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Semi-invariant subrings[☆]

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ABSTRACT

We say that a subring R_0 of a ring R is *semi-invariant* if R_0 is the ring of invariants in R under some set of ring endomorphisms of some ring containing R .

We show that R_0 is semi-invariant if and only if there is a ring $S \supseteq R$ and a set $X \subseteq S$ such that $R_0 = \text{Cent}_R(X) := \{r \in R : xr = rx \ \forall x \in X\}$; in particular, centralizers of subsets of R are semi-invariant subrings.

We prove that a semi-invariant subring R_0 of a semiprimary (resp. right perfect) ring R is again semiprimary (resp. right perfect) and satisfies $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$ for some $n \in \mathbb{N}$. This result holds for other families of semiperfect rings, but the semiperfect analogue fails in general. In order to overcome this, we specialize to Hausdorff *linearly topologized* rings and consider *topologically semi-invariant* subrings. This enables us to show that any topologically semi-invariant subring (e.g. a centralizer of a subset) of a semiperfect ring that can be endowed with a “good” topology (e.g. an inverse limit of semiprimary rings) is semiperfect.

Among the applications: (1) The center of a semiprimary (resp. right perfect) ring is semiprimary (resp. right perfect). (2) If M is a finitely presented module over a “good” semiperfect ring (e.g. an inverse limit of semiprimary rings), then $\text{End}(M)$ is semiperfect, hence M has a Krull–Schmidt decomposition. (This generalizes results of Bjork and Rowen; see Björk (1971) [5], Rowen (1986, 1987) [23,24].) (3) If ρ is a representation of a monoid or a ring over a module with a “good” semiperfect endomorphism ring (in the sense of (2)), then ρ has a Krull–Schmidt decomposition. (4) If S is a “good” commutative semiperfect ring and R is an S -algebra that is f.p. as an S -module, then R is semiperfect. (5) Let $R \subseteq S$ be rings

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and let M be a right S -module. If $\text{End}(M_R)$ is semiprimary (resp. right perfect), then $\text{End}(M_S)$ is semiprimary (resp. right perfect).

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1. Preface

Throughout, all rings are assumed to have a unity and ring homomorphisms are required to preserve it. Subrings are assumed to have the same unity as the ring containing them. Given a ring R , denote its set of invertible elements by R^\times , its Jacobson radical by $\text{Jac}(R)$, its set of idempotents by $E(R)$ and its center by $\text{Cent}(R)$. We let $\text{End}(R)$ (resp. $\text{Aut}(R)$) denote the set of ring homomorphisms (resp. isomorphisms) from R to itself. If $X \subseteq R$ is any set, then its right (left) annihilator in R will be denoted by $\text{ann}_R^r X$ ($\text{ann}_R^l X$). The subscript R will be dropped when understood from the context. A *semisimple* ring always means a semisimple artinian ring. For a prime number p , we let \mathbb{Z}_p , \mathbb{Q}_p and $\mathbb{Z}_{(p)}$ denote the p -adic integers, p -adic numbers and \mathbb{Z} localized (but not completed) at $p\mathbb{Z}$, respectively.

Let R be a ring and let $J = \text{Jac}(R)$. Recall that R is *semilocal* if R/J is semisimple. If in addition J is idempotent lifting, then R is called *semiperfect*. For a detailed discussion about semiperfect rings, see [25, §2.7] and [4]. Semiperfect rings play an important role in representation theory and module theory because of the Krull–Schmidt Theorem. Recall that an object A in an additive category \mathcal{A} is said to have a *Krull–Schmidt decomposition* if it is a sum of (non-zero) indecomposable objects and any two such decompositions are the same up to isomorphism and reordering.

Theorem 1.1 (*Krull–Schmidt, for Categories*). *Let \mathcal{A} be an additive category in which all idempotents split (e.g. an abelian category) and let $A \in \mathcal{A}$. If $\text{End}_{\mathcal{A}}(A)$ is semiperfect, then A has a Krull–Schmidt decomposition and the endomorphism ring of any indecomposable summand of A is local.*

Generalizations of this theorem and counterexamples of some natural variations are widely studied (e.g. see [14,3,1,12] and also [11]) and there has been considerable interest in finding rings over which all finitely presented modules have a Krull–Schmidt decomposition (e.g. [28, §6], [5,23,24,30]; Theorem 8.3(iii) below generalizes all these references except the last).

Example 1.2. Semiperfect rings naturally appear upon taking completions:

- (1) Let R be a semilocal ring and let $J = \text{Jac}(R)$. Then the J -adic completion of R , $\varprojlim \{R/J^n\}_{n \in \mathbb{N}}$, is well known to be semiperfect. If the natural map $R \rightarrow \varprojlim \{R/J^n\}_{n \in \mathbb{N}}$ is an isomorphism, then R is called *complete semilocal*. Such rings (especially noetherian or with Jacobson radical f.g. as a right ideal) appear in various areas (e.g. [19,16], [28, §6], [24]).
- (2) Let R be a commutative noetherian domain, let A be an R -algebra that is finitely generated as an R -module and let $P \in \text{Spec}(R)$. Then the completion of A at P is semiperfect (and noetherian). (See [20, §6]; This assertion can also be shown using the results of this paper.)

Let R be any ring and let $R_0 \subseteq R$ be a subring.

- (a) Call R_0 a *semi-invariant* subring if there is a ring $S \supseteq R$ and a set $\Sigma \subseteq \text{End}(S)$ such that $R_0 = R^\Sigma := \{r \in R : \sigma(r) = r \ \forall \sigma \in \Sigma\}$ (elements of Σ are not required to be injective nor surjective). The *invariant* subrings of R are the subrings for which we can choose $S = R$.
- (b) Call R_0 a *semi-centralizer* subring if there is a ring $S \supseteq R$ and a set $X \subseteq S$ such that $R_0 = \text{Cent}_R(X) := \{r \in R : rx = xr \ \forall x \in X\}$. If we can choose $S = R$, then R_0 is a *centralizer* subring.
- (c) Recall that R_0 is *rationaly closed* in R if $R^\times \cap R_0 = R_0^\times$. That is, elements of R_0 that are invertible in R are also invertible in R_0 .

Semi-centralizer and semi-invariant subrings are clearly rationally closed. The latter were studied (for semilocal R) in [9] and invariant subrings (w.r.t. an arbitrary set) were considered in [5]. However, the notion of semi-invariant subrings appears to be new.

The purpose of this paper is to study semi-invariant subrings of semiperfect rings where our motivation comes from the Krull–Schmidt Theorem and the following observations, verified in Sections 3:

- (1) For any ring R , a subring of R is semi-invariant if and only if it is semi-centralizer. In particular, all centralizers of subsets of R are semi-invariant subrings.
- (2) If $R \subseteq S$ are rings and M is a right S -module, then $\text{End}(M_S)$ is a semi-invariant subring of $\text{End}(M_R)$.
- (3) If M is a finitely presented right R -module, then $\text{End}(M_R)$ is a quotient of a semi-invariant subring of $M_n(R) \times M_m(R)$ for some n, m .

While in general semi-invariant subrings of semiperfect rings need not be semiperfect (see Examples 6.1–6.3 below), we show that this is true for special families of semiperfect rings, e.g. for semiprimary and right perfect rings (Theorem 4.6; see Section 2 for definitions.) In addition, if the ring in question is *pro-semiprimary*, i.e. an inverse limit of semiprimary rings (e.g. the rings of Example 1.2), then its *topologically semi-invariant* subrings (e.g. centralizer subrings; see Section 5 for definition) are semiperfect. This actually holds under milder assumptions regarding whether the ring can be endowed with a “good” topology; see Theorems 5.10 and 5.15.

Our results together with the previous observations and the Krull–Schmidt Theorem lead to numerous applications including:

- (1) The center and any maximal commutative subring of a semiprimary (resp. right perfect, semiperfect and pro-semiprimary) ring is semiprimary (resp. right perfect, etc.).
- (2) If R is a semiperfect pro-semiprimary ring, then all f.p. modules over R have a semiperfect endomorphism ring and hence admit a Krull–Schmidt decomposition. If moreover R is right noetherian, then the endomorphism ring of a f.g. right R -modules is pro-semiprimary. (This generalizes Swan [28, §6], Bjork [5] and Rowen [23,24] and also relates to works of Vámos [30], Facchini and Herbera [13]; see Remark 8.4 for more details.)
- (3) If S is a commutative semiperfect pro-semiprimary ring and R is an S -algebra that is Hausdorff (see Section 8) and f.p. as an S -module, then R is semiperfect. If moreover S is noetherian, then the Hausdorff assumption is superfluous and R is pro-semiprimary, hence the assertions of (2) apply. (The first statement is known to hold under mild assumptions for *Henselian* rings; see [30, Lm. 12].)
- (4) If ρ is a representation of a ring or a monoid over a module with a semiperfect pro-semiprimary endomorphism ring, then ρ has a Krull–Schmidt decomposition.
- (5) Let $R \subseteq S$ be rings and let M be a right S -module. If $\text{End}(M_R)$ is semiprimary (resp. right perfect), then so is $\text{End}(M_S)$. In particular, M has a Krull–Schmidt decomposition over S . (Compare with [13, Pr. 2.7].)

Additional applications concern bilinear forms and getting a “Jordan Decomposition” for endomorphisms of modules with semiperfect pro-semiprimary endomorphism ring. We also conjecture that (3) holds for non-commutative S under mild assumptions (see Section 10).

Other interesting byproducts of our work are the fact that a pro-semiprimary ring is an inverse limit of some of its semiprimary quotients and Theorem 9.6 below. (The former assertion fails once replacing semiprimary with right artinian; see Example 9.11 and the comment before it.)

Remark 1.3. It is still open whether all semiperfect pro-semiprimary rings are complete semilocal. However, this is true for noetherian rings; see Section 9.

Section 2 recites some definitions and well-known facts. Section 3 presents the basics of semi-invariant subrings; we present five equivalent characterizations of them and show that they naturally appear in various situations. As all our characterizations use the existence of *some* ambient ring, we

ask whether there is a definition avoiding this. In Section 4, we prove that various ring properties pass to semi-invariant subrings, e.g. being semiprimary and being right perfect. Section 5 develops the theory of T-semi-invariant subrings. The discussion leads to a proof that several properties, such as being pro-semiprimary and semiperfect, are inherited by T-semi-invariant subrings. Section 6 presents counterexamples. We show that semi-invariant subrings of semiperfect rings need not be semiperfect, even when the ambient ring is pro-semiprimary. In addition, we show that in general none of the properties discussed in Sections 4 and 5 pass to rationally closed subrings. The latter implies that there are non-semi-invariant rationally closed subrings. In Sections 7 and 8 we present applications of our results (most applications were briefly described above) and in Section 9 we specialize them to *strictly pro-right-artinian* rings (e.g. noetherian pro-semiprimary rings), which are better behaved. Section 10 describes some issues that are still open. Appendix A is concerned with providing conditions implying that the topologies $\{\tau_n^M\}_{n=1}^\infty$ defined at Section 8 coincide.

2. Preliminaries

This section recalls some definitions and known facts that will be used throughout the paper. Some of the less known facts include proofs for sake of completion. If no reference is specified, proofs can be found at [25], [4] or [18].

Let R be a semilocal ring. The ring R is called semiprimary (right perfect) if $\text{Jac}(R)$ is nilpotent (right T-nilpotent¹). Since any nil ideal is idempotent lifting, right perfect rings are clearly semiperfect.

Proposition 2.1 (Bass' Theorem P, partial). *Let R be a ring. Then R is right perfect \Leftrightarrow every right R -module has a projective cover $\Leftrightarrow R$ has DCC on principal left ideals.*

Proposition 2.2. *Let R be a ring. Then R is semiperfect \Leftrightarrow every right (left) f.g. R -module has a projective cover \Leftrightarrow there are orthogonal idempotents $e_1, \dots, e_r \in R$ such that $\sum_{i=1}^r e_i = 1$ and $e_i R e_i$ is local for all i .*

For the next lemma, and also for later usage, let us set some conventions about inverse limits of rings: By saying that $\{R_i, f_{ij}\}$ is an *inverse system of rings* indexed by I , we mean that: (1) I is a *directed set* (i.e. a partially ordered set such that for all $i, j \in I$ there is $k \in I$ with $i, j \leq k$), (2) R_i ($i \in I$) are rings and $f_{ij} : R_j \rightarrow R_i$ ($i \leq j$) are ring homomorphisms and (3) $f_{ii} = \text{id}_{R_i}$ and $f_{ij} f_{jk} = f_{ik}$ for all $i \leq j \leq k$ in I . When the maps $\{f_{ij}\}$ are obvious or of little interest, we will drop them from the notation, writing only $\{R_i\}_{i \in I}$. The *inverse limit* of $\{R_i, f_{ij}\}$ will be denoted by $\varprojlim \{R_i\}_{i \in I}$. It can be understood as the set of I -tuples $(a_i)_{i \in I} \in \prod_{i \in I} R_i$ such that $f_{ij}(a_j) = a_i$ for all $i \leq j$ in I .

Lemma 2.3. *Let $\{R_i, f_{ij}\}$ be an inverse system of rings and let $R = \varprojlim \{R_i\}$. Denote by f_i the standard map from R to R_i . Then:*

- (i) For all $n \in \mathbb{N}$, $\varprojlim \{M_n(R_i)\}_{i \in I} \cong M_n(\varprojlim \{R_i\}) = M_n(R)$.
- (ii) For all $e \in E(R)$, $\varprojlim \{e_i R_i e_i\}_{i \in I} \cong e R e$ where $e_i = f_i(e)$.

Proof. This is straightforward. \square

Proposition 2.4. (i) *Being semiprimary (resp. right perfect, semiperfect, semilocal, pro-semiprimary) is preserved under Morita equivalence.*

(ii) *Let R be a ring and $e \in E(R)$. Then R is semiprimary (resp. right perfect, semiperfect, semilocal) if and only if $e R e$ and $(1 - e)R(1 - e)$ are.*

¹ An ideal $I \trianglelefteq R$ is right T-nilpotent if for any sequence $x_1, x_2, x_3, \dots \in I$, the sequence $x_1, x_2 x_1, x_3 x_2 x_1, \dots$ eventually vanishes.

Proof. All statements regarding semiprimary, right perfect, semiperfect and semilocal rings are well known. The other statements follow from Lemma 2.3. \square

Part (ii) of Proposition 2.4 does not hold for pro-semiprimary rings. For instance, take $R = \left\{ \begin{bmatrix} a & v \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p, v \in \bigoplus_{i=1}^{\infty} \mathbb{Z}_p \right\}$ (where \mathbb{Z}_p are the p -adic integers) and let e be the matrix unit e_{11} .

Theorem 2.5 (Levitski). *Let R be a right noetherian ring. Then any nil subring of R is nilpotent.*

Let R be a ring. An element $a \in R$ is called *right π -regular* (in R) if the right ideal chain $aR \supseteq a^2R \supseteq a^3R \supseteq \dots$ stabilize.² If a is both left and right π -regular we will say it is π -regular. A ring that all of its elements are right π -regular is called π -regular. It was shown by Dischinger in [10] that the latter property is actually left–right symmetric.

Since π -regularity is not preserved under Morita equivalence (see [26]), it is convenient to introduce the following notion: A ring R is called π_{∞} -regular³ if $M_n(R)$ is π -regular for all n .

Proposition 2.6. (i) *Let R be a π -regular (π_{∞} -regular) ring and $e \in E(R)$. Then eRe is π -regular (π_{∞} -regular).*

(ii) *π_{∞} -regularity is preserved under Morita equivalence.*

Proof. (i) Assume R is π -regular, let $e \in R$ and let $a = eae \in eRe$. By definition, there is $b \in R$ and $n \in \mathbb{N}$ such that $a^n = a^{n+1}b$. Multiplying by e on the right we get $a^n = a^{n+1}ebe$, hence $a^n(eRe) = a^{n+1}(eRe)$.

Assume R is π_{∞} -regular and let $e \in R$. Let I denote identity matrix in $M_n(R)$. Then $(eI)M_n(R)(eI) = M_n(eRe)$. By the previous argument, the left-hand side is π -regular, hence we are through.

(ii) We only need to check that $M_n(R)$ is π_{∞} -regular for all $n \in \mathbb{N}$, which is obvious from the definition, and that eRe is π_{∞} -regular, which follows from (i). \square

Proposition 2.7. *Let R be a ring and let N denote its prime radical (i.e. the intersection of all prime ideals). Then R is π -regular (π_{∞} -regular) if and only if R/N is.*

Proof. See [25, §2.7]. (The argument is easily generalized to π_{∞} -regular rings.) \square

Remark 2.8. Any PI semilocal ring with nil Jacobson radical is π_{∞} -regular (see [23, Apx.]). However, there are semilocal rings with nil Jacobson radical that are not π -regular, see [26].

Remark 2.9. We have the following implications:

$$\text{right artinian} \Rightarrow \text{semiprimary} \Rightarrow \text{left/right perfect} \xRightarrow{(2.1)} \pi_{\infty}\text{-regular} \Rightarrow \pi\text{-regular}.$$

However, all these notions coincide for right noetherian rings. Indeed, assume R is π -regular and right noetherian and let $J = \text{Jac}(R)$. Then J is nil (see Lemma 4.4(i)), hence Theorem 2.5 implies $J^n = 0$ for some $n \in \mathbb{N}$. By Lemma 4.4(ii) below, R is semiperfect and in particular R/J is semisimple. As R is right noetherian, the right R/J -modules $\{J^{i-1}/J^i\}_{i=1}^n$ are f.g., hence their length as right R -modules is finite. It follows that R_R has a finite length, so R is right artinian.

