Journal of Algebra 371 (2012) 300–313

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Multiplicative Jordan decomposition in group rings and p-groups with all noncyclic subgroups normal

Chia-Hsin Liu

Department of Mathematics, National Taiwan Normal University, Taipei, 116, Taiwan, ROC

ARTICLE INFO

Article history:
Received 28 March 2012
Available online 1 September 2012
Communicated by Louis Rowen

MSC:
16U60
16S34

Keywords:
Integral group rings
Unit group
Semisimple element
Nilpotent element
Jordan decomposition
Noncyclic subgroup
2-Group
p-Group

ABSTRACT

Let p be a prime and let G be a finite p-group. We show that if the integral group ring $\mathbb{Z}[G]$ satisfies the multiplicative Jordan decomposition property, then every noncyclic subgroup of G is normal. This is used to simplify the work of Hales, Passi and Wilson on the classification of integral group rings of finite 2-groups with the multiplicative Jordan decomposition property.

1. Introduction

Let $\mathbb{Q}[G]$ be the group algebra of a finite group G over the field $\mathbb{Q}$ of rational numbers. A classical theorem tells us that for every α in $\mathbb{Q}[G]$, α has a unique additive Jordan decomposition $\alpha = \alpha_s + \alpha_n$, where α_s is a semisimple element, α_n is nilpotent and $\alpha_s\alpha_n = \alpha_n\alpha_s$. If α is a unit, then α_s is also invertible and $\alpha = \alpha_s(1 + \alpha_s^{-1}\alpha_n)$ is the product of a semisimple unit α_s and a commuting unipotent unit $\alpha_u = 1 + \alpha_s^{-1}\alpha_n$. This is the unique multiplicative Jordan decomposition of α. Note that if α is in the integral group ring $\mathbb{Z}[G]$, then α_s and α_u may not be in $\mathbb{Z}[G]$. Following [AHP98] and [HPW07], we say that $\mathbb{Z}[G]$ has the multiplicative Jordan decomposition property (MJD) if for every unit α

E-mail address: chliu@math.ntnu.edu.tw.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.07.010

of $\mathbb{Z}[G]$, its semisimple part α_s and unipotent part α_u are both contained in $\mathbb{Z}[G]$. For simplicity, we say that G satisfies MJD if $\mathbb{Z}[G]$ has the MJD property.

Significant progress on MJD problem has been made recently by Hales, Passi and Wilson in the paper [HPW07]. They showed that only a few families of finite groups can have MJD and in particular, they classified finite 2-groups with the MJD property. To do this, they used the classification list of groups of order 32 and 64 and checked every group in the list to see if it has MJD. After establishing this, they used mathematical induction to prove that for nonabelian 2-groups of order bigger than 64, only Hamiltonian 2-groups satisfy MJD. Note that there are 44 nonabelian groups of order 32 and more than two hundred nonabelian groups of order 64. Tremendous effort was required to check all these groups. The authors produced a great paper with lots of interesting ideas hidden in the computations. On the other hand, since the length of the paper is rather limited, not all the details were presented and hence it is difficult for the reader to check all nonabelian groups of order 32 and 64. But checking these groups is important for the proof since it is the base case of the mathematical induction. In this paper we take a different approach which does not need this kind of induction. Since it is less complicated, we can include more of the details.

If we look at the results in [HPW07], there are only a few groups with the MJD property. So we should use this MJD condition as early as possible to rule out certain groups. In this way, we do not have to get involved with the classifications of groups of order 32 and 64.

Our approach is based on the following important lemma established in [LP09].

Lemma 1.1. *Let G be a finite group such that $\mathbb{Z}[G]$ has MJD. If Y is a subgroup of G and N is a normal subgroup of G, then either $Y \supseteq N$ or YN is normal in G.*

This gives us a group theoretic condition about subgroups and normal subgroups of G. For convenience, we say that a finite group G has SN if for any subgroup Y of G and normal subgroup N of G, we have either $Y \supseteq N$ or YN is normal in G. We say that a finite group G has SSN if every subgroup of G has SN.

Since the MJD property is inherited by subgroups, the above lemma gives us

Lemma 1.2. *If G has MJD, then G has* SSN*.*

We will study groups with SSN and find out that they have interesting properties. The following result will be proved in Section 2 as Proposition 2.2.

Proposition 1.3. *If G is a finite* 2*-group and G has* SSN*, then every noncyclic subgroup of G is normal.*

2-groups with every noncyclic subgroup normal were classified in [Lim68]. Passman studied finite p-groups with this condition in [Pas70, Proposition 2.9] for any prime p. Recently, Božikov and Janko [BJ09] refined Passman's work and gave a complete classification of p-groups having all noncyclic subgroups normal. In their classification (Theorem 2.3), there are only a few classes of p-groups with this property. Together with Lemma 1.2 and Proposition 1.3, we are then able to simplify the work of [HPW07] on the MJD problem for 2-groups. Indeed, we reprove the following theorem in Section 2 by checking those classes of groups.

Theorem 1.4. *Let G be a nonabelian finite* 2*-group such that $\mathbb{Z}[G]$ has MJD.*

(1) *If $|G| \geqslant 64$, then G is Hamiltonian.*
(2) *If $|G| = 32$, then G is one of the following.*
  (a) $Q_8 \times C_2 \times C_2$.
  (b) $Q_8 \times C_4$.
  (c) $\langle a, b, c \mid a^8 = 1,\ a^2 = b^2 = c^2,\ a^b = a^5,\ ac = ca,\ bc = cb\rangle \cong Q_8 * C_8$.
  (d) *The central product* $D_8 * Q_8$.
(3) *If $|G| = 16$, then G is one of the following.*
  (a) $Q_8 \times C_2$.

(b) $Q_{16} = \langle a, b \mid a^8 = 1,\ b^2 = a^4,\ b^{-1}ab = a^7\rangle$.
(c) $P = \langle a, b \mid a^4 = b^4 = 1,\ b^{-1}ab = a^{-1}\rangle$.
(d) $D = \langle a, b, c \mid a^2 = b^2 = c^4 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac^2\rangle \cong Q_8 * C_4$.
(e) $D^{+}_{16} = \langle a, b \mid a^8 = 1,\ b^2 = 1,\ b^{-1}ab = a^5\rangle$.

(4) *If* $|G| = 8$, *then* G *is one of the following.*
(a) Q_8.
(b) D_8.

Note that all the groups listed in Theorem 1.4 have MJD by [AHP98,Par02,HPW07].

After [HPW07] solved the MJD problem for finite 2-groups, [LP09] and [LP10] studied the analogous problem for finite 3-groups. It is therefore natural to ask if Proposition 1.3 can be extended to 3-groups. Indeed, we show that the same result actually holds for finite p-groups for all odd primes p. This is done in Section 3.

2. MJD for 2-groups

The following lemma has been used extensively in [HPW07,LP09,LP10]. Note that part (1) follows from the uniqueness of Jordan decomposition and part (2) is from [HPW07, Corollary 9].

Lemma 2.1. *Let G have the MJD property.*

(1) *If H is a subgroup of G, then H has MJD.*
(2) *If α is a nilpotent element of $\mathbb{Z}[G]$ and e is a central idempotent of $\mathbb{Q}[G]$, then $\alpha e \in \mathbb{Z}[G]$.*

In fact, part (2) above is the key ingredient in the proof of Lemma 1.1.

