Q,σ) gives a Cartan datum $(I,\cdot)$ in the sense of [16, 1.1.1], where the bilinear form $\mathbb{Z}I\times\mathbb{Z}I\to\mathbb{Z}$, $(\mathbf{x},\mathbf{y})\mapsto\mathbf{x}\cdot\mathbf{y}$ is defined by

$$\mathbf{x}\cdot\mathbf{y}=\langle\mathbf{x},\mathbf{y}\rangle+\langle\mathbf{y},\mathbf{x}\rangle,\quad \text{for } \mathbf{x},\mathbf{y}\in\mathbb{Z}I.$$

Thus, for each $i\in I$, $i\cdot i=2\langle i,i\rangle=2d_i$. We will identify $C_{Q,\sigma}$ with its associated Cartan datum $(I,\cdot)$ and speak of the quantum algebra $\mathbf{U}^+$ associated with $(I,\cdot)$.

The algebra $\mathbf{U}^+$ admits an $\mathbb{N}I$-grading $\mathbf{U}^+=\bigoplus_{\mathbf{x}\in\mathbb{N}I}\mathbf{U}^+_{\mathbf{x}}$ such that $\mathbf{U}^+_{\mathbf{x}}$ is spanned by all monomials $E_{i_1}\cdots E_{i_s}$ with $i_1+\cdots+i_s=\mathbf{x}$. Given a homogeneous element $x\in\mathbf{U}^+_{\mathbf{x}}$, we write $|x|=\mathbf{x}$. For each $s\geqslant 2$, there is a twisted product on the s-fold tensor product $\mathbf{U}^+\otimes\cdots\otimes\mathbf{U}^+$ given by

$$(x_1\otimes\cdots\otimes x_s)(y_1\otimes\cdots\otimes y_s)=v^{\sum_{i>j}|x_i|\cdot|y_j|}x_1y_1\otimes\cdots\otimes x_sy_s,$$

where $\otimes=\otimes_{\mathbb{Q}(v)}$, and $x_1,\dots,x_s,y_1,\dots,y_s$ are homogeneous elements in $\mathbf{U}^+$. This algebra is called the s-fold graded tensor product of $\mathbf{U}^+$ and will be denoted by

$$\left(\mathbf{U}^+\right)^{\widetilde{\otimes}s}:=\underbrace{\mathbf{U}^+\,\widetilde{\otimes}\cdots\widetilde{\otimes}\,\mathbf{U}^+}_{s}.$$

Following [16, 1.2.2], there is a unique algebra homomorphism $\mathfrak{r}:\mathbf{U}^+\to\mathbf{U}^+\widetilde{\otimes}\mathbf{U}^+$ such that $\mathfrak{r}(E_i)=E_i\otimes 1+1\otimes E_i$ for each $i\in I$. Moreover,

$$(\mathfrak{r}\otimes 1)\mathfrak{r}=(1\otimes\mathfrak{r})\mathfrak{r}:\mathbf{U}^+\to\mathbf{U}^+\,\widetilde{\otimes}\,\mathbf{U}^+\,\widetilde{\otimes}\,\mathbf{U}^+.$$

In general, for each $s\geqslant 2$, there is an algebra homomorphism

$$\mathfrak{r}^{(s)}:\mathbf{U}^+\to\left(\mathbf{U}^+\right)^{\widetilde{\otimes}s},\quad E_i\mapsto\sum_{s_1+s_2=s-1}1^{\otimes s_1}\otimes E_i\otimes 1^{\otimes s_2}.$$

In particular, $\mathfrak{r}=\mathfrak{r}^{(2)}$.

For $\boldsymbol{i}=(i_1,\dots,i_t)\in I^t$ and $\boldsymbol{a}=(a_1,\dots,a_t)\in\mathbb{N}^t$, let $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ be the monomial defined in (1.0.1). Using [16, 1.4.2] and an inductive argument (see [22, Lem. 2.5]), we have for each $s\geqslant 2$,

$$\mathfrak{r}^{(s)}\big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big)=\sum_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_s=\boldsymbol{a}}v^{\eta_{\boldsymbol{i},\boldsymbol{a}}}E_{\boldsymbol{i}}^{(\boldsymbol{a}_1)}\otimes\cdots\otimes E_{\boldsymbol{i}}^{(\boldsymbol{a}_s)},\tag{2.0.2}$$

where $\boldsymbol{a}_r = (a_{r1}, \ldots, a_{rt}) \in \mathbb{N}^t$, for $1 \leqslant r \leqslant s$, and

$$\eta_{\boldsymbol{i},\boldsymbol{a}} = \sum_{\substack{1\leqslant m\leqslant t\\ 1\leqslant p<r\leqslant s}} \langle i_m, i_m\rangle a_{pm}a_{rm} + \sum_{\substack{1\leqslant p<r\leqslant s\\ 1\leqslant l<m\leqslant t}} (i_l \cdot i_m) a_{pm}a_{rl}.$$

The following results are taken from [16, Prop. 1.2.3] and [16, Lem. 1.4.4].

Lemma 2.1. *There is a unique bilinear inner product* $(-,-) : \mathbf{U}^+ \times \mathbf{U}^+ \to \mathbb{Q}(v)$ *such that*

(1) $(1,1) = 1$ *and* $(E_i, E_j) = \delta_{i,j}\frac{v_i^2}{v_i^2-1}$ *for all* $i, j \in I$;
(2) $(x, y'y'') = (\mathfrak{r}(x), y' \otimes y'')$ *for all* $x, y', y'' \in \mathbf{U}^+$;
(3) $(x'x'', y) = (x' \otimes x'', \mathfrak{r}(y))$ *for all* $x', x'', y \in \mathbf{U}^+$.

Here $v_i = v^{i\cdot i/2} = v^{d_i}$ *and* $(x' \otimes x'', y' \otimes y'') := (x', y')(x'', y'')$ *for all* $x', x'', y', y'' \in \mathbf{U}^+$. *Moreover, for each* $i \in I$ *and* $a \geqslant 0$,

$$\left(E_i^{(a)}, E_i^{(a)}\right) = \prod_{m=1}^{a} \frac{1}{(1 - v_i^{-2m})} = \frac{v_i^{a(a+1)/2}}{(v_i - v_i^{-1})^a [a]_i^!} \in \left(1 + v^{-1}\mathbb{Z}[\![v^{-1}]\!]\right) \cap \mathbb{Q}(v). \tag{2.1.1}$$

Observe that the sum in (2.0.2) is taken over the decompositions of $\boldsymbol{a} = \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_s$ each of which defines an $s \times t$ matrix $A = (a_{rm})$ with rows $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s$ satisfying $\mathrm{co}(A) = \boldsymbol{a}$, where $\mathrm{co}(A) = (\sum_{p=1}^{s} a_{p1}, \ldots, \sum_{p=1}^{s} a_{pt})$. We also put $\mathrm{ro}(A) = (\sum_{m=1}^{t} a_{1m}, \ldots, \sum_{m=1}^{t} a_{sm})$.

Definition 2.2. For any fixed $\boldsymbol{i} = (i_1, \ldots, i_t) \in I^t$ and $\boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{N}^t$, let $\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$ be the set of $t \times t$ matrices $A = (a_{rm})$ with entries a_{rm} in $\mathbb{N}$ satisfying the conditions $\mathrm{ro}(A) = \mathrm{co}(A) = \boldsymbol{a}$ and $a_{rm} = 0$ unless $i_r = i_m$. Define a quadratic form $\mathfrak{q} : \mathscr{M}_{\boldsymbol{i},\boldsymbol{a}} \to \mathbb{Z}$ by setting

$$\mathfrak{q}(A) = \sum_{\substack{1\leqslant m\leqslant t\\ 1\leqslant p<r\leqslant t}} \langle i_m, i_m\rangle a_{pm}a_{rm} + \sum_{\substack{1\leqslant p<r\leqslant t\\ 1\leqslant l<m\leqslant t}} (i_l \cdot i_m) a_{pm}a_{rl} + \sum_{\substack{1\leqslant r\leqslant t\\ 1\leqslant l<m\leqslant t}} \langle i_r, i_r\rangle a_{rm}a_{rl},$$

for all $A \in \mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$.

We illustrate the definition with an example. This example will be repeatedly used in an algorithm of computing tight monomials in the rank 2 case in Section 7.

Examples 2.3. Let $(I, \cdot)$ be a Cartan datum of Dynkin type with $I = \{1, 2\}$. Thus, $\mathbf{U}^+$ is a quantum algebra associated with a rank 2 Cartan matrix.

(1) If $\boldsymbol{i} \in \{(2,1,2), (1,2,1)\}$ and $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{N}^3$, then

$$\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}} = \left\{A_x \;\middle|\; 0 \leqslant x \leqslant \min\{a_1, a_3\}\right\}, \quad \text{where } A_x = \begin{bmatrix} a_1 - x & 0 & x \\ 0 & a_2 & 0 \\ x & 0 & a_3 - x \end{bmatrix},$$

and

$$\begin{aligned} \mathfrak{q}(A_x) &= \langle i_1, i_1\rangle a_{11}a_{31} + \langle i_3, i_3\rangle a_{31}a_{33} + (i_1 \cdot i_2)a_{22}a_{31} + (i_2 \cdot i_3)a_{13}a_{22} \\ &\quad + (i_1 \cdot i_3)a_{13}a_{31} + \langle i_1, i_1\rangle a_{11}a_{13} + \langle i_3, i_3\rangle a_{13}a_{33} \\ &= 2\langle i_1, i_1\rangle x(a_1 - x) + 2\langle i_3, i_3\rangle x(a_3 - x) + \big((i_1 \cdot i_2) + (i_2 \cdot i_3)\big)xa_2 + (i_1 \cdot i_3)x^2. \end{aligned}$$

(2) If $\boldsymbol{i} \in \{(2,1,2,1),(1,2,1,2)\}$ and $\boldsymbol{a} = (a_1,a_2,a_3,a_4) \in \mathbb{N}^4$, then

$$\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}} = \left\{A_{x,y} \;\middle|\; 0 \leqslant x \leqslant \min\{a_1,a_3\},\ 0 \leqslant y \leqslant \min\{a_2,a_4\}\right\},$$

where

$$A_{x,y} = \begin{bmatrix} a_1-x & 0 & x & 0 \\ 0 & a_2-y & 0 & y \\ x & 0 & a_3-x & 0 \\ 0 & y & 0 & a_4-y \end{bmatrix},$$

and

$$\begin{aligned} \mathfrak{q}(A_{x,y}) = {} & 2\big(\langle i_1,i_1\rangle x(a_1-x) + \langle i_2,i_2\rangle y(a_2-y) + \langle i_3,i_3\rangle x(a_3-x) + \langle i_4,i_4\rangle y(a_4-y)\big) \\ & + \big((i_1\cdot i_2)+(i_2\cdot i_3)\big)x(a_2-y) + (i_1\cdot i_3)x^2 + \big((i_1\cdot i_4)+(i_2\cdot i_3)\big)xy + (i_2\cdot i_4)y^2 \\ & + \big((i_2\cdot i_3)+(i_3\cdot i_4)\big)y(a_3-x). \end{aligned}$$

Lemma 2.1 together with (2.0.2) allows us to compute $(E_{\boldsymbol{i}}^{(\boldsymbol{a})}, E_{\boldsymbol{i}}^{(\boldsymbol{a})})$ in terms of $(E_{i_r}^{(a_r)}, E_{i_r}^{(a_r)})$ and $(E_{i_r}^{(a_{rm})}, E_{i_r}^{(a_{rm})})$ for all $A = (a_{rm}) \in \mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$. The proof of the following result is similar to that of [22, Th. 2.2] for the symmetric case. However, we provide a proof for completeness.