Throughout, we will use implicitly the next lemma. Notice that it implies that being semiprimary (resp. right perfect, semiperfect, semilocal) passes to quotients.

² This notion of π -regularity is sometimes called strong π -regularity.

³ This property is sometimes called completely π -regular.

Lemma 2.10. *Let R be a semilocal ring. Then any surjective ring homomorphism $\varphi : R \rightarrow S$ satisfies $\varphi(\text{Jac}(R)) = \text{Jac}(S)$.*

Proof. $\varphi(\text{Jac}(R))$ is an ideal of $\varphi(R) = S$ and $1 + \varphi(\text{Jac}(R)) = \varphi(1 + \text{Jac}(R)) \subseteq \varphi(R^\times) \subseteq S^\times$, hence $\varphi(\text{Jac}(R)) \subseteq \text{Jac}(S)$. On the other hand, $S/\varphi(\text{Jac}(R))$ is a quotient of $R/\text{Jac}(R)$ which is semisimple. Therefore, $S/\varphi(\text{Jac}(R))$ is semisimple, implying $\varphi(\text{Jac}(R)) \supseteq \text{Jac}(S)$. \square

3. Semi-invariant subrings

This section presents the basic properties of semi-invariant subrings. We begin by showing that for any ring the semi-invariant subrings are precisely the semi-centralizer subrings.

Proposition 3.1. *Let $R_0 \subseteq R$ be rings. The following are equivalent:*

- (a) *There is a ring $S \supseteq R$ and a set $\Sigma \subseteq \text{End}(S)$ such that $R_0 = R^\Sigma$.*
- (b) *There is a ring $S \supseteq R$ and a subset $X \subseteq S$ such that $R_0 = \text{Cent}_R(X)$.*
- (c) *There is a ring $S \supseteq R$ and $\sigma \in \text{Aut}(S)$ such that $\sigma^2 = \text{id}$ and $R_0 = R^{(\sigma)}$.*
- (d) *There is a ring $S \supseteq R$ and an inner automorphism $\sigma \in \text{Aut}(S)$ such that $\sigma^2 = \text{id}$ and $R_0 = R^{(\sigma)}$.*
- (e) *There are rings $\{S_i\}_{i \in I}$ and ring homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : R \rightarrow S_i$ such that $R_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r) \forall i \in I\}$.*

Note that condition (e) implies that the family of semi-invariant subrings is closed under intersection.

Proof. We prove (a) \Rightarrow (e) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b) \Rightarrow (a).

(a) \Rightarrow (e): Take $I = \Sigma$ and define $S_\sigma = S$, $\psi_\sigma^{(1)} = \sigma$ and $\psi_\sigma^{(2)} = \text{id}_R$.

(e) \Rightarrow (c): Let $\{S_i, \psi_i^{(1)}, \psi_i^{(2)}\}_{i \in I}$ be given. Without loss of generality we may assume that there is $i_0 \in I$ such that $S_{i_0} = R$ and $\psi_{i_0}^{(1)} = \psi_{i_0}^{(2)} = \text{id}_R$. Define $S = \prod_{(i,j) \in I \times \{1,2\}} S_{ij}$ where $S_{ij} = S_i$ and let $\Psi : R \rightarrow S$ be given by

$$\Psi(r) = (\psi_i^{(j)}(r))_{(i,j) \in I \times \{1,2\}} \in S.$$

The existence of i_0 above implies Ψ is injective. Let $\sigma \in \text{Aut}(S)$ be the automorphism exchanging the $(i, 1)$ and $(i, 2)$ components of S for all $i \in I$. Then one easily checks that $\sigma^2 = \text{id}$ and $\Psi(R)^{(\sigma)} = \Psi(R_0)$. We finish by identifying R with $\Psi(R)$.

(c) \Rightarrow (d): Let S, σ be given and let $S' = S[x; \sigma]$ denote the ring of σ -twisted polynomials with (left) coefficients in S . Observe that $(x^2 - 1) \in \text{Cent}(S')$ (since $\sigma^2 = \text{id}$), hence $S'' := S'/\langle x^2 - 1 \rangle$ is a free left S -module with basis $\{\bar{1}, \bar{x}\}$ (where \bar{a} is the image of $a \in S'$ in S''). Let $\tau \in \text{Aut}(S'')$ be conjugation by \bar{x} . Then $\tau^2 = \text{id}$ and $R^{(\tau)} = \{r \in R : \bar{x}r = r\bar{x}\} = \{r \in R : \sigma(r) = r\} = R^{(\sigma)} = R_0$.

(d) \Rightarrow (b): This is clear.

(b) \Rightarrow (a): Let S, X be given. Let $S' = S((t))$ be the ring of formal Laurent series $\sum_{n=k}^{\infty} a_n t^n$ ($k \in \mathbb{Z}$) with coefficients in S . The elements of S' commuting with X are precisely the elements that commute with $t^{-1} + X$ (as t^{-1} is central in S'). However, it is easily seen that all elements in $t^{-1} + X$ are invertible. For all $x \in X$, let $\sigma_x \in \text{End}(S')$ be the inner automorphism of S' given by conjugation with $t^{-1} + x$ and let $\Sigma = \{\sigma_x \mid x \in X\}$. Then $R^\Sigma = \text{Cent}_R(t^{-1} + X) = \text{Cent}_R(X) = R_0$. \square

Corollary 3.2. *Let R, W be rings and let $\varphi : R \rightarrow W$ be a ring homomorphism. Assume $W_0 \subseteq W$ is a semi-invariant subring of W . Then $\varphi^{-1}(W_0)$ is a semi-invariant subring of R .*

Proof. By Proposition 3.1(e), there are rings $\{S_i\}_{i \in I}$ and ring homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : W \rightarrow S_i$ such that $W_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r) \forall i \in I\}$. Define $\varphi_i^{(n)} = \psi_i^{(n)} \circ \varphi$ and note that $\varphi^{-1}(W_0) = \{r \in R : \varphi(r) \in W_0\} = \{r \in R : \psi_i^{(1)}(\varphi(r)) = \psi_i^{(2)}(\varphi(r)) \forall i \in I\} = \{r \in R : \varphi_i^{(1)}(r) = \varphi_i^{(2)}(r) \forall i \in I\}$. \square

The equivalent conditions of Proposition 3.1 require the existence of some ambient ring. This leads to the following question:

Question 1. *Is there an intrinsic definition of semi-invariant subrings?*

Informally, we ask for a definition that would make it easy to show that a given subring is *not* semi-invariant.

The next proposition is useful for producing examples of semi-invariant subrings.

Proposition 3.3. *Let $R \subseteq S$ be rings and let K be a central subfield of S . Then $R \cap K$ is a semi-invariant subring of R .*

Proof. Let $S' = S \otimes_K S$ and define $\varphi_1, \varphi_2 : S \rightarrow S'$ by $\varphi_1(s) = s \otimes 1$ and $\varphi_2(s) = 1 \otimes s$. As K is a central subfield, it is easy to check that $\{s \in S : \varphi_1(s) = \varphi_2(s)\} = K$, hence $\{s \in R : \varphi_1(s) = \varphi_2(s)\} = R \cap K$. We are done by Proposition 3.1(e). \square

Corollary 3.4. *Let K be a field. Then the semi-invariant subrings of K are precisely its subfields.*

Proof. Any semi-invariant subring $R \subseteq K$ satisfies $R^\times = R \cap K^\times = R \setminus \{0\}$, hence it is a field. The converse follows from the last proposition. \square

Remark 3.5. If K/L is an algebraic field extension, then L is an invariant subring of K if and only if K/L is Galois.

We finish this section by introducing two cases where semi-invariant subrings naturally appear.

Proposition 3.6. *Let $R \subseteq S$ be rings and let M be a right S -module. Then $\text{End}(M_S)$ is a semi-invariant subring of $\text{End}(M_R)$.⁴*

Proof. There is a ring homomorphism $\varphi : S^{\text{op}} \rightarrow \text{End}(M_{\mathbb{Z}})$ given by $\varphi(s^{\text{op}})(m) = ms$ for all $m \in M$. It is straightforward to check that $\text{End}(M_S) = \text{Cent}_{\text{End}(M_{\mathbb{Z}})}(\text{im } \varphi)$. As $\text{End}(M_S) \subseteq \text{End}(M_R)$, it follows that $\text{End}(M_S) = \text{Cent}_{\text{End}(M_R)}(\text{im } \varphi)$, hence $\text{End}(M_S)$ is a semi-centralizer subring of $\text{End}(M_R)$. \square

Proposition 3.7. *Let \mathcal{A} be an abelian category and let $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence in \mathcal{A} such that for any $c \in \text{End}(C)$ there are $b \in \text{End}(B)$ and $a \in \text{End}(A)$ with $cg = gb$ and $bf = fa$ (e.g.: if both A and B are projective, or if B is projective and f is injective). Then $\text{End}(C)$ is isomorphic to a quotient of:*

- (i) *A semi-invariant subring of $\text{End}(A) \times \text{End}(B)$.*
- (ii) *An invariant and a centralizer subring of $\text{End}(A \oplus B)$, provided f is injective.*

Proof. Let $B_0 = \text{im } f = \ker g$. Define R to be the subring of $\text{End}(B)$ consisting of maps $b \in \text{End}(B)$ for which there is $a \in \text{End}(A)$ with $bf = fa$. Then for all $b \in R$, $b(B_0) = b(\text{im } f) = \text{im}(fa) \subseteq \text{im } f = B_0$. Therefore, there is *unique* $c \in \text{End}(C)$ such that $cg = gb$. The map sending b to c is easily seen to be a ring homomorphism from R to $\text{End}(C)$ and the assumptions imply it is onto. Therefore, $\text{End}(C)$ is a quotient of R .

Let $S = \text{End}(A \oplus B)$. We represent elements of S as matrices $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with $x \in \text{End}(A)$, $y \in \text{Hom}(B, A)$, $z \in \text{Hom}(A, B)$, $w \in \text{End}(B)$. Let D denote the diagonal matrices in S (i.e. $\text{End}(A) \times \text{End}(B)$)

⁴ This proposition is a refinement of [13, Pr. 2.7], which asserts that under the same assumptions $\text{End}(M_S)$ is a rationally closed subring of $\text{End}(M_R)$.

and let $W = \text{Cent}_S\left(\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\right)$. Then for $a \in \text{End}(A)$ and $b \in \text{End}(B)$, $fa = bf$ if and only if $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W$. Define a ring homomorphism $\varphi : D \rightarrow \text{End}(B)$ by $\varphi\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = y$. Then $\varphi(\text{Cent}_D(\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix})) = \varphi(D \cap W) = R$. It follows that $\text{End}(C)$ is a quotient of R , which is a quotient of $D \cap W$, which is a semi-centralizer subring of $D = \text{End}(A) \times \text{End}(B)$. This settles (i). To see (ii), notice that if f is injective, then W consists of upper-triangular matrices, hence φ can be extended to W , which is a centralizer and an invariant subring of S since $W = \text{Cent}_S(\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix})$ and $\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \in S^\times$. \square

4. Properties inherited by semi-invariant subrings

In this section we prove that being semiprimary (right perfect, semiperfect and π_∞ -regular, semiperfect and π -regular) passes to semi-invariant subrings. We also present a supplementary result for algebras.

Our first step is introducing an equivalent condition for π -regularity of elements of a ring.

Lemma 4.1. *Let R be a ring and let $a \in R$ be a π -regular element. Define:*

$$A = \bigcap_{k=1}^{\infty} a^k R, \quad B = \bigcup_{k=1}^{\infty} \text{ann}^r a^k,$$

$$A' = \bigcap_{k=1}^{\infty} R a^k, \quad B' = \bigcup_{k=1}^{\infty} \text{ann}^\ell a^k.$$

Then there is $e \in E(R)$ such that $A = eR$, $B = fR$, $A' = Re$ and $B' = Rf$ where $f := 1 - e$. In particular, $R_R = A \oplus B$ and ${}_R R = A' \oplus B'$.

Proof. Let $n \in \mathbb{N}$ be such that $a^n R = a^k R$ and $R a^n = R a^k$ for all $k \geq n$. Notice that this implies $\text{ann}^r a^n = \text{ann}^r a^k$ and $\text{ann}^\ell a^n = \text{ann}^\ell a^k$ for all $k \geq n$.

We begin by showing $R_R = A \oplus B$. That ${}_R R = A' \oplus B'$ follows by symmetry. The argument is similar to the proof of Fitting's Lemma (see [25, §2.9]): Let $r \in R$. Then $a^n r \in a^n R = a^{2n} R$, hence there is $s \in R$ with $a^n r = a^{2n} s$. Observe that $a^n(r - a^n s) = 0$ and $a^n s \in a^n R$, so $r = a^n s + (r - a^n s) \in A + B$. Now suppose $r \in A \cap B$. Then $r = a^n s$ for some $s \in R$. However, $r \in B = \text{ann}^r a^n$ implies $s \in \text{ann}^r a^{2n} = \text{ann}^r a^n$, so $r = a^n s = 0$.

Since $R_R = A \oplus B$ there is $e \in R$ such that $e \in A$ and $f := 1 - e \in B$. It is well known that in this case $e^2 = e$, $A = eR$ and $B = fR$. This implies $B' = \text{ann}^\ell a^n = \text{ann}^\ell a^n R = \text{ann}^\ell eR = Rf$ and $A' = R a^n \subseteq \text{ann}^\ell \text{ann}^r R a^n = \text{ann}^\ell \text{ann}^r a^n = \text{ann}^\ell fR = Re$. As ${}_R R = A' \oplus B' = Re \oplus Rf$, we must have $A' = Re$. \square

Proposition 4.2. *Let R be a ring and $a \in R$. Then a is π -regular \Leftrightarrow there is $e \in E(R)$ such that*

- (A) $a = eae + faf$ where $f := 1 - e$.
- (B) eae is invertible in eRe .
- (C) faf is nilpotent.

In this case, the idempotent e is uniquely determined by a .

Proof. Assume a is π -regular and let e, f, A, B, A', B', n be as in Lemma 4.1. Then $ae \in aeR = aA = a(a^n R) = a^{n+1} R = A = eR$ and $af \in aB = a \text{ann}^r a^n \subseteq \text{ann}^r a^n = B = fR$. Therefore, $ae = eae$ and $af = faf$, hence $a = ae + af = eae + faf$. This implies $a^k = (eae)^k + (faf)^k$ for all $k \in \mathbb{N}$. As $a^n \in eR$, we have $a^n = ea^n$, hence $(eae)^n + (faf)^n = a^n = e(eae)^n + e(faf)^n = (eae)^n$ which implies $(faf)^n = 0$. In particular, for all $k \geq n$, $a^k = (eae)^k + (faf)^k = (eae)^k$. Since $e \in a^n R$, there is $x \in R$ such that

$e = a^n x = (eae)^n x$. Multiplying by e on the right yields $e = (eae)((eae)^{n-1}xe)$, hence eae is right invertible in eRe . By symmetry, eae is left also left invertible in eRe , hence we conclude that e satisfies (A)–(C).

Now assume there is $e \in E(R)$ satisfying (A)–(C) and let b be the inverse of a in eRe . Then $a^k = (eae)^k + (faf)^k$ for all $k \in \mathbb{N}$. Condition (C) now implies there is $n \in \mathbb{N}$ such that $a^k = (eae)^k$ for all $k \geq n$. Therefore, for all $k \geq n$, $a^n = (eae)^n = (eae)^k b^{k-n} = a^k b^{k-n} \in a^k R$ implying $a^n R = a^k R$. By symmetry, $Ra^n = Ra^k$ for all $k \geq n$, so a is π -regular.

Finally, assume that $e, e' \in E(R)$ satisfy conditions (A)–(C) and let $f = 1 - e$, $f' = 1 - e'$. By the previous paragraph a is π -regular, hence Lemma 4.1 implies $R = A \oplus B$ where $A = \bigcap_{k=1}^{\infty} a^k R$ and $B = \bigcup_{k=1}^{\infty} \text{ann}^r a^k R$. Let b be the inverse of eae in eRe and let $n \in \mathbb{N}$ be such that $(faf)^n = 0$. Then $e = (eae)^k b^k = a^k b^k \in a^k R$ for all $k \geq n$, hence $e \in A$, and $a^n f = (eae)^n f = 0$, hence $f \in B$. Similarly, $e' \in A$ and $f' \in B$. It follows that $e, e' \in A$ and $f, f' \in B$. Since $1 = e + f = e' + f'$ and $R = A \oplus B$, we must have $e = e'$. \square

Let R, a be as in Proposition 4.2. Henceforth, we call the unique idempotent e satisfying conditions (A)–(C) the *associated idempotent* of a (in R).

Corollary 4.3. (i) Let R be a ring, $R_0 \subseteq R$ a semi-invariant subring and let $a \in R_0$ be π -regular in R . Then a is π -regular in R_0 .

(ii) A semi-invariant subring of a π -regular (π_{∞} -regular) ring is π -regular (π_{∞} -regular).

