Journal of Algebra 430 (2015) 238–259

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Epimorphisms of pseudo-quadratic polar spaces

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ARTICLE INFO

Article history:
Received 31 October 2012
Available online xxxx
Communicated by Gernot Stroth

MSC:
20E42
51E12
51E24

Keywords:
Generalized polygon
Spherical building
Epimorphism

ABSTRACT

We classify the epimorphisms of the buildings $\mathsf{BC}_l(K, K_0, \sigma, L, q)$, $l \geq 2$, of pseudo-quadratic form type. This completes the classification of epimorphisms of irreducible spherical Moufang buildings of rank at least two.

1. Introduction

The aim of this paper is to complete the classification of epimorphisms of irreducible spherical Moufang buildings of rank at least two. For projective planes and spaces defined over a skew field or octonion division algebra K such a classification is given by the work of André [1], Faulkner and Ferrar [3] and Skornjakov [5]. It is shown there that such epimorphisms essentially correspond with the total subrings of K, i.e. subrings $R \subset K$

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http://dx.doi.org/10.1016/j.jalgebra.2015.01.028

such that $K = R \cup (R \setminus \{0\})^{-1}$. In [6] the second author derives some of the structure theory of epimorphisms of irreducible spherical Moufang buildings of rank at least two and uses this to show that when such a building is defined over a field (for a suitable definition), then these epimorphisms are closely related with affine buildings (and their non-discrete generalizations).

In view of these results, the only untreated case is that of the buildings $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ ($l \geq 2$) of pseudo-quadratic form type. We include the buildings $\mathsf{C}_l(K, K_0, \sigma)$ of involutory type in this class, which corresponds to the case $L = 0$. The main difference with the cases handled in [6] is that a total subring of a field always corresponds to a valuation of this field, while this is not true for skew fields in general. As a consequence one can no longer apply the rich theory of affine buildings, meaning that we have to construct the epimorphisms in a different, more ad hoc manner.

The precise statement of our classification can be found in Section 3. We note that in this paper we only consider type-preserving epimorphisms between (thick) buildings.

The first author is supported by the German Research Foundation (DFG), the second author by the Fund for Scientific Research – Flanders (FWO – Vlaanderen).

2. Polar spaces of pseudo-quadratic form type

In this section we define the polar spaces of interest in this paper. Our approach is based on [7, (16.5)]. Let K be a skew field and σ an involution of K, meaning σ is an anti-automorphism (so $(ab)^\sigma = b^\sigma a^\sigma$) with $\sigma^2 = \mathrm{id}$. Let

$$K_\sigma = \{a + a^\sigma | a \in K\},$$
$$K^\sigma = \{a | a \in K, a^\sigma = a\}.$$

Choose a $K_\sigma \subset K_0 \subset K^\sigma$ containing the element 1, such that for all $t \in K$ we have $t^\sigma K_0 t = K_0$. Such a set is called an *involutory set*. If the characteristic of K is different from 2, then $K_\sigma = K_0 = K^\sigma$. Let L be a right vector space over K. A map $f : L \times L \to K$ is a *skew-hermitian sesquilinear form* on L with respect to σ, if $f(a, b)^\sigma = -f(b, a)$ and $f(at, bu) = t^\sigma f(a, b)u$ for all $a, b \in L$ and $t, u \in K$. A map $q : L \to K$ is a *skew-hermitian pseudo-quadratic form* on L with respect to σ if f on L is a skew-hermitian sesquilinear form with respect to σ, such that the following two conditions are satisfied for all $a, b \in L$ and $t \in K$:

- $q(a + b) \equiv q(a) + q(b) + f(a, b) \bmod K_0$,
- $q(at) \equiv t^\sigma q(a)t \bmod K_0$.

If one moreover has that $q(a) \in K_0$ only if $a = 0$, then we say that q is *anisotropic*. If all of this is satisfied we say that the quintuple (K, K_0, σ, L, q) is an *anisotropic skew-hermitian pseudo-quadratic space*.

Remark 2.1. We will often omit the adjective "skew-hermitian" as we will not take other pseudo-quadratic spaces into consideration.

We are now able to define the rank l polar space $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ where $l \geq 2$ is an integer, and (K, K_0, σ, L, q) an anisotropic pseudo-quadratic space. Let X denote the right vector space $L \oplus K^{2l}$. The function

$$q_X : (v|a_1, \ldots, a_{2l}) \mapsto q(v) + a_1^\sigma a_2 + \cdots + a_{2l-1}^\sigma a_{2l}$$

with $(v|a_1, \ldots, a_{2l}) \in X$, is a pseudo-quadratic form on X. The associated skew-hermitian f_X is defined by $f_X((v|a_1, \ldots, a_{2l}), (w|b_1, \ldots, b_{2l})) := f(v, w) + b_1^\sigma a_2 - b_2^\sigma a_1 + \cdots + b_{2l-1}^\sigma a_{2l} - b_{2l}^\sigma a_{2l-1}$, where f is the sesquilinear form associated to q.

A subspace S of the vector space X is *singular* if $x \in S$ implies $q_X(x) \in K_0$. The polar space is now formed by the set of singular subspaces. The points will be the one-dimensional subspaces, the lines the two-dimensional subspaces, etc. The building of type C_l associated to this polar space is the flag complex of this incidence geometry.

3. Statement of the main result

We will see in the main theorem below that the total subrings essentially determine the type-preserving epimorphisms of buildings of pseudo-quadratic form type. Here a *total subring* of a skew field K is a subring R of K such that $K = R \cup (R \setminus \{0\})^{-1}$.

We denote the invertible elements of R by $R^\times$. Its complement $m := R \setminus R^\times$ is the unique maximal (two-sided) ideal of R (see for example [3, §2]). A direct corollary is that the quotient R/m is a skew field. We call this the *residue skew field* and denote it by K_R.

We will now state the main result.

Main result 1. *Let (K, K_0, σ, L, q) be an anisotropic skew-hermitian pseudo-quadratic space (where L is allowed to be 0). Every type-preserving epimorphism of the building $\mathsf{BC}_l(K, K_0, \sigma, L, q)$, $l \geq 2$, is completely determined (up to isomorphisms) by a total subring R of the skew field K and a left coset of $R^\times$ in the multiplicative group K^* satisfying the following three conditions.*

(C1) *The anti-automorphism $a \mapsto a^{\sigma s}$ of K stabilizes R,*
(C2) *$(u, t), (w, r) \in T : t, r \in sR \Rightarrow f(u, w) \in sR$,*
(C3) *$(u, t), (w, r) \in T : t \in sR, r \in sm \Rightarrow f(u, w) \in sm$,*

where s is an element of the left coset of $R^\times$, f the skew-hermitian form associated to q and m the unique maximal two-sided ideal of R. Conversely if R is a total subring of the skew field K and $sR^\times$ a left coset of $R^\times$ satisfying these three conditions, then there exists a type-preserving epimorphism of the polar space $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ determined exactly by this total subring and left coset.

The proof is split into two parts as described in Section 5.

Remark 3.1. We define a type-preserving epimorphism of a building as a surjective map on the set of chambers preserving s-equivalence for $s \in W$ (see [6, §2.1]). Another way of defining epimorphisms of polar spaces would be to consider surjective maps on the set of points mapping lines into lines. This does not pose a problem as these two notions are equivalent. One direction follows from [6, Lem. 4.2], the other from Lemmas 5.21, 5.22, 5.24 and Proposition 5.25.

4. Auxiliary results

This section gathers helpful results on spherical buildings, epimorphisms and polar spaces.