Proposition 2.2. *Let G be a finite* 2*-group with* SSN*. Then every noncyclic subgroup of G is normal.*

Proof. Suppose by way of contradiction that G has a noncyclic subgroup which is not normal. Then this subgroup must be contained in a maximal nonnormal subgroup Q of G. In particular, all subgroups of G properly larger than Q are normal. Choose a subgroup M of G containing Q such that $|M : Q| = 2$. Since $M \supsetneqq Q$ and Q is maximal nonnormal, we have $Q \lhd M \lhd G$. Let $N = \mathfrak{N}_G(Q)$, the normalizer of Q in G. Then $M \subseteq N \subsetneqq G$ since Q is not normal in G. Thus we can choose a subgroup T of G containing N such that $|T : N| = 2$. Note that both N and T are normal in G since $T \supsetneqq N \supsetneqq Q$.

Take $t \in T \setminus N$. Since $|T : N| = 2$, we have $t^2 \in N = \mathfrak{N}_G(Q)$ and hence $Q^{t^2} = Q$. Since $Q \subsetneqq M \lhd G$, we have $Q^t \subseteq M^t = M$. Furthermore, $Q \lhd N$ implies that $Q^t \lhd N^t = N$. Note that $t \notin N = \mathfrak{N}_G(Q)$ and hence $Q \neq Q^t$. Thus $QQ^t = M$ since $|M : Q| = 2$. Let $H = Q \cap Q^t$. Since $|M : Q^t| = |M : Q| = 2$, we see that $|Q : H| = |Q^t : H| = 2$. Since $Q \lhd N$ and $Q^t \lhd N$, we have $H \lhd N$. Moreover, $H^t = Q^t \cap Q^{t^2} = Q^t \cap Q = H$. Thus $H \lhd T$ since $T = \langle N, t\rangle$.

Finally, choose $b \in Q \setminus H$. Then $|Q : H| = 2$ implies that $Q = \langle b\rangle H$. Since G has SSN, it follows that T has SN. Thus, since $H \lhd T$, we get either $\langle b\rangle \supseteq H$ or $\langle b\rangle H \lhd T$. Now, we cannot have $\langle b\rangle H \lhd T$ since T is properly larger than the normalizer N of $Q = \langle b\rangle H$. Thus, we must have $\langle b\rangle \supseteq H$ and hence $Q = \langle b\rangle H = \langle b\rangle$ is cyclic. But then Q cannot contain a noncyclic subgroup, so this is the required contradiction. □

Recall that a group G is called Dedekind if all subgroups of G are normal. A group G is called Hamiltonian if G is nonabelian and Dedekind.

Let p be a prime. In [BJ09], Božikov and Janko gave the following classification of finite p-groups with all noncyclic subgroups normal. For convenience, we label those classes of groups as BJ1 to BJ9.

Theorem 2.3. *Let G be a finite p-group which is not Dedekind. If all noncyclic subgroups of G are normal, then G is one of the following groups.*

BJ1. *G is metacyclic minimal nonabelian and G is not isomorphic to Q_8. Namely,*

$$G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle$$

where $m \geqslant 2$, $n \geqslant 1$ and $|G| = p^{m+n}$.

BJ2. *$G = G_0 * Z$, the central product of a nonabelian group G_0 of order p^3 with a cyclic group Z, where $G_0 \cap Z = \mathfrak{Z}(G_0)$, the center of G_0, and if $p = 2$, then $|Z| > 2$.*

BJ3. *$p = 2$ and $G = Q \times Z$ where $Q \cong Q_8$ and Z is cyclic of order > 2.*

BJ4. *G is a group of order 3^4 and maximal class with $\Omega_1(G) = G' \cong C_3 \times C_3$.*

BJ5. *$G = \langle a, b \mid a^8 = b^8 = 1,\ a^b = a^{-1},\ a^4 = b^4 \rangle$, where $|G| = 2^5$.*

BJ6. *$G \cong Q_{16}$, the generalized quaternion group of order 2^4.*

BJ7. *$G = D_8 * Q_8$, an extraspecial 2-group of order 2^5.*

BJ8. *$G = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2,\ a^c = ab^2,\ b^c = ba^2 \rangle$, where G is the minimal non-metacyclic group of order 2^5. Note that $H = \langle a, b \rangle \cong C_4 \times C_4$ is an abelian normal subgroup of G with $G/H \cong C_2$.*

BJ9. *$G = \langle a, b, c, d \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2b^2,\ a^c = a^{-1},\ b^c = a^2b^{-1},\ d^2 = a^2,\ a^d = a^{-1}b^2,\ b^d = b^{-1},\ [c, d] = 1 \rangle$, where G is a special 2-group of order 2^6 in which every maximal subgroup is isomorphic to the minimal non-metacyclic group of order 2^5 in* BJ8*. Note that $H = \langle a, b \rangle \cong C_4 \times C_4$ is an abelian normal subgroup of G with $G/H \cong C_2 \times C_2$.*

Let G be a finite 2-group with MJD. Combining Lemma 1.2, Proposition 2.2 and Theorem 2.3, we see that if G is not Dedekind, then G must be a group listed in Theorem 2.3. It remains to determine which groups in this list have MJD.

Note that BJ4 is not a 2-group. Furthermore, BJ6 and BJ7 have MJD as shown in [Par02, Theorem 6] and [HPW07, p. 123]. We will consider the remaining groups in the rest of this section.

Now BJ1 was checked in [HPW07, pp. 126–128]. Here we take a different approach using the method developed in [LP09, Proposition 2.4]. We first require a few technical facts.

Lemma 2.4. *Let α, β be elements of the commutative ring R. Then for all integers $k \geqslant 0$, we have*

$$(1 + 4\alpha + 8\beta)^{2^k} = 1 + 4 \cdot 2^k\alpha + 8 \cdot 2^k\beta_k$$

for some $\beta_k \in R$.

Proof. Proceed by induction on k. The case $k = 0$ is obvious with $\beta_0 = \beta$. Now suppose that the result holds for k. Then

$$\begin{aligned}
(1 + 4\alpha + 8\beta)^{2^{k+1}} &= \left(1 + 4 \cdot 2^k\alpha + 8 \cdot 2^k\beta_k\right)^2 \\
&= \left(1 + 4 \cdot 2^k\alpha\right)^2 + 2\left(1 + 4 \cdot 2^k\alpha\right)\left(8 \cdot 2^k\beta_k\right) + \left(8 \cdot 2^k\beta_k\right)^2 \\
&= 1 + 4 \cdot 2^{k+1}\alpha + 16 \cdot 2^k2^k\alpha^2 + \left(1 + 4 \cdot 2^k\alpha\right)\left(8 \cdot 2^{k+1}\beta_k\right) + 64 \cdot 2^k2^k\beta_k^2 \\
&= 1 + 4 \cdot 2^{k+1}\alpha + 8 \cdot 2^{k+1}\beta_{k+1}
\end{aligned}$$

for some $\beta_{k+1} \in R$, as required. □

If X is a subset of G, write $\widehat{X}$ for the sum of the elements of X in $\mathbb{Z}[G]$. Furthermore, if H is a subgroup of G, write $e_H = \widehat{H}/|H|$ for the principal idempotent in $\mathbb{Q}[H]$ determined by H. Note that if $h \in H$, then $\widehat{H}(1 - h) = 0$ and hence $e_H(1 - h) = 0$.

