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Presentation of hyperbolic Kac–Moody groups over rings



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ABSTRACT

Tits has defined Kac–Moody and Steinberg groups over commutative rings, providing infinite dimensional analogues of the Chevalley–Demazure group schemes. Here we establish simple explicit presentations for all Steinberg and Kac–Moody groups whose Dynkin diagrams are hyperbolic and simply laced. Our presentations are analogues of the Curtis–Tits presentation of the finite groups of Lie type. When the ground ring is finitely generated, we derive the finite presentability of the Steinberg group, and similarly for the Kac–Moody group when the ground ring is a Dedekind domain of arithmetic type. These finite-presentation results need slightly stronger hypotheses when the rank is smallest possible, namely 4. The presentations simplify considerably when the ground ring is \mathbb{Z} , a case of special interest because of the conjectured role of the Kac–Moody group $E_{10}(\mathbb{Z})$ in superstring theory.

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Table 1

The simply-laced hyperbolic Dynkin diagrams. The *rank* means the number of nodes.

rank 4	
rank 5	
rank 6	
rank 7	
rank 8	
rank 9	
rank 10	

1. Introduction

Kac–Moody groups are infinite-dimensional generalizations of reductive Lie groups and algebraic groups. Over general rings, their final definition has not yet been found—it should be some sort of generalization of the Chevalley–Demazure group schemes. Given any root system, Tits defined a functor from commutative rings to groups and proved that it approximates any acceptable definition, and gives the unique best definition when the coefficient ring is a field [21]. His definition, by generators and relations, is very complicated. Even enumerating his relations in non-affine examples is difficult and in some cases impracticable [10,4].

In this paper we study the question of improving this in the case of the simplest non-affine Dynkin diagrams: the simply laced hyperbolic ones. The main result is that these, and related groups, have quite simple presentations, often finite. Our results are parallel those established for the affine case in [3]. Near the end of the Introduction we will remark on the situation beyond the affine and simply-laced hyperbolic cases.

An irreducible Dynkin diagram is called *hyperbolic* if it is not of affine or finite dimensional type, but its proper irreducible subdiagrams are. It is called *simply laced* if each pair of nodes is either unjoined, or joined by a single bond. One can classify the simply laced hyperbolic diagrams [16, §6.9], namely those in Table 1. The most important one is the last, known as E_{10} , because of its conjectural role in superstring theory (see below). We will pass between Dynkin diagrams and their generalized Cartan matrices whenever convenient.

For each generalized Cartan matrix A , Tits [21] defined the *Steinberg group*, a functor \mathfrak{St}_A from commutative rings to groups that generalizes Steinberg’s definition from

the classical finite dimensional case (the group G' on p. 78 of [20]). Morita and Rehmann [17] give another definition, but it agrees with Tits' for the diagrams in Table 1 because these are 2-spherical without isolated nodes [21, Remark a₄, p. 550]. By taking a quotient of \mathfrak{St}_A , Tits defined another functor \mathfrak{G}_A from commutative rings to groups, which we call the *Kac–Moody group*. We call \mathfrak{St}_A and \mathfrak{G}_A hyperbolic if A is hyperbolic. (Note: Tits actually defined a group functor $\tilde{\mathfrak{G}}_D$ for each root datum D . By \mathfrak{G}_A we mean $\tilde{\mathfrak{G}}_D$ where D is the root datum which has generalized Cartan matrix A and is “simply connected in the strong sense” [21, p. 551].)

Tits showed that his model of a Kac–Moody group is the natural one, at least for fields. Namely: any group functor with some obviously desirable properties admits a functorial homomorphism from \mathfrak{G}_A , which at every field is an isomorphism [21, Thm. 1', p. 553]. Tits does not call \mathfrak{G}_A a Kac–Moody group. We call it this just to have a name for the closest known approximation to whatever the ultimate definition of “the” Kac–Moody functors will be.