We start by giving root group sequences which describe the rank two Moufang spherical buildings, which are also known as *Moufang polygons*, which appear in buildings of pseudo-quadratic form type. Root group labelings then describe these buildings $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ $(l \geq 2)$. In Section 4.4 we show how the direct construction in Section 2 relates to the root group labeling. In Section 4.5 we summarize the results from [6] used in this paper. In particular we describe the interplay between epimorphisms and root group labelings.

4.1. The root group sequence of $\mathsf{A}_2(K)$

Let K be a skew field. Let U_i $(i \in \{1, 2, 3\})$ be groups parametrized by isomorphisms x_i from the additive group of K to U_i. The only non-trivial commutator relation is given by

$$[x_1(s), x_3(t)] = x_2(st)$$

for $s, t \in K$. This defines a root group sequence $\Theta_{\mathsf{A}_2(K)}$ (see [7, (16.1)]).

We also list the following identity (from [7, (32.5)]) which one will need in order to apply Lemma 4.2.

$$x_2(u)^{\mu(x_1(t))} = x_3(t^{-1}u) \tag{1}$$

In what follows we will work with $\mathsf{A}_2(K^{\mathsf{op}})$. The *opposite* skew field K^{op} is defined as the field with the same underlying set as K but with multiplication given by $a * b = ba$ (with $a, b \in K$).

4.2. The root group sequence of $\mathsf{BC}_2(K, K_0, \sigma, L, q)$

We use the notations from Section 2. Let (K, K_0, σ, L, q) be an anisotropic pseudo-quadratic space. Let T be the elements (w, t) in $L \times K$ such that $q(w) - t \in K_0$.

One derives that if $(w,t) \in T$ then $f(w,w) = t - t^\sigma$, where f is the sesquilinear form associated to q. The set T can be made into a group with multiplication $(w,t) \cdot (v,r) = (w+v, t+r+f(v,w))$ and inverse $(w,t)^{-1} = (-w, -t^\sigma)$. For proofs see [7, (11.24), (11.19)].

Let U_i ($i \in \{1,2,3,4\}$) be groups parametrized by the group T in case i is odd and by the additive group of K in case i is even, both via isomorphisms x_i. The non-trivial commutator relations are given by:

$$\begin{aligned} [x_1(w,t), x_3(v,r)^{-1}] &= x_2(f(w,v)), \\ [x_2(k), x_4(a)^{-1}] &= x_3(0, k^\sigma a + a^\sigma k), \\ [x_1(w,t), x_4(k)^{-1}] &= x_2(tk)x_3(wk, k^\sigma tk), \end{aligned}$$

for $(w,t),(v,r) \in T$ and $k,a \in K$. These relations define a root group sequence $\Theta_{\mathsf{BC}_2(K,K_0,\sigma,L,q)}$.

We end by giving the following equations from [7, (32.9)].

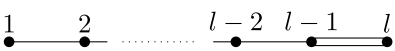
$$x_1(w,t)^{\mu(x_4(k))} = x_3(-wk, k^\sigma tk) \tag{2}$$

$$x_2(k)^{\mu(x_4(a))} = x_2(-a^{-\sigma}k^\sigma a) \tag{3}$$

$$x_4(a)^{\mu(x_1(w,t))} = x_2(ta) \tag{4}$$

4.3. *The root group labeling of* $\mathsf{BC}_l(K,K_0,\sigma,L,q)$

In this section we describe the root group labeling (u,Θ,θ) of the building $\mathsf{BC}_l(K,K_0,\sigma,L,q)$, following [8, (12.12), (12.16)]. We will not give every detail of it, only the parts relevant for our proof. Let Π be the following Coxeter diagram with numbered vertices.



For $i \in \{1,2,\dots,l-1\}$, let $u(i)$ be isomorphic with the additive group of the skew field K. We set $u(l)$ to be isomorphic with the group T. We parametrize the groups $u(i)$ ($i \in \{1,2,\dots,l\}$) by isomorphisms y_i from K or T (where applicable) to $u(i)$.

Let $\Theta_{i,i+1} = \Theta_{\mathsf{A}_2(K^{\mathrm{op}})}$ for $i \in \{1,2,\dots,l-2\}$ and $\Theta_{l,l-1} = \Theta_{\mathsf{BC}_2(K,K_0,\sigma,L,q)}$. This defines the root group labeling (u,Θ,θ) of the building $\mathsf{BC}_l(K,K_0,\sigma,L,q)$.

4.4. *Realization of the root group labeling*

We will now show how the direct construction of $\mathsf{BC}_l(K,K_0,\sigma,L,q)$ given in Section 2 realizes the root group labeling given in Section 4.3 in the sense of [8, (12.10)–(12.11)]. We do this by showing how the groups $u(i)$ from the root group labeling act on the vector space X.

$$
\begin{aligned}
(v|a_1,\ldots,a_{2l})^{y_1(k)} &= (v|a_1,\ldots,a_{2l-3}+ka_{2l-1},a_{2l-2},a_{2l-1},a_{2l}-k^\sigma a_{2l-2}),\\
&\ldots\\
(v|a_1,\ldots,a_{2l})^{y_{l-1}(k)} &= (v|a_1+ka_3,a_2,a_3,a_4-k^\sigma a_2,\ldots,a_{2l}),\\
(v|a_1,\ldots,a_{2l})^{y_l(w,t)} &= (v+wa_1|a_1,a_2-ta_1-f(w,v),a_3,\ldots,a_{2l}),
\end{aligned}
$$

where $k\in K$ and $(w,t)\in T$. The omitted coordinates are left invariant. These maps fix the chamber consisting of the subspaces

$$
\begin{aligned}
&\langle(0|0,\ldots,0,1)\rangle,\\
&\langle(0|0,\ldots,0,1),(0|0,\ldots,0,1,0,0)\rangle,\\
&\langle(0|0,\ldots,0,1),(0|0,\ldots,0,1,0,0),(0|0,\ldots,0,1,0,0,0,0)\rangle,\\
&\quad\ldots
\end{aligned}
$$

Remark 4.1. Note that the maps of the form

$$
\begin{aligned}
\zeta_i(m):(v|\ldots,a_{2i-2},a_{2i-1},a_{2i},a_{2i+1},\ldots)&\mapsto\\
&(v|\ldots,a_{2i-2},ma_{2i-1},m^{-\sigma}a_{2i},a_{2i+1},\ldots)
\end{aligned}
$$

with $m\in K^*$ and $i\in\{1,\ldots,l\}$ induce automorphisms of the polar space. This map ζ_i normalizes each of the groups $u(j)$ ($j\in\{1,\ldots,l\}$) and acts on the groups $u(j)$ ($j\in\{1,\ldots,l-1\}$) as follows.

$$
y_j(k)^{\zeta_i(m)}=\begin{cases} y_j(mk) & \text{if } j=l-i\\ y_j(km^{-1}) & \text{if } j=l-i+1\\ y_j(k) & \text{otherwise.}\end{cases}
$$

By combining these automorphisms one can assume without loss of generality that $y_j(1)\in v(j)\setminus w(j)$ for all $j\in\{1,\ldots,l-1\}$, where $v(j)$ and $w(j)$ are subgroups of $u(j)$ which will be introduced in Section 4.5.

4.5. A summary of results on epimorphisms of spherical Moufang buildings

In this section we summarize the results in [6] that we use in the current paper.

Let Δ, Δ' be two irreducible spherical Moufang buildings of rank at least two and ϕ a (type-preserving) epimorphism from Δ to Δ'.