Lemma 2.5. *Let G be a finite group. Suppose that G has a normal subgroup A and an element $g \in G$ such that $G/A = \langle Ag \rangle$ is cyclic of order 8. If $e_A = \widehat{A}/|A|$ and $e = e_A(1 - g^4)/2$, then e is a central idempotent in $\mathbb{Q}[G]$ with $e\mathbb{Q}[G]$ isomorphic to the cyclotomic field $\mathbb{Q}[\varepsilon]$, where $\varepsilon = eg$ is a primitive 8th root of unity.*

Proof. Since $e_A\mathbb{Q}[G]$ is naturally isomorphic to $\mathbb{Q}[G/A]$, we may assume that $A=1$. Then $G=\langle g\rangle$ is cyclic of order 8 and $\mathbb{Q}[G]\cong\mathbb{Q}[\zeta]/(\zeta^8-1)$. Since $\zeta^8-1=(\zeta-1)(\zeta+1)(\zeta^2+1)(\zeta^4+1)$, we have the algebra direct sum $\mathbb{Q}[G]=e_1\mathbb{Q}[G]+e_2\mathbb{Q}[G]+e_3\mathbb{Q}[G]+e_4\mathbb{Q}[G]$ where $e_1\mathbb{Q}[G]\cong\mathbb{Q}$, $e_2\mathbb{Q}[G]\cong\mathbb{Q}$, $e_3\mathbb{Q}[G]\cong\mathbb{Q}[i]$ with i a primitive 4th root of 1 and $e_4\mathbb{Q}[G]\cong\mathbb{Q}[\varepsilon]$. Next we note that $\mathbb{Q}[G]=e_B\mathbb{Q}[G]+(1-e_B)\mathbb{Q}[G]$ where $B=\langle g^4\rangle$. Since $e_B\mathbb{Q}[G]\cong\mathbb{Q}[G/B]\cong\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}[i]$, it follows that $(1-e_B)\mathbb{Q}[G]\cong\mathbb{Q}[\varepsilon]$. Finally, $1-e_B=1-(1+g^4)/2=(1-g^4)/2$, as desired. □

The following is the $p=2$ analog of [LP09, Proposition 2.4]. Instead of assuming that A is abelian, we only assume that C is normal in A.

Proposition 2.6. *Let G be a finite 2-group such that $Z=G'$ is central of order 2. Suppose that G has a normal subgroup A such that G/A is cyclic of order 8. If A has a normal subgroup C such that C is not normal in G, then $\mathbb{Z}[G]$ does not have MJD.*

Proof. By assumption, $G/A=\langle Ag\rangle$ for some $g\in G$, and Ag has order 8 in G/A. Since G' is central of order 2, $G/\mathfrak{Z}(G)$ has period 2 where $\mathfrak{Z}(G)$ is the center of G. It follows that g^2 is central in G. By Lemma 2.5, $e=e_A(1-g^4)/2$ is a central idempotent in $\mathbb{Q}[G]$ with $e\mathbb{Q}[G]$ equal to the cyclotomic field $\mathbb{Q}[\varepsilon]$, where $\varepsilon=eg$ is a primitive 8th root of unity. Using the algebra direct sum $\mathbb{Q}[G]=e\mathbb{Q}[G]+(1-e)\mathbb{Q}[G]$, we will construct a unit $u=u_1+u_2$ in $\mathbb{Z}[G]$ with semisimple unit u_1 in $e\mathbb{Q}[G]$, unipotent unit u_2 in $(1-e)\mathbb{Q}[G]$ such that both the semisimple part $s=u_1+(1-e)$ of u and the unipotent part $t=e+u_2$ of u are not in $\mathbb{Z}[G]$.

Since G/A is abelian, we have $Z=G'\subseteq A$. Since C is not normal in G and $|Z|=2$, we see that C does not contain $G'=Z$ and $C\cap Z=1$. So A contains $ZC\cong Z\times C$ and we have $|A|\geqslant 4$. For convenience, write $|A|=4n$ where $n=2^k$ for some integer $k\geqslant 0$. Now define $\alpha=g+g^{-1}\in\mathbb{Z}[G]$ and set

$$u_1=e(1-\alpha)^{4n}\in e\mathbb{Q}[G].$$

Since $g^8\in A$, we have $\widehat{A}(1-g^8)=0$ and hence $e(1+g^4)=0$. Thus $e(\varepsilon^2+\varepsilon^{-2})=e(g^2+g^{-2})=0$, so $e\alpha^2=2e$, $e(1-\alpha^2)=-e$ and $e(1-\alpha^2)^2=e$. It follows that u_1 is a unit in $e\mathbb{Q}[G]$ with inverse $v_1=e(1+\alpha)^{4n}\in e\mathbb{Q}[G]$. Of course, all units in $e\mathbb{Q}[G]\cong\mathbb{Q}[\varepsilon]$ are semisimple.

Next, we study $(1-e)\mathbb{Q}[G]$. Note that ZC is a normal subgroup of A and let T be a set of right coset representatives for ZC in A. Then CT is a full set of right coset representatives for Z in A. Moreover $\widehat{A}=\widehat{ZC}\widehat{T}=\widehat{T}\widehat{ZC}=\widehat{T}\widehat{Z}\widehat{C}$. Define

$$\gamma=\bigl(1-g^4\bigr)\widehat{C}\widehat{T}\alpha$$

and let

$$u_2=(1-e)(1-\gamma)\in(1-e)\mathbb{Q}[G].$$

Note that $1-g^4$ is central in $\mathbb{Q}[G]$ since g^2 is central in G. We will show that $(1-e)\gamma$ has square 0, so u_2 is a unipotent unit in $(1-e)\mathbb{Q}[G]$ with inverse $v_2=(1-e)(1+\gamma)$.

For convenience, set $\beta=(1-g^4)\widehat{C}\widehat{T}$, and let $h=g$ or g^{-1}. Since $G=\langle A,h\rangle$, $C\lhd A$ and C is not normal in G, we have $C^h\neq C$. Since $Z=G'$ has order 2, we see that $C^h\subseteq ZC$ and hence $CC^h=ZC$. It follows that $\widehat{C}\widehat{C}^h=m\widehat{Z}\widehat{C}$ where $m=|C|/2$. Since $C\lhd A$, we have $\widehat{C}\widehat{T}=\widehat{T}\widehat{C}$, and thus $(\widehat{C}\widehat{T})(\widehat{C}\widehat{T})^h=\widehat{T}\widehat{C}\widehat{C}^h\widehat{T}^h=m\widehat{T}\widehat{Z}\widehat{C}\widehat{T}^h=m\widehat{A}\widehat{T}^h$. It follows that $\beta\beta^h$ is divisible by $(1-g^4)\widehat{A}$, a scalar multiple of the idempotent e.

Therefore, $(1-e)\beta\beta^h=0$ and hence $(1-e)\beta h^{-1}\beta=0$. Since $\alpha=g+g^{-1}$, we have $(1-e)\beta\alpha\beta=0$ and consequently $(1-e)\gamma=(1-e)\beta\alpha$ has square 0. With this, it is easy to see that $u_2=(1-e)(1-\gamma)$ is a unipotent unit in $(1-e)\mathbb{Q}[G]$ with inverse $v_2=(1-e)(1+\gamma)$.

Now let $u=u_1+u_2$. By above paragraphs, u has inverse $v=v_1+v_2$ in $\mathbb{Q}[G]$. We want to show that both u and v are in $\mathbb{Z}[G]$ and hence that u is a unit in $\mathbb{Z}[G]$. For elements $\sigma,\tau\in\mathbb{Q}[G]$, let us use

$\sigma \equiv \tau$ to indicate the additive equivalence relation given by $\sigma - \tau \in \mathbb{Z}[G]$. Since $e\alpha^2 = 2e$, we have $e(1-\alpha)^4 = e(1-4\alpha+8(2-\alpha))$ so Lemma 2.4 implies that $u_1 = e(1-\alpha)^{4n} = e(1-4\alpha+8(2-\alpha))^n = e(1-4n\alpha+8n\delta)$ for suitable $\delta \in \mathbb{Z}[G]$. But $e = e_A(1-g^4)/2$ and $|A| = 4n$, so we see that $e(8n\delta) \in \mathbb{Z}[G]$ and hence $u_1 \equiv e(1-4n\alpha)$. Similarly, $v_1 \equiv e(1+4n\alpha)$.