Corollary 2.4. *Keep the notation above. For $\boldsymbol{i} = (i_1,\dots,i_t) \in I^t$ and $\boldsymbol{a} = (a_1,\dots,a_t) \in \mathbb{N}^t$, we have*

$$\begin{aligned} \big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}, E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big) &= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\mathfrak{q}(A)} \prod_{1\leqslant r,m\leqslant t} \big(E_{i_r}^{(a_{rm})}, E_{i_r}^{(a_{rm})}\big) \\ &= \prod_{r=1}^{t} \big(E_{i_r}^{(a_r)}, E_{i_r}^{(a_r)}\big) + \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}\setminus\{D_{\boldsymbol{a}}\}} v^{\mathfrak{q}(A)} \prod_{1\leqslant r,m\leqslant t} \big(E_{i_r}^{(a_{rm})}, E_{i_r}^{(a_{rm})}\big), \end{aligned}$$

where $D_{\boldsymbol{a}} := \mathrm{diag}(a_1,\dots,a_t)$.

Proof. By Lemma 2.1(3) and (2.0.2),

$$\begin{aligned} \big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}, E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big) &= \big(E_{i_1}^{(a_1)} \otimes \cdots \otimes E_{i_t}^{(a_t)}, \mathfrak{r}^{(t)}\big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big)\big) \\ &= \sum_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_t=\boldsymbol{a}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \big(E_{i_1}^{(a_1)} \otimes \cdots \otimes E_{i_t}^{(a_t)}, E_{\boldsymbol{i}}^{(\boldsymbol{a}_1)} \otimes \cdots \otimes E_{\boldsymbol{i}}^{(\boldsymbol{a}_t)}\big) \\ &= \sum_{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_t=\boldsymbol{a}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \prod_{1\leqslant r\leqslant t} \big(E_{i_r}^{(a_r)}, E_{\boldsymbol{i}}^{(\boldsymbol{a}_r)}\big), \end{aligned}$$

where $\boldsymbol{a}_r = (a_{r1},\dots,a_{rt})$ for $1 \leqslant r \leqslant t$. By Lemma 2.1(2), $(E_{i_r}^{(a_r)}, E_{\boldsymbol{i}}^{(\boldsymbol{a}_r)}) \neq 0$ unless $a_r = a_{r1} + \cdots + a_{rt}$ and $a_{rm} = 0$ if $i_m \neq i_r$. This implies that the matrix $A = (a_{rm})$ with rows $\boldsymbol{a}_1,\dots,\boldsymbol{a}_t$ is in $\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$, and $E_{\boldsymbol{i}}^{(\boldsymbol{a}_r)} = \frac{[a_r]^!_{i_r}}{[a_{r1}]^!_{i_r}\cdots[a_{rt}]^!_{i_r}} E_{i_r}^{(a_r)}$. Therefore,

$$\begin{aligned}
\big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}, E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big) &= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \prod_{1\leqslant r\leqslant t} \big(E_{i_r}^{(a_r)}, E_{\boldsymbol{i}}^{(\boldsymbol{a}_r)}\big) \\
&= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \prod_{1\leqslant r\leqslant t} \frac{[a_r]^!_{i_r}}{[a_{r1}]^!_{i_r}\cdots[a_{rt}]^!_{i_r}} \big(E_{i_r}^{(a_r)}, E_{i_r}^{(a_r)}\big) \\
&= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \prod_{1\leqslant r\leqslant t} \frac{[a_r]^!_{i_r}}{[a_{r1}]^!_{i_r}\cdots[a_{rt}]^!_{i_r}} \frac{v_{i_r}^{a_r(a_r+1)/2}}{(v_{i_r}-v_{i_r}^{-1})^{a_r}[a_r]^!_{i_r}} \\
&= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\eta_{\boldsymbol{i},\boldsymbol{a}}} \prod_{1\leqslant r\leqslant t} v_{i_r}^{\sum_{1\leqslant l<m\leqslant t} a_{rm}a_{rl}} \prod_{1\leqslant m\leqslant t} \frac{v_{i_r}^{a_{rm}(a_{rm}+1)/2}}{(v_{i_r}-v_{i_r}^{-1})^{a_{rm}}[a_{rm}]^!_{i_r}} \\
&= \sum_{A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}} v^{\mathfrak{q}(A)} \prod_{1\leqslant r,m\leqslant t} \big(E_{i_r}^{(a_{rm})}, E_{i_r}^{(a_{rm})}\big),
\end{aligned}$$

as required. The last equality follows from the fact that $\mathfrak{q}(D_{\boldsymbol{a}})=0$. $\square$

Let $\mathbf{B}$ be the canonical basis of $\mathbf{U}^+$; see Remark 4.5. Following [17], a monomial $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is called *tight* if it belongs to $\mathbf{B}$. We now can easily extend [22, Th. 3.2] to the general case.

Theorem 2.5. *Let $\mathbf{U}^+$ be the quantum algebra associated with a Cartan datum $(I,\cdot)$. For $\boldsymbol{i}=(i_1,\ldots,i_t)\in I^t$ and $\boldsymbol{a}=(a_1,\ldots,a_t)\in\mathbb{N}^t$, the monomial $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight if and only if $\mathfrak{q}(A)<0$ for all $A\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}\backslash\{D_{\boldsymbol{a}}\}$.*

Proof. By [16, Ch. 14] (see also [22, Prop. 3.1]),

$$E_{\boldsymbol{i}}^{(\boldsymbol{a})}\in\mathbf{B} \quad\Leftrightarrow\quad \big(E_{\boldsymbol{i}}^{(\boldsymbol{a})}, E_{\boldsymbol{i}}^{(\boldsymbol{a})}\big)\in\big(1+v^{-1}\mathbb{Z}[\![v^{-1}]\!]\big)\cap\mathbb{Q}(v).$$

Furthermore, by (2.1.1), for all $A=(a_{rm})\in\mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$ and $1\leqslant r,m\leqslant t$,

$$\big(E_{i_r}^{(a_{rm})}, E_{i_r}^{(a_{rm})}\big)\in\big(1+v^{-1}\mathbb{Z}[\![v^{-1}]\!]\big)\cap\mathbb{Q}(v).$$

The assertion then follows from Corollary 2.4. $\square$

The following result is very useful in the determination of tight monomials; see Section 7 for the rank 2 case.

Corollary 2.6. *Let $\boldsymbol{i}=(i_1,\ldots,i_t)\in I^t$ and $\boldsymbol{a}=(a_1,\ldots,a_t)\in\mathbb{N}^t$. Suppose $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight. Then the monomials $E_{i_r}^{(a_r)}E_{i_{r+1}}^{(a_{r+1})}\cdots E_{i_s}^{(a_s)}$, for all $1\leqslant r\leqslant s\leqslant t$, are also tight. Moreover, if $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight with $\boldsymbol{a}$ sincere, then $i_r\neq i_{r+1}$ for all $1\leqslant r<t$.*

Proof. Write $\boldsymbol{j}=(i_r,\ldots,i_s)$ and $\boldsymbol{b}=(a_r,\ldots,a_s)$. Then

$$E_{\boldsymbol{j}}^{(\boldsymbol{b})}=E_{i_r}^{(a_r)}E_{i_{r+1}}^{(a_{r+1})}\cdots E_{i_s}^{(a_s)}.$$

For each $B\in\mathscr{M}_{\boldsymbol{j},\boldsymbol{b}}$, define

$$\widetilde{B}=\begin{bmatrix} D_{\boldsymbol{a}'} & 0 & 0\\ 0 & B & 0\\ 0 & 0 & D_{\boldsymbol{a}''}\end{bmatrix},$$

where $\boldsymbol{a}' = (a_1, \ldots, a_{r-1})$ and $\boldsymbol{a}'' = (a_{s+1}, \ldots, a_t)$. It is easy to see that $\widetilde{B} \in \mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$ and $\mathfrak{q}(B) = \mathfrak{q}(\widetilde{B})$. Moreover, $\widetilde{B} = D_{\boldsymbol{a}}$ if and only if $B = D_{\boldsymbol{b}}$. Since $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight, we get by Theorem 2.5 that

$$\mathfrak{q}(B) = \mathfrak{q}(\widetilde{B}) < 0 \quad \text{for all } B \in \mathscr{M}_{\boldsymbol{j},\boldsymbol{b}} \backslash \{D_{\boldsymbol{b}}\}.$$

Applying Theorem 2.5 again, we conclude that $E_{\boldsymbol{j}}^{(\boldsymbol{b})}$ is tight, as desired.

With a similar argument, the last assertion follows from the fact that, for any positive integers a, b and $i \in I$, the monomial $E_i^{(a)} E_i^{(b)}$ is not tight. $\square$

3. The generic extension map associated with (Q, σ)

One of the main ingredients in describing the monomial basis property is the generic extension map $\wp$ from the set Ω of all words in the index set I of simple representations to the set of isoclasses of all representations over $\mathbb{F}_q$. Since representations of a Dynkin quiver Q over an arbitrary field are determined by their dimension vectors, we can simply define the map $\wp$ by sending a word $w = i_1 i_2 \ldots i_r$ to the generic extension $S_{i_1} * S_{i_2} * \cdots * S_{i_r}$ of simple representations $S_{i_1}, S_{i_2}, \ldots, S_{i_r}$ of Q over k. This definition does not make sense if k is replaced by the finite field $\mathbb{F}_q$. However, the theory developed in Section 1 can be used to generalize the definition of $\wp$.