We start by the rank two case. Let c be a chamber in some apartment Σ of Δ. With this choice of chamber and apartment there corresponds a root group sequence $(U_+,U_1,\ldots,U_n)$. Section 6.1 of [6] states that the epimorphism ϕ induces subgroups $W_i\lhd V_i\le U_i$ for every i. The subgroup V_i consists of those automorphisms $g\in U_i$ such that there exists an automorphism g' of Δ' making the following diagram commute. If this is the case then we say that g *descends*.

$$\begin{tikzcd} \Delta \arrow[r, "g"] \arrow[d, "\phi"'] & \Delta \arrow[d, "\phi"] \\ \Delta' \arrow[r, "g'"'] & \Delta' \end{tikzcd}$$

The subgroup W_i is then the subgroup of those elements in V_i such that the corresponding g' in the previous diagram is the identity automorphism.

The following three lemmas describe how these different subgroups are related.

Lemma 4.2. *Let $v_i \in V_i \setminus W_i$, then*

$$V_j^{\mu(v_i)} = V_{2i+n-j},$$
$$W_j^{\mu(v_i)} = W_{2i+n-j}$$

for each $i, j \in \{1, \ldots, n\}$ such that $2i + n - j \in \{1, \ldots, n\}$.

Proof. See [6, Cor. 6.7]. □

Lemma 4.3. *Choose $u_1 \in U_1$ and $u_n \in V_n \setminus W_n$. Let $[u_1, u_n^{-1}] = u_2 \ldots u_{n-1}$ (with $u_i \in U_i$), then*

$$u_1 \in V_1 \Leftrightarrow u_2 \in V_2,$$
$$u_1 \in W_1 \Leftrightarrow u_2 \in W_2.$$

Proof. See [6, Lem. 6.8]. □

Lemma 4.4. *Choose $u_1 \in V_1$ and $u_3 \in W_3$. If $[u_1, u_3] = u_2$ then $u_2 \in W_2$.*

Proof. This is a special case of [6, Cor. 6.5]. □

The arbitrary rank case can now be approached as follows. Choose a chamber c of the building Δ and let (u, Θ, θ) be a root group labeling associated with this choice of chamber (see [8, (12.10)–(12.11)]). The epimorphism ϕ again induces subgroups $w(i) \lhd v(i) \leq u(i)$ for every i as before. These subgroups determine the structure of ϕ, as shown by the following lemma.

Lemma 4.5. *If the subgroups $w(i) \lhd v(i) \leq u(i)$ are known for a root group labeling (u, Θ, θ) of a spherical Moufang building Δ, then the corresponding epimorphism of Δ is unique up to isomorphisms.*

Proof. See [6, Cor. 6.12]. □

For a root group sequence Θ_{ij} of the root group labeling the $u(i)$ and $u(j)$ form the extremal root groups U_1 and U_n of this root group sequence. The subgroups $w(i) \triangleleft v(i) \leq u(i)$ and $w(j) \triangleleft v(j) \leq u(j)$ correspond respectively to the subgroups $W_1 \triangleleft V_1 \leq U_1$ and $W_n \triangleleft V_n \leq U_n$.

Lemma 4.6. *If a certain label i corresponds with a rank one residue which is a projective line over a skew field K with $u(i)$ indexed by K via an isomorphism y_i, then there exists a total subring R of K with maximal ideal m and a constant $a \in K$ such that*

$$v(i) = \{y_i(k) | k \in Ra\},$$
$$w(i) = \{y_i(k) | k \in ma\}.$$

Proof. See [6, Lems. 7.2–7.3]. □

4.6. Some properties of polar spaces

In this section we state some properties of polar spaces of (pseudo-)quadratic form type needed later on.

Remark 4.7. In this section we will always suppose that our polar spaces are non-singular (i.e. there are no points collinear to all other points of the polar space) and not of hyperbolic type (i.e. those polar spaces corresponding to buildings of type D_n). These are exactly those polar spaces corresponding to a (thick) building of type C_l. Without this assumption Lemma 4.8 would fail.

Each set of mutually collinear points as well as each subspace of a rank l polar space is contained in a (maximal) subspace of (geometric) dimension $l - 1$. These maximal subspaces are called the *generators*.

Lemma 4.8. *A subspace of dimension $l - 2$ is contained in at least three generators.*

Proof. This follows from the thickness of the building associated with the polar space. □

The following two lemmas show how points and generators interact.

Lemma 4.9. *Given a generator π and a point p not in π, there is a unique generator ξ containing p and intersecting π in a subspace of co-dimension 1. This subspace consists exactly of the points of π collinear with p.*

Proof. This property is part of the incidence geometric definition of polar spaces, see for example [2, p. 556]. □

Lemma 4.10. *Let π be a t-dimensional subspace and p a point not in this subspace. The set of points in π collinear with p either forms a $(t-1)$-dimensional subspace, or every point of π is collinear with p. Moreover each $(t-1)$-dimensional subspace of π arises in this way.*

Proof. The first assertion follows directly from Lemma 4.9. In order to prove the second assertion let ζ be a $(t-1)$-dimensional subspace of π and embed π in a generator ξ. We then can find a subspace χ of co-dimension 1 in ξ such that the intersection of ξ and χ is exactly ζ. Lemma 4.8 allows us to find a generator ξ' containing χ and different from ξ. If p is a point of ξ' not in χ, then the points of ξ collinear with p have to be exactly the points of χ by Lemma 4.9. Restricting to the subspace π of ξ shows that ζ consists exactly of those points of π collinear with p. □

4.7. Collinearity in $\mathsf{BC}_l(K, K_0, \sigma, L, q)$

One checks that the points $\langle(v|a_1, a_2, \ldots, a_{2n})\rangle$ and $\langle(w|b_1, b_2, \ldots, b_{2n})\rangle$ of the space $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ are collinear if and only if

$$f(v, w) + b_1^\sigma a_2 - b_2^\sigma a_1 + \cdots + b_{2l-1}^\sigma a_{2l} - b_{2l}^\sigma a_{2l-1} = 0.$$

The left-hand side of this equation is the skew-hermitian sesquilinear form f_X associated to the pseudo-quadratic form q_X on X applied to the two vectors.

4.8. Polar spaces of quadratic form type

In this section we define the polar spaces $\mathsf{B}_l(K, L, q)$ of quadratic form type. We do this as these polar spaces will arise as images of epimorphisms in Section 5.2.

A *quadratic space* (K, L, q) is a triple consisting of a field K, a non-trivial vector space L over K, equipped with a *quadratic form* q. This is a map $q : L \to K$ such that there exists a (necessarily unique) bilinear form f on L satisfying the following two properties:

- $q(u + v) = q(u) + q(v) + f(u, v)$,
- $q(tu) = t^2 q(u)$,

for all $u, v \in L$. The quadratic form q is *anisotropic* (and (K, L, q) an *anisotropic quadratic space*) if $q(u) = 0$ if and only if $u = 0$ for $u \in L$.

We can now define the rank l polar space $\mathsf{B}_l(K, L, q)$ where $l \geq 2$ is an integer and (K, L, q) an anisotropic quadratic space. Let X denote the vector space $L \oplus K^{2l}$. The map

$$q_X : (v|a_1, \ldots, a_{2l}) \longmapsto q(v) + a_1 a_2 + \cdots + a_{2l-1} a_{2l}$$

is a quadratic form on X. A subspace S is called *singular* if it is mapped to zero by q_X. As in the pseudo-quadratic form case, the polar space is formed by the singular subspaces and the associated building is the flag complex of the polar space.