Next, since $\gamma \in \mathbb{Z}[G]$, we have $u_2 = (1-e)(1-\gamma) \equiv -e(1-\gamma)$. Since $\widehat{A}\widehat{C}\widehat{T} = (|A|/2)\widehat{A} = 2n\widehat{A}$ and $g^8 \in A$ implies $\widehat{A}(1-g^4)^2 = \widehat{A}(1+g^8-2g^4) = \widehat{A}(2-2g^4) = 2\widehat{A}(1-g^4)$, we see that $e\gamma = 4ne\alpha$ and $u_2 \equiv -e(1-4n\alpha)$. Similarly, $v_2 \equiv -e(1+4n\alpha)$. Thus

$$u = u_1 + u_2 \equiv e(1-4n\alpha) - e(1-4n\alpha) \equiv 0$$

and

$$v = v_1 + v_2 \equiv e(1+4n\alpha) - e(1+4n\alpha) \equiv 0.$$

So $u, v \in \mathbb{Z}[G]$ and hence u is a unit in $\mathbb{Z}[G]$. Now consider $s = u_1 + (1-e)$ and $t = e + u_2$. Then we have $u = st = ts$. Since u_1 is a semisimple unit in $e\mathbb{Q}[G]$ and u_2 is a unipotent unit in $(1-e)\mathbb{Q}[G]$, we see that $s = u_1 + (1-e)$ is the semisimple part of u and $t = e + u_2$ is the unipotent part of u by [LP09, Lemma 1.2]. In particular, $u = st$ is the multiplication Jordan decomposition of u.

Finally, we show that the semisimple part s is not in $\mathbb{Z}[G]$. Indeed, since g has order 8 modulo A,

$$\begin{aligned} s &= u_1 + (1-e) \equiv e(1-4n\alpha) + (1-e) = 1 - 4n\alpha e \equiv -4n\alpha e \\ &= (g+g^{-1})\widehat{A}(g^4-1)/2 = \frac{1}{2}(g^5+g^3-g-g^7)\widehat{A} \end{aligned}$$

and the latter element is not in $\mathbb{Z}[G]$ since every group element in gA has coefficient $-1/2$.

Thus u is a unit in $\mathbb{Z}[G]$ with semisimple part not in $\mathbb{Z}[G]$ and we conclude that G does not satisfy MJD. $\square$

Proposition 2.6 can be used to show that most of the groups in BJ1 do not have MJD. In particular, it covers the result of [HPW07, Proposition 22]. Indeed, we have the following corollary.

Corollary 2.7. *Let $G = \langle a, b \mid a^{2^m} = 1,\ b^{2^n} = 1,\ b^{-1}ab = a^{1+2^{m-1}} \rangle$ be a group in* BJ1. *If $m \geqslant 4$ and $n \geqslant 1$, then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Note that $G' = \langle a^{2^{m-1}} \rangle$ is central of order 2. Let $A = \langle a^8, b \rangle$ and $C = \langle b \rangle$. Since $m \geqslant 4$, we have $A \supseteq G'$. Thus A is normal in G and G/A is cyclic of order 8. Since a^2 is central in G, we see that A is abelian and hence C is normal in A. But C is not normal in G since $a^{-1}ba = ba^{2^{m-1}} \notin C$. By Proposition 2.6, $\mathbb{Z}[G]$ does not have MJD. $\square$

Before we proceed further, we state some simple facts we need. If H is a subgroup of G, we have the natural projection $\pi_H : \mathbb{Q}[G] \to \mathbb{Q}[H]$ given by $\pi_H(\sum_{g\in G} \alpha_g g) = \sum_{g \in H} \alpha_g g$. Obviously, if $\gamma \in \mathbb{Q}[H]$ and $\alpha, \beta \in \mathbb{Q}[G]$, then $\pi_H(\alpha+\beta) = \pi_H(\alpha) + \pi_H(\beta)$, $\pi_H(\gamma\alpha) = \gamma\pi_H(\alpha)$ and $\pi_H(\alpha\gamma) = \pi_H(\alpha)\gamma$. Moreover, if $\pi_H(\alpha) \notin \mathbb{Z}[H]$, then $\alpha \notin \mathbb{Z}[G]$.

Now we consider the $m = 2$ and $m = 3$ cases. The following lemma is actually covered by [HPW07, Lemmas 23, 24]. We include a proof since similar techniques will be used later when we prove Lemma 2.13.

Lemma 2.8. *Let $G = \langle a, b \mid a^{2^m} = 1,\ b^{2^n} = 1,\ b^{-1}ab = a^{1+2^{m-1}} \rangle$ be a group in* BJ1.

(1) *If $m = 2$ and $n \geqslant 4$, then $\mathbb{Z}[G]$ does not have MJD.*
(2) *If $m = 3$ and $n \geqslant 2$, then $\mathbb{Z}[G]$ does not have MJD.*

Proof. First of all, note that a^2 and b^2 are central in G.

For the case $m=2$ and $n\geqslant 4$, let $w=b^{2^{n-3}}$ and observe that w is central of order 8. Let

$$r=(a+w^2)b(1-a^2)(1-w^4) \quad \text{and}$$
$$s=(a+w)b(1-a^2)(1-w^2)(1+w^4).$$

Since $(1-w^4)(1+w^4)=1-w^8=0$, we have $rs=sr=0$. Next we want to show $r^2=s^2=0$.

Since $a^b=a^3=a^{-1}$, we get

$$(a+w^2)b(a+w^2)b=(a+w^2)b^2(a^{-1}+w^2)=b^2\big((1+w^4)+(a+a^{-1})w^2\big).$$

But $1+w^4$ is annihilated by $1-w^4$, and $a+a^{-1}=a^{-1}(a^2+1)$ is annihilated by $1-a^2$, so we see that $r^2=0$. Similarly, we have

$$(a+w)b(a+w)b=b^2\big((1+w^2)+(a+a^{-1})w\big)$$

and hence $s^2=0$ since $1+w^2$ is annihilated by $(1-w^2)(1+w^4)$.

Now let $\alpha=\frac{1}{2}(r(1+w)^3+s(1+w))$. It follows that $\alpha^2=0$ and we want to show that $\alpha\in\mathbb{Z}[G]$. For convenience, write $\sigma\equiv\tau$ for the additive and multiplicative equivalence relation defined on $\mathbb{Z}[G]$ by $\sigma-\tau\in 2\mathbb{Z}[G]$. Since $(1+w)\equiv(1-w)$ and $(1+w)^{2^k}\equiv 1+w^{2^k}\equiv 1-w^{2^k}$ for any positive integer k, we have

$$\begin{aligned}2\alpha&\equiv r(1+w)^3+s(1+w)\\&\equiv (a+w^2)b(1-a^2)(1+w)^4(1+w)^3+(a+w)b(1-a^2)(1+w)^2(1+w)^4(1+w)\\&\equiv (2a+w^2+w)b(1-a^2)(1+w)^7\\&\equiv wb(1-a^2)(1+w)^8\equiv wb(1-a^2)(1+w^8)\equiv 0\end{aligned}$$

and hence $\alpha\in\mathbb{Z}[G]$.

Let $e=(1-w^4)/2$. Then e is a central idempotent in $\mathbb{Q}[G]$ such that $er=r$ and $es=0$. It follows that $e\alpha=r(1+w)^3/2$. To prove that $\mathbb{Z}[G]$ does not have MJD, by Lemma 2.1(2), it suffices to show that $e\alpha\notin\mathbb{Z}[G]$. For this, let $B=\langle b\rangle$. Then

$$\begin{aligned}\pi_B(e\alpha)&=\pi_B\big(r(1+w)^3\big)/2\\&=\pi_B\big((a+w^2)(1-a^2)b(1-w^4)(1+w)^3\big)/2\\&=\pi_B\big(a+w^2-a^3-a^2w^2\big)b(1-w^4)(1+w)^3/2\\&=w^2b(1-w^4)(1+w)^3/2\notin\mathbb{Z}[B]\end{aligned}$$

and we have $e\alpha\notin\mathbb{Z}[G]$. This completes the case $m=2$ and $n\geqslant 4$.