Proof. (i) Let $S \supseteq R$ and $\Sigma \subseteq \text{End}(S)$ be such that $R_0 = R^{\Sigma}$ and let $a \in R_0$ be π -regular in R . Let e be the associated idempotent of a in R . Then e is clearly the associated idempotent of a in S (hence a is π -regular in S). However, it is straightforward to check that $\sigma(e)$ satisfies conditions (A)–(C) (in S) for all $\sigma \in \Sigma$ (since $\sigma(a) = a$), so the uniqueness of e forces $e \in S^{\Sigma} \cap R = R^{\Sigma} = R_0$. Therefore, a is π -regular in R_0 .

(ii) The π -regular case follows from (i). The π_{∞} -regular case follows once noting that if R_0 is a semi-invariant subring of R , then $M_n(R_0)$ is a semi-invariant subring of $M_n(R)$ for all $n \in \mathbb{N}$. \square

Lemma 4.4. Let R be a π -regular ring. Then:

- (i) $\text{Jac}(R)$ is nil.
- (ii) R is semiperfect $\Leftrightarrow R$ does not contain an infinite set of orthogonal idempotents.

Proof. For $a \in R$, let e_a denote the associated idempotent of a and let $f_a = 1 - e_a$.

(i) Let $a \in \text{Jac}(R)$ and let b be the inverse of $e_a a e_a$ in $e_a R e_a$. Then $e_a = b(e_a a e_a) \in \text{Jac}(R)$, hence $e_a = 0$, implying $a = f_a a f_a$ is nilpotent.

(ii) That R is semiperfect clearly implies R does not contain an infinite set of orthogonal idempotents, so assume the converse. Let $a \in R$. Observe that if $e_a = 0$ then a is nilpotent and if $e_a = 1$ then a is invertible. Therefore, if $e_a \in \{0, 1\}$ for all $a \in R$, then R is local and in particular, semiperfect.

Assume there is $a \in R$ with $e := e_a \notin \{0, 1\}$. We now apply an inductive argument to deduce that eRe and $(1-e)R(1-e)$ are semiperfect, thus proving R is semiperfect (by Proposition 2.4). The induction process must stop because otherwise there is a sequence of idempotents $\{e_k\}_{k=0}^{\infty} \subseteq R$ such that $e_k \in e_{k-1} R e_{k-1}$ and $e_k \notin \{0, e_{k-1}\}$. This implies $\{e_{k-1} - e_k\}_{k=1}^{\infty}$ is an infinite set of non-zero orthogonal idempotents, which cannot exist by our assumptions. \square

Lemma 4.5. Let $R_0 \subseteq R$ be rings. If R is semiperfect and both R_0 and R are π -regular, then R_0 is semiperfect and $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$ for some $n \in \mathbb{N}$. If in addition R is semiprimary (right perfect), then so is R_0 .

Proof. By Lemma 4.4(ii), R does not contain an infinite set of orthogonal idempotents. Therefore, this also applies to R_0 , so the same lemma implies R_0 is semiperfect. Let φ denote the standard projection from R to $R/\text{Jac}(R)$. By Lemma 4.4(i), $\varphi(\text{Jac}(R_0))$ is nil. Therefore, by Theorem 2.5 (applied to $\varphi(R)$, which is semisimple), $\varphi(\text{Jac}(R_0))$ is nilpotent, hence there is $n \in \mathbb{N}$ such that $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$. If

moreover R is semiprimary (right perfect), then $\text{Jac}(R)$ is nilpotent (right T-nilpotent). The inclusion $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$ then implies $\text{Jac}(R_0)$ is nilpotent (right T-nilpotent), so R_0 is semiprimary (right perfect). \square

Theorem 4.6. *Let R be a ring and let R_0 be a semi-invariant subring of R . If R is semiprimary (resp. right perfect, semiperfect and π_∞ -regular, semiperfect and π -regular), then so is R_0 . In addition, there is $n \in \mathbb{N}$ such that $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$.*

Proof. Recall that being right perfect implies being π -regular by Proposition 2.1. Given that, the theorem follows from Corollary 4.3 and Lemma 4.5. \square

Corollary 4.7. *Let $R \subseteq S$ be rings and let M be a right S -module. If $\text{End}(M_R)$ is semiprimary (resp. right perfect, semiperfect and π_∞ -regular, semiperfect and π -regular), then so is $\text{End}(M_S)$ and there exists $n \in \mathbb{N}$ such that $\text{Jac}(\text{End}(M_S))^n \subseteq \text{Jac}(\text{End}(M_R))$.*

Proof. This follows from the last theorem and Proposition 3.6. \square

Remark 4.8. Camps and Dicks proved in [9] that a *rationaly closed* subring of a semilocal ring is semilocal, thus implying the semilocal analogues of Theorem 4.6 and Corollary 4.7, excluding the part regarding the Jacobson radical (which indeed fails in this case; see Example 6.7). In fact, the semilocal analogue of Corollary 4.7 was noticed in [13, Pr. 2.7]. However, we cannot use this analogue with the Krull–Schmidt Theorem (as we do in Section 7 with our results) because modules with semilocal endomorphism ring need not have a Krull–Schmidt decomposition, as shown in [14] and [3].

Nevertheless, as there are plenty of weaker Krull–Schmidt Theorems for modules that do not require $\text{End}(M_R)$ to be semiperfect (mainly due to Facchini et al.; e.g. [3,12]), it might be that if M, R, S are as in Corollary 4.7 and $\text{End}(M_R)$ is merely semiperfect, then M has a Krull–Schmidt decomposition over S (despite the fact $\text{End}(M_S)$ need not be semiperfect). To the author's best knowledge, this topic is still open.

We finish this section with a supplementary result for algebras.

Proposition 4.9. *Let $R \subseteq S$ be rings and $\Sigma \subseteq \text{End}(S)$. Assume there is a division ring $D \subseteq R$ such that $\sigma(D) \subseteq D$ for all $\sigma \in \Sigma$. Then $\dim_{D^\Sigma} R^\Sigma \leq \dim_D R$.*

Proof. Consider the left D -vector space $V = DR^\Sigma$. Let $\{v_i\}_{i \in I} \subseteq R^\Sigma$ be a left D -basis for V . We claim that $\{v_i\}_{i \in I}$ is a left D^Σ -basis for R^Σ . Indeed, let $v \in R^\Sigma$. Then there are unique $\{d_i\}_{i \in I} \subseteq D$ (almost all 0) such that $v = \sum_i d_i v_i$. However, for all $\sigma \in \Sigma$, $v = \sigma(v) = \sum_i \sigma(d_i) v_i$, so $\sigma(d_i) = d_i$ for all $i \in I$, hence $v \in \sum_{i \in I} D^\Sigma v_i$. Therefore, $\dim_{D^\Sigma} R^\Sigma = \dim_D V \leq \dim_D R$. \square

Remark 4.10. An *invariant* subring of a f.d. algebra need not be left nor right artinian, even when invariants are taken w.r.t. to the action of a finite cyclic group. This was demonstrated by Bjork in [5, §2]. In particular, the assumption $\sigma(D) \subseteq D$ for all $\sigma \in \Sigma$ in the last proposition is essential. However, Bjork also proved that if Σ is a *finite group* acting on a f.d. algebra over a *perfect* field, then the invariant subring (w.r.t. Σ) is artinian; see [5, Th. 2.4]. For a detailed discussion about when a subring of an artinian ring is artinian, see [5] and [6].

5. Topologically semi-invariant subrings

In this section we specialize the notions of semi-invariance and π -regularity to certain topological rings. As a result we obtain a topological analogue of Theorem 4.6 (Theorem 5.10), which is used to prove that *topologically semi-invariant* subrings of semiperfect pro-semiprimary rings are semiperfect and pro-semiprimary (Theorem 5.15). Note that once restricted to *discrete* topological

rings, some of the results of this section reduce to results from the previous sections. However, the latter are not superfluous since we will rely on them. For a general reference about topological rings, see [31].

Definition 5.1. A topological ring R is called *linearly topologized* (abbreviated: LT) if it admits a local basis (i.e. a basis of neighborhoods of 0) consisting of two-sided ideals. In this case the topology on R is called *linear*.

Let us set some general notation: For any topological ring R , let \mathcal{I}_R denote its set of open ideals. Then R is LT if and only if \mathcal{I}_R is a local basis. We use $\text{Hom}_c(\text{End}_c)$ to denote continuous homomorphisms (endomorphisms). The category of Hausdorff linearly topologized rings will be denoted by \mathcal{LT}_2 , where $\text{Hom}_{\mathcal{LT}_2}(A, B) = \text{Hom}_c(A, B)$ for all $A, B \in \mathcal{LT}_2$.⁵ A subring of a topological ring is assumed to have the induced topology. In particular, if $R \in \mathcal{LT}_2$ then so is any subring of R . The following facts will be used freely throughout the paper. For proofs, see [31, §3].

- (1) Let $(G, +)$ be an abelian topological group and let \mathcal{B} be a local basis of G . Then for any subset $X \subseteq G$, $\overline{X} = \bigcap_{U \in \mathcal{B}} (X + U)$.
- (2) Under the previous assumptions, G is Hausdorff $\Leftrightarrow \overline{\{0\}} = \bigcap_{U \in \mathcal{B}} U = \{0\}$.
- (3) Given a ring R and a filter base of ideals \mathcal{B} , there exists a unique ring topology on R with local basis \mathcal{B} . This topology makes R into an LT ring.

Example 5.2. (i) Any ring assigned with the discrete topology is LT.

(ii) \mathbb{Z}_p (with the p -adic topology) is LT but \mathbb{Q}_p is not.

(iii) Let R be an LT ring and let $n \in \mathbb{N}$. We make $M_n(R)$ into an LT ring by assigning it the unique ring topology with local basis $\{M_n(I) \mid I \in \mathcal{I}_R\}$.

(iv) If R is LT and $e \in E(R)$, then eRe is LT w.r.t. the induced topology.

(v) Let $\{R_i\}_{i \in I}$ be LT rings. Then $\prod_{i \in I} R_i$ is LT w.r.t. the product topology.

(vi) Let R be an inverse limit of LT rings $\{R_i\}_{i \in I}$, with the topology induced from the product topology on $\prod_{i \in I} R_i$.⁶ Then by (v) R is LT.

(vii) If R is LT and $J \trianglelefteq R$, then R/J with the quotient topology is LT. Indeed, $\{I/J \mid J \subseteq I \in \mathcal{I}_R\}$ is a local basis for that topology. The ring R/J is Hausdorff if and only if J is closed, and discrete if and only if J is open.

The last example implies that \mathcal{LT}_2 is closed under products, inverse limits and forming matrix rings (with the appropriate topologies). We will say that a property Q of LT rings is preserved under Morita equivalence if whenever $R \in \mathcal{LT}_2$ has Q , then so does $M_n(R)$ and eRe for $e \in E(R)$ s.t. eR is a progenerator.⁷

Definition 5.3. Let $R \in \mathcal{LT}_2$. A subring $R_0 \subseteq R$ is called a *topologically semi-invariant* (abbrev.: T-semi-invariant) subring if there is $R \subseteq S \in \mathcal{LT}_2$ and a set $\Sigma \subseteq \text{End}_c(S)$ such that $R_0 = R^\Sigma$. The subring R_0 is called a *topologically semi-centralizer* (abbrev.: T-semi-centralizer) subring if there is $R \subseteq S \in \mathcal{LT}_2$ and a set $X \subseteq S$ such that $R_0 = \text{Cent}_R(X)$.

A T-semi-invariant subring is always closed. In addition, there is an analogue of Proposition 3.1 for T-semi-invariant rings.

Proposition 5.4. Let R_0 be a subring of $R \in \mathcal{LT}_2$. The following are equivalent:

⁵ The subscript “2” in \mathcal{LT}_2 stands for the second separation axiom T_2 (i.e. being Hausdorff).

⁶ With this topology R is indeed the inverse limit of $\{R_i\}_{i \in I}$ in category of topological rings, i.e. it admits the required universal property.

⁷ Caution: There is a notion of Morita equivalence for (right) LT rings, but we will not use it in this paper; see [15] and related articles.

- (a) There is $R \subseteq S \in \mathcal{LT}_2$ and a set $\Sigma \subseteq \text{End}_c(S)$ such that $R_0 = R^\Sigma$.
- (b) There is $R \subseteq S \in \mathcal{LT}_2$ and a subset $X \subseteq S$ such that $R_0 = \text{Cent}_R(X)$.
- (c) There is $R \subseteq S \in \mathcal{LT}_2$ and $\sigma \in \text{Aut}_c(S)$ with $\sigma^2 = \text{id}$ and $R_0 = R^{(\sigma)}$.
- (d) There is $R \subseteq S \in \mathcal{LT}_2$ and an inner automorphism $\sigma \in \text{Aut}_c(S)$ such that $\sigma^2 = \text{id}$ and $R_0 = R^{(\sigma)}$.
- (e) There are LT Hausdorff rings $\{S_i\}_{i \in I}$ and continuous ring homomorphisms $\psi_i^{(1)}, \psi_i^{(2)} : R \rightarrow S_i$ such that $R_0 = \{r \in R : \psi_i^{(1)}(r) = \psi_i^{(2)}(r) \forall i \in I\}$.

Proof. This is essentially the proof of Proposition 3.1, but we need to endow the rings constructed throughout the proof with topologies making them into LT Hausdorff rings that contain R as a topological ring. This is briefly done below; the details are left to the reader.

(e) \Rightarrow (c): Endow $S = \prod_{(i,j) \in I \times \{1,2\}} S_{ij}$ with the product topology.

(c) \Rightarrow (d): Observe that $\mathcal{B} = \{J \cap \sigma(I) \mid I \in \mathcal{I}_S\}$ is a local basis of S and $\sigma(J) = J$ for all $J \in \mathcal{B}$. Assign $S' = S[x; \sigma]$ the unique ring topology with local basis $\{J[x; \sigma] \mid J \in \mathcal{B}\}$, where $J[x; \sigma]$ denotes the set of polynomials with (left) coefficients in J , and give $S'' = S' / \langle x^2 - 1 \rangle$ the quotient topology.

(b) \Rightarrow (a): Give $S(\langle t \rangle)$ the unique ring topology with local basis $\{I(\langle t \rangle) \mid I \in \mathcal{I}_S\}$, where $I(\langle t \rangle)$ denotes the set of polynomials with coefficients in I . \square

We now generalize the notion of π -regularity for topological rings. Our definition is inspired by Proposition 4.2.

Definition 5.5. Let $R \in \mathcal{LT}_2$ and $a \in R$. The element a is called *quasi- π -regular* in R if there is an idempotent $e \in E(R)$ such that:

- (A) $a = eae + faf$ where $f := 1 - e$.
- (B) eae is invertible in eRe .
- (C') $(faf)^n \xrightarrow{n \rightarrow \infty} 0$ (w.r.t. the topology on R).

Call R *quasi- π -regular* if all its elements are quasi- π -regular.

Since we only consider LT rings, condition (C') means that for any $I \in \mathcal{I}_R$ there is $n \in \mathbb{N}$ such that $(faf)^n \in I$. This implies that quasi- π -regularity coincide with π -regularity for discrete topological rings (take $I = \{0\}$) and that if a is quasi- π -regular in R then $a + I$ is π -regular in R/I for all $I \in \mathcal{I}_R$. In particular, if R is quasi- π -regular then R/I is π -regular. We will call the idempotent e satisfying conditions (A), (B) and (C') the *associated idempotent* of a . The following lemma shows that it is unique.

Lemma 5.6. Let $R \in \mathcal{LT}_2$ and $a \in R$ a quasi- π -regular element. Then the idempotent e satisfying conditions (A), (B) and (C') is uniquely determined by a .

Proof. Assume both e and e' satisfy conditions (A), (B), (C') and let $I \in \mathcal{I}_R$. Then $e + I$ and $e' + I$ are associated idempotents of $a + I$ in R/I , hence $e + I = e' + I$, or equivalently $e - e' \in I$. It follows that $e - e' \in \bigcap_{I \in \mathcal{I}_R} I = \{0\}$ (since R is Hausdorff), so $e = e'$. \square

Remark 5.7. (i) In the assumptions of the previous lemma it is also possible to show that $eR = \bigcap_{n=1}^{\infty} a^n R$ and $(1 - e)R = \{r \in R : a^n r \xrightarrow{n \rightarrow \infty} 0\}$.

(ii) If we do not restrict to LT Hausdorff rings, the associated idempotent need not be unique. For example, in \mathbb{Q}_p both 0 and 1 are associated idempotents of p . (It is not known if Lemma 5.6 holds under the assumption that R is *right LT*, i.e. has a local basis of right ideals).

(iii) If one assigns a semiperfect ring R with $\bigcap_{n=1}^{\infty} \text{Jac}(R)^n = \{0\}$ the $\text{Jac}(R)$ -adic topology, then R becomes a Hausdorff LT ring and for any $a \in R$ there is an idempotent e satisfying conditions (B) and (C') (but such e need not be unique even when R is simple). However, condition (A) might be impossible to satisfy for some a . Indeed, the ring R constructed in Example 6.1 below, which is

isomorphic to $M_4(\mathbb{Z}_{(3)})$, is a semiperfect ring having no ring topology making it into a quasi- π -regular Hausdorff LT ring. As $\mathbb{Z}_{(3)}$ is quasi- π -regular w.r.t. the 3-adic topology (since it is local), it follows that quasi- π -regularity is not preserved under Morita equivalence. (This also follows from the comment before Proposition 2.6.)