Let R be a commutative ring. Tits' definition of $\mathfrak{St}_A(R)$ is by a presentation with a generator $X_\alpha(t)$ for each real root α of the Kac–Moody algebra \mathfrak{g}_A and each $t \in R$. Whenever two real roots α, β form a *prenilpotent pair* (defined in Section 2), Tits imposes a relation $[X_\alpha(t), X_\beta(u)] = \cdots$ for each pair $t, u \in R$. The right side is a product of other generators $X_\gamma(v)$, generalizing the classical Chevalley relations; see Section 2 for the details in the cases we need. Unless A has finite-dimensional type, there are infinitely many Weyl-group orbits of prenilpotent pairs, yielding infinitely many distinct kinds of relations. Our main result is a new, much simpler, presentation, given entirely in terms of the Dynkin diagram:

Theorem 1 (*Presentation of Steinberg and Kac–Moody groups*). *Suppose R is a commutative ring and A is a simply laced hyperbolic Dynkin diagram, with I being its set of nodes. Then the Steinberg group $\mathfrak{St}_A(R)$ has a presentation with generators S_i and $X_i(t)$, with i varying over I and t over R , and relations listed in Table 2.*

The Kac–Moody group $\mathfrak{G}_A(R)$ is the quotient of $\mathfrak{St}_A(R)$ by the extra relations $\tilde{h}_i(a)\tilde{h}_i(b) = \tilde{h}_i(ab)$, for any single $i \in I$ and all units a, b of R , where $\tilde{h}_i(a) := \tilde{s}_i(a)\tilde{s}_i(-1)$ and $\tilde{s}_i(a) := X_i(a)S_iX_i(1/a)S_i^{-1}X_i(a)$.

Our generating set coincides with the one in [9], and the presentation works just as well for the simply laced spherical or affine Dynkin diagrams without A_1 components; see [3]. When $R = \mathbb{Z}$ the presentation simplifies considerably. We give it explicitly because this entire paper grew from trying to understand the Kac–Moody group $\mathfrak{G}_{E_{10}}(\mathbb{Z})$:

Corollary 2 (*Presentation over \mathbb{Z}*). *If A is simply laced hyperbolic, then the Steinberg group $\mathfrak{St}_A(\mathbb{Z})$ has a presentation with generators S_i and X_i , where i varies over the simple roots, and the relations listed in Table 3. The Kac–Moody group $\mathfrak{G}_A(\mathbb{Z})$ is the quotient of $\mathfrak{St}_A(\mathbb{Z})$ by the relation $\tilde{h}_i(-1)^2 = 1$, for any single $i \in I$. \square*

Table 2

Defining relations for $\mathfrak{St}_A(R)$ when A is simply laced hyperbolic. The generators are $X_i(t)$ and S_i where i varies over the nodes of the Dynkin diagram and t and u vary over R . See [Theorem 1](#) for the additional relations needed to define $\mathfrak{St}_A(R)$, and [Corollary 2](#) for simplifications in the special case $R = \mathbb{Z}$.

$\left. \begin{aligned} X_i(t)X_i(u) &= X_i(t+u) \\ [S_i^2, X_i(t)] &= 1 \\ S_i &= X_i(1)S_iX_i(1)S_i^{-1}X_i(1) \end{aligned} \right\} \text{all } i$	
$\left. \begin{aligned} S_iS_j &= S_jS_i \\ [S_i, X_j(t)] &= 1 \\ [X_i(t), X_j(u)] &= 1 \end{aligned} \right\} \text{all unjoined } i \neq j$	
$\left. \begin{aligned} S_iS_jS_i &= S_jS_iS_j \\ S_i^2S_jS_i^{-2} &= S_j^{-1} \\ X_i(t)S_jS_i &= S_jS_iX_j(t) \\ S_i^2X_j(t)S_i^{-2} &= X_j(t)^{-1} \\ [X_i(t), S_iX_j(u)S_i^{-1}] &= 1 \\ [X_i(t), X_j(u)] &= S_iX_j(tu)S_i^{-1} \end{aligned} \right\} \text{all joined } i \neq j$	

Table 3

Defining relations for $\mathfrak{St}_A(\mathbb{Z})$ when A is simply laced hyperbolic; see [Corollary 2](#).