The case $m=3$ and $n\geqslant 2$, is similar, so we only sketch the proof. Let $u=a^2$ and note that u is central of order 4. Moreover, let

$$r=a(u^2+b)(1-u^2)(1+b^2)\widehat{T} \quad \text{and}$$
$$s=a(u+b)(1-u^2)(1-b^2)\widehat{T}$$

where $T=\langle b^4\rangle$ has order 2^{n-2}. Then $rs=sr=0$ since $b^4\in T$. Furthermore, $aba^{-1}=u^2b$ implies that

$$a\bigl(u^2+b\bigr)a\bigl(u^2+b\bigr)=\bigl(u^2+aba^{-1}\bigr)a^2\bigl(u^2+b\bigr)=a^2\bigl(\bigl(1-b^2\bigr)+\bigl(1+u^2\bigr)\bigl(b^2+b\bigr)\bigr)$$

and

$$a(u+b)a(u+b)=\bigl(u+aba^{-1}\bigr)a^2(u+b)=a^2\bigl(u^2\bigl(1+b^2\bigr)+u\bigl(u^2+1\bigr)b\bigr).$$

It follows that $r^2=s^2=0$.

Now let $\alpha=\frac{1}{2}(r(1+u)+s(1+u))$. Then $\alpha^2=0$ and as above, we can show that $\alpha\in\mathbb{Z}[G]$. Finally, let $e=(1+b^2)\widehat{T}/2^{n-1}$. Then e is a central idempotent in $\mathbb{Q}[G]$ such that $er=r$ and $es=0$. Moreover, $e\alpha=r(1+u)/2\notin\mathbb{Z}[G]$ and we conclude that $\mathbb{Z}[G]$ does not have MJD by Lemma 2.1(2). $\square$

We still need to verify one more special case, namely $m=2$ and $n=3$. This group is actually the group 32.21 discussed in [HPW07, p. 121]. We use the units given in the forthcoming errata to the above paper and we thank Professor Hales for sending us a preliminary version of this work.

Lemma 2.9. *If $G=\langle a,b\mid a^4=1,\ b^8=1,\ b^{-1}ab=a^3\rangle$, then G does not have MJD.*

Proof. Note that a^2 and b^2 are central. Let

$$\alpha=1+2\bigl(1-b^4\bigr)\widehat{T}$$

where $T=\langle a\rangle$ and let

$$\beta=\bigl(2+a^2\bigr)\bigl(1-b^4\bigr)\delta$$

where $\delta=(1+a)(1-ab^2)$. Clearly α is central in $\mathbb{Z}[G]$ and we want to show that $u=\alpha+b\beta$ has inverse $v=\alpha-b\beta$ in $\mathbb{Z}[G]$.

Since $(1-b^4)^2=2(1-b^4)$ and $\widehat{T}^2=4\widehat{T}$, it is easy to see that $\alpha^2=1+36(1-b^4)\widehat{T}$. To compute $b\beta b\beta=b^2\beta^b\beta$, we first compute $\delta^b\delta$. Note that $\delta=(1-a^2b^2)+a(1-b^2)$, so

$$\begin{aligned}\delta^b\delta&=\bigl[\bigl(1-a^2b^2\bigr)+a^3\bigl(1-b^2\bigr)\bigr]\cdot\bigl[\bigl(1-a^2b^2\bigr)+a\bigl(1-b^2\bigr)\bigr]\\&=\bigl(1-a^2b^2\bigr)^2+\bigl(1-b^2\bigr)^2+\bigl(1-a^2b^2\bigr)\bigl(a+a^3\bigr)\bigl(1-b^2\bigr)\\&=\bigl(1+b^4-2a^2b^2\bigr)+\bigl(1+b^4-2b^2\bigr)+\bigl(a+a^3\bigr)\bigl(1-b^2\bigr)^2\\&=\bigl(1+b^4\bigr)\bigl(2+a+a^3\bigr)-2b^2\widehat{T}\end{aligned}$$

and hence $(1-b^4)\delta^b\delta=-2b^2(1-b^4)\widehat{T}$. Since $a\widehat{T}=\widehat{T}$ and $b^4(1-b^4)=-(1-b^4)$, it follows that

$$\begin{aligned}b\beta b\beta=b^2\beta^b\beta&=b^2\bigl(2+a^2\bigr)^2\bigl(1-b^4\bigr)^2\delta^b\delta\\&=-2\bigl(2+a^2\bigr)^2\bigl(1-b^4\bigr)^2b^4\widehat{T}=36\bigl(1-b^4\bigr)\widehat{T}\end{aligned}$$

and we get $uv=vu=\alpha^2-b\beta b\beta=1$.

Next, let $e=(1-a^2)/2$ and $f=1-e=(1+a^2)/2$. Then e and f are orthogonal central idempotents in $\mathbb{Q}[G]$. Since $f\mathbb{Q}[G]\cong\mathbb{Q}[G/G']$ and G/G' is commutative, it follows that fu is a semisimple unit in $f\mathbb{Q}[G]$. Furthermore, since $e\widehat{T}=0$, we have $e\alpha=e$ and $(eb\beta)^2=eb\beta b\beta=0$. So $eu=e+(eb\beta)$ is a unipotent unit in $e\mathbb{Q}[G]$. Let $s=e+fu$ and $t=eu+f$. Then $u=st=ts$ and by [LP09, Lemma 1.2], we see that s is the semisimple part of u and t is the unipotent part of u. Since $ea^2=-e$, we see that

$$t = eu + f = 1 + eb\beta = 1 + \frac{1}{2}(1-a^2)b(2+a^2)(1-b^4)\big((1-a^2b^2) + a(1-b^2)\big)$$
$$= 1 + \frac{1}{2}(1-a^2)b(1-b^4)\big((1+b^2) + a(1-b^2)\big) \notin \mathbb{Z}[G]$$

and hence G does not have MJD. □

Now we are ready to check BJ1.

Lemma 2.10. *If G is a 2-group with MJD in* BJ1*, then G is one of the following groups.*

(1) $G = \langle a, b \mid a^8 = 1,\ b^2 = 1,\ a^b = a^5\rangle$ *of order* 16.
(2) $G = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ a^b = a^3\rangle$ *of order* 8.
(3) $G = \langle a, b \mid a^4 = 1,\ b^4 = 1,\ a^b = a^3\rangle$ *of order* 16.

Proof. Let $G = \langle a, b \mid a^{2^m} = 1,\ b^{2^n} = 1,\ b^{-1}ab = a^{1+2^{m-1}}\rangle$ where $m \geqslant 2$ and $n \geqslant 1$. Suppose that G has MJD. By Corollary 2.7, we see that $m = 2$ or $m = 3$. By Lemma 2.8, we see that if $m = 3$, then $n = 1$. This gives us (1). Moreover, if $m = 2$, then $n = 1, 2$ or 3. But Lemma 2.9 eliminates the case $m = 2$ and $n = 3$. The remaining cases are (2) and (3). □

To check groups in BJ2, we first consider the following special case.