It light of the last remark, it is convenient to call an LT Hausdorff ring R *quasi- π_∞ -regular* if $M_n(R)$ is quasi- π -regular for all n . This property is preserved under Morita equivalence and turns out to be related with *Henselianity* (see Section 8). However, to avoid cumbersome notation, we will not mention it in this section. All statements henceforth can be easily seen to hold once replacing (quasi-) π -regular with (quasi-) π_∞ -regular.

Corollary 5.8. (i) Let $R \in \mathcal{LT}_2$, let R_0 be a T -semi-invariant subring of R and let $a \in R_0$ be quasi- π -regular in R . Then a is quasi- π -regular in R_0 .

(ii) A T -semi-invariant subring of a quasi- π -regular ring is quasi- π -regular.

Proof. This is similar to the proof of Corollary 4.3. \square

Lemma 5.9. Let $R \in \mathcal{LT}_2$ be quasi- π -regular. Then:

- (i) For all $a \in \text{Jac}(R)$, $a^n \xrightarrow{n \rightarrow \infty} 0$. (That is, $\text{Jac}(R)$ is “topologically nil”.)
- (ii) R is semiperfect $\Leftrightarrow R$ does not contain an infinite set of orthogonal idempotents.

Proof. (i) Let $I \in \mathcal{I}_R$. Then $a + I \in (\text{Jac}(R) + I)/I \subseteq \text{Jac}(R/I)$, so by Lemma 4.4(i) applied to R/I (which is π -regular), there is $n \in \mathbb{N}$ such that $a^n \in I$.

(ii) We only show the non-trivial implication. For $a \in R$, let e_a denote the associated idempotent of a . Note that $e_a = 1$ implies $a \in R^\times$ and $e_a = 0$ implies $a^n \xrightarrow{n \rightarrow \infty} 0$.

Assume $e_a \in \{0, 1\}$ for all $a \in R$. We claim R is local. This is clear if $R = \{0\}$. Otherwise, let $a \in R$ and assume by contradiction that $e_a = e_{1-a} = 0$. Let $R \neq I \in \mathcal{I}_R$ (here we need $R \neq \{0\}$). Then there is $n \in \mathbb{N}$ such that $a^n, (1-a)^n \in I$, implying $(1-a^n)^n = (1-a)^n(1+a+\dots+a^{n-1})^n \in I$. We can write $1 = (1-a^n)^n + a^n h(a)$ for some $h(x) \in \mathbb{Z}[x]$, thus getting $1 \in I$, in contradiction to the assumption $I \neq R$. Therefore, one of e_a, e_{1-a} is 1, hence one of $a, 1-a$ is invertible.

Now assume there is $a \in R$ with $e := e_a \notin \{0, 1\}$. Then we can induct on eRe and $(1-e)R(1-e)$ as in the proof of Lemma 4.4(ii). However, we need to verify that eRe is quasi- π -regular (w.r.t. the induced topology). Let $b \in eRe$. It enough to show $e_b \in eRe$, i.e. $e_b = ee_b e$. As $R \in \mathcal{LT}_2$, this is equivalent to $e_b + I = ee_b e + I$ for all $I \in \mathcal{I}_R$. Indeed, since R/I is π -regular, so is $e(R/I)e$ (by Proposition 2.6(i)), hence $b + I$ has an associated idempotent $\varepsilon \in e(R/I)e$. However, it easy to see that ε is also the associated idempotent of $b + I$ in R/I , so necessarily $\varepsilon = e_b + I$. As $\varepsilon = (e + I)\varepsilon(e + I)$, it follows that $e_b + I = ee_b e + I$. \square

We can now state and prove a T -semi-invariant analogue of Theorem 4.6.

Theorem 5.10. Let R_0 be a T -semi-invariant subring of a semiperfect and quasi- π -regular ring $R \in \mathcal{LT}_2$. Then R_0 is semiperfect and quasi- π -regular and there is $n \in \mathbb{N}$ such that $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$.

Proof. That R_0 is quasi- π -regular and semiperfect follows from Corollary 5.8 and Lemma 5.9(ii). Now let $I \in \mathcal{I}_R$. Then both R/I and $(R_0 + I)/I$ are semiperfect and π -regular (since $(R_0 + I)/I \cong R_0/(R_0 \cap I)$ and $R_0 \cap I$ is open in R_0). Therefore, by Lemma 4.5, there is $n_I \in \mathbb{N}$ such that $\text{Jac}((R_0 + I)/I)^{n_I} \subseteq \text{Jac}(R/I)^{n_I}$. As $\text{Jac}(R/I) = (\text{Jac}(R) + I)/I$, this implies $\text{Jac}(R_0)^{n_I} \subseteq \text{Jac}(R) + I$. However, $(R/I)/(\text{Jac}(R/I)) \cong R/(\text{Jac}(R) + I)$ is a quotient of $R/\text{Jac}(R)$ which is semisimple, hence the index

of nilpotence of any of its subsets is bounded (when finite) by $\text{length}(R/\text{Jac}(R))$.⁸ Therefore, there is $n \in \mathbb{N}$ such that for all $I \in \mathcal{I}_R$, $\text{Jac}(R_0)^n \subseteq \text{Jac}(R) + I$ or equivalently, $\text{Jac}(R_0)^n \subseteq \bigcap_{I \in \mathcal{I}_R} (\text{Jac}(R) + I) = \overline{\text{Jac}(R)}$. Thus, we are done by the following lemma. \square

Lemma 5.11. *Let $R \in \mathcal{LT}_2$ be quasi- π -regular. Then R^\times and $\text{Jac}(R)$ are closed.*

Proof. Let $a \in \overline{R^\times}$ and let e be its associated idempotent. Then for any $I \in \mathcal{I}_R$ there is $a_I \in R^\times$ such that $a - a_I \in I$. Clearly $e + I$ is the associated idempotent of $a + I = a_I + I$ in R/I . However, $a_I + I \in (R/I)^\times$ and thus $1 + I$ is its associated idempotent. It follows that $e + I = 1 + I$ for all $I \in \mathcal{I}_R$, hence $e = 1$ and $a \in R^\times$.

Now assume $a \in \overline{\text{Jac}(R)}$. It is enough to show that for all $b \in R$, $1 + ab \in R^\times$. Let $I \in \mathcal{I}_R$ and let $a_I \in \text{Jac}(R)$ be such that $a - a_I \in I$. Then $1 + a_I b \in R^\times$ and $(1 + ab) - (1 + a_I b) \in I$. Therefore, $1 + ab \in \bigcap_{I \in \mathcal{I}_R} (R^\times + I) = \overline{R^\times} = R^\times$. \square

Remark 5.12. (i) The assumption that R is quasi- π -regular in the last lemma is essential; see Example 9.2 (take $n = 0$). In addition, $\text{Jac}(R)^2$ need not be closed even when R is quasi- π -regular; see Example 9.11.

(ii) If R is quasi- π -regular and semiperfect, then R^\times and $\text{Jac}(R)$ are also open. Indeed, by the previous lemma $\text{Jac}(R) = \overline{\text{Jac}(R)} = \bigcap_{I \in \mathcal{I}_R} (\text{Jac}(R) + I)$, hence $\text{Jac}(R)$ is an intersection of open ideals. Since $R/\text{Jac}(R)$ is artinian, $\text{Jac}(R)$ is the intersection of finitely many such ideals, thus open. The set R^\times is open since it is a union of cosets of $\text{Jac}(R)$.

In order to apply Theorem 5.10 to pro-semiprimary rings, we need to recall some facts about complete topological rings. While the exact definition (see [31, §§7–8]) is of little use to us, we will need the following results. Let $R \in \mathcal{LT}_2$, then:

- (1) R is complete if and only if R is isomorphic to an inverse limit of an inverse system of discrete topological rings $\{R_i\}_{i \in I}$. In this case, if φ_i is the standard map from R to R_i , then $\{\ker \varphi_i \mid i \in I\}$ is a local basis of R .
- (2) If R is complete and \mathcal{B} is a local basis consisting of ideals, then $R \cong \varprojlim \{R/I\}_{I \in \mathcal{B}}$. (Note that R/I is discrete for all $I \in \mathcal{B}$.)

We will also use the fact that a closed subring of complete ring is complete. (This can be verified directly for rings in \mathcal{LT}_2 using the previous facts.)

We now specialize the definition of pro-semiprimary rings given in Section 1 to topological rings. For a ring property \mathcal{P} , a topological ring R will be called *pro- \mathcal{P}* if R is isomorphic as a topological ring to the inverse limit of an inverse system of discrete rings satisfying \mathcal{P} . If in addition the standard map from R to each of these rings is onto,⁹ then R will be called *strictly pro- \mathcal{P}* . Clearly any pro- \mathcal{P} ring is complete and lies in \mathcal{LT}_2 . An LT ring R is strictly pro- \mathcal{P} if and only if it is complete and admits a local basis of ideals \mathcal{B} such that R/I has \mathcal{P} for all $I \in \mathcal{B}$. Notice that if \mathcal{P} is preserved under Morita equivalence, then so does being pro- \mathcal{P} and being strictly pro- \mathcal{P} (because the isomorphisms of Lemma 2.3 are also topological isomorphisms).

Remark 5.13. Any inverse limit of (non-topological) rings satisfying \mathcal{P} can be endowed with a linear ring topology making it into a pro- \mathcal{P} ring, but this topology usually depends on the inverse system used to construct the ring. However, when $\mathcal{P} = \text{semiprimary}$ and the ring is right noetherian, the topology is uniquely determined and always coincide with the Jacobson topology! See Section 9.

⁸ Actually, the index of nilpotence is bounded in any right noetherian ring R . Indeed, the prime radical of R , denoted N , is nilpotent and R/N is a semiprime Goldie ring. Therefore, by Goldie's Theorem, R/N embeds in a semisimple ring and thus has a bounded index of nilpotence.

⁹ This is not trivial since the maps in the inverse system are not assumed to be onto.

Once recalling Remark 2.9, the following lemma implies that pro-semiprimary rings are quasi- π -regular.

Lemma 5.14. *Let $\{R_i, f_{ij}\}$ be an I -indexed inverse system of π -regular rings and let $R = \varprojlim \{R_i\}_{i \in I}$. Then R is quasi- π -regular.*

Proof. We identify R with the set of compatible I -tuples in $\prod_{i \in I} R_i$ (i.e. tuples $(x_i)_{i \in I}$ satisfying $f_{ij}(x_j) = x_i$ for all $i \leq j$ in I). Let $a = (a_i)_{i \in I} \in R$ and let $e_i \in E(R_i)$ be the associated idempotent of a_i in R_i . The uniqueness of e_i implies that $e = (e_i)_{i \in I}$ is compatible and hence lie in R . We claim that e is the associated idempotent of a in R . Conditions (A) and (C') are straightforward, so we only check (B): Let b_i be the inverse of $e_i a_i e_i$ in $e_i R e_i$. The for all $i \leq j$ in I , $f_{ij}(b_j)$ is also an inverse of $e_i a_i e_i$ in $e_i R e_i$, hence $f_{ij}(b_j) = b_i$. Therefore, $b := (b_i)_{i \in I}$ is compatible and lie in R . Clearly $b = ebe$ and $b(eae) = (eae)b = e$ (since this holds in each coordinate), so condition (B) is satisfied. We thus conclude that R is quasi- π -regular. \square

The converse of Lemma 5.9 is almost true; if $R \in \mathcal{L}\mathcal{T}_2$ is quasi- π -regular, then R is dense in a pro- π -regular ring, namely $\varprojlim \{R/I\}_{I \in \mathcal{I}_R}$. The following theorem implies that T-semi-invariant subrings of semiperfect pro-semiprimary rings are semiperfect and pro-semiprimary (w.r.t. the induced topology).

Theorem 5.15. *Assume $R = \varprojlim \{R_i\}_{i \in I}$ where each R_i is π -regular. Denote by J_i the kernel of the standard map $R \rightarrow R_i$ and let R_0 be a T-semi-invariant subring of R . Then:*

- (i) R_0 is quasi- π -regular and $R_0 = \varprojlim \{R_0/(J_i \cap R_0)\}_{i \in I}$.
- (ii) If R does not contain an infinite set of orthogonal idempotents, then R_0 is semiperfect and there is $n \in \mathbb{N}$ such that $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$.
- (iii) For all $i \in I$, $R_0/(J_i \cap R_0)$ is π -regular. If moreover R_i is semiprimary (right perfect, semiperfect), then so is $R_0/(J_i \cap R_0)$. In particular, if R is pro-semiprimary (pro-right-perfect, pro- π -regular-and-semiperfect), then so is R_0 .

Proof. By Lemma 5.14, R is quasi- π -regular, so the first assertion of (i) is Corollary 5.8(ii). As for the second assertion, R is complete and R_0 is closed in R , hence R_0 is complete. Since $\{R_0 \cap J_i \mid i \in I\}$ is a local basis of R_0 , $R_0 = \varprojlim \{R_0/(J_i \cap R_0)\}_{i \in I}$. (ii) follows from Lemma 5.9(ii) and Theorem 5.10. As for (iii), $R_0/(J_i \cap R_0)$ is π -regular as a quotient of a quasi- π -regular ring with an open ideal. The rest follows from Lemma 4.5 (applied to $R_0/(J_i \cap R_0)$ identified as a subring of R_i). \square

Let $\mathcal{P} \in \{\text{semiprimary, right perfect, } \pi\text{-regular and semiperfect, } \pi\text{-regular}\}$. Then Theorem 5.15 implies that pro- \mathcal{P} rings are strictly pro- \mathcal{P} (take $R_0 = R$). In fact, we can prove an even stronger result:

Corollary 5.16. *In the previous notation, the inverse limit of a small category of pro- \mathcal{P} rings is strictly pro- \mathcal{P} .*

Proof. Let \mathcal{C} be a small category of pro- \mathcal{P} rings and let $\{R_i\}_{i \in I}$ be the objects of \mathcal{C} . Then $R = \varprojlim \mathcal{C}$ can be identified with the set of I -tuples $(x_i)_{i \in I} \in \prod_{i \in I} R_i$ such that $f(x_j) = x_i$ for all $i, j \in I$ and $f \in \text{Hom}_{\mathcal{C}}(R_j, R_i)$. Clearly $S := \prod_{i \in I} R_i$ is pro- \mathcal{P} . If we can prove that R is a T-semi-invariant subring of S , then we are through by Theorem 5.15. Indeed, let π_i denote the projection from S to R_i . For all $i, j \in I$ and $f \in \text{Hom}_{\mathcal{C}}(R_j, R_i)$ define $\varphi_f^{(1)}, \varphi_f^{(2)} : S \rightarrow R_i$ by $\varphi_f^{(1)} = \pi_i$, $\varphi_f^{(2)} = f \circ \pi_j$. Then $R = \{x \in S : \varphi_f^{(1)}(x) = \varphi_f^{(2)}(x) \ \forall f\}$, hence R is a T-semi-invariant subring of S by Proposition 5.4(e). \square

In some sense, Corollary 5.16 includes Theorem 4.6 and part of Theorem 5.15 because a T-semi-invariant subring can be understood as the inverse limit of a category with two objects. (Indeed, if $R \subseteq S \in \mathcal{L}\mathcal{T}_2$ and Σ is a submonoid of $\text{End}_c(S)$, then take $\text{Ob}(\mathcal{C}) = \{R, S\}$ with $\text{End}_{\mathcal{C}}(S) = \Sigma$, $\text{End}_{\mathcal{C}}(R) = \{\text{id}_R\}$, $\text{Hom}_{\mathcal{C}}(S, R) = \phi$ and $\text{Hom}_{\mathcal{C}}(R, S) = \{i\}$ where $i : R \rightarrow S$ is the inclusion map).

Corollary 5.17. *If R is pro-semiprimary, then for any $J \in \mathcal{I}_R$ there is $n \in \mathbb{N}$ such that $\text{Jac}(R)^n \subseteq J$. In particular, $\bigcap_{n=1}^{\infty} \text{Jac}(R)^n = \{0\}$.*

Proof. Assume $R = \varprojlim \{R_i\}_{i \in I}$ with each R_i semiprimary and let J_i be as in Theorem 5.15. Since $\{J_i \mid i \in I\}$ is local basis, there is $i \in I$ such that $J_i \subseteq J$. By Theorem 5.15(iii), R/J_i is semiprimary, hence there is $n \in \mathbb{N}$ such that $\text{Jac}(R/J_i)^n = 0$. As $\text{Jac}(R/J_i) \supseteq (\text{Jac}(R) + J_i)/J_i$, we get $\text{Jac}(R)^n \subseteq J_i \subseteq J$. \square

Remark 5.18. We will show in Proposition 8.6 that Henselian rank-1 valuation rings are quasi- π_∞ -regular. In particular, non-complete such rings (e.g. the \mathbb{Q} -algebraic elements in \mathbb{Z}_p) are examples of non-complete quasi- π_∞ -regular rings. In addition, the author suspects that the following are also explicit examples of non-complete quasi- π_∞ -regular rings: (1) The ring of power series $\sum_{i=0}^{\infty} a_i t^i \in \mathbb{Z}_p[[t]]$ with $a_i \rightarrow 0$ endowed with the t -adic topology (such rings are common in rigid geometry). (2) The ring in the comment after Lemma 2.3 w.r.t. its Jacobson topology.