$\left. \begin{aligned} [S_i^2, X_i] &= 1 \\ S_i &= X_iS_iX_iS_i^{-1}X_i \end{aligned} \right\} \text{all } i$	
$\left. \begin{aligned} S_iS_j &= S_jS_i \\ [S_i, X_j] &= 1 \\ [X_i, X_j] &= 1 \end{aligned} \right\} \text{all unjoined } i \neq j$	
$\left. \begin{aligned} S_iS_jS_i &= S_jS_iS_j \\ S_i^2S_jS_i^{-2} &= S_j^{-1} \\ X_iS_jS_i &= S_jS_iX_j \\ S_i^2X_jS_i^{-2} &= X_j^{-1} \\ [X_i, S_iX_jS_i^{-1}] &= 1 \\ [X_i, X_j] &= S_iX_jS_i^{-1} \end{aligned} \right\} \text{all joined } i \neq j$	

Remark. $X_i(u)$ in [Theorem 1](#) corresponds to X_i^u here; in particular $X_i = X_i(1)$. Also, one can show that $\tilde{h}_i(-1) = S_i^{-2}$ in \mathfrak{St}_A . So one could rewrite the relation $\tilde{h}_i(-1)^2 = 1$ as $S_i^4 = 1$.

The next result follows from the evident fact that each relation in [Table 2](#) involves at most 2 subscripts. If R is a field then the \mathfrak{St}_A case is a special case of a result of Abramenko and Mühlherr [\[1,18\]](#); see also [\[11\]](#).

Corollary 3 (*Curtis–Tits property of the presentation*). *Let R be a commutative ring and A a simply laced hyperbolic Dynkin diagram. Consider the Steinberg groups $\mathfrak{St}_B(R)$ and the obvious maps between them, as B varies over the singletons and pairs of nodes of A .*

The direct limit of this family of groups equals the Steinberg group $\mathfrak{St}_A(R)$. The same result holds with \mathfrak{G}_A in place of \mathfrak{St}_A throughout. \square

As one might expect, this result allows one to deduce finite-presentation results about $\mathfrak{St}_A(R)$ from similar results about the groups $\mathfrak{St}_B(R)$. The following theorem follows immediately from [Theorem 8](#) in the current paper (a restatement of [Theorem 1](#)) and Thm. 1.4 of [\[2\]](#). Of course, any finite presentation result will require some hypothesis on R . But conceptually one might think of the presentation in [Table 2](#) as “finite over R ” for any commutative ring R . By this we mean that the generators and relations have finitely many forms, with some of the forms being parameterized by elements of R (or pairs of elements).

Theorem 4 (*Finite presentation*). *In the setting of [Corollary 3](#), $\mathfrak{St}_A(R)$ is finitely presented as a group if either*

- (i) *R is finitely generated as a module over some subring generated by finitely many units, or*
- (ii) *$\mathrm{rk} A > 4$ and R is finitely generated as a ring.*

In either case, if the unit group of R is finitely generated as an abelian group, then $\mathfrak{G}_A(R)$ is also finitely presented as a group. \square

Many mathematicians have worked on the question of whether S -arithmetic groups in algebraic groups over adèle rings are finitely presented. This was finally resolved in all cases by Behr [\[6,7\]](#). Since Kac–Moody groups are infinite-dimensional analogues of algebraic groups, it is natural to ask whether their “ S -arithmetic groups” are finitely presented. The following result answers this, at least insofar as \mathfrak{G}_A is an analogue of an algebraic group. It is an immediate application of [Theorem 4](#).