Lemma 2.11. *Let $G = Q_8 * C_{16}$ with $Q_8 \cap C_{16} = \mathfrak{Z}(Q_8)$. Then G does not have MJD.*

Proof. Write $Q_8 = \langle a, b \mid a^4 = 1 = b^4,\ a^2 = b^2,\ a^b = a^{-1}\rangle$ and $C_{16} = \langle t \mid t^{16} = 1\rangle$ so that in G we have $at = ta$, $bt = tb$, $a^2 = t^8$. Let $x = at$. Then $x^2 = a^2t^2$, $x^4 = a^4t^4 = t^4$, $x^8 = t^8 = a^2 = b^2$ and x has order 16. Since both t and a^2 are central, we see that x^2 is central. Let $y = bx^4 = bt^4$ so we have $y^2 = b^2x^8 = b^4 = 1$. Moreover, $x^y = x^b = a^bt = a^{-1}t = a^{-2}x = x^8x = x^9$. Thus $H = \langle x, y\rangle$ is a group of order 32 with relations $x^{16} = 1$, $y^2 = 1$ and $x^y = x^9$. We see that H is a group in BJ1 and hence $\mathbb{Z}[H]$ does not have MJD by Lemma 2.10. Therefore, $\mathbb{Z}[G]$ does not have MJD by Lemma 2.1(1). □

Lemma 2.12. *If G is a 2-group with MJD in* BJ2*, then G is one of the following.*

(1) $G = Q_8 * C_8$, *a group of order* 32.
(2) $G = Q_8 * C_4$, *a group of order* 16.

Proof. Let G be a 2-group in BJ2, so that $G = G_0 * Z$ where G_0 is nonabelian of order 8 and Z is cyclic with $|Z| \geqslant 4$. The groups $Q_8 * Z$ and $D_8 * Z$ are easily seen to be isomorphic, so we can assume that $G = Q_8 * Z$. If $|G| \geqslant 64$, then G has a subgroup H isomorphic to $Q_8 * C_{16}$. By Lemma 2.11, H does not have MJD and therefore Lemma 2.1(1) implies that G does not have MJD. In particular, if G has MJD, then $|G| \leqslant 32$ and the result follows. □

Next we consider a special case in BJ3.

Lemma 2.13. *Let $G = Q_8 \times C_8$. Then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Let $Q_8 = \langle a, b \mid a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1}\rangle$ and $C_8 = \langle t \mid t^8 = 1\rangle$. For convenience, write $z = a^2 = b^2$. Note that t and z are central in G and $ba = abz$.

Let $r = (a + bt)(1 - t^2)(1 + t^4)(1 - z)$ and $s = (a + bt^2)(1 - t^4)(1 - z)$. We want to show that $r^2 = s^2 = rs = sr = 0$. To this end, observe that $(a + bt)^2 = a^2 + b^2t^2 + abt + bat = z(1 + t^2) + abt(1 + z)$. Since $(1 + t^2)$ is annihilated by $(1 - t^2)(1 + t^4)$ and $(1 + z)$ is annihilated by $(1 - z)$, we see that

$r^2 = 0$. Similarly, since $(a+bt^2)^2 = z(1+t^4) + abt^2(1+z)$, we can get $s^2 = 0$. Moreover, $rs = 0 = sr$ since $(1-t^4)(1+t^4) = 0$.

Now let $\alpha = \frac{1}{2}(r(1-t) + s(1-t)^3)$. It follows that $\alpha^2 = 0$ and we want to show that $\alpha \in \mathbb{Z}[G]$. For convenience, let $\sigma \equiv \tau$ be the additive and multiplicative equivalence relation defined on $\mathbb{Z}[G]$ by $\sigma - \tau \in 2\mathbb{Z}[G]$. Since $(1+t) \equiv (1-t)$ and $(1+t)^{2^n} \equiv 1 + t^{2^n}$ for any positive integer n, we have

$$\begin{aligned}
2\alpha &\equiv r(1+t) + s(1+t)^3 \\
&\equiv (a+bt)(1+t)^2(1+t)^4(1-z)(1+t) + \bigl(a+bt^2\bigr)(1+t)^4(1-z)(1+t)^3 \\
&\equiv (a+bt)(1+t)^7(1-z) + \bigl(a+bt^2\bigr)(1+t)^7(1-z) \\
&\equiv \bigl(2a+bt+bt^2\bigr)(1+t)^7(1-z) \\
&\equiv bt(1+t)^8(1-z) \equiv bt\bigl(1+t^8\bigr)(1-z) \equiv 0
\end{aligned}$$

and therefore $\alpha \in \mathbb{Z}[G]$.

Let $e = (1+t^4)/2$. Then e is a central idempotent and we have $er = r$ and $es = 0$. It follows that $e\alpha = r(1-t)/2$. To prove that $\mathbb{Z}[G]$ does not have MJD, by Lemma 2.1(2), it suffices to show that $e\alpha \notin \mathbb{Z}[G]$. For this, let $T = \langle t \rangle$. Then

$$\begin{aligned}
\pi_T\bigl(a^{-1}e\alpha\bigr) &= \pi_T\bigl(a^{-1}r(1-t)/2\bigr) \\
&= \pi_T\bigl(\bigl(1+a^{-1}bt\bigr)\bigl(1-t^2\bigr)\bigl(1+t^4\bigr)(1-z)\bigr)\cdot(1-t)/2 \\
&= \pi_T\bigl(\bigl(1+a^{-1}bt\bigr)\bigl(1-a^2\bigr)\bigr)\cdot\bigl(1-t^2\bigr)\bigl(1+t^4\bigr)(1-t)/2 \\
&= \bigl(1-t^2\bigr)\bigl(1+t^4\bigr)(1-t)/2 \notin \mathbb{Z}[T]
\end{aligned}$$

and we have $a^{-1}e\alpha \notin \mathbb{Z}[G]$. Therefore, $e\alpha \notin \mathbb{Z}[G]$, as required. □

Lemma 2.14. *If G is a group with MJD in* BJ3*, then $G \cong Q_8 \times C_4$.*

Proof. Since G is a group with MJD in BJ3, $G = Q \times Z$ where $Q \cong Q_8$ and Z is a cyclic 2-group of order > 2. If $|G| \geqslant 64$, then G has a subgroup $H \cong Q_8 \times C_8$. By Lemma 2.1(1), H has MJD and this contradicts Lemma 2.13. It follows that $|G| = 32$ and $G \cong Q_8 \times C_4$. □

Next we consider the group in BJ5.

Lemma 2.15. *Let $G = \langle a, b \mid a^8 = b^8 = 1,\ a^4 = b^4,\ a^b = a^{-1}\rangle$ be the group in* BJ5*. Then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Clearly b^2 is central in G. Since b has order 8 and $a^4 = b^4$, we see that $a^4(1-b^4) = b^4(1-b^4) = -(1-b^4)$. Furthermore, $e = (1-a^4)/2 = e(1-b^4)/2$ is a central idempotent in $\mathbb{Q}[G]$ with $ea^4 = eb^4 = -e$ and $ea^2 = -ea^{-2}$.

Now, let $\gamma = a^2b(1-b^2) + (1-a^2)$ and $\beta = \gamma(1+b^2) = a^2b(1-b^4) + (1-a^2)(1+b^2)$. Then $\beta^b = a^{-2}b(1-b^4) + (1-a^{-2})(1+b^2)$ and $ba = a^{-1}b$ imply that

$$\begin{aligned}
a\beta^b a &= aa^{-2}ba\bigl(1-b^4\bigr) + a^2\bigl(1-a^{-2}\bigr)\bigl(1+b^2\bigr) \\
&= a^{-2}b\bigl(1-b^4\bigr) + \bigl(a^2-1\bigr)\bigl(1+b^2\bigr) \\
&= a^2a^4b\bigl(1-b^4\bigr) - \bigl(1-a^2\bigr)\bigl(1+b^2\bigr) \\
&= -a^2b\bigl(1-b^4\bigr) - \bigl(1-a^2\bigr)\bigl(1+b^2\bigr) = -\beta
\end{aligned}$$

and hence $b^{-1}\beta ba = -a^{-1}\beta$. It follows that $\beta bab = -ba^{-1}\beta b = -ab\beta b$ and we have $\beta b(1-ab) = (1+ab)\beta b$.