From now on, we assume that Q is a (connected) *Dynkin quiver*, that is, the underlying graph of Q is a Dynkin graph (of type A, D or E). Suppose σ is an automorphism of Q. By a well-known result in [13,12], the correspondence $M \mapsto \mathbf{dim}\, M$ induces a bijection between the set of isoclasses of indecomposable $\mathcal{A}^F$-modules and the set of positive roots $\Phi^+ = \Phi^+(Q, \sigma)$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{g}(C_{Q,\sigma})$ associated with $C_{Q,\sigma}$. For each $\alpha \in \Phi^+$, let $M_q(\alpha)$ denote the corresponding indecomposable $\mathcal{A}^F$-module. Thus, $\mathbf{dim}\, M_q(\alpha) = \alpha$. By the Krull–Remak–Schmidt theorem, every $\mathcal{A}^F$-module M is isomorphic to

$$M(\lambda) = M_q(\lambda) := \bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) M_q(\alpha)$$

for some function $\lambda : \Phi^+ \to \mathbb{N}$. Hence, the isoclasses of $\mathcal{A}^F$-modules are indexed by the set

$$\mathfrak{P} = \mathfrak{P}(Q, \sigma) := \{\lambda \mid \Phi^+ \to \mathbb{N}\},$$

which is clearly independent of q. For convenience, we will view each $\alpha \in \Phi^+$ as the function $\Phi^+ \to \mathbb{N}$, $\beta \mapsto \delta_{\alpha\beta}$ in $\mathfrak{P}$.

It is shown in [20] that for any two $\mathcal{A}$-modules M and N, there is a unique (up to isomorphism) extension G of M by N (i.e., $0 \to N \to G \to M \to 0$) with minimal dimension of its endomorphism algebra $\mathrm{End}_{\mathcal{A}}(G)$, which is denoted by $G = M * N$ and called *generic extension* of M by N. Moreover, for given $\mathcal{A}$-modules L, M, N,

(1) $(L * M) * N \cong L * (M * N)$,
(2) $L * 0 \cong L \cong 0 * L$.

Thus, there is a monoid structure on the set $\mathcal{M}_Q$ of isoclasses of $\mathcal{A}$-modules with multiplication $[M] * [N] = [M * N]$ and identity $1 = [0]$. This monoid $\mathcal{M}_Q$ has been studied in [20]. We have the following result (see [9, Prop. 11.1]).

Lemma 3.1. *Let M and N be F-stable $\mathcal{A}$-modules. Then $M * N$ is also F-stable.*

Proof. Since the Frobenius twist functor $(\ )^{[1]} : \mathcal{A}\text{-}\mathbf{mod} \to \mathcal{A}\text{-}\mathbf{mod}$ is an equivalence of categories, we get $(M * N)^{[1]} \cong M^{[1]} * N^{[1]}$. From $M^{[1]} \cong M$ and $N^{[1]} \cong N$ it follows that $(M * N)^{[1]} \cong M * N$, that is, $M * N$ is F-stable. $\square$

In views of this lemma, the set $\mathcal{M}_{Q,\sigma}$ of the isoclasses of F-stable $\mathcal{A}$-modules becomes a submonoid of $\mathcal{M}_Q$. For each vertex $i \in I$, define $S\{i\} = S_i \otimes_{\mathbb{F}_q} k$. It is a semisimple $\mathcal{A}$-module. Then, $S\{i\}$, $i \in I$, form a complete set of simple F-stable $\mathcal{A}$-modules.

The following lemma can be proved by using an argument similar to that in the proof of [20, Prop. 3.3]; see [9, Prop. 11.2].

Lemma 3.2. *The monoid $\mathcal{M}_{Q,\sigma}$ is generated by $[S\{i\}]$, $i \in I = \Gamma_0$.*

Let $\mathcal{M}_{\mathcal{A}^F}$ be the set of isoclasses of $\mathcal{A}^F$-modules, i.e., $\mathcal{M}_{\mathcal{A}^F} = \{[M_q(\lambda)] \mid \lambda \in \mathfrak{P}\}$. Applying the correspondence $M \mapsto M^F$ to $\mathcal{M}_{Q,\sigma}$ yields

$$\mathcal{M}_{\mathcal{A}^F} = \{[M^F] \mid [M] \in \mathcal{M}_{Q,\sigma}\}.$$

Thus, there is a map $\wp$ from Ω to $\mathfrak{P}$ (or equivalently, to $\mathcal{M}_{\mathcal{A}^F}$) defined by

$$\big[M_q\big(\wp(w)\big)\big] = \big[\big(S\{i_1\} * \cdots * S\{i_m\}\big)^F\big], \quad \text{for all } w = i_1 i_2 \cdots i_m \in \Omega. \tag{3.2.1}$$

This map $\wp : \Omega \to \mathfrak{P}$, $w \mapsto \wp(w)$ is called the *generic extension map* associated with (Q,σ). By Lemma 3.2, $\wp$ is surjective, and it induces a partition $\Omega = \bigcup_{\lambda \in \mathfrak{P}} \wp^{-1}(\lambda)$.

Remark 3.3. Associated with the generic extension map $\wp$, one defines a *generic extension graph* as in [11, Def. 2.3]. It would be interesting to compare the generic extension graph with the corresponding crystal graph; see [11] for the type A case.

4. The monomial basis property

Keep the notation introduced in the previous sections. Thus, (Q,σ) is a Dynkin quiver with automorphism, and $\mathcal{A} = kQ$ (resp., $F = F_{Q,\sigma;q}$) is the associated path algebra (resp., Frobenius morphism). Let $\mathbf{U}^+$ be the quantum algebra over $\mathbb{Q}(v)$ associated with (Q,σ) with $\mathcal{Z}$-subalgebra U^+.

For $\mathcal{A}^F$-modules $M, N_1, \ldots, N_m$, let $F^M_{N_1,\ldots,N_m}$ denote the number of filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{m-1} \supseteq M_m = 0$$

such that $M_{s-1}/M_s \cong N_s$ for all $1 \leqslant s \leqslant m$. Ringel [25] shows that for $\lambda, \mu_1, \ldots, \mu_m \in \mathfrak{P}$, there is an integral polynomial $\varphi^\lambda_{\mu_1,\ldots,\mu_m}(T) \in \mathbb{Z}[T]$, called *a Hall polynomial*, such that for any finite field $\mathbb{F}_q$ of q elements,

$$\varphi^\lambda_{\mu_1,\ldots,\mu_m}(q) = F^{M_q(\lambda)}_{M_q(\mu_1),\ldots,M_q(\mu_m)}.$$

By definition, the (*twisted generic*) *Ringel–Hall algebra* $\mathfrak{H} = \mathfrak{H}_{\mathcal{Z}}(Q,\sigma)$ of (Q,σ) is the free $\mathcal{Z}$-module with basis $\{u_\lambda = u_{[M(\lambda)]} \mid \lambda \in \mathfrak{P}\}$, and the multiplication is defined by

$$u_\lambda u_\mu = v^{\langle\lambda,\mu\rangle} \sum_{\pi \in \mathfrak{P}} \varphi^\pi_{\lambda,\mu}(v^2) u_\pi,$$

where $\langle\lambda,\mu\rangle = \langle \mathbf{dim}\, M(\lambda), \mathbf{dim}\, M(\mu)\rangle$. For each $i \in I$, we write $u_i = u_{[S_i]}$.

Ringel [24,26] proves that there is a $\mathcal{Z}$-algebra isomorphism

$$U^+ \xrightarrow{\sim} \mathfrak{H}, \quad E_i^{(m)} \mapsto u_i^{(m)} := \frac{u_i^m}{[m]^!_i} \quad (i \in I). \tag{4.0.1}$$

In what follows, we simply identify U^+ with $\mathfrak{H}$ under this isomorphism. In particular, $\{u_\lambda\}_{\lambda\in\mathfrak{P}}$ forms a $\mathcal{Z}$-basis for U^+.

For $w = i_1 i_2 \cdots i_m \in \Omega$ and $\lambda \in \mathfrak{P}$, let $\varphi_w^\lambda(T)$ denote the Hall polynomial $\varphi^\lambda_{i_1,\dots,i_m}(T)$. Thus, for each finite field $\mathbb{F}_q$, $\varphi_w^\lambda(q) = F^{M_q(\lambda)}_{S_{i_1},\dots,S_{i_m}}$. Let further w be written in the *tight* form $w = j_1^{m_1} \cdots j_l^{m_l}$, where $j_i \neq j_{i+1}$ for all $1 \leqslant i < l$. We then denote $\varphi^\lambda_{m_1 j_1,\dots,m_l j_l}(T)$ by $\gamma_w^\lambda(T)$, i.e., $\gamma_w^\lambda(q) := F^{M_q(\lambda)}_{m_1 S_{j_1},\dots,m_l S_{j_l}}$ is the number of reduced filtrations

$$M_q(\lambda) = L_0 \supset L_1 \supset \cdots \supset L_{l-1} \supset L_l = 0$$

satisfying $L_{i-1}/L_i \cong m_i S_{j_i}$ for all $1 \leqslant i \leqslant l$. A word w is called *weakly distinguished* if $\gamma_w^{\wp(w)}(T) = T^d$ for some $d \in \mathbb{N}$. We remark that distinguished words considered in [6,7] are defined by the condition $\gamma_w^{\wp(w)}(T) = 1$. The weak version considered here will become apparent in Section 6 as the words associated with tight monomials are necessarily weakly distinguished.

To each word $w = j_1^{m_1} \cdots j_l^{m_l} \in \Omega$ in the alphabet I, where $j_i \neq j_{i+1}$ for all $1 \leqslant i < l$, we attach a monomial

$$\mathfrak{m}^{(w)} := E_{j_1}^{(m_1)} E_{j_2}^{(m_2)} \cdots E_{j_l}^{(m_l)} \in U^+.$$

If we associate the word

$$w_{\boldsymbol{i},\boldsymbol{a}} = \underbrace{i_1 \cdots i_1}_{a_1} \cdots \underbrace{i_t \cdots i_t}_{a_t} \in \Omega \tag{4.0.2}$$

with $\boldsymbol{i} = (i_1, \dots, i_t) \in I^t$ and $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$, then $\mathfrak{m}^{(w_{\boldsymbol{i},\boldsymbol{a}})} = E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ in the notation of (1.0.1) *whenever* $i_j \neq i_{j+1}$ for all $1 \leqslant j < t$. The following result answers the monomial basis question in the finite type case.

Theorem 4.1. *Let Q be a Dynkin quiver with automorphism σ and $\mathbf{U}^+$ (resp. U^+) the quantum algebra over $\mathbb{Q}(v)$ (resp. $\mathcal{Z}$) associated with (Q, σ). For each $\lambda \in \mathfrak{P}(Q, \sigma)$, choose an arbitrary word $w_\lambda \in \wp^{-1}(\lambda)$.*

(1) *The set $\{E_{w_\lambda} \mid \lambda \in \mathfrak{P}\}$ is a $\mathbb{Q}(v)$-basis of $\mathbf{U}^+$.*
(2) *If, moreover, all w_λ are weakly distinguished, then the set $\{\mathfrak{m}^{(w_\lambda)} \mid \lambda \in \mathfrak{P}\}$ is a $\mathcal{Z}$-basis of U^+.*

This theorem generalizes [5, Th. 1.1]. Note that various integral monomial bases described in part (2) have already been introduced in the simply-laced case; see, for example, [15,27,21,5]. When all w_λ are distinguished, part (2) is given in [9, Th. 11.13].