Corollary 5 (*Finite presentation in arithmetic contexts*). *Suppose K is a global field, meaning a finite extension of \mathbb{Q} or $\mathbb{F}_q(t)$. Suppose S is a nonempty finite set of places of K , including all infinite places in the number field case. Let R be the ring of S -integers in K .*

Suppose A is a simply laced hyperbolic Dynkin diagram. Then $\mathfrak{G}_A(R)$ and $\mathfrak{St}_A(R)$ are finitely presented, except perhaps in the case that $\mathrm{rk} A = 4$, K is a function field and $|S| = 1$. \square

Higher-dimensional finiteness properties are also very interesting. Their analysis for most S -arithmetic subgroups of algebraic groups has recently been completed by Bux, Köhl and Witzel [\[8\]](#). One should be able to combine their results with [Corollary 3](#) to obtain higher-dimensional finiteness properties in the setting of [Corollary 5](#).

A major motivation for this work came from the conjectural appearance of integral forms of hyperbolic Kac–Moody groups as symmetries of supergravity and superstring

theories [5]. E_{10} is the “overextended” version of E_8 , and the corresponding overextended versions of E_6 and E_7 also appear in Table 1. Hull and Townsend conjectured that $\mathfrak{G}_{E_{10}}(\mathbb{Z})$ is the discrete “U-duality” group of Type II superstring theories [15]. And by analogy with $SL_2(\mathbb{Z})$, Damour and Nicolai conjectured that $\mathfrak{G}_{E_{10}}(\mathbb{Z})$ is the “modular group” for certain automorphic forms that are expected to arise in the context of 11-dimensional supergravity [13]. The role of E_{10} in the physics conjectures is somewhat mysterious and not well understood. We began this work by pondering how to give a “workable” definition of $\mathfrak{G}_{E_{10}}(\mathbb{Z})$. We hope that our explicit finite presentation will provide insight into these conjectures.

Our other major motivation was to bring Kac–Moody groups into the world of geometric and combinatorial group theory, leading to many new open questions such as those raised in [3].

The methods of this paper use hyperbolic geometry in an essential way. In particular, the proof of Theorem 8 relies on distance estimates in hyperbolic space. Therefore our proofs do not extend to general Kac–Moody groups. The simply-laced hypothesis could probably be removed at the cost of additional hypotheses on R , since double and triple bonds are known to cause complications over \mathbb{F}_2 . See [1] for this, in particular for the suggestion that a Kac–Moody group over \mathbb{F}_2 might fail to be finitely presented when all the nodes of its Dynkin diagram are joined to each other by double bonds. It is not clear yet how well the results of this paper will extend to the general case. But the first author has been able to treat some Kac–Moody groups beyond the hyperbolic cases of this paper. These results will appear separately.

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2. The Curtis–Tits presentation for Steinberg groups

We fix a simply laced hyperbolic Dynkin diagram A and a commutative ring R . We will briefly review Tits’ definition of $\mathfrak{St}_A(R)$, recall the “pre-Steinberg group” \mathfrak{PSt}_A from [2], prove that $\mathfrak{PSt}_A \rightarrow \mathfrak{St}_A$ is an isomorphism (Theorem 8), and deduce Theorem 1.

Regarding A as a generalized Cartan matrix, it is symmetric. So we may regard it as the inner product matrix of the simple roots and then extend linearly to the root lattice Λ . We indicate this inner product by “ \cdot ”. By the *norm*, α^2 , of an element $\alpha \in \Lambda$, we mean $\alpha \cdot \alpha$. The simple roots α_i have norm 2, and reflections w_{α_i} in the α_i are isometries of Λ . The Weyl group W is the group generated by the w_{α_i} . The W -images of the simple roots are called real roots, and we write Φ for the set of them. Since the inner product is W -invariant, all real roots have norm 2.