6. Counterexamples

This section consists of counterexamples. In particular, we show that:

- (1) If R is a semiperfect ring and $\Sigma \subseteq \text{End}(R)$, then R^Σ need not be semiperfect even when Σ is a finite group and even when Σ consists of a single automorphism. Similarly, if $X \subseteq R$ is a set, then $\text{Cent}_R(X)$ need not be semiperfect even when X consists of a single element.
- (2) The semiperfect analogue of Corollary 4.7 is not true in general.
- (3) A semi-invariant subring of a semiperfect pro-semiprimary ring need not be semiperfect even when closed (in contrast to T-semi-invariant subrings).
- (4) Rationally closed subrings of a f.d. algebra need not be semiperfect. In particular, Theorem 4.6 do not generalize to rationally closed subrings.
- (5) No two of the families of semi-invariant, invariant, centralizer and rationally closed subrings coincide in general.

We note that (1) is also true if we replace semiperfect with artinian. This was treated at the end of Section 4.

We begin with demonstrating (1). Our examples use Azumaya algebras and we refer the reader to [27] for definition and details.

Example 6.1. Let S be a discrete valuation ring with maximal ideal πS , residue field $k = S/\pi S$ and fraction field F , and let A be an Azumaya algebra over S . Recall that this implies $A/\pi A$ is a central simple k -algebra and $\text{Jac}(A) = \pi A$. Assume the following holds:

- (a) $D = F \otimes_S A$ is a division ring.
- (b) $A/\pi A$ has zero divisors.

In addition, assume there is a set $X \subseteq A^\times$ generating A as an S -algebra (such X always exists). Note that conditions (a) and (b) imply that A is not semiperfect because A contains no non-trivial idempotents while $A/\pi A = A/\text{Jac}(A)$ does contain such idempotents, hence $\text{Jac}(A)$ is not idempotent lifting.

Define $R = A \otimes_S A^{\text{op}}$ and let $\Sigma = \{\sigma_x\}_{x \in X}$ where σ_x is conjugation by $1 \otimes x^{\text{op}}$. Then R is an Azumaya S -algebra which is an S -order inside $D \otimes_F D^{\text{op}} \cong M_r(F)$. This is well known to imply that $R \cong \text{End}(P_S)$ for some faithful finite projective S -module P (in fact, P is free since S is local). Therefore, R is Morita equivalent to S , hence semiperfect. On the other hand, $R^\Sigma = \text{Cent}_R(\{1 \otimes x^{\text{op}} \mid x \in X\}) = \text{Cent}_R(S \otimes A^{\text{op}}) = A \otimes S \cong A$, so R^Σ is not semiperfect.

An explicit choice for S, A, F, D is $S = \mathbb{Z}_{(3)}$ ($\pi = 3$), $F = \mathbb{Q}$, $D = (-1, -1)_{\mathbb{Q}} = \mathbb{Q}[i, j \mid ij = -ji, i^2 = j^2 = -1]$ and $A = S[i, j]$. If we take $X = \{i, j\}$ then Σ will consist of two inner automorphisms which are easily seen to generate an automorphism group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$.

Example 6.2. Let S, π, A, F, D be as in Example 6.1 and assume there is a cyclic Galois extension K/F such that:

- (c) K/F is totally ramified at π .
- (d) $K \otimes_F D$ splits (i.e. $K \otimes_F D \cong M_t(K)$).

Write $\text{Gal}(K/F) = \langle \sigma \rangle$.

Let T denote the integral closure of S in K . Then $\sigma(T) = T$. We claim that $T \otimes_S A$ is semiperfect, but $(T \otimes_S A)^{\{\sigma \otimes 1\}}$ is not. Indeed, $T^{\{\sigma\}} = T \cap F = S$, so $(T \otimes_S A)^{\{\sigma \otimes 1\}} = S \otimes A \cong A$ which is not semiperfect as explained in the previous example. On the other hand, $T \otimes A$ is an Azumaya T -algebra and a T -order in $K \otimes D \cong M_t(K)$. Again, this implies $T \otimes A$ is Morita equivalent to T . But T is local because K/F is totally ramified at π , hence $T \otimes A$ is semiperfect.

If we take S, A, F, D as in the previous example, then $T = S[\sqrt{-3}]$, $K = \mathbb{Q}[\sqrt{-3}]$ will satisfy (c) and (d). Indeed, dimension constraints imply $K \otimes D$ is either a division ring or $M_2(K)$, but $\sqrt{-3} + i + j + ij \in K \otimes D$ has reduced norm 0, so the latter option must hold.

Example 6.3. Start with a semiperfect ring R and $\sigma \in \text{End}(R)$ such that $R^{(\sigma)}$ is not semiperfect (e.g. those of Example 6.2). Let $R' = R[[t; \sigma]]$ be the ring of σ -twisted formal power series with left coefficients in R (i.e. $\sigma(r)t = t r$ for all $r \in R$) and let $x = 1 + t \in (R')^\times$. We claim R' is semiperfect, but $\text{Cent}_{R'}(x)$ is not. Indeed, $\text{Cent}_{R'}(t) = \text{Cent}_{R'}(t) = R^{(\sigma)}[[t]]$, so we are done by applying the following proposition for $R[[t; \sigma]]$ and $R^{(\sigma)}[[t]]$.

Proposition 6.4. For any ring W and $\tau \in \text{End}(W)$, W is semiperfect if and only if $W[[t; \tau]]$ is.

Proof. Let $V = W[[t; \tau]]$ and let $J = \text{Jac}(W) + Vt \trianglelefteq V$. Then $V/J \cong W/\text{Jac}(W)$. Since the right-hand side has zero Jacobson radical, $J \supseteq \text{Jac}(V)$. However, as $1 + J \subseteq V^\times$, we have $J \subseteq \text{Jac}(V)$, thus $\text{Jac}(V) = J$. The isomorphism $V/J \cong W/\text{Jac}(W)$ now implies that V is semilocal $\Leftrightarrow W$ is semilocal. We finish by observing that Vt is idempotent lifting (this immediate as $V = W \oplus Vt$), hence J is idempotent lifting in $V \Leftrightarrow J/Vt$ is idempotent lifting in $V/Vt \Leftrightarrow \text{Jac}(W)$ is idempotent lifting in W . \square

We now show (2), relying on the previous examples.

Example 6.5. Let R be a semiperfect ring and let $X \subseteq R$ be such that $\text{Cent}_R(X)$ is not semiperfect (the existence of such R and X was shown in previous examples). Let $Y = \{y_a \mid a \in X\}$ be a set of formal variables and let $S = R\langle Y \rangle$ be the ring of non-commutative polynomials in Y over R (Y commutes with R). We can make R into a right S -module by considering the standard right action of R onto itself and extending it to S by defining $r \cdot y_a = ar$ for all $a \in X$. Let M denote the right S -module obtained thusly. Identify R with $\text{End}(M_S) = \text{End}(R_R)$ via $r \mapsto (m \mapsto rm) \in \text{End}(R_R)$. It is straightforward to check that $\text{End}(M_S)$ now corresponds to $\text{Cent}_R(X)$. Therefore, $\text{End}(M_R) \cong R$ is semiperfect but $\text{End}(M_S) \cong \text{Cent}_R(X)$ is not semiperfect.

The next example demonstrates (3).

Example 6.6. Let p, q be distinct primes. Endow $R = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Q}$ with the product topology (the topology on \mathbb{Q} is the discrete topology). Then R is clearly semiperfect and pro-semiprimary. Define $K = \{(a, a, a) \mid a \in \mathbb{Q}\}$ and let $R_0 = R \cap K$. Then R_0 is a semi-invariant subring of R by Proposition 3.3 (take $S = \mathbb{Q}_p \times \mathbb{Q}_q \times \mathbb{Q}$) and it is routine to check R_0 is closed. However, R_0 is not semiperfect. Indeed, it is isomorphic to $T = M^{-1}\mathbb{Z}$ where $M = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. The ring T is not semiperfect because it has no non-trivial idempotents while $T/\text{Jac}(T) \cong T/pqT \cong T/pT \times T/qT$ has such, so $\text{Jac}(T)$ cannot be idempotent lifting.

The following example shows that rationally closed subrings of a f.d. algebra need not be semiperfect. As f.d. algebras are semiprimary, this shows that Theorem 4.6 fails for rationally closed subrings.

Example 6.7. Let $K = \mathbb{Q}(x)$ and $R = K \times K \times K$. Define $S' = \{f/g \mid f, g \in \mathbb{Q}[x], g(0) \neq 0, g(1) \neq 0\}$ and observe that S' is not semiperfect since it is a domain but $S'/\text{Jac}(S')$ has non-trivial idempotents. (Indeed, $S'/\text{Jac}(S') = S'/\langle x(x-1) \rangle \cong S'/\langle x \rangle \times S'/\langle x-1 \rangle \cong \mathbb{Q} \times \mathbb{Q}$). Define $\varphi: S' \rightarrow R$ to be the \mathbb{Q} -algebra homomorphism obtained by sending x to $a := (0, 1, x) \in R$ and let $S = \text{im } \varphi$. It is easy to verify that φ is well defined and injective, hence S is not semiperfect. However, S is rationally closed in R . To see this, let $q(x) \in S'$ and assume $q(a) \in R^\times$. Then $q(0), q(1) \neq 0$. This implies $q(x) \in (S')^\times$, hence $q(a) = \varphi(q(x)) \in S^\times$.

We finish by demonstrating (5). The subring S of the last example cannot be semi-invariant, for otherwise we would get a contradiction to Theorem 4.6. In particular, S is not an invariant subring nor a centralizer subring. Next, let $R = \mathbb{Q}[\sqrt[3]{2}, \sqrt{3}]$. Then the only centralizer subring of R is R itself, the invariant subrings of R are R and $\mathbb{Q}[\sqrt[3]{2}]$ (Remark 3.5) and the semi-invariant subrings of R are the four subfields of R (Corollary 3.4). In particular, R admits a semi-invariant non-invariant subring and an invariant non-centralizer subring.

7. Applications

This section presents applications of the previous results. In order to avoid cumbersome phrasing, we introduce the following families of ring-theoretic properties:

$$\mathcal{P}_{\text{disc}} = \left\{ \begin{array}{l} \text{semiprimary, right perfect, left perfect, } \pi_\infty\text{-regular and semiperfect,} \\ \pi_\infty\text{-regular, } \pi\text{-regular and semiperfect, } \pi\text{-regular} \end{array} \right\},$$

$$\mathcal{P}_{\text{top}} = \left\{ \begin{array}{l} \text{pro-}\mathcal{P}, \text{ pro-}\mathcal{P} \text{ and semiperfect,} \\ \text{quasi-}\mathcal{Q}, \text{ quasi-}\mathcal{Q} \text{ and semiperfect} \end{array} \middle| \begin{array}{l} \mathcal{P} \in \mathcal{P}_{\text{disc}} \\ \mathcal{Q} \in \{\pi_\infty\text{-regular, } \pi\text{-regular}\} \end{array} \right\}$$

(For example, “quasi- π_∞ -regular and semiperfect” lies in \mathcal{P}_{top} .) Note that the properties in $\mathcal{P}_{\text{disc}}$ apply to rings while the properties in \mathcal{P}_{top} apply to LT rings. Nevertheless, we will sometimes address non-topological rings as satisfying one of the properties of \mathcal{P}_{top} , meaning that they satisfy it w.r.t. some linear ring topology. We also define \mathcal{P}_{mor} (resp. \mathcal{P}_{sp}) to be the set of properties in $\mathcal{P}_{\text{disc}} \cup \mathcal{P}_{\text{top}}$ which are preserved under Morita equivalence (resp. imply that the ring is semiperfect). Recall that a property in \mathcal{P}_{top} is preserved under Morita equivalence if this holds in the sense of Section 5 (and not in the sense of [15]). For example, “ π -regular” and “quasi- π -regular” do not lie in \mathcal{P}_{mor} nor in \mathcal{P}_{sp} , “pro-semiprimary and semiperfect” lies in both \mathcal{P}_{sp} and \mathcal{P}_{mor} , and “pro-semiprimary” lies in \mathcal{P}_{mor} , but not in \mathcal{P}_{sp} .

Theorems 4.6, 5.10 and 5.15 and Corollaries 4.3(ii) and 5.8(ii) can be now summarized as follows:

Theorem 7.1. Let R be a ring and R_0 a subring.

- (i) If R has $\mathcal{P} \in \mathcal{P}_{\text{disc}}$ and R_0 is semi-invariant, then R_0 has \mathcal{P} .
- (ii) If $R \in \mathcal{LT}_2$ has $\mathcal{P} \in \mathcal{P}_{\text{top}}$ and R_0 is T -semi-invariant, then R_0 has \mathcal{P} (w.r.t. the induced topology).
- (iii) In both (i) and (ii), if $\mathcal{P} \in \mathcal{P}_{\text{sp}}$, then $\text{Jac}(R_0)^n \subseteq \text{Jac}(R)$ for some $n \in \mathbb{N}$.

Our first application follows from the fact that a centralizer subring is always (T-)semi-invariant:

Corollary 7.2. Let $\mathcal{P} \in \mathcal{P}_{\text{disc}} \cup \mathcal{P}_{\text{top}}$ and let R be a ring satisfying \mathcal{P} . Then $\text{Cent}(R)$ and any maximal commutative subring of R satisfy \mathcal{P} .

Proof. $\text{Cent}(R)$ is the centralizer of R and a maximal commutative subring of R is itself's centralizer. Now apply Theorem 7.1. \square

Surprisingly, the author could not find in the literature results that are similar to the previous corollary, except the fact that the center of a right artinian ring is semiprimary. (This follows from

a classical result of Jacobson, stating that the endomorphism ring of any module of finite length is semiprimary, together with the fact that the center of a ring R is isomorphic to $\text{End}_{(R)}(R_R)$.)

The next applications concern endomorphism rings of finitely presented modules. We will only treat here the non-topological properties (i.e. $\mathcal{P}_{\text{disc}}$). The topological analogues of the results to follow require additional notation and are thus postponed to the next section.

Theorem 7.3. *Let R be a ring satisfying $\mathcal{P} \in \mathcal{P}_{\text{disc}} \cap \mathcal{P}_{\text{mor}}$ and let M be a finitely presented right R -module. Then $\text{End}(M_R)$ satisfies \mathcal{P} .*

Proof. There is an exact sequence $R^n \rightarrow R^m \rightarrow M \rightarrow 0$ with $n, m \in \mathbb{N}$. Since R^n, R^m are projective, we may apply Proposition 3.7 to deduce that $\text{End}(M)$ is a quotient of a semi-invariant subring of $\text{End}(R^n) \times \text{End}(R^m) \cong M_n(R) \times M_m(R)$. The latter has \mathcal{P} because any $\mathcal{P} \in \mathcal{P}_{\text{disc}} \cap \mathcal{P}_{\text{mor}}$ is preserved under Morita equivalence and under taking finite products. Since all ring properties in $\mathcal{P}_{\text{disc}}$ pass to quotients, we are done by Theorem 7.1. \square

Corollary 7.4. *Let $\varphi : S \rightarrow R$ be a ring homomorphism. Consider R as a right S -module via φ and assume it is finitely presented. Then if S satisfies $\mathcal{P} \in \mathcal{P}_{\text{disc}} \cap \mathcal{P}_{\text{mor}}$, then so does R .*

Proof. By Theorem 7.3, $\text{End}(R_S)$ has \mathcal{P} . Therefore, by Corollary 4.7, $R \cong \text{End}(R_R)$ has \mathcal{P} . \square

Remark 7.5. Theorem 7.3 actually follows from results of Bjork, who proved the semiprimary case and part of the left/right perfect cases [5, Ths. 4.1–4.2], and Rowen, who proved the left/right perfect and the semiperfect and π_∞ -regular cases [23, Cr. 11 and Th. 8(iii)]. Our approach suggests a single simplified proof to all the cases. Note that we cannot replace “finitely presented” with “finitely generated” in Theorem 7.3; in [6, Ex. 2.1], Bjork presents a right artinian ring with a cyclic left module having a non-semilocal endomorphism ring. See also [13, Ex. 3.5].

By arguing as in the proof of Theorem 7.3 one can also obtain:

Theorem 7.6. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in an abelian category \mathcal{A} such that B is projective.*

- (i) *If $\text{End}(A)$ and $\text{End}(B)$ has $\mathcal{P} \in \mathcal{P}_{\text{disc}}$, then $\text{End}(C)$ has \mathcal{P} .*
- (ii) *If $\text{End}(A \oplus B)$ has $\mathcal{P} \in \mathcal{P}_{\text{sp}}$, then $\text{End}(C)$ is semiperfect.*

Next, we turn to representations over modules with “good” endomorphism ring. By a representation of a monoid (ring) G over a right R -module M , we mean a monoid (ring) homomorphism $\rho : G \rightarrow \text{End}(M)$ (so G acts on M via ρ).