5. Proof of the main result

We split the proof of the main result in two parts. In Section 5.1 we will prove:

Theorem 5.1. *A (type-preserving) epimorphism of a polar space* $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ *is completely determined (up to isomorphisms) by a total subring* R *of the skew field* K *and a left coset of* $R^\times$ *in the multiplicative group* K^* *satisfying the following three conditions.*

(C1) *The anti-automorphism* $a \mapsto a^{\sigma s}$ *of* K *stabilizes* R*,*
(C2) $(u, t), (w, r) \in T : t, r \in sR \Rightarrow f(u, w) \in sR$*,*
(C3) $(u, t), (w, r) \in T : t \in sR, r \in sm \Rightarrow f(u, w) \in sm$*,*

where s *is an element of the left coset of* $R^\times$*,* f *the skew-hermitian form associated to* q *and* m *the unique maximal two-sided ideal of* R*.*

Section 5.2 is devoted to the proof of the following theorem.

Theorem 5.2. *Consider the polar space* $\mathsf{BC}_l(K, K_0, \sigma, L, q)$*. If* R *is a total subring of the skew field* K *and* $sR^\times$ *a left coset of* $R^\times$ *in the multiplicative group* K^* *satisfying the following three conditions.*

(C1) *The anti-automorphism* $a \mapsto a^{\sigma s}$ *of* K *stabilizes* R*,*
(C2) $(u, t), (w, r) \in T : t, r \in sR \Rightarrow f(u, w) \in sR$*,*
(C3) $(u, t), (w, r) \in T : t \in sR, r \in sm \Rightarrow f(u, w) \in sm$*,*

where f *is the skew-hermitian form associated to* q *and* m *the unique maximal two-sided ideal of* R*, then there exists a (type-preserving) epimorphism of the polar space* $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ *for which Theorem 5.1 gives rise to the same total subring and left coset.*

The main result follows by combining Theorems 5.1 and 5.2.

5.1. Proof of Theorem 5.1

Let Δ be the building $\mathsf{BC}_l(K, K_0, \sigma, L, q)$, f the skew-hermitian form associated to q, the triple (u, Θ, θ) its root group labeling as given in Section 4.3 and ϕ a type-preserving epimorphism from Δ to another building Δ' of type C_l.

By Section 4.5 we know that this epimorphism is essentially described by subgroups $w(i) \lhd v(i) \leq u(i)$ for $i \in \{1, \dots, l\}$. Remark 4.1 allows us to assume without loss of generality that $y_i(1) \in v(i) \setminus w(i)$ for $i \in \{1, \ldots, l-1\}$. We also fix an element $y_l(v,s) \in v(l) \setminus w(l)$.

By Lemma 4.6 we know that there exists a total subring R of K with maximal ideal m and a constant $a \in K$ such that

$$v(1) = \{y_1(k) | k \in Ra\},$$
$$w(1) = \{y_1(k) | k \in ma\}.$$

The next lemma extends these expressions for other $u(i)$, and shows that one can assume that $a = 1$.

Lemma 5.3. *For every $i \in \{1, \ldots, l-1\}$ one has*

$$v(i) = \{y_i(k) | k \in R\},$$
$$w(i) = \{y_i(k) | k \in m\}.$$

Proof. We proof this by induction. We first consider the case $i = 1$. As $y_1(1) \in v(1) \setminus w(1)$ it follows that a^{-1} (and so also a) is a unit of R, and that the statement is true for $i = 1$.

Now suppose that the statement is true for some $j \in \{1, \ldots, l-2\}$. From Section 4.3 we know that $\Theta_{j,j+1} = \Theta_{\mathsf{A}_2(K^{\mathrm{op}})}$. Hence we can identify the subgroups $w(j) \lhd v(j) \leq u(j), w(j+1) \lhd v(j+1) \leq u(j+1)$ with groups $W_1 \lhd V_1 \leq U_1, W_3 \lhd V_3 \leq U_3$, respectively, as outlined in Section 4.5, and U_1 and U_3 as in Section 4.1. These identifications imply that

$$V_1 = \{x_1(k) | k \in R\},$$
$$W_1 = \{x_1(k) | k \in m\},$$
$$x_3(1) \in V_3 \subset W_3.$$

Applying Lemma 4.3 and the commutator relation $[x_1(b), x_3(1)^{-1}] = x_2(-b)$ for b in K we see that

$$V_2 = \{x_2(k) | k \in R\},$$
$$W_2 = \{x_2(k) | k \in m\}.$$

From Eq. (1) in Section 4.1 we know that $x_2(u)^{\mu(x_1(1))} = x_3(u)$, so Lemma 4.2 yields

$$V_3 = \{x_3(k) | k \in R\},$$
$$W_3 = \{x_3(k) | k \in m\},$$

which is, via the identifications, exactly what we need to prove. □

The next lemma determines the subgroups $w(l)$ and $v(l)$.

Lemma 5.4. *The subgroups $v(l)$ and $w(l)$ are described by*

$$v(l) = \{y_l(w,t) | (w,t) \in T, t \in sR\},$$
$$w(l) = \{y_l(w,t) | (w,t) \in T, t \in sm\}.$$

Proof. As $\Theta_{l,l-1} = \Theta_{\mathsf{BC}_2(K,K_0,\sigma,L,q)}$ (see Section 4.3), one can identify the subgroups $w(l-1) \lhd v(l-1) \leq u(l-1)$ and $w(l) \lhd v(l) \leq u(l)$ with groups $W_4 \lhd V_4 \leq U_4$ and $W_1 \lhd V_1 \leq U_1$, respectively, as outlined in Section 4.5, where U_1 and U_4 are as in Section 4.2.

Lemma 5.3 implies that V_4 and W_4 can be expressed as

$$V_4 = \{x_4(k) | k \in R\},$$
$$W_4 = \{x_4(k) | k \in m\}.$$

By Lemma 4.2, Eq. (4) (see Section 4.2) and $y_l(v,s) \in v(l) \setminus w(l)$ one obtains that

$$V_2 = \{x_2(k) | k \in sR\},$$
$$W_2 = \{x_2(k) | k \in sm\}.$$

It is now possible to describe the relevant subgroups of U_1 using Lemma 4.3 and the commutator relation $[x_1(w,t), x_4(1)^{-1}] = x_2(t)x_3(w,t)$ found in Section 4.2. One derives that $x_1(w,t) \in V_1$ or W_1 if and only if $t \in sR$ or sm respectively, so

$$V_1 = \{x_1(w,t) | (w,t) \in T, t \in sR\},$$
$$W_1 = \{x_1(w,t) | (w,t) \in T, t \in sm\},$$

which is what we need to show. □

At this point we have determined all of the subgroups $w(i) \lhd v(i) \leq u(i)$ for $i \in \{1, \ldots, l\}$. These subgroups are completely encoded by the total subring R and an element $s \in K^*$ (or more exactly a left coset of $R^\times$ in the multiplicative group K^*). These subgroups determine on their turn the epimorphism by Lemma 4.5.

In the remainder of this section we will derive the properties that these R and s satisfy.

Lemma 5.5. *The map $a \mapsto s^{-1}a^\sigma s^\sigma$ of K stabilizes $R^\times$.*

Proof. We use the same setting of Lemma 5.4. Let a be an invertible element of R. By Eq. (3) we have $x_2(s)^{\mu(x_4(a^{-1}))} = x_2(-a^\sigma s^\sigma a^{-1})$. Hence $-a^\sigma s^\sigma a^{-1} \in sR^\times$ by Lemma 4.2. Therefore $s^{-1}a^\sigma s^\sigma \in R^\times$. □

Proposition 5.6. *The anti-automorphism $a \mapsto a^{\sigma s}$ of K stabilizes R.*

Proof. As this map is the combination of an automorphism and anti-automorphism it is clear that it is an anti-automorphism. Let $a \in R^\times$. Then $a^{\sigma s} = s^{-1}a^\sigma s = (s^{-1}a^\sigma s^\sigma)(s^{-\sigma}s)$. The first factor and the inverse of the second factor are of the form as in Lemma 5.5, so both of them and their product lie in $R^\times$. So the anti-automorphism $a \mapsto a^{\sigma s}$ maps $R^\times$ into $R^\times$.