The group $\mathfrak{St}_A(R)$ is defined as a certain quotient of the free product $*_{\alpha \in \Phi} \mathfrak{U}_\alpha$, where \mathfrak{U}_α is a copy of the additive group of R . A standard difficulty in Lie theory is that it is impossible to distinguish a single “best” isomorphism $\mathfrak{U}_\alpha \cong R$. Instead, there is a natural

pair of parameterization $R \rightarrow \mathfrak{U}_\alpha$, differing by inversion. (Tits refers to the “double bases” of the root spaces [21, §3.3].) For each $\alpha \in \Phi$ we fix one of these isomorphisms and call it X_α , so the $X_\alpha(t)$ with $t \in R$ are the elements of \mathfrak{U}_α , with the obvious group operation.

The sign in Lemma 6 below depends on this choice, but only in a way that won’t affect any of our arguments. We remark also that our presentation in Table 2 does not involve a choice of X_α for every $\alpha \in \Phi$. We made such a choice only for the simple roots α_i . This choice for the simple roots does not distinguish any “natural” choices for the other roots. For example, if i and j are joined, then $t \mapsto S_i X_j(t) S_i^{-1}$ and $t \mapsto S_j X_i(t) S_j^{-1}$ are the two possibilities for $X_{\alpha_i + \alpha_j}$, which by symmetry are equally preferable. Happily, we do not need our choices to be natural in any way: one may choose the $X_{\alpha \in \Phi}$ arbitrarily.

Describing the relations requires a preliminary definition. Let (α, β) be a pair of real roots. Then (α, β) is called a *prenilpotent pair* if there exist $w, w' \in W$ such that

$$w\alpha, w\beta \in \Phi_+ \text{ and } w'\alpha, w'\beta \in \Phi_-.$$

One can show that a pair of real roots $\{\alpha, \beta\}$ is prenilpotent if and only if $\alpha \neq -\beta$ and

$$(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$$

is a finite set (see for example [12]).

The relations in $\mathfrak{St}_A(R)$ are the following: for each prenilpotent pair α, β and each pair $t, u \in R$, there is a relation

$$[X_\alpha(t), X_\beta(u)] = \prod_{\gamma \in \Phi \cap (\mathbb{N}\alpha + \mathbb{N}\beta) - \{\alpha, \beta\}} X_\gamma(v), \quad (1)$$

where the v ’s on the right side depend on α, β, t, u , the ordering on the γ ’s, and the chosen isomorphisms from R to $\mathfrak{U}_\alpha, \mathfrak{U}_\beta$ and the \mathfrak{U}_γ ’s. (A consequence of prenilpotency is that the product has only finitely many factors.) The following lemma describes the prenilpotent pairs in our situation, and makes these relations explicit. $\mathfrak{St}_A(R)$ is the quotient of $*_{\alpha \in \Phi} \mathfrak{U}_\alpha$ by all of these relations.

Lemma 6. *Suppose A is simply laced and hyperbolic. Then distinct real roots $\alpha, \beta \in \Phi$ form a prenilpotent pair just if $\alpha \cdot \beta \geq -1$. The corresponding relations in $\mathfrak{St}_A(R)$ are*

$$[X_\alpha(t), X_\beta(u)] = \begin{cases} X_{\alpha+\beta}(\pm tu) & \text{if } \alpha \cdot \beta = -1 \\ 1 & \text{otherwise} \end{cases}$$

for all $t, u \in R$.

We remark that if $\alpha \cdot \beta = -1$ then $\alpha + \beta$ is also a real root, so the first case makes sense. The sign in $X_{\alpha+\beta}(\pm tu)$ depends on the choices of isomorphisms $X_\alpha, X_\beta, X_{\alpha+\beta}$ from R to $\mathfrak{U}_\alpha, \mathfrak{U}_\beta, \mathfrak{U}_{\alpha+\beta}$. But we will not use the relation itself, merely the fact that $\mathfrak{U}_{\alpha+\beta} = [\mathfrak{U}_\alpha, \mathfrak{U}_\beta]$.