Corollary 7.7. *Let R be a ring and let ρ be a representation of a monoid (or a ring) G over a right R -module M . Assume that one of the following holds*

- (i) *$\text{End}(M)$ has $\mathcal{P} \in \mathcal{P}_{\text{disc}} \cup \mathcal{P}_{\text{top}}$.*
- (ii) *There is a sub-monoid (subring) $H \subseteq G$ such that $\text{End}(\rho|_H)$ has $\mathcal{P} \in \mathcal{P}_{\text{disc}}$.*
- (iii) *$\text{End}(M)$ is LT and Hausdorff and there is a sub-monoid (or a subring) $H \subseteq G$ such that $\text{End}(\rho|_H)$ has $\mathcal{P} \in \mathcal{P}_{\text{top}}$ w.r.t. the induced topology.*

Then $\text{End}(\rho)$ has \mathcal{P} . Moreover, if $\mathcal{P} \in \mathcal{P}_{\text{sp}}$, then ρ has a Krull–Schmidt decomposition $\rho \cong \rho_1 \oplus \cdots \oplus \rho_t$ and $\text{End}(\rho_i)$ is local and has \mathcal{P} for all $1 \leq i \leq t$.

Proof. (i) follows from (ii) and (iii) if we take H to be the trivial monoid (basic subring). To see (ii) (resp. (iii)), notice that $\text{End}(\rho) = \text{Cent}_{\text{End}(\rho|_H)}(\rho(G))$. Therefore, $\text{End}(\rho)$ is a semi-invariant (resp. T-semi-invariant) subring of $\text{End}(\rho|_H)$, hence by Theorem 7.1, $\text{End}(\rho)$ has \mathcal{P} . If $\mathcal{P} \in \mathcal{P}_{\text{sp}}$, then $\text{End}(\rho)$

is semiperfect. The Krull–Schmidt Theorem now implies ρ has a Krull–Schmidt decomposition $\rho \cong \rho_1 \oplus \cdots \oplus \rho_t$ and $\text{End}(\rho_i)$ is local for all i . Finally, $\text{End}(\rho_i) \cong e \text{End}(\rho) e$ for some $e \in E(\text{End}(\rho))$. Since for any ring R and $e \in E(R)$, R has \mathcal{P} implies eRe has \mathcal{P} , we are through. \square

Assume R is a ring and M is a right R -module such that $\text{End}(M_R)$ is semiperfect and quasi- π -regular (see Theorem 8.3 below for cases when this happens). Then the endomorphisms of M have a “Jordan decomposition” in the following sense: If $f \in \text{End}(M_R)$, then we can consider M as a right $R[x]$ -module by letting x act as f . Clearly $\text{End}(M_{R[x]}) = \text{Cent}_{\text{End}(M_R)}(f)$, so by Theorem 5.10, $\text{End}(M_{R[x]})$ is semiperfect. Therefore, $M_{R[x]}$ has a Krull–Schmidt decomposition $M = M_1 \oplus \cdots \oplus M_t$. (Notice that each M_i is an f -invariant submodule of M .) This decomposition plays the role of a Jordan decomposition for f , since the isomorphism classes of M_1, \dots, M_t (as $R[x]$ -modules) determine the conjugation class of f . In particular, studying endomorphisms of M can be done by classifying LE-modules over $R[x]$.

Finally, the results of this paper can be applied in a rather different manner to bilinear forms: Let $*$ be an anti-endomorphism of a ring R (i.e. an additive, unity-preserving map that reverses order of multiplication). Then $\sigma = *^2$ is an endomorphism of R and $*$ becomes an involution on the invariant subring $R^{(\sigma)}$. As some claims on $(R, *)$ can be reduced to claims on $(R^{(\sigma)}, *|_{R^{(\sigma)}})$, our results become a useful tool for studying the former. Recalling that bilinear (resp. sesquilinear) forms correspond certain anti-endomorphisms and quadratic (resp. hermitian) forms correspond to involutions (see [17, Ch. I]), these ideas, taken much further, can be used to reduce the isomorphism problem of bilinear forms to the isomorphism problem of hermitian forms. This was actually done (using other methods) for bilinear forms over fields by Riehm [22], who later generalized this with Shrader-Frechette to sesquilinear forms over semisimple algebras [21]. The author can improve these results for bilinear (sesquilinear) forms over various semiperfect pro-semiprimary rings (e.g. f.g. algebras over \mathbb{Z}_p). This will be published elsewhere.

8. Modules over linearly topologized rings

In this section we extend Theorem 7.3 and other applications to LT rings. This is done by properly topologizing modules and endomorphisms rings of modules over LT rings.

Let R be an LT ring and let M be a right R -module. Then M can be made into a topological R -module by taking $\{x + MJ \mid J \in \mathcal{I}_R\}$ as basis of neighborhoods of $x \in M$. (That M is indeed a topological module follows from [31, Th. 3.6].) Notice that any homomorphism of modules is continuous w.r.t. this topology. Furthermore, $\text{End}(M)$ can be linearly topologized by taking $\{\text{Hom}_R(M, MJ) \mid J \in \mathcal{I}_R\}$ as a local basis.¹⁰ We will refer to the topologies just defined on M and $\text{End}(M)$ as their *standard topologies*. In general, that R is Hausdorff does not imply M or $\text{End}(M)$ are Hausdorff. (E.g.: For any distinct primes $p, q \in \mathbb{Z}$, the \mathbb{Z} -module \mathbb{Z}/q is not Hausdorff w.r.t. the p -adic topology on \mathbb{Z} .) Observe that $\overline{\{0_{\text{End}(M)}\}} = \bigcap_{J \in \mathcal{I}_R} \text{Hom}(M, MJ) = \text{Hom}(M, \bigcap_{J \in \mathcal{I}_R} MJ) = \text{Hom}(M, \overline{\{0_M\}})$, so M is Hausdorff implies $\text{End}(M)$ is Hausdorff.

Now let E_\bullet be a finite resolution of M , i.e. E_\bullet consists of an exact sequence $E_{n-1} \rightarrow \cdots \rightarrow E_0 \rightarrow E_{-1} = M \rightarrow 0$.¹¹ The maps $E_i \rightarrow E_{i-1}$ will be denoted by d_i . We say that E_\bullet has the *lifting property* if any $f_{-1} \in \text{End}(M)$ can be extended to a chain complex homomorphism $f_\bullet : E_\bullet \rightarrow E_\bullet$. (Recall that f_\bullet consists of a sequence $\{f_i\}_{i=-1}^{n-1}$ such that $f_i \in \text{End}(E_i)$ and $d_i f_i = f_{i-1} d_i$ for all i .) In other words, E_\bullet has the lifting property if and only if the following commutative diagram can be completed for every $f_{-1} \in \text{End}(M)$.

¹⁰ This topology is the uniform convergence topology (w.r.t. the natural uniform structure of M). If M is f.g. then this topology coincides with the pointwise convergence topology (i.e. the topology induced from the the product topology on M^M).

¹¹ We do not require the map $E_{n-1} \rightarrow E_{n-2}$ to be injective.

$$\begin{array}{ccccccc}
 E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & E_0 \longrightarrow M \\
 \downarrow f_{n-1} & & & & \downarrow f_1 & & \downarrow f_0 \\
 E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & E_0 \longrightarrow M
 \end{array}$$

For example, any projective resolution has the lifting property. We define a linear ring topology τ_E on $\text{End}(M)$ as follows: For all $J \in \mathcal{I}_R$, define $B(J, E)$ to be the set of maps $f_{-1} \in \text{End}(M, MJ)$ that extend to a chain complex homomorphism $f_\bullet : E_\bullet \rightarrow E_\bullet$ such that $\text{im } f_i \subseteq E_i J$ for all $-1 \leq i < n$. The lifting property implies $B(J, E) \trianglelefteq \text{End}(M)$ and it is clear that $\mathcal{B}_E := \{B(J, E) \mid J \in \mathcal{I}_R\}$ is a filter base. Therefore, there is a unique ring topology on $\text{End}(M)$, denoted τ_E , having \mathcal{B}_E as a local basis.

It turns out that if E_\bullet is a projective resolution, then τ_E only depends on the length of E , i.e. the number n . Indeed, if P_\bullet, P'_\bullet are two projective resolutions of length n of M , then the map $\text{id}_M : M \rightarrow M$ gives rise to chain complex homomorphisms $\alpha_\bullet : P_\bullet \rightarrow P'_\bullet$ and $\beta_\bullet : P'_\bullet \rightarrow P_\bullet$ with $\alpha_{-1} = \beta_{-1} = \text{id}_M$. Now, if $J \in \mathcal{I}_R$ and $f_{-1} \in B(J, P)$, then there is $f'_\bullet : P'_\bullet \rightarrow P_\bullet$ such that $\text{im } f'_i \subseteq P_i J$ for all i . Define $f'_\bullet = \alpha_\bullet f_\bullet \beta_\bullet$. Then $\text{im } f'_i \subseteq \alpha_i(P_i J) \subseteq P'_i J$ for all i and $f'_{-1} = \text{id}_M f_{-1} \text{id}_M = f_{-1}$, so $f_{-1} \in B(J, P')$. By symmetry, we get $B(J, P) = B(J, P')$ for all $J \in \mathcal{I}_R$, hence $\tau_P = \tau_{P'}$.

The topology of $\text{End}(M)$ obtained from a projective resolution of length n will be denoted by τ_n^M and the closure of the zero ideal in that topology will be denoted by I_n^M . Note that $\tau_1^M \subseteq \tau_2^M \subseteq \cdots$ and that τ_1^M is the standard topology on $\text{End}(M)$. (Indeed, if $P_\bullet : P_0 \rightarrow M \rightarrow 0$ is a projective resolution of length 1, then any $f \in \text{Hom}(M, MJ)$ can be lifted to $f_0 : P_0 \rightarrow P_0 J$ because the map $P_0 J \rightarrow MJ$ is onto, hence $B(P, J) = \text{Hom}(M, MJ)$.) More generally, for any resolution E_\bullet of M , τ_E contains the standard topology on $\text{End}(M)$. Therefore, if M is Hausdorff, then τ_E is Hausdorff. In Appendix A we provide sufficient conditions for $\tau_1^M, \tau_2^M, \dots$ to coincide.

With this terminology at hand, we can generalize Proposition 3.7:

Proposition 8.1. *Let R be an LT ring and let $E : A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of right R -modules satisfying the lifting property (w.r.t. C) and such that A and B are Hausdorff. Assign $\text{End}(A)$ and $\text{End}(B)$ the natural topology and endow $\text{End}(C)$ with τ_E . Then $\text{End}(C)$ is isomorphic as a topological ring to a quotient of a T -semi-invariant subring of $\text{End}(A) \times \text{End}(B)$.*

Proof. We use the notation of the proof of Proposition 3.7. By that proof, $\text{End}(C)$ is isomorphic to a quotient of $\text{Cent}_D(\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix})$. It is easy to check that the embedding $D \hookrightarrow S$ is a topological embedding, hence $\text{End}(C)$ is isomorphic to a quotient of a T -semi-invariant subring of D . That the quotient topology on $\text{End}(C)$ is indeed τ_E is routine. \square

We are now in position to generalize previous results.

Lemma 8.2. *Let $R \in \mathcal{L}\mathcal{T}_2$ and $\mathcal{P} \in \mathcal{P}_{\text{disc}}$. Then R is pro- \mathcal{P} if and only if R is complete and R/I has \mathcal{P} for all $I \in \mathcal{I}_R$.*

Proof. If R is complete and R/I has \mathcal{P} for all $I \in \mathcal{I}_R$, then $R \cong \varprojlim \{R/I\}_{I \in \mathcal{I}_R}$, so R is pro- \mathcal{P} . On the other hand, if R is pro- \mathcal{P} , then it is complete. In addition, it is strictly pro- \mathcal{P} (Corollary 5.16), hence there is a local basis of ideals \mathcal{B} such that R/I has \mathcal{P} for all $I \in \mathcal{B}$. Now, let $I \in \mathcal{I}_R$. Then there is $I_0 \in \mathcal{B}$ contained in I . Now, R/I is a quotient of R/I_0 . As the latter has \mathcal{P} , so does R/I . \square

Recall that a topological ring is *first countable* if it admits a countable local basis. If R is pro- \mathcal{P} , then this is equivalent to saying that R is the inverse limit of countably many discrete rings satisfying \mathcal{P} .

Theorem 8.3. *Let $R \in \mathcal{L}\mathcal{T}_2$ be a ring and let M be a f.p. right R -module.*

- (i) *If R is first countable and satisfies $\mathcal{P} \in \mathcal{P}_{\text{top}} \cap \mathcal{P}_{\text{mor}}$, then $\text{End}(M)/I_2^M$ satisfies \mathcal{P} when $\text{End}(M)$ is endowed with τ_2^M . In particular, if M is Hausdorff, then $\text{End}(M)$ has \mathcal{P} .*

- (ii) Assume R is quasi- π_∞ -regular and let $i \in \{1, 2\}$. Then $\text{End}(M)/I_i^M$ is quasi- π_∞ -regular when $\text{End}(M)$ is endowed with τ_i^M . In particular, if M is Hausdorff, then $\text{End}(M)$ is quasi- π_∞ -regular w.r.t. τ_1^M .
- (iii) If R is semiperfect and quasi- π_∞ -regular, then $\text{End}(M)$ is semiperfect.

Proof. (i) The argument in the proof of Theorem 7.3 shows that $\text{End}(M)$ is a quotient of an LT Hausdorff ring satisfying \mathcal{P} , which we denote by W (use Proposition 8.1 instead of Proposition 3.7). I_2^M is a closed ideal of $\text{End}(M)$ and therefore $\text{End}(M)/I_2^M$ is a quotient of W by a closed ideal. We finish by claiming that for any closed ideal $I \trianglelefteq W$, W/I satisfies \mathcal{P} . We will only check the case $\mathcal{P} = \text{pro-}\mathcal{Q}$ for $\mathcal{Q} \in \mathcal{P}_{\text{disc}}$. The other cases are straightforward or follow from the $\text{pro-}\mathcal{Q}$ case. Indeed, any open ideal of W/I is of the form J/I for some $J \in \mathcal{I}_W$, hence by Lemma 8.2, $(W/I)/(J/I) \cong W/J$ satisfies \mathcal{Q} . In addition, that R is first countable implies W is first countable, hence by the Birkhoff–Kakutani Theorem, W is metrizable. By [8, p. 163] a Hausdorff quotient of a complete metric ring is complete, hence W/I is complete. Therefore, by Lemma 8.2 (applied to W/I), W/I is $\text{pro-}\mathcal{Q}$.

(ii) The case $i = 2$ follows from the argument of (i) since being π -regular passes to quotients by closed ideals (the first countable assumption is not needed). As for $i = 1$, Since I_1^M is closed in τ_1^M , it is also closed in τ_2^M . Therefore, $\text{End}(M)/I_1^M$ is quasi- π_∞ -regular when M is equipped with τ_2^M . We are done by observing that if a ring is quasi- π_∞ -regular w.r.t. a given topology, then it is quasi- π_∞ -regular w.r.t. any linear Hausdorff sub-topology.

(iii) By (i), $\text{End}(M)$ is a quotient of a semiperfect ring, namely W . \square

Remark 8.4. Part (iii) of Theorem 8.3 was proved in [24] for complete semilocal rings with Jacobson radical f.g. as a right ideal and in [23] for semiperfect π_∞ -regular rings. Both conditions are included in being semiperfect and quasi- π_∞ -regular. In addition, Vámos proved in [30, Lms. 13–14] that all *finitely generated* or torsion-free of finite rank modules rank over a Henselian integral domain¹² have semiperfect endomorphism ring. Results of similar flavor were also obtained in [13], where it is shown that the endomorphism ring of a f.p. (resp. f.g.) module over a semilocal (resp. commutative semilocal) ring is semilocal.

Corollary 8.5. Let S be a commutative LT ring and let R be an S -algebra s.t. R is f.p. and Hausdorff as an S -module. Then:

- (i) If S is quasi- π_∞ -regular, then R is quasi- π_∞ -regular (w.r.t. to some linear ring topology). If moreover S is semiperfect, then so is R .
- (ii) If S satisfies $\mathcal{P} \in \mathcal{P}_{\text{top}} \cap \mathcal{P}_{\text{mor}}$ w.r.t. a given topology which is also first countable, then R satisfies \mathcal{P} .

Proof. We only prove (ii); (i) is similar. By Theorem 7.3, $\text{End}(R_S)$ satisfies \mathcal{P} . For all $r \in R$, define $\hat{r} \in \text{End}(R_S)$ by $\hat{r}(x) = xr$ and observe that $\text{Cent}_{\text{End}(R_S)}(\{\hat{r} \mid r \in R\}) \cong \text{End}(R_R) = R$, hence R has \mathcal{P} by Theorem 7.1. \square

Let C be a commutative local ring. Azumaya proved in [2, Th. 22] that C is Henselian if and only if every commutative C -algebra R with R_C f.g. is semiperfect. This was improved by Vámos to non-commutative C -algebras in which all non-units are integral over C ; see [30, Lm. 12]. Given the previous corollary, Azumaya and Vámos' results suggest that the notions of Henselian and quasi- π_∞ -regular might sometimes coincide. This is verified in the following proposition.

Proposition 8.6. Let R be a rank-1 valuation ring. Then R is Henselian if and only if R is quasi- π_∞ -regular w.r.t. the topology induced by the valuation.

¹² A commutative ring R is called *Henselian* if R is local and Hensel's Lemma applies to R .