The non-invertible elements m of R form an ideal, so $1 + m \subset R^\times$. This implies that $R^\times$ generates R as a ring and that $a \mapsto a^{\sigma s}$ maps R into R.

The inverse of the map $a \mapsto a^{\sigma s}$ is given by the map $a \mapsto s^{-\sigma}a^\sigma s^\sigma$. One shows analogously, using the decomposition $s^{-\sigma}a^\sigma s^\sigma = (s^{-1}s^\sigma)s^{-\sigma}a^\sigma s^\sigma$, that this inverse maps R into R. Hence we conclude that the anti-automorphism $a \mapsto a^{\sigma s}$ of K stabilizes R. $\square$

Proposition 5.7.

$$\forall (u,t),(w,r) \in T : t, r \in sR \Rightarrow f(u,w) \in sR.$$

Proof. Let $(u,t),(w,r) \in T$ such that $t, r \in sR$. Lemma 5.4 implies that $y_l(u,t)$, $y_l(w,r) \in v(l)$. As $v(l)$ is a subgroup it follows that the product $y_l(w,r) \cdot y_l(u,t)$ also lies in $v(l)$, hence $t + r + f(u,w) \in sR$ (see Section 4.2 and again Lemma 5.4). Because R is a ring this is equivalent to $f(u,w) \in sR$. $\square$

Proposition 5.8.

$$\forall (u,t),(w,r) \in T : t \in sR, r \in sm \Rightarrow f(u,w) \in sm.$$

Proof. We use the same setting of Lemma 5.4. Let $(u,t),(w,r) \in T$ with $t \in sR$, $r \in sm$. Note that $x_1(u,t) \in V_1$ and $x_1(w,r) \in W_1$. We start by determining V_3 using Lemma 4.2, Eq. (2) and $x_4(1) \in V_4 \setminus W_4$. Combining this yields that $x_3(w,r) \in W_3$. Lemma 4.4 now implies that $[x_1(u,t), x_3(w,r)^{-1}] = x_2(f(u,w)) \in W_2$ which is equivalent to $f(u,w) \in sm$. $\square$

As Propositions 5.6, 5.7 and 5.8 prove Conditions (C1)–(C3), this concludes the proof of Theorem 5.1.

5.2. Proof of Theorem 5.2

In this section we construct epimorphisms of the polar space $\mathsf{BC}_l(K, K_0, \sigma, L, q)$. We do this starting from a total subring $R \subset K$ and a left coset $sR^\times$ of $R^\times$ in the multiplicative group K^* satisfying the conditions outlined in Theorem 5.2.

As before we will let m denote the set of non-invertible elements of R and K_R the corresponding residue skew field.

5.2.1. Structure of K_R

We start by showing that one can choose the representative s in the left coset in a special way.

Lemma 5.9. *The left coset $sR^\times$ contains an element r such that we are in exactly one of the following two cases:*

Case I. $r \in K_0$ *and* $a \mapsto a^{\sigma r}$ *is an involution,*
Case II. $K_0 \cap rR^\times = \varnothing$, $r^{-1}r^\sigma + 1 \in m$, K_R *is a field and* $a \mapsto a^{\sigma r}$ *induces the identity on* K_R.

Proof. Note that any element r in $K_0 \cap sR^\times$ is fixed by σ, implying that $a \mapsto a^{\sigma r}$ is an involution. So such an element directly satisfies Case I. Hence we may suppose that $K_0 \cap sR^\times$ is empty. This implies that we have two possibilities for $1 + s^{-1}s^\sigma = s^{-1}(s+s^\sigma) \in s^{-1}K_\sigma \subset s^{-1}K_0$. It can either be an element of m, or an element of $K \setminus R$. Suppose the latter holds. Then $s^{-1}s^\sigma$ also belongs to $K \setminus R$, and the inverse $s^{-\sigma}s$ belongs to m. But $(s^{-1}s^\sigma)^{\sigma s} = s^{-\sigma}s$ which contradicts Condition (C1). So $1 + s^{-1}s^\sigma \in m$ and consequently $s^{-1}s^\sigma \in R$. Now for a given $a \in R$ we have $s^{-1}(s^\sigma a + a^\sigma s) \in s^{-1}K_0$ and hence $s^{-1}s^\sigma a + a^{\sigma s} \notin R^\times$ since $K_0 \cap sR^\times = \varnothing$. Since $s^{-1}s^\sigma \in R$ and, by Condition (C1), also $a^{\sigma s} \in R$, it follows that $s^{-1}s^\sigma a + a^{\sigma s} \in m$. Combined with $1 + s^{-1}s^\sigma \in m$ this implies that $a \equiv a^{\sigma s} \bmod m$, or that $a \mapsto a^{\sigma s}$ is the identity automorphism of K_R. Because it is also an anti-automorphism this yields that K_R is a field.

Finally we remark that $r \in K_0$ and $K_0 \cap rR^\times = \varnothing$ are mutually exclusive, so exactly one of the two cases holds. □

From now on suppose that s is as in one of the two cases described by this lemma. We denote the anti-automorphism induced on K_R by the anti-automorphism $a \mapsto a^{\sigma s}$ by σ_R.

Lemma 5.10. *Under the assumption of Case I, $(K_R, \overline{K_0}, \sigma_R)$ is an involutory set, where $\overline{K_0} := s^{-1}K_0 \cap R \bmod m$.*

Proof. The involution σ_R fixes all the elements of $\overline{K_0}$. Since $s \in K_0$ (by hypothesis), the set $\overline{K_0}$ contains 1 and $b + b^{\sigma s} = s^{-1}(sb + (sb)^\sigma) \in s^{-1}K_0$ for all $b \in K$. Hence $a + a^{\sigma_R} \in \overline{K_0}$ for all $a \in K_R$.

If $c \in s^{-1}K_0 \cap R$ and $d \in R^\times$ then $d^{\sigma s}cd \in R$ because $d^{\sigma s} \in R$ by our assumptions. Also $d^{\sigma s}cd \in s^{-1}K_0$ as K_0 is an involutory set and hence contains $d^\sigma scd$. This implies that if $t \in K_R$, then $t^{\sigma_R}\overline{K_0}t = \overline{K_0}$. Hence $(K_R, \overline{K_0}, \sigma_R)$ is an involutory set. □

5.2.2. Structures on L

Consider the following two subsets of L.

$$L' := \{v \in L | (\exists a \in R)(q(v) \equiv sa \bmod K_0)\} \text{ and}$$
$$L'' := \{v \in L | (\exists a \in m)(q(v) \equiv sa \bmod K_0)\}.$$

Lemma 5.11. *The sets L' and L'' are additive abelian subgroups of L.*

Proof. We only proof that L' is a subgroup of L as the proof for L'' is completely analogous. Let $v, w \in L'$. As $q(-v) \equiv q(v) \bmod K_0$ it is clear that that L' is closed under taking inverses. By construction of L' we can find $a, b \in sR$ such that $q(v) \equiv sa \bmod K_0$ and $q(w) \equiv sb \bmod K_0$. By the definition of a skew-hermitian pseudo-quadratic form we have that

$$q(v + w) \equiv sa + sb + f(v, w) \bmod K_0.$$

Condition (C2) asserts that $f(v, w) \in sR$. Hence $sa + sb + f(v, w) \in sR$ and L' is indeed a subgroup of L. □

The next lemma investigates how these subgroups behave under scalar products.