For convenience, write $\gamma = 1 + a^2(\delta - 1)$ where $\delta = b(1-b^2)$. Then $\gamma^b = 1 + a^{-2}(\delta - 1)$ and $e\gamma^b = e(1 - a^2(\delta - 1))$. Note that $e\delta^2 = eb^2(1 + b^4 - 2b^2) = e(-2b^4) = 2e$. Thus, using $ba^2 = a^{-2}b$, we have $a^2\delta a^2 = \delta$ and hence

$$\begin{aligned} e\gamma^b\gamma &= e\big(1 - a^2(\delta - 1)\big)\big(1 + a^2(\delta - 1)\big) = e\big(1 - a^2(\delta - 1)a^2(\delta - 1)\big) \\ &= e\big(1 - \big(\delta - a^4\big)(\delta - 1)\big) = e\big(1 - (\delta + 1)(\delta - 1)\big) = e\big(2 - \delta^2\big) = 0. \end{aligned}$$

Next, let $\alpha = b(1-ab)\beta$. Then we have

$$\begin{aligned} \alpha^2 &= b(1-ab)\beta b(1-ab)\beta = b(1-ab)(1+ab)\beta b\beta \\ &= b\big(1 - (ab)^2\big)\gamma b\gamma\big(1 + b^2\big)^2 = b\big(1 - b^2\big)\big(1 + b^2\big)\big(1 + b^2\big)b\gamma^b\gamma \\ &= 2b^2\big(1 + b^2\big)e\gamma^b\gamma = 0 \end{aligned}$$

since $(ab)^2 = abab = aa^{-1}b^2 = b^2$.

Finally, we have shown that the element $\alpha = b(1-ab)(a^2b(1-b^2) + (1-a^2))(1+b^2)$ is a nilpotent element in $\mathbb{Z}[G]$. Furthermore, $e = (1-a^4)/2$ is a central idempotent in $\mathbb{Q}[G]$. By Lemma 2.1(2), it follows that $\mathbb{Z}[G]$ does not have MJD if we can show that $\alpha e \notin \mathbb{Z}[G]$. Of course, it is enough to show that $b^{-1}\alpha e \notin \mathbb{Z}[G]$. To this end, let $A = \langle a\rangle$. Since b^2 is central and $b^4 = a^4$, we have

$$\begin{aligned} \pi_A\big(b^{-1}\alpha e\big) &= \pi_A\big((1-ab)\big(a^2b\big(1-b^2\big) + \big(1-a^2\big)\big)\big(1+b^2\big)e\big) \\ &= \pi_A\big((1-ab)\big(a^2b\big(1-a^4\big) + \big(1+b^2\big)\big(1-a^2\big)\big)\big) \cdot e \\ &= \pi_A\big((1-ab)a^2b\big) \cdot \big(1-a^4\big)e + \pi_A\big((1-ab)\big(1+b^2\big)\big) \cdot \big(1-a^2\big)e \\ &= \big(1-a^2\big)e = \big(1-a^2\big)\big(1-a^4\big)/2 \notin \mathbb{Z}[A] \end{aligned}$$

and it follows that $b^{-1}\alpha e \notin \mathbb{Z}[G]$, as required. $\square$

The group in BJ8 is actually the group 32.40 in the list used by [HPW07, p. 120]. In the latter paper a nilpotent element is offered without proof. Here we choose a different nilpotent element and give a detailed argument.

Lemma 2.16. *Let $G = \langle a, b, c \mid a^4 = b^4 = [a,b] = 1,\ c^2 = a^2,\ a^c = ab^2,\ b^c = ba^2\rangle$ be the group in* BJ8. *Then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Let $f = (1 - a^2b^2)/2$, $\beta = 2f(1+a)(1+b)$ and $\alpha = \beta c$. Since both a^2 and b^2 are central of order 2, we see that f is a central idempotent in $\mathbb{Q}[G]$ and $f(1 + a^2b^2) = 0$. Note that $\alpha, \beta \in \mathbb{Z}[G]$ and we want to show that $\alpha^2 = 0$. Since $\alpha^2 = c\beta^c\beta c$, it is enough to show that $\beta^c\beta = 0$.

Let $H = \langle a, b\rangle$. Then H is an abelian normal subgroup of G and the map $N : \mathbb{Q}[H] \to \mathbb{Q}[H]$ given by $N(\gamma) = \gamma^c\gamma$ is a multiplicative homomorphism. Note that

$$f \cdot N(1+a) = f\big(1 + a^c\big)(1+a) = f\big(1 + ab^2 + a + a^2b^2\big) = f\big(a + ab^2\big) = fa\big(1 + b^2\big)$$

and similarly, $f \cdot N(1+b) = fb(1+a^2)$. Then we have

$$\begin{aligned}\beta^c\beta = N(\beta) &= 4\cdot N(f)\cdot N(1+a)\cdot N(1+b) = 4f\cdot N(1+a)\cdot f\cdot N(1+b)\\ &= fa\left(1+b^2\right)fb\left(1+a^2\right) = abf\left(1+a^2+b^2+a^2b^2\right)\\ &= abf\left(1+a^2b^2\right)\left(1+a^2\right) = 0\end{aligned}$$

and hence $\alpha^2 = 0$.

Now let $e = (1+a^2)/2$ and observe that e is a central idempotent in $\mathbb{Q}[G]$. We want to show that $\alpha e \notin \mathbb{Z}[G]$. If this is the case, then G does not have MJD by Lemma 2.1(2). Since $\alpha e \in \mathbb{Z}[G]$ if and only if $\beta e = e\alpha c^{-1} \in \mathbb{Z}[G]$, we only have to show that $\beta e \notin \mathbb{Z}[G]$. For the latter, let $A = \langle a\rangle$. Then

$$\pi_A(\beta e) = \pi_A\left(\left(1-a^2b^2\right)(1+b)\right)\cdot(1+a)\left(1+a^2\right)/2 = (1+a)\left(1+a^2\right)/2 \notin \mathbb{Z}[A]$$

and it follows that $\beta e \notin \mathbb{Z}[G]$. □

Lemma 2.17. *Let G be the group in* BJ9. *Then $\mathbb{Z}[G]$ does not have MJD.*

Proof. Let G be the group in BJ9. Since G has a subgroup isomorphic to the group in BJ8, we see that $\mathbb{Z}[G]$ does not have MJD by Lemma 2.16 and Lemma 2.1(1). □

Finally, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let G be a nonabelian finite group with MJD. We see that (1), (2)(a), (3)(a) and (4)(a) are Hamiltonian 2-groups. So assume now that G is not Dedekind. Then G has SSN by Lemma 1.2 and hence every noncyclic subgroup of G is normal by Proposition 2.2. It follows from Theorem 2.3 that G must be in one of BJ1 through BJ9. If G is in BJ1, Lemma 2.10 shows that G can be (3)(e), (3)(c) or (4)(b). If G is in BJ2, Lemma 2.12 shows that G can be (2)(c) or (3)(d). If G is in BJ3, Lemma 2.14 shows that G is (2)(b). BJ4 is not a 2-group. By Lemma 2.15, Lemma 2.16 and Lemma 2.17, G cannot be BJ5, BJ8, BJ9. Since the cases BJ6 and BJ7 give us (3)(b) and (2)(d), the proof is complete. □

3. Groups with all noncyclic subgroups normal

Proposition 2.2 shows that in any finite 2-group with SSN, every noncyclic subgroup is normal. It is natural to ask if the same result holds for p-groups with p an odd prime. This is done in Proposition 3.3. To start with, we quote the following easy observation from the proof in [LP09, Lemma 2.6].