Proof. Assume R is quasi- π_∞ -regular. Observe that any free R -module is Hausdorff w.r.t. the standard topology, hence Corollary 8.5(i) implies that any R -algebra A such that A_R is free of finite rank is semiperfect. Thus, by [2, Th. 19], R is Henselian.

Conversely, assume R is Henselian. Denote by ν the (additive) valuation of R . Since ν is of rank 1, we may assume ν take values in $(\mathbb{R}, +)$. For every $\delta \in \mathbb{R}$, let $I_\delta = \{x \in R \mid \nu(x) > \delta\}$. Then $\{M_n(I_\delta) \mid \delta \in [0, \infty)\}$ is a local basis for $M_n(R)$. Let $a \in M_n(R)$. By the Cayley–Hamilton Theorem, a is integral over R , hence $R[a]$ is a f.g. R -module. Let $J = \text{Jac}(R) \cdot R[a]$. Then $J \trianglelefteq R[a]$ and it is well known that $J \subseteq \text{Jac}(R[a])$. The ring $R[a]/J$ is artinian, hence $a + J$ has an associated idempotent $\varepsilon \in E(R[a]/J)$ (i.e. ε satisfies conditions (A)–(C) of Lemma 4.2). By [2, Th. 22], J is idempotent lifting, hence there is $e \in E(R[a])$ such that $e + J = \varepsilon$. Let $f = 1 - e$. Then $a = eae + faf$ (since $R[a]$ is commutative). Furthermore, $eae + J$ is invertible in $\varepsilon(R[a]/J)\varepsilon$, hence eae is invertible in $eR[a]e$ and in particular in $eM_n(R)e$. Next, $(faf)^k \in J \subseteq M_n(\text{Jac}(R)) = M_n(I_0)$ for some $k \in \mathbb{N}$. This means $(faf)^k \in M_n(I_\delta)$ for some $0 < \delta \in \mathbb{R}$, which implies $(faf)^m \xrightarrow{m \rightarrow \infty} 0$. Thus, e satisfies conditions (A),(B) and (C') w.r.t. a and we may conclude that $M_n(R)$ is quasi- π -regular for all $n \in \mathbb{N}$. \square

Using the ideas in the proof of Theorem 8.3, we can also obtain:

Theorem 8.7. Let R be an LT ring and let $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of right R -modules such that B is projective and A and B are Hausdorff. Endow $\text{End}(A)$ and $\text{End}(B)$ with their standard topologies and $\text{End}(C)$ with τ_E and let I_E denote the closure of the zero ideal in $\text{End}(C)$. Then:

- (i) If $\text{End}(A)$ and $\text{End}(B)$ are first countable and satisfy $\mathcal{P} \in \mathcal{P}_{\text{top}}$, then so does $\text{End}(C)/I_E$.
- (ii) If $\text{End}(A)$ and $\text{End}(B)$ are quasi- π -regular, then so does $\text{End}(C)/I_E$.
- (iii) If $\text{End}(A)$ and $\text{End}(B)$ are quasi- π -regular and semiperfect, then $\text{End}(C)$ is semiperfect.

In light of the previous results, one might wonder under what conditions all right f.p. modules over an LT ring are Hausdorff. This is treated in the next section and holds, in particular, for right noetherian pro-semiprimary rings and rank-1 complete valuation rings.

We finish this section by noting that we can take a different approach for complete Hausdorff modules. For the following discussion, a right R -module module M is called *complete* if the natural map $M \rightarrow \varprojlim \{M/MJ\}_{J \in \mathcal{I}_R}$ is an isomorphism.¹³

Proposition 8.8. (i) Let R be a complete first countable Hausdorff LT ring. Then any Hausdorff f.g. right R -module is complete.

(ii) Let $\mathcal{P} \in \mathcal{P}_{\text{disc}}$ and let R be an LT ring such that R/I has \mathcal{P} for all $I \in \mathcal{I}_R$. Let M be a complete right R -module such that M/JM is f.p. as a right R/J -module for all $J \in \mathcal{I}_R$ (e.g. if M is f.p., or if M is f.g. and R is strictly pro-right-artinian). Then $\text{End}(M)$ is pro- \mathcal{P} w.r.t. τ_1^M . If moreover R is semiperfect and M is f.g., then $\text{End}(M)$ is semiperfect.

Proof. (i) This is a well-known argument: Let \mathcal{B} be a countable local basis of R consisting of ideals. Without loss of generality, we may assume $\mathcal{B} = \{J_n\}_{n=1}^\infty$ with $J_1 \supseteq J_2 \supseteq \dots$. Let M be a f.g. Hausdorff R module and let $\{x_1, \dots, x_n\}$ be a set of generators of M . Since M is Hausdorff, it is enough to show that any sum $\sum_{i=1}^\infty m_i$ with $m_i \in MJ_i$ converges in M . Indeed, write $m_i = \sum_{j=1}^n x_j r_{ij}$ with $r_{i1}, \dots, r_{in} \in J_i$. Then $\sum_{i=1}^\infty r_{ij}$ converges in R for all j , hence $\sum_{i=1}^\infty m_i$ converges to $\sum_{j=1}^n x_j r_j$ where $r_j = \sum_{i=1}^\infty r_{ij}$.

(ii) Throughout, J denotes an open ideal of R . We first note that if M is f.p. then there is an exact sequence $R^n \rightarrow R^m \rightarrow M \rightarrow 0$ for some $n, m \in \mathbb{N}$. Tensoring it with R/J we get $(R/J)^n \rightarrow (R/J)^m \rightarrow M/MJ \rightarrow 0$, implying M/J is a f.p. R/J -module. Next, if M is f.g. and R is strictly pro-right-artinian, then M/MJ is a f.g. module over R/J which is right artinian, hence M/J is f.p. over R/J .

¹³ Completeness can also be defined for non-Hausdorff topological abelian groups; see [31].

Now, since M/MJ is f.p. over R/J , $\text{End}(M/MJ)$ satisfies \mathcal{P} by Theorem 7.3. There is a natural map $\text{End}(M) \rightarrow \text{End}(M/MJ)$ whose kernel is $\text{Hom}(M, MJ)$. Assign $\text{End}(M)$ the standard topology. Then since M is complete, $\text{Hom}(M, M) \cong \varprojlim \{\text{End}(M/MJ)\}_{J \in \mathcal{I}}$ as topological rings and therefore, $\text{End}(M)$ is pro- \mathcal{P} .

Finally, assume M is f.g. and R is semiperfect. Then by Proposition 2.2, M admits a projective cover P which is easily seen to be finitely generated. Assume $M = M_1 \oplus \cdots \oplus M_t$. Then each M_i is f.g. and thus has a projective cover P_i . Necessarily $P \cong P_1 \oplus \cdots \oplus P_t$. By Proposition 2.4, $\text{End}(P_R)$ is semiperfect, hence there is a finite upper bound on the cardinality of sets of orthogonal idempotents in it. This means t is bounded and hence, $\text{End}(M)$ cannot contain an infinite set of orthogonal idempotents. By Lemma 5.9(ii), this implies $\text{End}(M)$ is semiperfect. \square

9. LT rings with Hausdorff finitely presented modules

In this section we present sufficient conditions on an LT ring guaranteeing all right f.p. modules are Hausdorff (w.r.t. the standard topology). The discussion leads to an interesting consequence about noetherian pro-semiprimary rings.

We begin by noting a famous result that solves the problem for many noetherian rings with the Jacobson topology. For proof and details, see [25, Th. 3.5.28].

Theorem 9.1 (Jategaonkar–Schelter–Cauchon). *Let R be an almost fully bounded noetherian ring whose primitive images are artinian (e.g. a noetherian PI ring). Assign R the $\text{Jac}(R)$ -adic topology or any stronger linear ring topology. Then any f.g. right R -module is Hausdorff.*

Example 9.2. The assumption that all powers of $\text{Jac}(R)$ are open in the last theorem cannot be dropped: Let R be a Dedekind domain with exactly two prime ideals P and Q . Then R is noetherian, almost fully bounded and any primitive image of R is artinian. Let $n \in \mathbb{N} \cup \{0\}$ and let $\mathcal{B} = \{P^m Q^n \mid m \in \mathbb{N}\}$. Assign R the unique topology with local basis \mathcal{B} . Clearly $\text{Jac}(R)^k = P^k Q^k$ is open for all $1 \leq k \leq n$. However, $\text{Jac}(R)^{n+1} = P^{n+1} Q^{n+1} = \bigcap_{m=1}^{\infty} (P^{n+1} Q^{n+1} + P^m Q^n) = \bigcap_{m=1}^{\infty} (P^{\min\{n+1, m\}} Q^n) = P^{n+1} Q^n$, so $\text{Jac}(R)^{n+1}$ is not closed. In particular, by (*) below, $R/\text{Jac}(R)^{n+1}$ is a f.g. non-Hausdorff R -module.

When considering quasi- π -regular rings, there is actually no point in taking a topology stronger than the Jacobson topology in Theorem 9.1, because for right noetherian rings the latter is the largest topology making the ring quasi- π -regular.

Proposition 9.3. *Let R be an LT semilocal ring and let τ be the topology on R . Assume R is quasi- π -regular w.r.t. τ and R/I is semiprimary for all $I \in \mathcal{I}_R$ (e.g. if R is right noetherian or pro-semiprimary w.r.t. τ). Then R is quasi- π -regular w.r.t. the Jacobson topology and the latter contains τ .*

Proof. We first note that if R is right noetherian, then for all $I \in \mathcal{I}_R$, R/I is right noetherian and π -regular, hence by Remark 2.9, R/I is semiprimary. If R is pro-semiprimary, then R/I is semiprimary for all $I \in \mathcal{I}_R$ by Lemma 8.2.

Let τ_{Jac} denote the Jacobson topology and let $a \in R$. Then a has an associated idempotent e w.r.t. τ . Let $f = 1 - e$ and observe that $\text{Jac}(R)$ is open by Remark 5.12(ii). Then there is $n \in \mathbb{N}$ such that $(faf)^n \in \text{Jac}(R)$ and it follows that $(faf)^n \xrightarrow{n \rightarrow \infty} 0$ w.r.t. τ_{Jac} . Therefore, e is the associated idempotent of a w.r.t. τ_{Jac} , hence R is quasi- π -regular provided we can verify τ_{Jac} is Hausdorff. This holds since $\tau_{\text{Jac}} \supseteq \tau$, by the proof of Corollary 5.17 (which still works under our weaker assumptions). \square

Stronger linear topologies are “better” since they have more Hausdorff modules. Note that the topology of an arbitrary quasi- π_{∞} -regular ring can be stronger than the Jacobson topology. For example, take any non-semiprimary perfect ring R with $\bigcap_{m \in \mathbb{N}} \text{Jac}(R) = \{0\}$ (e.g. $R = \mathbb{Q}[x_1, x_2, x_3, \dots \mid x_m^2 = x_n x_m = 0 \ \forall n > 2m]$) and give it the discrete topology.

The next result will rely on following observation:

(*) Let R be an LT ring. If M is a right R -module and N is a submodule, then $\overline{N/N} = \overline{N}/N$. In particular, M/N is Hausdorff if and only if N is closed.

Indeed, $\overline{N/N} = \bigcap_{J \in \mathcal{I}_R} (M/N)J = \bigcap_{J \in \mathcal{I}_R} (MJ + M)/N = (\bigcap_{J \in \mathcal{I}_R} (MJ + M))/N = \overline{N}/N$. We will also need the following theorem. For proof, see [7, §7.4].

Theorem 9.4. Let $\{X_i, f_{ij}\}$ be an I -indexed inverse system of non-empty sets. Assume that for each $i \in I$ we are given a family of subsets $T_i \subseteq P(X_i)$ such that for all $i \leq j$ in I we have:

- (a) $X_i \in T_i$ and T_i is closed under (arbitrary large) intersection.
- (b) Finite Intersection Property: If $L \subseteq T_i$ is such that the intersection of finitely many of the elements of L is non-empty, then $\bigcap_{A \in L} A \neq \emptyset$.
- (c) For all $A \in T_j$, $f_{ij}(A) \in T_i$.
- (d) For all $x \in X_i$, $f_{ij}^{-1}(x) \in T_j$.

Then $\varprojlim \{X_i\}_{i \in I}$ is non-empty.¹⁴

Lemma 9.5. Let R be a ring and let M be a right R -module. Let $\{M_i\}_{i \in I}$ be a family of submodules of M and let $\{x_i\}_{i \in I}$ be elements of M . Then $\bigcap_{i \in I} (x_i + M_i)$ is either empty or a coset of $\bigcap_{i \in I} M_i$.

Proof. This is straightforward. \square

Theorem 9.6. Let R be strictly pro-right-artinian. Then any f.g. submodule of a Hausdorff right R -module is closed.

Proof. Let \mathcal{B} be a local basis of ideals such that R/J is right artinian for all $J \in \mathcal{B}$. Assume M is a Hausdorff right R -module, let $m_1, \dots, m_k \in M$ and $N = \sum_{i=1}^k m_i R$. We will show that $m \in \overline{N}$ implies $m \in N$.

Let $m \in \overline{N}$. For every $J \in \mathcal{B}$ define

$$X_J = \left\{ (a_1, \dots, a_k) \in (R/J)^k : \sum_i (m_i + MJ)a_i = m + MJ \right\}.$$

Observe that $m \in \overline{N} = \bigcap_{J \in \mathcal{B}} (N + MJ)$, hence for all $J \in \mathcal{B}$ there are $b_1, \dots, b_k \in R$ and $z \in MJ$ such that $\sum m_i b_i = m + z$, implying $X_J \neq \emptyset$. For all $J \subseteq I$ in \mathcal{B} , let f_{IJ} denote the map from $(R/J)^k$ to $(R/I)^k$ given by sending $(b_1 + J, \dots, b_k + J)$ to $(b_1 + I, \dots, b_k + I)$. Then $f_{IJ}(X_J) \subseteq X_I$. It is easy to check that $\{X_I, f_{IJ}|_{X_J}\}$ is an inverse system of sets.

For all $J \in \mathcal{B}$, define T_J to be the set consisting of the empty set together with all cosets of (right) R -submodules of $(R/J)^k$ contained in X_J . We claim that conditions (a)–(d) of Theorem 9.4 hold. Indeed, X_J is easily seen to be a coset of a submodule of $(R/J)^k$, thus $X_J \in T_J$. In addition, by Lemma 9.5, T_J is closed under intersection, so (a) holds. Since R/J is right artinian, so is $(R/J)^k$ (as a right R -module). Lemma 9.5 then implies that cosets of submodules of $(R/J)^k$ satisfy the DCC, hence (b) holds. Conditions (c) and (d) are straightforward. Therefore, we may apply Theorem 9.4 to deduce that $\varprojlim X_J$ is non-empty.

¹⁴ This can be compared to the following topological fact: An inverse limit of an inverse system of non-empty Hausdorff compact topological spaces is non-empty and compact.

Let $x \in \varprojlim \{X_J\}_{J \in \mathcal{B}}$. Then x consists of tuples $\{(a_1^{(J)}, \dots, a_k^{(J)}) \in (R/J)^k\}_{J \in \mathcal{B}}$ that are compatible with the maps $\{f_{IJ}\}$. As R is complete, there are $b_1, \dots, b_k \in R$ such that $a_i^{(J)} = b_i + J$ for all $1 \leq i \leq k$ and $J \in \mathcal{B}$. It follows that $m - \sum_i m_i b_i \in \bigcap_{J \in \mathcal{B}} MJ$. As M is Hausdorff, the right-hand side is $\{0\}$, so $m = \sum_i m_i b_i \in N$. \square

Remark 9.7. Theorem 9.6 and its consequences actually hold for the larger class of strictly pro-right-finitely-cogenerated rings. A module M over a ring R is called *finitely cogenerated*¹⁵ (abbrev.: f.cog.) if its submodules satisfy the Finite Intersection Property (condition (b) in Theorem 9.4). This is equivalent to $\text{soc}(M)$ being f.g. and essential in M (see [18, Pr. 19.1]). A ring R called *right f.cog.* if R_R is finitely cogenerated. (For example, any right pseudo-Frobenius ring is right f.cog.) Among the examples of strictly pro-right-finitely-cogenerated rings are complete rank-1 valuation rings. Indeed, if $\nu: R \rightarrow \mathbb{R}$ is an (additive) valuation, and R is complete w.r.t. ν , then $R = \varprojlim \{R/\{x \in R \mid \nu(x) > n\}\}_{n \in \mathbb{N}}$. For a detailed discussion about f.cog. modules and rings, see [29] and [18, §19].

Notice that a complete semilocal ring is strictly pro-right-artinian if and only if its Jacobson radical is f.g. as a right module. The latter condition is commonly used when studying complete semilocal rings (e.g. [24]). In particular, Hinohara proved Theorem 9.6 for complete semilocal rings satisfying it [16, Lm. 3]. (Other authors usually assume the ring is right noetherian.) By $(*)$ we now get:

Corollary 9.8. *Let R be a strictly pro-right-artinian ring. Then any f.p. right R -module is Hausdorff.*

We can now prove that under mild assumptions, strictly pro-right-artinian rings are complete semilocal.