Lemma 5.12. *The subgroups L' and L'' are R-modules, in particular we have that $L'.R = L'$ and $L''.R = L''$. Moreover we have that $L'.m \subset L''$.*

Proof. Let $v \in L'$ and $t \in R$. By construction of L' there exists an element $a \in R$ such that $q(v) \equiv sa \bmod K_0$. Then $q(vt) \equiv t^\sigma sat \bmod K_0$ as q is a skew-hermitian pseudo-quadratic form. From Condition (C1) it follows that $s^{-1}t^\sigma s \in R$, so $s^{-1}t^\sigma sat \in R$ or equivalently $t^\sigma sat \in sR$. This implies that $vt \in L'$. Hence $L'.R = L'$. The proofs for $L''.R = L''$ and $L'.m \subset L''$ are completely analogous. □

Let $\overline{L}$ be the quotient L'/L''. If $v \in L'$ and $k \in R$ then $(v + L'').(k + m) \subset vk + L''$ (by Lemma 5.12), so this group can be interpreted as a right vector space over the residue skew field K_R.

We construct two functions on $\overline{L}$. As first function we define

$$\overline{f} : \overline{L} \times \overline{L} \to K_R : (v + L'', w + L'') \mapsto s^{-1}f(v, w) + m.$$

Note that this is indeed a well-defined map into K_R by Conditions (C2) and (C3). As second function we define $\bar{q} : \overline{L} \to K_R$ mapping elements $v + L''$ of $\overline{L}$ to $t + m \in K_R$ (with $t \in R$) such that $q(v) \equiv st \bmod K_0$. Note that such an element $t + m$ always exists by definition of L', but that $\bar{q}$ is not necessarily well-defined as there might be several $t + m \in K_R$ satisfying this condition, which may depend on the choice of representative of $v + L''$. However the next four lemmas show that there is indeed a unique choice (albeit modulo $\overline{K_0}$ in the first case), and derive properties for $\overline{f}$ and $\bar{q}$. We start with the following observation.

Lemma 5.13. *Let $v \in L'$. If $q(v) \equiv st \equiv st' \bmod K_0$ with $t, t' \in R$, then $t \equiv t' \bmod \overline{K_0}$.*

Proof. This follows directly from the definition of $\overline{K_0}$. □

Lemma 5.14. *Let $v, w \in L'$ such that $w + L'' = v + L''$. If $t \in R$ is such that $q(v) \equiv st \bmod K_0$, then there exists a $t' \in t + m$ such that $q(w) \equiv st' \bmod K_0$.*

Proof. Let v, w, t be as in the statement of the lemma, note that $w - v \in L''$. By definition of L'' there exists an $a \in m$ such that $(w - v, sa) \in T$, or equivalently $q(w - v) \equiv sa \bmod K_0$. Note that $f(v, w - v) \in sm$ by Condition (C3). As q is a pseudo-hermitian form we have

$$\begin{aligned} q(w) &\equiv q(v + (w - v)) \bmod K_0 \\ &\equiv q(v) + q(w - v) + f(v, w - v) \bmod K_0 \\ &\equiv st + sa + f(v, w - v) \bmod K_0. \end{aligned}$$

As $sa + f(v, w - v)$ is in sm, the element $t' := t + a + s^{-1} f(v, w - v)$ is in $t + m$ and satisfies $q(w) \equiv st' \bmod K_0$. □

Lemma 5.15. *Under the assumption of Case I, the map $\overline{f}$ is a skew-hermitian sesquilinear function, and $\bar{q}$ is well-defined modulo $\overline{K_0}$ and an anisotropic skew-hermitian pseudo-quadratic form on $\overline{L}$ with respect to the involution σ_R, the involutory set $\overline{K_0}$, and $\overline{f}$.*

Proof. From Lemmas 5.13 and 5.14 and the construction of $\bar{q}$ and $\overline{K_0}$, it follows that the function $\bar{q}$ is well-defined modulo $\overline{K_0}$.

The remainder of the statement of the lemma follows from straightforward calculations using the properties of skew-hermitian sesquilinear and pseudo-quadratic forms, Conditions (C1)–(C3) and the fact that $s \in K_0$ (and hence fixed by σ) by Lemma 5.9. □

Lemma 5.16. *Under the assumption of Case II, we have that $\overline{f}$ is a symmetric bilinear function, and that $\bar{q}$ is a well-defined quadratic form on $\overline{L}$ with $\overline{f}$ as associated bilinear function.*

Proof. For an element $v \in L'$ let $t, t' \in R$ be two elements such that $q(v) \equiv st \equiv st' \bmod K_0$. Because $K_0 \cap sR^\times = \varnothing$ (see Lemma 5.9), the difference $st - st' \in K_0$ lies in sm, implying that $t + m = t' + m$ and that, in the light of Lemma 5.14, $\bar{q}(v + L'')$ is well-defined.

Let $v + L'', w + L''$ be two elements of $\overline{L}$. Making use of Lemma 5.9 and Condition (C1) we can derive the following.

$$\begin{aligned} \overline{f}(v, w) &= \overline{f}(v, w)^{\sigma_R} \\ &= (s^{-1} f(v, w) + m)^{\sigma s} \\ &= s^{-1} f(v, w)^{\sigma} s^{-\sigma} s + m \\ &= s^{-1} f(v, w)^{\sigma} + m \end{aligned}$$

$$= s^{-1}f(w,v) + m$$
$$= \overline{f}(w,v)$$

We can conclude that $\overline{f}$ is symmetric. The other part of the statement follows easily. □

5.2.3. Constructing the epimorphism

We will now construct an epimorphism ρ from the polar space $\mathsf{BC}_l(K, K_0, \sigma, L, q)$ to the polar space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ when we are in Case I, and to $\mathsf{B}_l(K_R, \overline{L}, \bar{q})$ in Case II. We use the notations from Section 2.

We call a vector $(v|a_1, \ldots, a_{2l}) \in X$ *normed* if all coefficients a_i $(i \in \{1, \ldots, 2l\})$ lie in the subring R, and at least one is an invertible element of R. The next lemma deals with the existence of a normed scalar multiple of a given vector.

Lemma 5.17. *If $w := (v|a_1, \ldots, a_{2l}) \in X$ is a vector such that $(a_1, \ldots, a_{2l}) \neq (0, \ldots, 0)$, then there exists an element $t \in K$ such that the scalar product wt is normed.*

Proof. Given such a non-zero vector $w := (v|a_1, \ldots, a_{2l})$, one can assume without loss of generality (by taking an appropriate scalar product) that the set $J := \{i \in \{1, \ldots, 2l\} | a_j \in K \setminus R\}$ is non-empty. Choose $j \in J$, and let w' be the vector wa_j^{-1}. As $a_j^{-1} \in m$, this implies that if a coordinate of w was already in R, then this holds for the corresponding coordinate of w' as well. Note that the coordinate of w' corresponding with j is 1, so repeating this algorithm a finite number of times yields the desired scalar multiple. □

The following lemma shows how the different choices of normed scalar multiples are related.