We will use the following special features of the hyperbolic case. First the signature of Λ is $(\operatorname{rk} A - 1, 1)$, so the vectors in $\Lambda \otimes \mathbb{R}$ of norm < 0 fall into two components. The fundamental chamber

$$C := \{x \in \Lambda \otimes \mathbb{R} : x \cdot \alpha_i \leq 0 \text{ for all } i\}$$

meets only one of these components, which we call the future cone F . The projectivization of F is a copy of real hyperbolic space of dimension $n := \operatorname{rk} A - 1$, for which we write H^n . Second, C lies in the closure \bar{F} , and its projectivization PC is a hyperbolic simplex together with its ideal vertices. The reason for this is that the Coxeter diagram underlying A is that of a finite-covolume hyperbolic reflection group; see [16, §§6.8–6.9]. The Weyl group W is defined as the group generated by the reflections in the simple roots, and the Tits cone is defined as the union of the W -images of C . We can now state the third special property: the interior of the Tits cone is exactly the future cone. One direction is obvious: since $C \subseteq \bar{F}$ and W preserves the open set F , the interior of the Tits cone lies in F . For the other direction, one must show that the W -translates of $C \cap F$ cover F . This is part of Poincaré’s polyhedron theorem. A very clean treatment in the case of reflection groups appears in [14, Thm. 60].

Proof of Lemma 6. First we use the coincidence of F with the Tits cone’s interior to rephrase prenilpotency as follows. *Claim: real roots α, β form a prenilpotent pair if and only if some vector of F has positive inner product with both of them, and some other vector of F has negative inner product with both of them.* This follows from the following observation. Fix a vector v in the interior of C and recall that Φ^+ consists of the real roots having negative product with it, and similarly for Φ^- . Then for any $w \in W$, w sends α, β into Φ^+ (resp. Φ^-) if and only if α, β have negative (resp. positive) inner product with $w^{-1}(v) \in F$.

Now suppose $\alpha, \beta \in \Phi$. Since $\Lambda \otimes \mathbb{R}$ has signature $(n, 1)$, α^\perp and β^\perp meet in F just if the inner product matrix of α and β is positive definite, i.e., just if $\alpha \cdot \beta \in \{0, \pm 1\}$. (Here \perp indicates the orthogonal complement in $\Lambda \otimes \mathbb{R}$.) In this case α^\perp and β^\perp are transverse at a point of $\alpha^\perp \cap \beta^\perp \cap F$. Obviously we may choose a nearby element of F on the positive side of both, and another element of F on the negative side of both. So in this case the pair is prenilpotent.

If $\alpha \cdot \beta \leq -2$ then their positive half-spaces in F are disjoint. Therefore it is impossible to choose a point of F that is on the positive side of both α^\perp and β^\perp . So the pair is not prenilpotent.

If $\alpha \cdot \beta \geq 2$ then one positive half-space in F lies inside the other, so obviously there is a point in the intersection, and similarly in the intersection of the negative half-spaces. So the pair is prenilpotent. This finishes the proof of the first claim.

If $\alpha \cdot \beta = -1$, then α, β are simple roots for an A_2 root system and the displayed relation is the corresponding Chevalley relation. If $\alpha \cdot \beta \geq 0$ then the only roots γ in

$\mathbb{N}\alpha + \mathbb{N}\beta$ are α and β (indeed these are the only vectors of norm ≤ 2). Since the product on the right side of the Chevalley relation (1) is empty, the relation is $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$. \square

We mentioned above that $PC \subseteq \overline{H^n}$ is a hyperbolic simplex. Its facets have a curious geometric property that turns out to be the key to our proof of Theorem 1. We have not seen anything like it in Kac–Moody theory before. Recall that the hyperbolic distance between two points of H^n , represented by vectors $x, y \in F$, is $\cosh^{-1} \sqrt{(x \cdot y)^2 / x^2 y^2}$.