Lemma 3.1. *Let G have* SN *and let N be a noncyclic normal subgroup of G. Then G/N is a Dedekind group. In particular, if G/N has odd order, then this factor group is abelian.*

We first consider nonnormal subgroups in finite p-groups with SN.

Lemma 3.2. *Let G be a finite p-group with p an odd prime and suppose that G has* SN. *Let Q be a nonnormal subgroup of G.*

(1) *If W is a normal subgroup of G with $W \subseteq Q$, then either $W = 1$ or Q is cyclic.*
(2) *Q is either cyclic or elementary abelian.*

Proof. (1) We first show that any normal subgroup C of G contained in Q is cyclic. Indeed, if C is not cyclic, then by Lemma 3.1, G/C is abelian and hence Q/C is a normal subgroup of G/C. It follows that Q is normal in G, a contradiction. Thus C is cyclic and, in particular, W is cyclic.

Now, if $W \neq 1$, we want to show that Q is cyclic. To this end, let W_p be the unique subgroup of order p in W, so that W_p is normal and hence central in G. Now let V be any subgroup of order p

in Q. If $V \neq W_p$, then V cannot contain W_p, and SN implies that $VW_p \cong V \times W_p$ is a noncyclic normal subgroup of G contained in Q, a contradiction. Hence $V = W_p$, and Q contains a unique subgroup of order p. Since p is odd, we see that Q is cyclic.

(2) Let K be a subgroup of G containing Q and maximal with the property that K is not normal in G. If Q is not cyclic, then K is not cyclic and we show that K is elementary abelian. Since K is not normal in G, we can find a subgroup M of G containing K such that $|M : K| = p$. Since K is maximal nonnormal in G, we see that $M \lhd G$ and hence $\Phi(M) \lhd G$, where $\Phi(M)$ is the Frattini subgroup of M. Now K is a maximal subgroup of M, so $K \supseteq \Phi(M)$, and since K is not cyclic, part (1) implies that $\Phi(M) = 1$. Thus M is an elementary abelian p-group and hence so are K and Q. □

We can now prove the main result of this section.

Proposition 3.3. *Let G be a finite p-group with p an odd prime. If G has* SSN, *then every nonnormal subgroup of G is cyclic.*

Proof. Suppose by way of contradiction that there exists a nonnormal subgroup Q of G such that Q is not cyclic. Since G has SSN, Lemma 3.2(2) implies that Q is elementary abelian. Let Z be a central subgroup of G of order p. Then $Z \cap Q$ is a normal subgroup of G contained in Q. Since Q is not cyclic, we have $Z \cap Q = 1$ by Lemma 3.2(1). Thus $A = QZ \cong Q \times Z$ is elementary abelian.

Since Q is not normal in G, there exists some element $g \in G$ such that $Q^g \neq Q$. We now show that for any $x \in A$, we have $x^g x^{-1} \in Z$. Indeed, if $x \in Z$, then obviously $1 = x^g x^{-1} \in Z$. If $x \in A$ is not in Z, then $X = \langle x \rangle$ cannot contain Z because x has order p. Since G has SN and $Z \lhd G$, it follows that $XZ \lhd G$ and hence the group XZ/Z of order p is a normal subgroup of the finite p-group G/Z. It follows that XZ/Z is central in G/Z and we conclude that $Zx^g = Zx$ and $x^g x^{-1} \in Z$.

We can now define $\phi : A \to Z$ by $\phi(a) = a^g a^{-1}$. Since the maps $a \mapsto a^g$ and $a \mapsto a^{-1}$ are both endomorphisms of the abelian group A, it follows that their product $a \mapsto a^g a^{-1}$ is also an endomorphism of A, and hence ϕ is a homomorphism from A to Z. Thus, if $C = \ker(\phi) = \{a \in A \mid a^g a^{-1} = 1\} = \mathfrak{C}_A(g)$, we see that $|A/C|$ divides $|Z| = p$. But $Q \neq Q^g$, so $C \neq A$ and hence $|A : C| = p$. Furthermore, $Z \subseteq C$ so $A = QZ \subseteq QC \subseteq A$ and $QC = A$.

Finally, let $L = \langle A, g \rangle$ be the subgroup of G generated by A and g. Since A is abelian and $C \subseteq A$, we see that C centralizes both A and g, and hence C is central in L. Now G has SSN, so L has SN. Furthermore, $Q \cap C$ is central and hence normal in L. Thus since Q is not normal in L and Q is not cyclic, Lemma 3.2(1) implies that $Q \cap C = 1$. It follows that $A = QC = Q \times C$, so $|Q| = |A : C| = p$, clearly a contradiction. □

We remark that there exist groups which have SN but do not have SSN. For example, let $V = \langle a, b \mid a^{p^2} = 1,\ b^p = 1,\ a^b = a^{p+1} \rangle$ and $W = \langle u, v \mid u^{p^2} = 1,\ v^p = 1,\ u^v = u^{p+1} \rangle$ be nonabelian groups of order p^3, and let $G = V * W$ be the central product of V and W with center $Z = \mathfrak{Z}(V) = \mathfrak{Z}(W)$. Since Z has order p, every nonidentity normal subgroup N of G contains $Z = G'$. Thus for any subgroup Y of G, we have $YN \supseteq G'$, so $YN \lhd G$ and G has SN. On the other hand, let $H = \langle a, b, v \rangle$. Then $\langle v \rangle \lhd H$ and $\langle b \rangle$ does not contain $\langle v \rangle$. Since $\langle b \rangle \langle v \rangle$ is not normal in H, we see that H does not have SN and hence G does not have SSN. In fact, $\langle b, v \rangle$ is a noncyclic nonnormal abelian subgroup of G, so G does not satisfy the conclusion of either Proposition 2.2 or Proposition 3.3.

Acknowledgments

The author would like to thank Professor Passman for helpful discussions. We remark that many of the ideas and techniques used in this paper are inspired by the paper [HPW07]. The author would like to thank Professor Hales, Passi and Wilson for their wonderful work on the MJD problem.

References

[AHP98] Satya R. Arora, A.W. Hales, I.B.S. Passi, The multiplicative Jordan decomposition in group rings, J. Algebra 209 (2) (1998) 533–542, MR 1659903 (99j:16015).

[BJ09] Zdravka Božikov, Zvonimir Janko, A complete classification of finite p-groups all of whose noncyclic subgroups are normal, Glas. Mat. Ser. III 44 (64) (1) (2009) 177–185, MR 2525662 (2010g:20029).

[HPW07] Alfred W. Hales, Inder Bir S. Passi, Lawrence E. Wilson, The multiplicative Jordan decomposition in group rings. II, J. Algebra 316 (1) (2007) 109–132, MR 2354855 (2008g:16041).

[Lim68] F.N. Liman, 2-Groups with normal noncyclic subgroups, Math. Notes 4 (1968) 535–539, http://dx.doi.org/10.1007/BF01429816.

[LP09] Chia-Hsin Liu, D.S. Passman, Multiplicative Jordan decomposition in group rings of 3-groups, J. Algebra Appl. 8 (4) (2009) 505–519, MR 2555517 (2010m:16034).

[LP10] Chia-Hsin Liu, D.S. Passman, Multiplicative Jordan decomposition in group rings of 2, 3-groups, J. Algebra Appl. 9 (3) (2010) 483–492, MR 2659731 (2011j:16046).

[Par02] M.M. Parmenter, Multiplicative Jordan decomposition in integral group rings of groups of order 16, Comm. Algebra 30 (10) (2002) 4789–4797, MR 1940463 (2003k:16036).

[Pas70] D.S. Passman, Nonnormal subgroups of p-groups, J. Algebra 15 (1970) 352–370, MR 0258959 (41 #3604).