The proof of part (1) of the theorem will be given in the next section. We make some preparations and prove part (2) in the rest of the section.

The degeneration order relation plays a key role in the proof and in the construction of canonical bases. We maintain the assumption that Q is a Dynkin quiver with automorphism σ and $\mathcal{A} = kQ$ is the path algebra of Q over k.

Definition 4.2. Given two $\mathcal{A}$-modules M, N of the same dimension vector, we say that M degenerates to N, or that N is a *degeneration* of M, and write $N \leqslant_{\mathrm{dg}} M$, if for all $X \in A$-**mod**,

$$\dim_k \mathrm{Hom}_A(X, N) \geqslant \dim_k \mathrm{Hom}_A(X, M).$$

Furthermore, the relation $\leqslant_{\mathrm{dg}}$ defines a partial order on the set $\mathcal{M}_Q$ of isoclasses of $\mathcal{A}$-modules, called the *degeneration order* (see [23,2]).

The poset $(\mathcal{M}_Q, \leqslant_{\mathrm{dg}})$ is compatible with the monoid structure on $\mathcal{M}_Q$ (see [20]). More precisely, for given $\mathcal{A}$-modules M, N, M', N', we have

$$M \leqslant_{\mathrm{dg}} M', \qquad N \leqslant_{\mathrm{dg}} N' \quad \Rightarrow \quad M * N \leqslant_{\mathrm{dg}} M' * N'. \tag{4.2.1}$$

Restricting the order $\leqslant_{\mathrm{dg}}$ to the submonoid $\mathcal{M}_{Q,\sigma}$ induces a partial order relation on $\mathcal{M}_{\mathcal{A}^F}$ (and hence on $\mathfrak{P} = \mathfrak{P}(Q,\sigma)$). In other words, $\mathfrak{P}$ has a partial order defined by

$$\lambda \leqslant \mu \quad \Leftrightarrow \quad M_q(\lambda) \otimes k \leqslant_{\mathrm{dg}} M_q(\mu) \otimes k, \quad \lambda, \mu \in \mathfrak{P}.$$

The next result follows directly from the definition and (4.2.1).

Proposition 4.3. *For each $w = i_1 i_2 \cdots i_m \in \Omega$, we have*

$$E_w = v^{\varepsilon_1(w)} \sum_{\lambda \leqslant \wp(w)} \varphi_w^\lambda\big(v^2\big) u_\lambda, \tag{4.3.1}$$

where $\varepsilon_1(w) = \sum_{1 \leqslant r < s \leqslant m} \langle \mathbf{dim}\, S_{i_r}, \mathbf{dim}\, S_{i_s} \rangle$.

Now, for each word $w = j_1^{m_1} \cdots j_l^{m_l} \in \Omega$ in tight form,

$$\mathfrak{m}^{(w)} = E_{j_1}^{(m_1)} \cdots E_{j_l}^{(m_l)} = \left(\prod_{r=1}^{l} [m_r]^!_{j_r} \right)^{-1} u_{j_1}^{m_1} \cdots u_{j_l}^{m_l}.$$

Since $\prod_{r=1}^l [m_r]^!_{j_r} = v^{-\varepsilon_2(w)} \prod_{r=1}^l [\![m_r]\!]^!_{j_r}$, where $\varepsilon_2(w) := \sum_{r=1}^l m_r(m_r-1) d_{j_r}/2$ and $[\![m]\!]^!_i = [\![1]\!]_i [\![2]\!]_i \cdots [\![m]\!]_i$ with $[\![a]\!]_i = \frac{v_i^{2a}-1}{v_i^2-1}$, we have, by Proposition 4.3, that

$$\mathfrak{m}^{(w)} = \left(\prod_{r=1}^{l} [\![m_r]\!]^!_{j_r} \right)^{-1} v^{\varepsilon_2(w)} u_w = v^{\varepsilon_1(w)+\varepsilon_2(w)} \sum_{\lambda \leqslant \wp(w)} \gamma_w^\lambda\big(v^2\big) u_\lambda. \tag{4.3.2}$$

For $\lambda \in \mathfrak{P}$, let

$$E_\lambda = v^{\dim \mathrm{End}(M(\lambda)) - \dim M(\lambda)} u_\lambda. \tag{4.3.3}$$

If w is a *directed* distinguished word in $\wp^{-1}(\lambda)$ (see [5, §5] and [9, §11.2] for definition and existence and see Section 7 for certain examples), then [5, 6.6] and [9, Lem. 11.31] imply that

$$\varepsilon_1(w) + \varepsilon_2(w) = \dim \mathrm{End}\big(M(\lambda)\big) - \dim M(\lambda). \tag{4.3.4}$$

In particular, combining (4.3.2) and (4.3.3) yields

$$\mathfrak{m}^{(w)} = E_{\wp(w)} + \sum_{\mu < \wp(w)} f_{\mu,\wp(w)} E_\mu, \tag{4.3.5}$$

where $f_{\mu,\wp(w)} \in \mathcal{Z}$. Thus, if we choose w_λ to be directed distinguished for every $\lambda \in \mathfrak{P}$, then (4.3.5) gives

$$E_\lambda = \mathfrak{m}^{(w_\lambda)} + \sum_{\mu<\lambda} g_{\mu,\lambda}\mathfrak{m}^{(w_\mu)}. \tag{4.3.6}$$

Now (4.3.5) and (4.3.6) imply

$$\iota(E_\lambda) = E_\lambda + \sum_{\mu<\lambda} r_{\mu,\lambda}E_\mu, \tag{4.3.7}$$

where ι is the $\mathbb{Z}$-algebra involution on U^+ defined by

$$\iota : U^+ \to U^+, \quad E_i^{(m)} \mapsto E_i^{(m)}, \quad v \mapsto v^{-1}.$$

Corollary 4.4. *For a weakly distinguished word $w \in \wp^{-1}(\lambda)$ with $\gamma_w^\lambda(T) = T^d$, (4.3.5) continues to hold. In particular, for any weakly distinguished word $w \in \wp^{-1}(\lambda)$,*

$$\varepsilon_1(w) + \varepsilon_2(w) + 2d = \dim \operatorname{End}\big(M(\lambda)\big) - \dim M(\lambda).$$

Proof. By (4.3.2), $\mathfrak{m}^{(w)} = v^s E_{\wp(w)} + \sum_{\lambda<\wp(w)} h_{\mu,\wp(w)}E_\lambda$, where

$$s = \varepsilon_1(w) + \varepsilon_2(w) + 2d - \dim \operatorname{End}\big(M(\lambda)\big) + \dim M(\lambda).$$

Applying ι and (4.3.7) yields $\mathfrak{m}^{(w)} = v^{-s} E_{\wp(w)} + \sum_{\lambda<\wp(w)} h'_{\mu,\wp(w)}E_\lambda$. Hence, $s = 0$, giving the first assertion. The last assertion is a direct consequence of the first one. $\square$

This proves part (2) of Theorem 4.1.

Remark 4.5. The above result shows that one may use a monomial basis associated with weakly distinguished words described in Theorem 4.1(2) to get the relation (4.3.7), and hence, to construct the canonical basis $\mathbf{B} = \{c_\lambda \mid \lambda \in \mathfrak{P}\}$ for U^+ which is defined uniquely by the conditions

$$\iota(c_\lambda) = c_\lambda, \quad c_\lambda \in E_\lambda + \sum_{\mu<\lambda} v^{-1}\mathbb{Z}\big[v^{-1}\big]E_\mu.$$

In particular, a monomial $\mathfrak{m}^{(w)}$ lies in $\mathbf{B}$ (i.e., $\mathfrak{m}^{(w)}$ is a *tight monomial*) if and only if $\mathfrak{m}^{(w)} \in E_\lambda + \sum_{\mu<\lambda} v^{-1}\mathbb{Z}[v^{-1}]E_\mu$.

5. Proof of Theorem 4.1(1)

By Proposition 4.3, it suffices to prove the following statement:

$$(*) \quad \text{For any given } \lambda \in \mathfrak{P}(Q,\sigma), \text{ we have } \varphi_w^\lambda(T) \neq 0 \text{ for all } w \in \wp^{-1}(\lambda).$$

If $\sigma = \mathrm{id}$, this result is a weaker version of [5, Prop. 6.2].

Proof. As before, let $\mathcal{A} = kQ$ be the path algebra of Q. It is well known that each $\mathcal{A}$-module identifies with a representation of Q over k. For a representation $V = (V_i, V_\rho)$ of Q over k, define ${}^\sigma V = (W_i, W_\rho)$ by setting $W_{\sigma(a)} = V_a$ and $W_{\sigma(\rho)} = V_\rho$ for all $a \in Q_0$ and all $\rho \in Q_1$. In other words, as an $\mathcal{A}$-module, ${}^\sigma V$ is the module obtained by twisting the $\mathcal{A}$-action on V via the algebra automorphism of $\mathcal{A}$ induced by σ.

For each $\pi \in \mathfrak{P}$, set

$$M_q(\pi)_k = M_q(\pi) \otimes_{\mathbb{F}_q} k,$$

which is an F-stable $\mathcal{A}$-module. Let $\mathbf{dim}\, M_q(\pi)_k = (c_a) \in \mathbb{N}^{Q_0}$. Up to isomorphism, we can identify $M_q(\pi)_k$ with a representation (V_a, V_ρ) of Q with $V_a = k^{c_a}$ and $V_\rho \in k^{c_{\mathrm{h}\rho} \times c_{\mathrm{t}\rho}}$ for all $a \in Q_0$ and $\rho : \mathrm{t}\rho \to \mathrm{h}\rho \in Q_1$. Then $M_q(\pi)_k = \bigoplus_{a \in Q_0} V_a$, and there is a standard Frobenius map F_π on $M_q(\pi)_k$ by taking $(x_1, \ldots, x_{c_a}) \in V_a$ to $(x_1^q, \ldots, x_{c_a}^q) \in V_a$. Since $M_q(\pi)_k$ is defined over $\mathbb{F}_q$, we may suppose that all entries in matrices V_ρ, $\rho \in Q_1$, are chosen in $\mathbb{F}_q$. Then $M'_q(\pi) = \{x \in M_q(\pi)_k \mid F_\pi(x) = x\}$ becomes an $\mathbb{F}_q Q$-module. On the other hand, the Frobenius twist $M_q(\pi)_k^{[F_\pi]} = (W_a, W_\rho)$ (relative to the Frobenius morphism $F_{Q,\sigma;q}$ on $\mathcal{A}$) is given by $W_{\sigma(a)} = V_a$ and $W_{\sigma(\rho)} = V_\rho$ for $a \in Q_0$ and $\rho \in Q_1$. Hence, $(M_q(\pi)_k)^{[F_\pi]}$ identifies with ${}^\sigma(M_q(\pi)_k)$. The F-stability of $M_q(\pi)_k$ implies that $(M_q(\pi)_k)^{[F_\pi]} = {}^\sigma(M_q(\pi)_k)$ is isomorphic to $M_q(\pi)_k$. We fix an isomorphism $\psi_\pi : {}^\sigma(M_q(\pi)_k) \to M_q(\pi)_k$, which is again defined over $\mathbb{F}_q$ since both modules are so. Then $F'_\pi := \psi_\pi F_\pi$ is also a Frobenius map on $M_q(\pi)_k$ such that $(M_q(\pi)_k)^{[F'_\pi]} = M_q(\pi)_k$ as $\mathcal{A}$-modules. Then the fixed-point module $(M_q(\pi)_k)^{F'_\pi} = \{x \in M_q(\pi)_k \mid F'_\pi(x) = x\}$ is an $\mathcal{A}^F$-module, which can be identified with $M_q(\pi)$.