Lemma 5.18. *If $w := (v|a_1, \ldots, a_{2l}) \in X$ is a vector such that $(a_1, \ldots, a_{2l}) \neq (0, \ldots, 0)$ and $t, t' \in K$ are elements such that the scalar products wt and wt' are normed, then $t^{-1}t' \in R^\times$.*

Proof. We will prove this by contradiction. Without loss of generality one may assume that $t^{-1}t' \in K \setminus R$. By the definition of being normed, there exists $j \in \{1, \ldots, 2l\}$ such that $a_jt \in R^\times$. Then $a_jt' = (a_jt)(t^{-1}t')$ lies in $K \setminus R$, which is impossible for a normed vector. □

Let the vector $(v|a_1, a_2, \ldots, a_{2l-1}, a_{2l})$ represent a point of $\mathsf{BC}_l(K, K_0, \sigma, L, q)$. Note that as q is anisotropic we have that $(a_1, a_2, \ldots, a_{2l-1}, a_{2l}) \neq (0, \ldots, 0)$. By Lemma 5.17 we can choose this vector such that $(v|a'_1, a'_2, \ldots, a'_{2l-1}, a'_{2l})$ is normed, with a'_i equal to a_i when i is odd, and equal to $s^{-1}a_i$ when i is even $(i \in \{1, \ldots, 2l\})$. We now have that

$$\begin{aligned} a_1^\sigma a_2 + \cdots + a_{2n-1}^\sigma a_{2n} &= {a'_1}^\sigma s a'_2 + \cdots + {a'}^\sigma_{2n-1} s a'_{2n} \\ &= s({a'_1}^{\sigma s} a'_2 + \cdots + {a'}^{\sigma s}_{2n-1} a'_{2n}) \in sR. \end{aligned}$$

As the 1-dimensional space spanned by the vector is a point of the polar space, we have that $q(v) + a_1^\sigma a_2 + \cdots + a_{2n-1}^\sigma a_{2l} \in K_0$. This implies that $v \in L'$. In particular we have that $\bar{q}(v + L'') = -({a'_1}^{\sigma s} a'_2 + \cdots + {a'}_{2l-1}^{\sigma s} a'_{2l}) + m$ (in Case I this is modulo $\overline{K_0}$, see Lemma 5.15).

Note that $(v + L''|a'_1 + m, a'_2 + m, \dots, a'_{2l} + m)$ is non-zero as we assumed that $(v|a'_1, a'_2, \dots, a'_{2l-1}, a'_{2l})$ is normed. Hence $\langle (v + L''|a'_1 + m, a'_2 + m, \dots, a'_{2l} + m)\rangle$ is a point of the polar space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ in Case I, and a point of $\mathsf{B}_l(K_R, \overline{L}, \bar{q})$ in Case II. Lemma 5.18 shows that this point does not depend on the choice of the vector representing the point of $\mathsf{BC}_l(K, K_0, \sigma, L, q)$.

We denote the map we defined from the points of the space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ to the points of $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ or $\mathsf{B}_l(K_R, \overline{L}, \bar{q})$ by ρ. We claim that ρ is the desired epimorphism.

Lemma 5.19. *The map ρ is surjective.*

Proof. Let $(v + L''_0|a'_1 + m, a'_2 + m, \dots, a'_{2l} + m) \in \overline{L} \times K_r^{2l}$ represent a point of the polar space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ or $\mathsf{B}_l(K_R, \overline{L}, \bar{q})$ depending on the case. By definition of these polar spaces and by the definition of $\bar{q}$ and $\overline{K_0}$ we have that $s^{-1}q(v) + {a'_1}^{\sigma s} a'_2 + \cdots + {a'}_{2l-1}^{\sigma s} a'_{2l} \in s^{-1}K_0 + m$. Without loss of generality we may assume that there is a $j \in \{1, \dots, 2l\}$ such that $a'_j = 1$, this because at least one of the a'_i ($i \in \{1, \dots, 2l\}$) is non-zero modulo m as $\bar{q}$ is anisotropic.

Set $a_1 := a'_1$, $a_2 := sa'_2$ and so on. Then

$$\begin{aligned} q_X(v|a_1, \dots, a_{2l}) &= q(v) + a_1^\sigma a_2 + \cdots + a_{2l-1}^\sigma a_{2l} \\ &= s(s^{-1}q(v) + {a'_1}^{\sigma s} a'_2 + \cdots + {a'}_{2l-1}^{\sigma s} a'_{2l}) \in K_0 + sm. \end{aligned}$$

Choose a $t \in m$ such that $q_X(v|a_1, \dots, a_{2l}) \equiv st \mod K_0$. Let $b_i := a_i$ for all $i \in \{1, \dots, 2l\}$ except for $b_{j-1} = a_{j-1} - t^{\sigma s^\sigma}$ when j is even and $b_{j+1} = a_{j+1} - st$ when j is odd. For each possibility one obtains:

$$\begin{aligned} q_X(v|b_1, \dots, b_{2l}) &= q(v) + b_1^\sigma b_2 + \cdots + b_{2l-1}^\sigma b_{2l} \\ &= q(v) + a_1^\sigma a_2 + \cdots + a_{2l-1}^\sigma a_{2l} - st \\ &= q_X(v|a_1, \dots, a_{2l}) - st \in K_0, \end{aligned}$$

by construction of t. Hence $\langle (v|b_1, \dots, b_{2l})\rangle$ is a point of $\mathsf{BC}_l(K, K_0, \sigma, L, q)$, for which one easily verifies that its image under ρ is the point $\langle (v+L''|a'_1+m, a'_2+m, \dots, a'_{2l}+m)\rangle$. □

Lemma 5.20. *The map ρ preserves collinearity.*

Proof. From the construction, the definition of $\overline{f}$, and Section 4.7. □

The next series of lemmas proves that a collinearity-preserving surjective map induces an epimorphism of the buildings associated to the polar spaces. In order to simplify notations we denote the polar space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ by Π and the polar space $\mathsf{BC}_l(K_R, \overline{K_0}, \sigma_R, \overline{L}, \bar{q})$ or $\mathsf{B}_l(K_R, \overline{L}, \bar{q})$ (depending on the case) by Π'.

Lemma 5.21. *Let π, π' be subspaces of respectively Π and Π' such that the points of π are mapped into π' by ρ. If ξ' is a subspace of co-dimension n in π', then there exists a subspace ξ of co-dimension at most n in π whose image under ρ is contained in ξ'.*

Proof. We can assume that π' is non-empty. We prove the claim for a subspace ξ' of co-dimension 1 in π', the general case then follows by induction. Embed π' in a generator χ' of Π'. By Lemma 4.10 there exists a point p' of Π' such that the intersection of π' and the set of points collinear with p' in χ' is exactly ξ'. Let p be a point in Π mapped to p'. Let ξ be the subspace of π consisting of those points in π collinear with p, which is of co-dimension 0 or 1 in π (see Lemma 4.10). As ρ preserves collinearity it follows that ξ has the desired properties. $\square$

Lemma 5.22. *Let π, π' be same-dimensional subspaces of respectively Π and Π' such that the points of π are mapped into π' by ρ. Then the set of points of π is mapped surjectively to the set of points of π'. In particular the set of points in π cannot be mapped into a lower-dimensional subspace of Π'.*

Proof. Let p' be a point of π', so the set $\{p'\}$ is 0-dimensional subspace of π'. By Lemma 5.21 there exists a subspace ξ of π such that ξ is of dimension at least zero and mapped into the set $\{p'\}$ by ρ. As ξ is necessarily non-empty the same can be said about its image, so there exists a point of π mapped to p'.

The second assertion follows from embedding the lower-dimensional subspace in a subspace of the same dimension as π and then applying the first assertion. $\square$

Lemma 5.23. *The map ρ maps lines of Π to subsets of lines of Π'.*

Proof. Let p_1 and p_2 be two collinear points of Π such that $p_1^\rho \neq p_2^\rho$. We need to prove that if p_3 is a point on the line through p_1 and p_2, then this point is mapped to a point on the line through p_1^ρ and p_2^ρ. We may assume that $p_1^\rho \neq p_3^\rho \neq p_2^\rho$. Let w_1, w_2 and w_3 be vectors satisfying the norming condition as in the definition of ρ, representing respectively p_1, p_2 and p_3.