Lemma 7. *Suppose ϕ is any facet of the projectivized Weyl chamber PC and p is any point of ϕ . Then there is a facet ϕ' of PC that makes angle $\pi/3$ with ϕ , such that the hyperbolic distance $d(p, \phi \cap \phi')$ is at most $\cosh^{-1} \sqrt{4/3} \approx 0.549$.*

Proof. We applied the following argument to each of the 18 possibilities for A . We write I for the set of A 's nodes, and for $i \in I$ we write ϕ_i for the corresponding facet of PC . We also fix elements $\omega_i \in \Lambda \otimes \mathbb{Q}$ with $\omega_i \cdot \alpha_j = -1$ or 0 according to whether $i, j \in I$ are equal or not. In Lie terminology these are the fundamental weights. Geometrically, ω_i represents the vertex of PC opposite ϕ_i .

We applied the following argument to each of the $\text{rk } A$ many possibilities for $\phi := \phi_i$. We write J for the set of $j \in I$ that are joined to i . We define q as the point of H^n represented by the sum of the $\omega_j \in J$. It lies in the interior of the face of ϕ that is opposite (in ϕ) to the face $\phi \cap (\cap_{j \in J} \phi_j)$ of ϕ . For each $j \in J$ we write K_j for the convex hull of q and $\phi \cap \phi_j$. By the property of q just mentioned, PC is the union of the K_j . For every $j \in J$ we found by inner product computations that $\cosh^2(d(q, \phi \cap \phi_j)) \leq 4/3$. In surprisingly many cases we found equality. We used the PARI/GP package [19] for the calculations.

Now, given $p \in \phi$, it lies in K_j for some $j \in J$, and we set $\phi' = \phi_j$. By K_j 's definition, q is its point furthest from $\phi \cap \phi'$. So

$$d(p, \phi \cap \phi') \leq d(q, \phi \cap \phi') \leq \cosh^{-1} \sqrt{4/3}. \quad \square$$

At the beginning of this section we mentioned the *pre-Steinberg group* \mathfrak{PSt}_A . It is a group functor defined in [2], by the same definition as the Steinberg group, except that we only impose the Chevalley relations for prenilpotent pairs α, β that are *classically prenilpotent*. This means that $(\mathbb{Q}\alpha + \mathbb{Q}\beta) \cap \Phi$ is finite and $\alpha + \beta \neq 0$. Arguing as in the proof of Lemma 6 shows that this is equivalent to $\alpha = \beta$ or $\alpha \cdot \beta \in \{0, \pm 1\}$. After stating the following theorem about \mathfrak{PSt}_A , we will show how Theorem 1 reduces to it. Then we will prove it.

Theorem 8. *The natural map $\mathfrak{PSt}_A(R) \rightarrow \mathfrak{St}_A(R)$ is an isomorphism.*

Proof of Theorem 1, given Theorem 8. Theorem 1.2 of [2] gives an explicit presentation for $\mathfrak{PSt}_A(R)$, namely the one in Table 2. So Theorem 8 is identical to the first part of Theorem 1.

For the second part, we recall that Tits defined $\mathfrak{G}_A(R)$ as the quotient of $\mathfrak{St}_A(R)$ by the relations $\tilde{h}_i(a)\tilde{h}_i(b) = \tilde{h}_i(ab)$ for all $i \in I$ and all units a, b of R . In fact imposing these relations for a single i automatically gives the others too. This follows from the fact that all roots are equivalent under the Weyl group, because A is simply laced. \square

Proof of Theorem 8. We must show that the Chevalley relations for classically prenilpotent pairs imply all the other Chevalley relations. We will abbreviate $\mathfrak{PSt}_A(R)$ and $\mathfrak{St}_A(R)$ to \mathfrak{PSt} and \mathfrak{St} .