We first show the following:

Claim. *For $i \in I$, $\lambda, \mu \in \mathfrak{P}$ with $\mu \neq 0$, if $M_q(\lambda)_k \cong S\{i\} * M_q(\mu)_k$, then $\varphi^\lambda_{i,\mu}(T) \neq 0$.*

From the definition, we have (see, for example, [28, Prop. 1])

$$F^{M_q(\lambda)}_{S_i, M_q(\mu)} = \frac{|\mathcal{E}^{M_q(\lambda)}_{S_i, M_q(\mu)}|}{|\operatorname{Aut}_{\mathcal{A}^F}(M_q(\mu))||\operatorname{Aut}_{\mathcal{A}^F}(S_i)|},$$

where $\mathcal{E}^{M_q(\lambda)}_{S_i, M_q(\mu)}$ denotes the set of pairs (f, g) of $\mathcal{A}^F$-module homomorphisms such that

$$0 \to M_q(\mu) \xrightarrow{f} M_q(\lambda) \xrightarrow{g} S_i \to 0$$

is an exact sequence. The fact $M_q(\lambda)_k \cong S\{i\} * (M_q(\mu)_k)$ implies that

$$\mathbf{dim}_k M_q(\lambda)_k = \mathbf{dim}_k M_q(\mu)_k + \mathbf{dim}_k S\{i\} \quad \text{and} \quad \mathbf{dim}_{\mathbb{F}_q} M_q(\lambda) = \mathbf{dim}_{\mathbb{F}_q} M_q(\mu) + \mathbf{dim}_{\mathbb{F}_q} S_i.$$

Since, up to isomorphism, S_i is the unique $\mathcal{A}^F$-module of dimension vector i, any injective homomorphism $f : M_q(\mu) \to M_q(\lambda)$ induces an exact sequence

$$0 \to M_q(\mu) \xrightarrow{f} M_q(\lambda) \to S_i \to 0.$$

Thus, we obtain

$$F^{M_q(\lambda)}_{S_i, M_q(\mu)} = \frac{|\mathscr{X}|}{|\operatorname{Aut}_{\mathcal{A}^F}(M_q(\mu))|},$$

where $\mathscr{X}$ is the set of all injective $\mathcal{A}^F$-module homomorphisms $f : M_q(\mu) \to M_q(\lambda)$. Consequently, to show the claim, it suffices to prove that $\mathscr{X}$ is not empty for sufficient large q.

As indicated above, we have Frobenius maps F_λ, F'_λ on $M_q(\lambda)_k$ and Frobenius maps F_μ, F'_μ on $M_q(\mu)_k$ satisfying $F'_\lambda = \psi_\lambda F_\lambda$ and $F'_\mu = \psi_\mu F_\mu$. They induce a standard Frobenius map

$$F_{\lambda,\mu} : \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big) \to \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big), \quad f \mapsto F_\lambda f F_\mu^{-1}$$

and a Frobenius map

$$F'_{\lambda,\mu} : \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big) \to \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big), \quad f \mapsto F'_\lambda f {F'_\mu}^{-1}.$$

From the construction, via restriction of maps we can identify

$$\mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big)^{F'_{\lambda,\mu}} = \mathrm{Hom}_{\mathcal{A}^F}\big(M_q(\mu), M_q(\lambda)\big),$$

whereas

$$\mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big)^{F_{\lambda,\mu}} = \mathrm{Hom}_{\mathbb{F}_q Q}\big(M'_q(\mu), M'_q(\lambda)\big).$$

Further, we have $F'_{\lambda,\mu} = \psi F_{\lambda,\mu}$, where

$$\psi : \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big) \to \mathrm{Hom}_{\mathcal{A}}\big(M_q(\mu)_k, M_q(\lambda)_k\big), \quad f \mapsto \psi_\lambda f \psi_\mu^{-1}.$$

Let $\mathscr{Y}$ be the set of all injective $\mathcal{A}$-module homomorphisms: $M_q(\mu)_k \to M_q(\lambda)_k$. It is an open subset of the affine space $\mathcal{V} := \mathrm{Hom}_{\mathcal{A}}(M_q(\mu)_k, M_q(\lambda)_k)$. Since $M_q(\lambda)_k \cong S\{i\} * M_q(\mu)_k$, $\mathscr{Y}$ is not empty. Moreover, $\mathscr{Y}$ is stable under both $F_{\lambda,\mu}$ and $F'_{\lambda,\mu}$, and thus,

$$\mathscr{X} = \mathscr{Y}^{F'_{\lambda,\mu}} := \big\{y \in \mathscr{Y} \bigm| F'_{\lambda,\mu}(y) = y\big\}.$$

For the standard Frobenius map $F_{\lambda,\mu}$ on $\mathcal{V}$, we have by the theorem of Lang and Weil [14] that

$$\big|\mathscr{Y}^{F_{\lambda,\mu}}\big| \approx q^t,$$

where $t = \dim_k \mathcal{V}$ and $\approx$ means asymptotical behaviour. The standard Frobenius map $F_{\lambda,\mu}$ on $\mathcal{V}$ induces a Frobenius map F on $\mathrm{GL}_k(\mathcal{V})$ by taking $g \to F_{\lambda,\mu} g F_{\lambda,\mu}^{-1}$. By the known Lang theorem, there is $\psi_1 \in \mathrm{GL}_k(\mathcal{V})$ such that

$$\psi = \psi_1 F\big(\psi_1^{-1}\big) = \psi_1 F_{\lambda,\mu} \psi_1^{-1} F_{\lambda,\mu}^{-1}.$$

Hence, $F'_{\lambda,\mu} = \psi F_{\lambda,\mu} = \psi_1 F_{\lambda,\mu} \psi_1^{-1}$. Then $\mathscr{Y}_1 := \mathscr{Y} \cap \psi_1^{-1}(\mathscr{Y})$ is open in $\mathcal{V}$ and $\psi_1(\mathscr{Y}_1) = \mathscr{Y} \cap \psi_1(\mathscr{Y})$. It is easy to see that $\mathscr{Y}_1$ and $\psi_1(\mathscr{Y}_1)$ are stable under $F_{\lambda,\mu}$ and $F'_{\lambda,\mu}$, respectively, and that

$$\big|\mathscr{Y}_1^{F_{\lambda,\mu}}\big| = \big|\psi_1(\mathscr{Y}_1)^{F'_{\lambda,\mu}}\big|.$$

By $|\mathscr{Y}_1^{F_{\lambda,\mu}}| \approx |\mathscr{Y}^{F_{\lambda,\mu}}| \approx q^t$, we finally get that

$$\big|\mathscr{Y}^{F'_{\lambda,\mu}}\big| \approx \big|\psi_1(\mathscr{Y}_1)^{F'_{\lambda,\mu}}\big| \approx q^t,$$

that is, $|\mathscr{X}| \approx q^t$. We conclude that $\varphi^\lambda_{i,\mu}(T) \neq 0$, proving the claim.

Let $w = i_1 i_2 \ldots i_m \in \Omega$ with $m \geqslant 1$. If $m = 1$, then clearly $\varphi_w^{\wp(w)}(T) = 1 \neq 0$. Now suppose $m \geqslant 2$ and set $w_1 = i_2 \ldots i_m$, $\lambda = \wp(w)$ and $\mu = \wp(w_1)$. Then $M_q(\lambda)_k \cong S_{i_1} * M_q(\mu)_k$, and hence, by the claim above, $\varphi^{\lambda}_{i,\mu}(T) \neq 0$. On the other hand, by induction, we may suppose that $\varphi^{\mu}_{w_1}(T) \neq 0$. Thus, there is some prime power q such that

$$F^{M_q(\lambda)}_{S_{i_1},M_q(\mu)} \neq 0 \quad \text{and} \quad F^{M_q(\mu)}_{S_{i_2},\dots,S_{i_m}} \neq 0.$$

This implies that

$$F^{M_q(\lambda)}_{S_{i_1},S_{i_2},\dots,S_{i_m}} = \sum_{\pi \in \mathfrak{P}} F^{M_q(\lambda)}_{S_{i_1},M_q(\pi)} F^{M_q(\pi)}_{S_{i_2},\dots,S_{i_m}} \neq 0.$$

Therefore, $\varphi^{\wp(w)}_{w}(T) \neq 0$. $\square$

6. Tight monomials and weakly distinguished words

We are now ready to establish a relationship between tight monomials and weakly distinguished words. Assume again that Q is a Dynkin quiver with automorphism σ. Thus, $C_{Q,\sigma}$ is a classical Cartan matrix (of finite type). We keep all the notation in the previous sections. In particular, $\mathbf{U}^+$ is the quantum algebra associated with (Q,σ), which is identified with $\mathfrak{H} \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ via $E_i \mapsto u_i = u_i \otimes 1$, $\wp$ is the generic extension map, and $\{c_\lambda \mid \lambda \in \mathfrak{P}\}$ is the canonical basis of $\mathbf{U}^+$.