Because p_1, p_2 and p_3 lie on a line, there exist non-zero constants t_1 and t_2 in K such that $w_3 = w_1 t_1 + w_2 t_2$. If we can show that t_1 and t_2 lie in the total subring R, we are done (by construction of ρ). One can assume without loss of generality that $t_2 t_1^{-1} \in R$. As $w_3 t_1^{-1} = w_1 + w_2 t_2 t_1^{-1}$, one has that $t_1 \in R$ since otherwise t_1^{-1} would be an element of m, implying that p_1 and p_2 are mapped to the same point by ρ. It follows that $t_2 \in R t_1 \subset R$. $\square$

Lemma 5.24. *The map ρ maps subspaces of Π into subsets of same-dimensional subspaces of Π'.*

Proof. Let π be a subspace of Π, we prove the lemma by induction on the dimension t of π. The result is immediate for $t = -1, 0$, and is proven in Lemma 5.23 for $t = 1$. Now suppose the result holds for all subspaces of dimension at most $t - 1$. Let ξ be a $(t-1)$-dimensional subspace of π. The image of the set of points of ξ is exactly the point set of a $(t-1)$-dimensional subspace ξ' of Π' by the induction hypothesis and Lemma 5.22. This lemma also implies that there exists a point p in π mapped outside ξ'. As ρ preserves collinearity every point of ξ' is collinear with p^ρ. Hence ξ' and p^ρ span a t-dimensional subspace π' of Π'. Each point of π lies on a line meeting both p and ξ, so Lemma 5.23 yields such a point is mapped to a point on a line meeting p^ρ and ξ^ρ, which is hence contained in π'. This proves the lemma. $\square$

Proposition 5.25. *The map ρ induces an epimorphism between the buildings associated to the polar spaces Π and Π'.*

Proof. The combination of Lemmas 5.22 and 5.24 states that for each subspace π of Π there exists a unique, same-dimensional subspace π' of Π' such that the image under ρ of the point set of π is exactly the point set of π'. Hence one can extend ρ to subspaces by setting $\pi^\rho = \pi'$. This extension to subspaces clearly preserves the incidence relation (which is containment) between the subspaces.

We now prove that ρ is surjective on generators. Pick a generator π in Π, and consider a generator χ in Π' intersecting π^ρ in a subspace of co-dimension one. Let p' be a point of χ not in the intersection. By surjectivity there exists a point p of Π such that $p^\rho = p'$. Note it is impossible that p lies in π. By Lemma 4.9 there exists a unique generator ξ containing p and intersecting π in a subspace of dimension one less. The only possibility for the image of ξ is χ.

The dual polar space associated to Π' is connected (see for instance [4, Thm. 8.1.5(1)]), hence repeating this argument yields that we obtain each generator as an image.

In order to have an epimorphism of buildings we only need to prove that this induces a surjective map between the sets of chambers of the buildings associated to Π and Π', or equivalently that for each maximal flag F' of subspaces of the polar space Π' there is a maximal flag F of subspaces of Π mapped to it by ρ. The previous paragraph assures that one can find a generator π of Π mapped to the generator in F'. Subsequently one can use Lemmas 5.21 and 5.22 to find a subspace of co-dimension 1 in π which is mapped to the corresponding subspace in F'. Repeating this argument yields that ρ is surjective on maximal flags of subspaces, as desired. $\square$

In order to finish the proof of Theorem 5.2 we only need to check that the total subring and left coset one obtains from applying Theorem 5.1 to the epimorphism ρ are indeed R and $sR^\times$. In order to achieve this we need to check which portion of the groups $u(i)$ $(i \in \{1, \ldots, l\})$ descends in the sense of Section 4.5. Recall that these groups were explicitly described in Section 4.4.

Let $(v|a_1, \ldots, a_{2l})$ be a vector of X representing a point of $\mathsf{BC}_l(K, K_0, \sigma, L, q)$, satisfying the norming condition appearing in the definition of the epimorphism ρ. So the vector $(v|a'_1, a'_2, \ldots)$, with a'_j equal to a_j when j is odd, and equal to $s^{-1}a_j$ when j is even ($j \in \{1, \ldots, 2l\}$), is normed. Notice, as before, that this implies that $v \in L'$.

We start with the groups $u(i)$ with $i \in \{1, \ldots, l-1\}$. We perform the calculations for $i = l-1$, the argument for the other possible values is completely analogous. For $k \in K$ one has

$$\begin{aligned}(v|a_1, \ldots, a_{2l})^{y_{l-1}(k)} &= (v|a'_1, sa'_2, \ldots, sa'_{2l})^{y_{l-1}(k)} \\ &= (v|a'_1 + ka'_3, sa'_2, a'_3, sa'_4 - k^\sigma sa'_2, \ldots, sa'_{2l}) \\ &= (v|a'_1 + ka'_3, sa'_2, a'_3, s(a'_4 - k^{\sigma s}a'_2), \ldots, sa'_{2l}).\end{aligned}$$

We claim that if $k \in R$, then $(v|a'_1 + ka'_3, a'_2, a'_3, a'_4 - k^{\sigma s}a'_2, \ldots, a'_{2l})$ is again normed. To see this note that all the entries lie in R, as $(v|a'_1, a'_2, \ldots)$ is normed and Condition (C1). Secondly observe that if all the entries $a'_1 + ka'_3, a'_2, a'_3, a'_4 - k^{\sigma s}a'_2, \ldots, a'_{2l}$ would lie in m then the same holds for $a'_1, a'_2, \ldots, a'_{2l}$ (as these can be obtained by linear combinations with coefficients in R from the first list of entries). Hence there is an invertible element among these.

So if $k \in R$ we end up with a vector again satisfying the norming condition appearing in the definition of ρ. From this it follows that the automorphism $y_{l-1}(k)$ descends if $k \in R$. If $k \in K \backslash R$ the situation is as follows: now $y_{l-1}(k)$ maps both vectors $(0|1, 0, 0, \ldots)$ and $(0|0, 0, 1, 0, \ldots)$, where the points they represent have different images under ρ, to vectors representing points which are mapped to the same point $\langle(0|1, 0, 0, \ldots)\rangle$ by ρ. Hence $y_{l-1}(k)$ descends if and only if $k \in R$. In the light of Section 5.1 and Lemma 5.3 this implies that the total subring of K given by Theorem 5.1 is exactly the subring R.

For the group $u(l)$ we do a similar calculation. For $(w, t) \in T$ one has

$$\begin{aligned}(v|a_1, \ldots, a_{2l})^{y_l(w,t)} &= (v|a'_1, sa'_2, \ldots, sa'_{2l})^{y_l(w,t)} \\ &= (v + wa'_1|a'_1, sa'_2 - ta'_1 - f(w, v), a'_3, \ldots, sa'_{2l}).\end{aligned}$$

By Condition (C2) we have that if $t \in sR$ (and hence $w \in L'$) then this vector again satisfies the norming condition and $y_l(w, t)$ descends. If $t \in K \setminus sR$, then both points $\langle(0|1, 0, 0, \ldots)^{y_l(w,t)}\rangle$ and $\langle(0|0, 1, 0, \ldots)^{y_l(w,t)}\rangle$ are mapped to $\langle(0|0, 1, 0, \ldots)\rangle$ by ρ, while $\langle(0|1, 0, 0, \ldots)\rangle^\rho$ and $\langle(0|1, 0, 0, \ldots)\rangle^\rho$ are different points. We obtain that $y_l(w, t)$ descends if and only if $t \in sR$. Comparing this with Lemma 5.4 we see that Theorem 5.1 yields the left coset $sR^\times$, as desired.

This concludes the proof of Theorem 5.2.

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