Corollary 9.9. *Let R be a strictly pro-right-artinian ring. If $J \subseteq \text{Jac}(R)$ is an ideal that is f.g. as a right ideal, then R is complete in the J -adic topology (i.e. $R \cong \varprojlim \{R/J^n\}_{n \in \mathbb{N}}$). If moreover R/J is right artinian, then the topology on R is the Jacobson topology! In particular, if $\text{Jac}(R)$ is f.g. as a right ideal, then the topology on R is the Jacobson topology and R is complete semilocal.*

Proof. Let \mathcal{B} be a local basis of ideals of R such that R/I is right artinian for all $I \in \mathcal{B}$. We identify R with its natural copy in $\prod_{I \in \mathcal{B}} R/I$. Since J is f.g. as a right ideal, then so are its powers. Therefore, by Theorem 9.6, J^n is closed for all $n \in \mathbb{N}$.

Let φ denote the standard map from R to $\varprojlim \{R/J^n\}_{n \in \mathbb{N}}$. Define a map $\psi: \varprojlim \{R/J^n\}_{n \in \mathbb{N}} \rightarrow R$ as follows: Let $r \in \varprojlim \{R/J^n\}_{n \in \mathbb{N}}$ and let r_n denote the image of r in R/J^n . By Corollary 5.17, for all $I \in \mathcal{B}$, there is $n \in \mathbb{N}$ (depending on I) such that $J^n \subseteq I$. Let r_I denote the image of r_n in R/I . It is easy to check that r_I is independent of n and that $\hat{r} := (r_I)_{I \in \mathcal{B}} \in R$. Define $\psi(r) = \hat{r}$.

It is straightforward to check that $\psi \circ \varphi = \text{id}$. Therefore, we are done if we show that ψ is injective. Let $y \in \ker \psi$ and let $y_n + J^n$ be the image of y in R/J^n . Then for all $I \in \mathcal{B}$, $J^n \subseteq I$ implies $y_n \in I$. This means $y_n \in \bigcap_{J^n \subseteq I \in \mathcal{B}} I = \overline{J^n} = J^n$, so $y_n + J^n = 0 + J^n$ for all $n \in \mathbb{N}$, hence $y = 0$.

Now assume R/J is right artinian. Then $\text{Jac}(R)^k \subseteq J \subseteq \text{Jac}(R)$ for some $k \in \mathbb{N}$, hence the Jacobson topology and the J -adic topology coincide. By Proposition 9.3, the topology on R is contained in the Jacobson topology, so we only need to show the converse. Let $n \in \mathbb{N}$. It is enough to show that J^n is open. Indeed, since J_R is f.g., then so is $(J^i/J^{i+1})_R$ ($i \geq 0$). As $(R/J)_R$ has finite length, $(J^i/J^{i+1})_R$ has finite length. Thus, $(R/J^n)_R$ have finite length as well. Since J^n is closed, J^n is an intersection of open ideals. As $(R/J^n)_R$ is of finite length, J^n is the intersection of finitely many of those ideals, hence open. \square

Corollary 9.10. *Let R be a right noetherian pro- π -regular ring. Then the topology on R is the Jacobson topology, R is strictly pro-right-artinian w.r.t. it and any right ideal of R is closed. In particular, R is semilocal complete.*

¹⁵ Other names used in the literature are “co-finitely generated”, “finitely embedded” or “essentially artinian”.

Proof. By Lemma 8.2, R/I is π -regular for all $I \in \mathcal{I}_R$, hence Remark 2.9 implies R/I is right artinian for all $I \in \mathcal{I}_R$ (since R/I is right noetherian). Therefore, R is pro-right-artinian, with $\text{Jac}(R)_R$ finitely generated. Now apply the previous corollary. \square

The next example demonstrates that Theorem 9.6 fails for pro-artinian rings (and in particular for pro-semiprimary rings). It also implies that there are pro-artinian rings that are not strictly pro-right-artinian.

Example 9.11. Let $S = \mathbb{Q}(x)[t \mid t^3 = 0]$. For all $n \in \mathbb{N}$ define $R_n = \mathbb{Q}(x^{2^n}) + \mathbb{Q}(x)t + \mathbb{Q}(x)t^2 \subseteq S$ and $I_n = \mathbb{Q}(x^{2^n})t^2 \subseteq S$. Then R_n is an artinian ring and $I_n \triangleleft R_n$. For $n \leq m$ define a map $f_{nm} : R_m/I_m \rightarrow R_n/I_n$ by $f_{nm}(x + I_m) = x + I_n$. Then $\{R_n/I_n, f_{nm}\}$ is an inverse system of artinian rings. Let $R = \varprojlim \{R_n/I_n\}_{n \in \mathbb{N}}$. Then R can be identified with $\mathbb{Q} + \mathbb{Q}(x)t + Vt^2$ where V is the \mathbb{Q} -vector space $\varprojlim \{\mathbb{Q}(x)/\mathbb{Q}(x^{2^n})\}_{n \in \mathbb{N}}$ (R does not embed in S). Observe that $\mathbb{Q}(x)$ is dense in V , but $\mathbb{Q}(x) \neq V$ since V is not countable (it contains a copy of all power series $\sum a_n x^{2^n} \in \mathbb{Q}[[x]]$). Therefore, the ideal $tR = \mathbb{Q}t + \mathbb{Q}(x)t^2$ is not closed in R and by $(*)$, R/tR is a non-Hausdorff f.p. module. We also note that $\text{Jac}(R)^2 = \mathbb{Q}(x)t^2$ is not closed (but $\text{Jac}(R)$ must be closed by Proposition 5.11).

We conclude by specializing the results of the previous section to first countable strictly pro-right-artinian rings. (We are guaranteed that all f.p. modules are Hausdorff in this case). By Corollary 9.10, this family include all right noetherian pro-semiprimary rings. More general statements can be obtained by applying Remark 9.7.

Corollary 9.12. (i) Let R be a first countable pro-right-artinian ring and let M be a f.p. right R -module. Then $\text{End}(M_R)$ is pro-semiprimary and first countable (w.r.t. τ_2^M). If R is semiperfect (e.g. if R is right noetherian), then $\text{End}(M_R)$ is semiperfect.

(ii) Let S be a commutative first countable pro-right-artinian ring and let R be an S -algebra s.t. R is f.p. as an S -module. Then R is pro-semiprimary (w.r.t. some topology). If S is semiperfect (e.g. if S is right noetherian), then R is semiperfect.

10. Further remarks

It is likely that the theory of semi-invariant subrings developed in Section 5 can be extended to right linearly topologized rings, i.e. topological rings having a local basis consisting of right ideals. This actually has the following remarkable implication (compare with Corollary 7.4 and Corollary 8.5):

Conjecture 10.1. Let $S \in \mathcal{L}\mathcal{T}_2$ be a semiperfect quasi- π_∞ -regular ring and let $\varphi : S \rightarrow R$ be a ring homomorphism. Assume that:

- (a) When considered as a right S -module via φ , R is f.p. and Hausdorff.
- (b) For all $r \in R$ and $I \in \mathcal{I}_S$, there is $J \in \mathcal{I}_S$ such that $R\varphi(J)r \subseteq R\varphi(I)$.¹⁶

Then R is semiperfect and quasi- π_∞ -regular (w.r.t. some topology).

The proof should be as follows: For any right S -module M , let W denote the ring of continuous \mathbb{Z} -homomorphisms from M to itself. Then W can be made into a right LT ring by taking $\{B(J) \mid J \in \mathcal{I}_S\}$ as a local basis where $B(J) = \{f \in W : \text{im } f \subseteq MJ\}$ (this is the topology of uniform convergence).¹⁷

¹⁶ This equivalent to saying that the topology on R spanned by cosets of the left ideals $\{R\varphi(I) \mid I \in \mathcal{I}_S\}$ is a ring topology; see [31, §3].

¹⁷ Caution: Not any filter base of right ideals gives rise to a ring topology. By [31, §3], we need to check that for all $f \in W$ and $I \in \mathcal{I}_S$ there is $J \in \mathcal{I}_S$ such that $fB(J) \subseteq B(I)$. Indeed, we can take any J with $f(MJ) \subseteq MI$ and such J exists since f is continuous.

Clearly W contains $\text{End}(M_S)$ as a topological ring (endow $\text{End}(M_S)$ with τ_1^M). Now take $M = R$ (where R is viewed as a right S -module via φ). Then condition (a) implies $\text{End}(R_S)$ is semiperfect and quasi- π_∞ -regular w.r.t. τ_1^R (Theorem 8.3). Condition (b) implies that for all $r \in R$, the map $\hat{r}: x \mapsto xr$ from R to itself is continuous and hence lie in W . Since we assume the results of Section 5 extend to right LT rings, $R \cong \text{End}(R_R) = \text{Cent}_{\text{End}(R_S)}(\{\hat{r} \mid r \in R\})$ is a T-semi-invariant subring of $\text{End}(R_S)$, so R is semiperfect and quasi- π_∞ -regular.

Examples of rings satisfying conditions (a) and (b) can be produced by taking R to be: (1) a twisted group algebra $S^\alpha G$ where G is a finite group and $\alpha: G \rightarrow \text{Aut}_c(G)$ is a group homomorphism or (2) a “crossed product”, i.e. $R = \text{CrossProd}(S, \psi, G)$ where S is commutative, G is finite and act on S via continuous automorphisms and $\psi \in H^2(G, S^\times)$. (Further examples can be produced by taking quotients). However, we can show directly that the conjecture holds in these special cases. Indeed, that G is finite implies $\mathcal{B} = \{\bigcap_{g \in G} g(I) \mid I \in \mathcal{I}_R\}$ is a local basis of S and we have $RJ \subseteq JR$ for all $J \in \mathcal{B}$. For any right R -module M let $W' = \{f \in W: f(MJ) \subseteq MJ \forall J \in \mathcal{B}\}$ (with W as in the previous paragraph). Then W' is a linearly topologized ring w.r.t. the topology induced from W (as seen by taking the local basis $\{W' \cap B(J) \mid J \in \mathcal{B}\}$). In addition, when $M = R_S$, \hat{r} of the previous paragraph lies in W' (since $RJ \subseteq JR$ for all $J \in \mathcal{B}$). Therefore, repeating the argument of the last paragraph with W' instead of W , we get that R is semiperfect and quasi- π_∞ -regular.

The author could not find nor contradict the existence of the following:

- (1) A pro-semiprimary ring that is not complete semilocal (i.e. complete w.r.t. its Jacobson topology).
- (2) A complete semilocal ring, endowed with the Jacobson topology, with a non-Hausdorff f.p. module.

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Appendix A. When do $\tau_1^M, \tau_2^M, \dots$ coincide?

This appendix is dedicated to the question of when the topology obtained from a resolution is the standard topology. For that purpose we briefly recall the Artin–Rees property for ideals. For details and proofs, see [25, §3.5D].

Let R be a right noetherian ring. An ideal $I \trianglelefteq R$ is said to satisfy the *Artin–Rees property* (abbreviated: AR-property) if for any right ideal $A \leq R$ there is $n \in \mathbb{N}$ such that $I^n \cap A \subseteq AI$. This is well known to imply that for any f.g. right R -module M and a submodule N , there is $n \in \mathbb{N}$ such that $MI^n \cap N \subseteq NI$. For example, by [25, p. 462, Ex. 19], every *polycentral* ideal (e.g. an ideal generated by central elements) satisfies the AR-property. In addition, if R is *almost bounded* (e.g. a PI ring), then all ideals of R are satisfy the AR-property.

Now let R be any LT ring. A right R -module M is said to satisfy the *topological Artin–Rees property* (abbreviated: TAR-property) if for any submodule $N \subseteq M$ and any $I \in \mathcal{I}_R$ there is $J \in \mathcal{I}_R$ such that $MJ \cap N \subseteq NI$. (Equivalently, the induced topology and the standard topology coincide for any submodule of M .) For example, if R is right noetherian, $J \trianglelefteq R$ and R is given the J -adic topology, then all f.g. right R -modules satisfy the TAR-property if and only if J satisfies the AR-property.

Proposition A.1. *Let R be an LT ring, let M be a right R -module and let $P: P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow P_{-1} = M \rightarrow 0$ be a projective resolution of M . Assume that that P_0, \dots, P_{n-1} have the TAR-property. Then τ_P is the natural topology on $\text{End}(M)$.*

Proof. Denote by d_i the map $P_i \rightarrow P_{i-1}$ and let $B_i = \ker d_i$. We will prove that for all $I \in \mathcal{I}_R$ there is $J \in \mathcal{I}_R$ such that $\text{Hom}(M, MJ) \subseteq B(I, P)$. Given $I \in \mathcal{I}_R$ we define a sequence of open ideals $I_{n-1}, I_{n-2}, \dots, I_{-1}$ as follows: Let $I_{n-1} = I$. Given I_i , take I_{i-1} to be an open ideal such that $P_{i-1}I_{i-1} \cap B_{i-1} \subseteq B_{i-1}I_i$ and $I_{i-1} \subseteq I_i$ (the existence of I_{i-1} follows from the TAR-property). We claim that $\text{Hom}(M, MI_{-1}) \subseteq B(I, P)$. To see this, let $f_{-1} \in \text{Hom}(M, MI_{-1})$ and assume we have constructed maps $f_i \in \text{Hom}(P_i, P_i I_i)$ for all $-1 \leq i < k$ such that $d_i f_i = f_{i-1} d_i$. Then it is enough to show there is $f_k \in \text{Hom}(P_k, P_k I_k)$ such that $d_k f_k = f_{k-1} d_k$. The argument to follow is illustrated in the following diagram:

$$\begin{array}{ccccccc}
 P_k & \xrightarrow{d_k} & B_{k-1} & \hookrightarrow & P_{k-1} & \xrightarrow{d_{k-1}} & \dots \\
 \vdots & & \downarrow f_{k-1} & & \downarrow f_{k-1} & & \\
 & & P_{k-1}I_{k-1} \cap B_{k-1} & \hookrightarrow & P_{k-1}I_{k-1} & \xrightarrow{d_{k-1}} & \dots \\
 & & \downarrow & & & & \\
 P_k I_k & \xrightarrow{d_k} & B_{k-1} I_k & & & &
 \end{array}$$

That $d_{k-1} f_{k-1} = f_{k-2} d_{k-1}$ implies $f_{k-1}(B_{k-1}) \subseteq B_{k-1}$. As $\text{im } f_{k-1} \subseteq P_{k-1}I_{k-1}$, we get that $f_{k-1}(B_{k-1}) \subseteq P_{k-1}I_{k-1} \cap B_{k-1} \subseteq B_{k-1}I_k$ (by the definition of I_{k-1}). Since the map $P_k I_k \rightarrow B_{k-1}I_k$ is onto (because $\text{im}(d_k) = B_{k-1}$), we can lift $f_{k-1} d_k : P_k \rightarrow B_{k-1}I_k$ to module homomorphism $f_k : P_k \rightarrow P_k I_k$, as required. \square

Remark A.2. The proof still works if we replace the assumption that P_{n-1} is projective with d_{n-1} is injective.

Corollary A.3. Let R be an LT right noetherian ring admitting local basis of ideals \mathcal{B} such that:

- (1) all ideals in \mathcal{B} have the AR-property (e.g. if all ideals in \mathcal{B} are generated by central elements or if R is PI), and
- (2) all powers of ideals in \mathcal{B} are open. Then $\tau_1^M = \tau_2^M = \dots$ for any f.g. right R -module M .

Proof. Let $n \in \mathbb{N}$. Since R is right noetherian, any f.g. R -module admits a resolution of length n consisting of f.g. projective modules. The assumptions (1) and (2) are easily seen to imply that any f.g. R -module satisfies the TAR-property. Therefore, by Proposition A.1, τ_n^M is the natural topology on $\text{End}(M)$. \square

Example A.4. Condition (2) in Corollary A.3 is essential even when all ideals of R have the AR-property: Assign \mathbb{Z} the unique topology with local basis $\mathcal{B} = \{2 \cdot 3^n \mathbb{Z} \mid n \geq 0\}$ and let $M = \mathbb{Z}/4 \times \mathbb{Z}/2$. Since $MI = 2M$ for all $I \in \mathcal{B}$, the standard topology on $\text{End}(M)$ is obtained from the local basis $\{\text{Hom}(M, 2M)\}$. (M is not Hausdorff). Let $I = 2\mathbb{Z} \in \mathcal{B}$ and consider the projective resolution

$$P : 4\mathbb{Z} \times 2\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow M \rightarrow 0.$$

Define $f_{-1} : M \rightarrow MI = 2M$ by $f(x + 4\mathbb{Z}, y + 2\mathbb{Z}) = (2y + 4\mathbb{Z}, 0)$. Then any lifting $f_0 \in \text{End}(\mathbb{Z} \times \mathbb{Z})$ of f_{-1} must satisfy $f_0(0, 1) = (4x + 2, 2y)$ for some $x, y \in \mathbb{Z}$. This means that any lifting $f_1 \in \text{End}(4\mathbb{Z} \times 2\mathbb{Z})$ of f_0 (there is only one such lifting) satisfies $f_1(0, 2) = (8x + 4, 4y) \notin 8\mathbb{Z} \times 4\mathbb{Z} = (4\mathbb{Z} \times 2\mathbb{Z})I$. Therefore, $f_1 \notin \text{Hom}(4\mathbb{Z} \times 2\mathbb{Z}, (4\mathbb{Z} \times 2\mathbb{Z})I)$, implying $f_{-1} \notin B(P, I)$. But this means that $B(P, I) \subsetneq \text{Hom}(M, 2M)$, hence $\tau_2^M \neq \tau_1^M$.

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