Proposition 6.1. *Let $\boldsymbol{i}, \boldsymbol{j} \in I^t$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^t$. If both $E^{(\boldsymbol{a})}_{\boldsymbol{i}}$ and $E^{(\boldsymbol{b})}_{\boldsymbol{j}}$ are tight and $\wp(w_{\boldsymbol{i},\boldsymbol{a}}) = \wp(w_{\boldsymbol{j},\boldsymbol{b}})$, then $E^{(\boldsymbol{a})}_{\boldsymbol{i}} = E^{(\boldsymbol{b})}_{\boldsymbol{j}}$.*

Proof. Let $\lambda = \wp(w_{\boldsymbol{i},\boldsymbol{a}}) = \wp(w_{\boldsymbol{j},\boldsymbol{b}})$. Applying (4.3.5) gives that

$$E^{(\boldsymbol{a})}_{\boldsymbol{i}} = \sum_{\mu \leqslant \lambda} \phi_{\lambda,\mu}\big(v, v^{-1}\big) E_\mu \quad \text{and} \quad E^{(\boldsymbol{b})}_{\boldsymbol{j}} = \sum_{\mu \leqslant \lambda} \phi'_{\lambda,\mu}\big(v, v^{-1}\big) E_\mu,$$

where $\phi_{\lambda,\mu}(v, v^{-1}), \phi'_{\lambda,\mu}(v, v^{-1}) \in \mathbb{Z}[v, v^{-1}]$. By the statement $(*)$ in Section 4, $\phi_{\lambda,\lambda}(v, v^{-1}) \neq 0$ and $\phi'_{\lambda,\lambda}(v, v^{-1}) \neq 0$. Since both $E^{(\boldsymbol{a})}_{\boldsymbol{i}}$ and $E^{(\boldsymbol{b})}_{\boldsymbol{j}}$ are tight, we must have $E^{(\boldsymbol{a})}_{\boldsymbol{i}} = c_\lambda = E^{(\boldsymbol{b})}_{\boldsymbol{j}}$. $\square$

Call $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$ *sincere* if $a_i \neq 0$ for all i.

Theorem 6.2. *Let $\boldsymbol{i} = (i_1, \dots, i_t) \in I^t$ and $\boldsymbol{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$. If $E^{(\boldsymbol{a})}_{\boldsymbol{i}}$ is tight, then the word $w_{\boldsymbol{i},\boldsymbol{a}}$ defined in* (4.0.2) *is weakly distinguished. More precisely, we have $i_r \neq i_{r+1}$, for all $1 \leqslant r < t$, and the Hall polynomial $\varphi^{\wp(w_{\boldsymbol{i},\boldsymbol{a}})}_{a_1 i_1,\dots,a_t i_t}(T) = T^d$ with*

$$d = \frac{1}{2}\Bigg(\sum_{1 \leqslant r < p \leqslant t} a_p a_r \langle i_p, i_r\rangle + \dim \operatorname{Ext}^1\big(M(\lambda), M(\lambda)\big)\Bigg).$$

Proof. Without loss of generality, we assume that $\boldsymbol{a}$ is sincere. Write $\lambda = \wp(w_{\boldsymbol{i},\boldsymbol{a}})$. By the definition of the multiplication in $\mathfrak{H}(Q,\sigma)$,

$$\begin{aligned} E^{(\boldsymbol{a})}_{\boldsymbol{i}} &= E^{(a_1)}_{i_1} \cdots E^{(a_t)}_{i_t} = u^{(a_1)}_{i_1} \cdots u^{(a_t)}_{i_t} \\ &= \Bigg(\prod_{r=1}^{t} v_{i_r}^{a_r(a_r-1)}\Bigg) u_{[a_1 S_{i_1}]} \cdots u_{[a_t S_{i_t}]} \end{aligned}$$

$$= \left(\prod_{r=1}^{t} v_{i_r}^{a_r(a_r-1)}\right) v^{\sum_{1\leqslant p<r\leqslant t} a_p a_r \langle i_p, i_r\rangle} \sum_{\mu\leqslant\lambda} \varphi^{\mu}_{a_1 i_1,\ldots,a_t i_t}(v^2) u_\mu$$

$$= \left(\prod_{r=1}^{t} v_{i_r}^{a_r(a_r-1)}\right) v^{\sum_{1\leqslant p<r\leqslant t} a_p a_r \langle i_p, i_r\rangle} \sum_{\mu\leqslant\lambda} \varphi^{\mu}_{a_1 i_1,\ldots,a_t i_t}(v^2) v^{\dim M(\mu)-\dim \mathrm{End}(M(\mu))} E_\mu .$$

Since $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight, we have $E_{\boldsymbol{i}}^{(\boldsymbol{a})} = c_\lambda$. Therefore,

$$\left(\prod_{r=1}^{t} v_{i_r}^{a_r(a_r-1)}\right) v^{\sum_{1\leqslant p<r\leqslant t} a_p a_r \langle i_p, i_r\rangle} v^{\dim M(\lambda)-\dim \mathrm{End}(M(\lambda))} \varphi^{\lambda}_{a_1 i_1,\ldots,a_t i_t}(v^2) = 1,$$

that is, $\varphi^{\wp(w_{\boldsymbol{i},\boldsymbol{a}})}_{a_1 i_1,\ldots,a_t i_t}(T) = T^{d'}$, where

$$d' = \frac{1}{2}\left(-\sum_{r=1}^{t} a_r(a_r-1)\langle i_r, i_r\rangle - \sum_{1\leqslant p<r\leqslant t} a_p a_r \langle i_p, i_r\rangle - \dim M(\lambda) + \dim \mathrm{End}\big(M(\lambda)\big)\right).$$

Since $\dim M(\lambda) = \sum_{r=1}^{t} a_r \dim S_{i_r} = \sum_{r=1}^{t} a_r \langle i_r, i_r\rangle$ and

$$\sum_{1\leqslant p,r\leqslant t} a_p a_r \langle i_p, i_r\rangle = \big\langle \mathbf{dim}\, M(\lambda), \mathbf{dim}\, M(\lambda)\big\rangle$$
$$= \dim \mathrm{End}\big(M(\lambda)\big) - \dim \mathrm{Ext}^1\big(M(\lambda), M(\lambda)\big),$$

it follows that $d' = d$. $\square$

Remark 6.3. It is very likely that a Hall polynomial of the form $\gamma_w^\lambda(T)$ (obtained by counting the number of reduced filtrations) cannot be a nonzero power of T. Thus, we conjecture that tight monomials are monomials associated with distinguished words. In fact, examples in Section 8 show that tight monomials are monomials associated with *directed* distinguished words (see [9, §11.2] for a definition).

7. Computing tight monomials: the rank 2 Dynkin case

We now develop an algorithm for computing tight monomials in the Dynkin case of rank 2. As in Example 2.3, we consider in this section a quantum algebra $\mathbf{U}^+$ associated with a rank 2 Cartan datum $(I, \cdot)$ of Dynkin type. Thus, $I = \{1, 2\}$. By (4.0.1), we can identify $\mathbf{U}^+$ with the Ringel–Hall algebra $\mathfrak{H} \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ via $E_i \mapsto u_i = u_i \otimes 1$.

For $\boldsymbol{i} = (i_1, \ldots, i_i) \in I^t$ and $\boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{N}^t$, we are going to determine the tightness of the monomial

$$E_{\boldsymbol{i}}^{(\boldsymbol{a})} = E_{i_1}^{(a_1)} \cdots E_{i_t}^{(a_t)} \in \mathbf{U}^+.$$

Without loss of generality, we suppose that all a_i are positive, i.e., $\boldsymbol{a}$ is sincere.

First, it is direct to see from Remark 4.5 that $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight whenever $t \leqslant 2$. In other words, the monomials

$$E_1^{(a_1)}, E_2^{(a_1)}, E_1^{(a_1)} E_2^{(a_2)}, E_2^{(a_1)} E_1^{(a_2)} \quad (a_1 > 0,\ a_2 > 0)$$

are tight. Now we determine those for $t \geqslant 3$. The algorithm given below repeatedly uses the sets $\mathscr{M}_{\mathbf{i},\boldsymbol{a}}$ and the quadratic forms described in Example 2.3.

7.1. Type A_2

The canonical basis in this case is known and consists of tight monomials; see, e.g., [15, 3.4]. As a starting point of the algorithm, we give an independent construction. The Euler form in this case is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_2 y_1$ for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}I$. Thus, the associated quiver Q has two vertices 1,2 and an arrow from 2 to 1.

If $\boldsymbol{t} = \mathbf{3}$, by Theorem 6.2, it suffices to consider $\boldsymbol{i} \in \{(2,1,2), (1,2,1)\}$, and by Example 2.3(1),

$$\mathrm{q}(A_x) = -2x^2 + 2(a_1 + a_3 - a_2)x,$$

for all $A_x \in \mathscr{M}_{\mathbf{i},\boldsymbol{a}}$ with $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{N}^3$. By Theorem 2.5, if $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ is tight, then $\mathrm{q}(A_x) < 0$ for all $0 < x \leqslant \min\{a_1, a_3\}$. Setting $x = 1$ implies $a_1 + a_3 \leqslant a_2$. Hence, we obtain

$$E_2^{(a_1)} E_1^{(a_2)} E_2^{(a_3)} \quad \bigl(\text{resp., } E_1^{(a_1)} E_2^{(a_2)} E_1^{(a_3)}\bigr) \text{ is tight} \quad \Leftrightarrow \quad a_1 + a_3 \leqslant a_2.$$

If $\boldsymbol{t} \geqslant \mathbf{4}$, then all monomials $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ are not tight. This can be proved by an argument similar to the $t \geqslant 5$ case below. However, an application of the generic extension map defined in Section 3 shows that the founded tight monomials above already form the whole canonical basis.

Consider an arbitrary representation M of Q labelled by (a, b, c) in the sense that $M \cong aS_1 \oplus bS_{12} \oplus cS_2$, where S_{12} is the indecomposable module with dimension vector $(1, 1)$. If we choose $\boldsymbol{a} = (b, b + c, a)$, then $a_1 + a_3 \leqslant a_2$ is equivalent to $a \leqslant c$ and $\wp(2^b 1^{b+c} 2^a) = (a, b, c)$. If we choose $\boldsymbol{a} = (c, a + b, b)$, then $a_1 + a_3 \leqslant a_2$ is equivalent to $a \geqslant c$ and $\wp(1^c 2^{a+b} 1^b) = (a, b, c)$. Moreover, $a = c$ if and only if $\wp(1^c 2^{a+b} 1^b) = \wp(2^b 1^{b+c} 2^a)$. Hence, we have proved the following (cf., [9, Prop. 11.35]).

Proposition 7.1.1. *The set*

$$\bigl\{E_2^{(a_1)} E_1^{(a_2)} E_2^{(a_3)} \bigm| \boldsymbol{a} \in \mathbb{N}^3,\ a_1 + a_3 \leqslant a_2\bigr\} \cup \bigl\{E_1^{(a_1)} E_2^{(a_2)} E_1^{(a_3)} \bigm| \boldsymbol{a} \in \mathbb{N}^3,\ a_1 + a_3 \leqslant a_2\bigr\}$$

is a complete set of tight monomials of type A_2 (and forms the canonical basis).

7.2. Type B_2

The canonical basis in this case is constructed in [29]. However, not all tight monomials are identified in this construction; see Proposition 8.2 below.

Consider the following quiver Q of type A_3

$$\begin{tikzcd}[row sep=small] & \bullet \arrow[dl] \\ \bullet & \\ & \bullet \arrow[ul] \end{tikzcd}$$

and let σ be the automorphism of Q defined by exchanging two arrows. Then

$$\mathcal{A}^F = (kQ)^{F_{Q,\sigma;q}} \cong \begin{bmatrix} \mathbb{F}_q & \mathbb{F}_{q^2} \\ 0 & \mathbb{F}_{q^2} \end{bmatrix}.$$

Up to isomorphism, there are two simple $\mathcal{A}^F$-modules S_1 and S_2 with $\dim_{\mathbb{F}_q} S_1 = 1$ and $\dim_{\mathbb{F}_q} S_2 = 2$.

Then the Euler form $\langle -,-\rangle : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ of (Q,σ) is given by

$$\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + 2x_2y_2 - 2x_2y_1 \quad \text{for } \mathbf{x},\mathbf{y} \in \mathbb{Z}I.$$

Also, the associated Cartan datum $(I,\cdot)$ is defined by

$$\mathbf{x}\cdot\mathbf{y} = \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle = 2x_1y_1 + 4x_2y_2 - 2x_2y_1 - 2x_1y_2,$$

and the associated Cartan matrix $C_{Q,\sigma} = \left[\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right]$ has type B_2.

If $t = 3$, by Theorem 6.2, it suffices to consider $\boldsymbol{i} \in \{(2,1,2),(1,2,1)\}$, and by Example 2.3(1),

$$\mathfrak{q}(A_x) = \begin{cases} -4x^2 + 4(a_1 + a_3 - a_2)x, & \text{if } \boldsymbol{i} = (2,1,2); \\ -2x^2 + 2(a_1 + a_3 - 2a_2)x, & \text{if } \boldsymbol{i} = (1,2,1), \end{cases}$$

for all $A_x \in \mathscr{M}_{\mathbf{i},\boldsymbol{a}}$ with $\boldsymbol{a} = (a_1,a_2,a_3) \in \mathbb{N}^3$. Since $0 \leqslant x \leqslant \min\{a_1,a_3\}$, by Theorem 2.5,

$$\begin{cases} E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)} \text{ is tight} & \Leftrightarrow \quad a_1 + a_3 \leqslant a_2; \\ E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)} \text{ is tight} & \Leftrightarrow \quad a_1 + a_3 \leqslant 2a_2. \end{cases}$$

If $t = 4$, again by Theorem 6.2, we only need to consider $\boldsymbol{i} = (2,1,2,1)$ or $(1,2,1,2)$. By Example 2.3(2), for each $A_{x,y} \in \mathscr{M}_{\mathbf{i},\boldsymbol{a}}$,

$$\mathfrak{q}(A_{x,y}) = \begin{cases} -(2x-y)^2 - y^2 + 4(a_1 - a_2 + a_3)x + 2(a_2 - 2a_3 + a_4)y, & \text{if } \boldsymbol{i} = (2,1,2,1); \\ -x^2 - (x-2y)^2 + 2(a_1 - 2a_2 + a_3)x + 4(a_2 - a_3 + a_4)y, & \text{if } \boldsymbol{i} = (1,2,1,2). \end{cases}$$

Applying Theorem 2.5 gives that

$$\begin{cases} E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)} \text{ is tight} & \Leftrightarrow \quad a_1 + a_3 \leqslant a_2 \quad \text{and} \quad a_2 + a_4 \leqslant 2a_3; \\ E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)} \text{ is tight} & \Leftrightarrow \quad a_1 + a_3 \leqslant 2a_2 \quad \text{and} \quad a_2 + a_4 \leqslant a_3. \end{cases} \tag{7.0.1}$$

If $t \geqslant 5$, then all monomials $E_{\boldsymbol{i}}^{(\boldsymbol{a})}$ are not tight. By Theorem 6.2 and Corollary 2.6, it suffices to show the case for $\boldsymbol{i} = (2,1,2,1,2)$ or $(1,2,1,2,1)$.

First, suppose that $E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}E_2^{(a_5)}$ is tight. Again, by Corollary 2.6, both $E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}$ and $E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}E_2^{(a_5)}$ are tight. By (7.0.1), we obtain

$$a_1 + a_3 \leqslant a_2, \qquad a_2 + a_4 \leqslant 2a_3, \qquad a_3 + a_5 \leqslant a_4.$$

Then the equalities $a_2 + a_4 \leqslant 2a_3$ and $a_3 + a_5 \leqslant a_4$ imply $a_2 \leqslant a_3$. This contradicts the inequality $a_1 + a_3 \leqslant a_2$. Hence, $E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}E_2^{(a_5)}$ is not tight. Second, suppose $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}E_1^{(a_5)}$ is tight. Then both $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}$ and $E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}E_1^{(a_5)}$ are tight. Thus, by (7.0.1),

$$a_1 + a_3 \leqslant 2a_2, \qquad a_2 + a_4 \leqslant a_3, \qquad a_3 + a_5 \leqslant 2a_4.$$

It follows that

$$2a_2 + 2a_4 \geqslant a_1 + a_3 + a_3 + a_5 \geqslant 2a_2 + 2a_4 + a_1 + a_5,$$

a contradiction. Hence, $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}E_1^{(a_5)}$ is not tight.

In a summary, we obtain the following result.

Proposition 7.2.1. *The following is a complete list of tight monomials of type B_2:*

(1) 1, $E_1^{(a_1)}$, $E_2^{(a_1)}$, $E_1^{(a_1)}E_2^{(a_2)}$, $E_2^{(a_1)}E_1^{(a_2)}$,
(2) $E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}$ $(a_1+a_3 \leqslant a_2)$, $E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}$ $(a_2+a_4 \leqslant 2a_3)$,
(3) $E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}$ $(a_1+a_3 \leqslant a_2$ *and* $a_2+a_4 \leqslant 2a_3)$,
(4) $E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}$ $(a_1+a_3 \leqslant 2a_2$ *and* $a_2+a_4 \leqslant a_3)$, *where* $a_1, a_2, a_3, a_4 \in \mathbb{N}\backslash\{0\}$.

7.3. Type G_2

In this case, if $\boldsymbol{i} = (2,1,2,1,2,1)$ or $\boldsymbol{i} = (1,2,1,2,1,2)$ and $\boldsymbol{a} = (a_i) \in \mathbb{N}^6$, then each matrix $A \in \mathscr{M}_{\boldsymbol{i},\boldsymbol{a}}$ has the form

$$A = \begin{bmatrix} x_{11} & 0 & x_{12} & 0 & x_{13} & 0 \\ 0 & y_{11} & 0 & y_{12} & 0 & y_{13} \\ x_{21} & 0 & x_{22} & 0 & x_{23} & 0 \\ 0 & y_{21} & 0 & y_{22} & 0 & y_{23} \\ x_{31} & 0 & x_{32} & 0 & x_{33} & 0 \\ 0 & y_{31} & 0 & y_{32} & 0 & y_{33} \end{bmatrix},$$

where $x_{ij}, y_{ij} \in \mathbb{N}$ satisfy

$$\sum_{j=1}^{3} x_{ij} = \sum_{j=1}^{3} x_{ji} = a_{2i-1} \quad \text{and} \quad \sum_{j=1}^{3} y_{ij} = \sum_{j=1}^{3} y_{ji} = a_{2i} \quad \text{for } 1 \leqslant i \leqslant 3.$$

Thus, this case can be computed by extending the algorithm above to the cases of $t = 3, 4, 5, 6$ and $t \geqslant 7$, and is much more complicated.[1]

8. Identification of tight monomials of type B_2

It is proved in [9, §11.7] that tight monomials of type A_2 all arise from directed distinguished words. In this last section, we first establish a similar result for the tight monomials of type B_2 given in Proposition 7.2.1 and then identify them with the canonical basis elements described by Xi in [29].

Continue the type B_2 case in the previous section and let

$$\Phi^+(Q,\sigma) = \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$$

be the set of positive roots, where α_1 and α_2 are simple roots. The Auslander–Reiten quiver of $\mathcal{A}^F = (kQ)^{F_{Q,\sigma;q}}$ in this case has the form

$$\begin{tikzcd} & {[P_2]} \arrow[dr] & & {[S_2]} \\ {[S_1]} \arrow[ur] & & {[I_1]} \arrow[ur] & \end{tikzcd}$$

[1] Using the method developed in this paper, Xiaoming Wang has completely determined all tight monomials for quantum groups of types G_2, A_3, and those associated with the Cartan matrices $\left[\begin{smallmatrix} 2 & -p \\ -1 & 2 \end{smallmatrix}\right]$ $(p \geqslant 4)$. See her papers: *Tight monomials for type G_2 and A_3*, Comm. Algebra (to appear); *Tight monomials for quantum enveloping algebras of rank-2 Kac–Moody Lie algebras*, J. Pure Appl. Algebra (to appear).

where P_2 and I_1 are the projective cover and injective envelope of S_2 and S_1, respectively. Then the dimension vectors of P_2 and I_1 are $2\alpha_1+\alpha_2$ and $\alpha_1+\alpha_2$, respectively. Following [9, §11.2], there are exactly five directed partitions Φ^* of $\Phi^+(Q,\sigma)$ given as follows:

(1) $\Phi^{(1)}=\{\alpha_1, 2\alpha_1+\alpha_2\}$, $\Phi^{(2)}=\{\alpha_1+\alpha_2, \alpha_2\}$;
(2) $\Phi^{(1)}=\{\alpha_1\}$, $\Phi^{(2)}=\{2\alpha_1+\alpha_2, \alpha_1+\alpha_2\}$, $\Phi^{(3)}=\{\alpha_2\}$;
(3) $\Phi^{(1)}=\{\alpha_1, 2\alpha_1+\alpha_2\}$, $\Phi^{(2)}=\{\alpha_1+\alpha_2\}$, $\Phi^{(3)}=\{\alpha_2\}$;
(4) $\Phi^{(1)}=\{\alpha_1\}$, $\Phi^{(2)}=\{2\alpha_1+\alpha_2\}$, $\Phi^{(3)}=\{\alpha_1+\alpha_2, \alpha_2\}$;
(5) $\Phi^{(1)}=\{\alpha_1\}$, $\Phi^{(2)}=\{2\alpha_1+\alpha_2\}$, $\Phi^{(3)}=\{\alpha_1+\alpha_2\}$, $\Phi^{(4)}=\{\alpha_2\}$.

Each $\lambda\in\mathfrak{P}(Q,\sigma)=\{\mu \mid \mu:\Phi^+(Q,\sigma)\to\mathbb{N}\}$ can be also written as a 4-tuple $(a,b,c,d)\in\mathbb{N}^4$ with

$$a=\lambda(\alpha_1),\qquad b=\lambda(2\alpha_1+\alpha_2),\qquad c=\lambda(\alpha_1+\alpha_2),\qquad d=\lambda(\alpha_2).$$

In other words, the corresponding $\mathcal{A}^F$-module $M_q(\lambda)$ is given by

$$M_q(\lambda)=aS_1\oplus bP_2\oplus cI_1\oplus dS_1.$$

For each $\lambda=(a,b,c,d)$, the above five directed partitions define five directed distinguished words in the fibre $\wp^{-1}(\lambda)$:

$$w_{\lambda,1}=2^b1^{a+2b}2^{c+d}1^c,\qquad w_{\lambda,2}=1^a2^{b+c}1^{2b+c}2^d,\qquad w_{\lambda,3}=2^b1^{a+2b}2^c1^c2^d,$$
$$w_{\lambda,4}=1^a2^b1^{2b}2^{c+d}1^c,\qquad w_{\lambda,5}=1^a2^b1^{2b}2^c1^c2^d.$$

Since the monomials corresponding to $w_{\lambda,i}$ for $i=3,4,5$ are not tight for sincere λ, we only consider the sets of monomials in $\mathbf{U}^+$ corresponding to $w_{\lambda,1}, w_{\lambda,2}$ for all λ. Let

$$\begin{aligned}\mathfrak{M}&=\big\{\mathfrak{m}_{a,b,c,d}:=E_2^{(b)}E_1^{(a+2b)}E_2^{(c+d)}E_1^{(c)}\bigm| a,b,c,d\in\mathbb{N}\big\},\\ \mathfrak{M}'&=\big\{\mathfrak{m}'_{a,b,c,d}:=E_1^{(a)}E_2^{(b+c)}E_1^{(2b+c)}E_2^{(d)}\bigm| a,b,c,d\in\mathbb{N}\big\}.\end{aligned}\tag{8.0.2}$$

By Theorem 4.1, both $\mathfrak{M}$ and $\mathfrak{M}'$ form bases for U^+.

Proposition 8.1. *The tight monomials given in Proposition* 7.2.1 *form a subset of* $\mathfrak{M}\cup\mathfrak{M}'$.

Proof. First, consider the tight monomials given in Proposition 7.2.1(3), (4). Since the inequalities $a_1+a_3\leqslant a_2$ and $a_2+a_4\leqslant 2a_3$ imply $a_4\leqslant a_3$ and $2a_1\leqslant a_2$, while $a_1+a_3\leqslant 2a_2$ and $a_2+a_4\leqslant a_3$ imply $a_2\leqslant a_3\leqslant 2a_2$, it follows that

$$\begin{cases}\text{(a)}\quad \mathfrak{m}_{a,b,c,d}=E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}, & \text{if } a=a_2-2a_1,\ b=a_1,\ c=a_4,\ d=a_3-a_4;\\ \text{(b)}\quad \mathfrak{m}'_{a,b,c,d}=E_1^{(a_1)}E_2^{(a_2)}E_1^{(a_3)}E_2^{(a_4)}, & \text{if } a=a_1,\ b=a_3-a_2,\ c=2a_2-a_3,\ d=a_4.\end{cases}\tag{8.1.1}$$

For those given in Proposition 7.2.1(2), we have by (8.1.1)(a)

$$\begin{cases}E_2^{(a_1)}E_1^{(a_2)}E_2^{(a_3)}=\mathfrak{m}_{a,b,0,d}=E_2^{(b)}E_1^{(a+2b)}E_2^{(d)};\\ E_1^{(a_2)}E_2^{(a_3)}E_1^{(a_4)}=\mathfrak{m}_{a,0,c,d}=E_1^{(a)}E_2^{(c+d)}E_1^{(c)}.\end{cases}\tag{8.1.2}$$

Finally, one checks easily that there exist a,b,c,d,e,f such that

$$E_2^{(a_1)}E_1^{(a_2)}=\begin{cases}\mathfrak{m}_{a,b,0,0}(=E_2^{(b)}E_1^{(a+2b)}), & \text{if } a_2>2a_1;\\ \mathfrak{m}_{0,0,c,d}(=E_2^{(c+d)}E_1^{(c)}), & \text{if } a_1>a_2;\\ \mathfrak{m}'_{0,e,f,0}(=E_2^{(e+f)}E_1^{(2e+f)}), & \text{if } a_1\leqslant a_2\leqslant 2a_1.\end{cases}$$

The remaining cases are clear. □

We now identify the tight monomials given in Proposition 7.2.1 with the canonical basis elements computed by Xi. For nonnegative integers a,b,c,d, the monomials

$$E_1^{(a)}E_2^{(d)},$$
$$E_2^{(b)}E_1^{(a+2b)}E_2^{(c+d)}E_1^{(c)}\quad (c+d\leqslant a+b \text{ and } a+2b\leqslant c+2d),$$
$$E_1^{(a)}E_2^{(b+c)}E_1^{(2b+c)}E_2^{(d)}\quad (a\leqslant c \text{ and } d\leqslant b),$$

are all the tight monomials appeared as the canonical basis elements described in [29, Th. 2.2]. Thus, the tight monomials in Proposition 7.2.1(1), (3), (4) appear in monomial form in Xi's basis. However, most of the tight monomials in Proposition 7.2.1(2) do not appear in monomial form. We now use the representation-theoretic approach to identify them.

We follow the notation introduced in (8.1.2). Thus, by Proposition 7.2.1(2), $\mathfrak{m}_{a,0,c,d}=E_1^{(a)}E_2^{(c+d)}E_1^{(c)}$ (resp., $\mathfrak{m}_{a,b,0,d}=E_2^{(b)}E_1^{(a+2b)}E_2^{(d)}$) is tight if $a\leqslant c+2d$ (resp., $d\leqslant a+b$). For $m\in\mathbb{N}$ and $n\in\mathbb{Z}$, let $\left[{m \atop n}\right]_{v^2}$ be obtained from $\left[{m \atop n}\right]$ by replaced v by v^2.

Proposition 8.2. *Let a,b,c,d be nonnegative integers and keep the notation above.*

(1) *If $a\leqslant c$ and $c\geqslant 1$, then*

$$\mathfrak{m}_{a,0,c,d}=\sum_{0\leqslant l\leqslant c}(-1)^l\begin{bmatrix} d+l-1\\ l\end{bmatrix}_{v^2}E_1^{(a)}E_2^{(c-l)}E_1^{(c)}E_2^{(d+l)}.$$

(2) *If $1\leqslant c\leqslant a\leqslant c+d$, then*

$$\mathfrak{m}_{a,0,c,d}=\sum_{0\leqslant m,l\leqslant c}(-1)^{m+l}\begin{bmatrix} a-c+m-1\\ m\end{bmatrix}\begin{bmatrix} d+m+l-1\\ l\end{bmatrix}_{v^2}E_1^{(a+m)}E_2^{(c-l)}E_1^{(c-m)}E_2^{(d+l)}.$$

(3) *If $d\leqslant b$ and $b\geqslant 1$, then*

$$\mathfrak{m}_{a,b,0,d}=\sum_{0\leqslant m\leqslant 2b}(-1)^l\begin{bmatrix} a+m-1\\ m\end{bmatrix}E_1^{(a+m)}E_2^{(b)}E_1^{(2b-m)}E_2^{(d)}.$$

(4) *If $1\leqslant b\leqslant d\leqslant b+a/2$, then*

$$\mathfrak{m}_{a,b,0,d}=\sum_{\substack{0\leqslant m\leqslant 2b\\ 0\leqslant l\leqslant b}}(-1)^{m+l}\begin{bmatrix} a+2l-m-1\\ m\end{bmatrix}\begin{bmatrix} d-b+l-1\\ l\end{bmatrix}_{v^2}E_1^{(a+m)}E_2^{(b-l)}E_1^{(2b-m)}E_2^{(d+l)}.$$

Proof. We only prove (1). Statements (2)–(4) can be proved similarly.

Let $\lambda = (a, 0, c, d) \in \mathfrak{P}(Q, \sigma)$, i.e., $M(\lambda) = aS_1 \oplus cI_1 \oplus dS_2$. Since $\mathfrak{m}_{a,0,c,d}$ is tight, we have

$$\mathfrak{m}_{a,0,c,d} \equiv E_\lambda \mod v^{-1}\mathscr{L},$$

where $\mathscr{L}$ is the $\mathbb{Z}[v^{-1}]$-submodule of U^+ spanned by E_μ, $\mu \in \mathfrak{P}(Q, \sigma)$; see Remark 4.5.

On the other hand, we denote the element in the right-hand side of the equality in (1) by $X_{a,c,d}$. By [29, Th. 2.2], $X_{a,c,d}$ is a canonical basis element and

$$X_{a,c,d} \equiv E_1^{(a)}\big(E'_{12}\big)^{(c)}E_2^{(d)} \mod v^{-1}\mathscr{L},$$

where $(E'_{12})^{(c)} = (E'_{12})^c/[c]^!$ with

$$E'_{12} = E_2E_1 - v^{-2}E_1E_2.$$

By the definition of the multiplication in $\mathfrak{H}_{\mathcal{Z}}(Q, \sigma)$, we have

$$\begin{aligned} E'_{12} &= E_2E_1 - v^{-2}E_1E_2 = u_2u_1 - v^{-2}u_1u_2 \\ &= v^{-2}(u_{[I_1]} + u_{[S_1\oplus S_2]}) - v^{-2}u_{[S_1\oplus S_2]} = v^{-2}u_{[I_1]}. \end{aligned}$$

Thus,

$$\big(E'_{12}\big)^{(c)} = v^{-2c}u_{[I_1]}/[c]^! = v^{-2c}v^{c^2-c}u_{[cI_1]} = v^{c^2-3c}u_{[cI_1]}.$$

It then follows that

$$\begin{aligned} E_1^{(a)}\big(E'_{12}\big)^{(c)}E_2^{(d)} &= v^{a^2-a+c^2-3c+2d^2-2d}u_{[aS_1]}u_{[cI_1]}u_{[dS_2]} \\ &= v^{a^2-a+c^2-3c+2d^2-2d+ac+2cd}u_\lambda. \end{aligned}$$

Since

$$\dim \operatorname{End}\big(M(\lambda)\big) = a^2 + c^2 + 2d^2 + ac + 2cd \quad \text{and} \quad \dim M(\lambda) = a + 3c + 2d.$$

Thus, by (4.3.3),

$$E_1^{(a)}\big(E'_{12}\big)^{(c)}E_2^{(d)} = v^{\dim \operatorname{End}(M(\lambda)) - \dim M(\lambda)}u_\lambda = E_\lambda$$

and, hence,

$$\mathfrak{m}_{a,0,c,d} \equiv E_\lambda \equiv E_1^{(a)}\big(E'_{12}\big)^{(c)}E_2^{(d)} \equiv X_{a,c,d} \mod v^{-1}\mathscr{L}.$$

We finally conclude that $\mathfrak{m}_{a,0,c,d} = X_{a,c,d}$, as required. $\square$

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