By Lemma 6, any prenilpotent pair has inner product ≥ -1 . Therefore it suffices to prove the following by induction on $k \geq -1$: for every prenilpotent pair $\alpha, \beta \in \Phi$ with $\alpha \cdot \beta = k$, the Chevalley relations of α and β in \mathfrak{St} already hold in \mathfrak{PSt} . If $\alpha = \beta$ then the Chevalley relations say merely that \mathfrak{U}_α is commutative, which follows from the multiplication rules in \mathfrak{U}_α . So we will suppose $\alpha \neq \beta$. If $k \in \{0, \pm 1\}$ then the pair is classically prenilpotent, so this holds by definition of \mathfrak{PSt} .

So suppose $k := \alpha \cdot \beta \geq 2$. By Lemma 6, the Chevalley relation we must establish is $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$. We will exhibit roots α', α'' with

$$\alpha' + \alpha'' = \alpha, \quad \alpha' \cdot \beta > 0, \quad \text{and} \quad \alpha'' \cdot \beta > 0. \quad (2)$$

It follows that both these inner products are less than k . By induction we get $[\mathfrak{U}_{\alpha'}, \mathfrak{U}_\beta] = [\mathfrak{U}_{\alpha''}, \mathfrak{U}_\beta] = 1$. Since $\alpha = \alpha' + \alpha''$, we have $\mathfrak{U}_\alpha = [\mathfrak{U}_{\alpha'}, \mathfrak{U}_{\alpha''}]$. Since \mathfrak{U}_β commutes with $\mathfrak{U}_{\alpha'}$ and $\mathfrak{U}_{\alpha''}$, it also commutes with \mathfrak{U}_α , as desired.

It remains to construct α' and α'' . We distinguish two cases: $k = 2$ and $k > 2$. First suppose $k = 2$. What is special about this case is that the span of α and β is degenerate: $\nu := \alpha - \beta$ is orthogonal to both and has norm 0. It will suffice to exhibit α' with $\alpha' \cdot \alpha = \alpha' \cdot \beta = 1$, for then we can take $\alpha'' := \alpha - \alpha'$ and apply the previous paragraph.

Since $\bar{F} \cup (-\bar{F})$ is the set of vectors of norm ≤ 0 , we have $\nu \in \pm \bar{F}$. By exchanging α, β we may suppose without loss that $\nu \in \bar{F}$. We fix a vector $x \in \Lambda$ in the interior of C and consider its inner products with the images of ν under the Weyl group W . These inner products are integral because $x \in \Lambda$, and nonpositive because $x \cdot F \subseteq (-\infty, 0)$. Therefore they achieve their maximum, which is to say: after replacing α, β by their images under an element of W we may suppose without loss that $\nu \cdot x = \max_{w \in W} w(\nu) \cdot x$. This maximality forces $\nu \cdot \alpha_i \leq 0$ for all i , for if $\nu \cdot \alpha_i > 0$ then reflection in α_i increases ν 's inner product with x . So $\nu \in C$. That is, it represents an ideal vertex of PC .

The simple roots orthogonal to ν correspond to the nodes of an irreducible affine subdiagram A^ν of A , and the W -stabilizer of ν is the Weyl group of A^ν . By replacing α, β by their images under an element of this stabilizer, we may suppose without loss that α is a simple root corresponding to a node of A^ν . Since every node of A^ν is joined to some other node of A^ν , α lies in some A_2 root system in A^ν . Inside this A_2 is a root α' having inner product 1 with α . Since α' is orthogonal to ν , it also has inner product 1 with β . This finishes the case $k = 2$.

Now for the inductive step: fixing $k \geq 3$, we seek roots α', α'' with the properties (2). Regarding α^\perp and β^\perp as hyperplanes in H^n , we define $p, q \in H^n$ as follows. First, p is the point of α^\perp closest to β^\perp . Second, q is a point of α^\perp , which is orthogonal to some real root α' with $\alpha' \cdot \alpha = 1$, and closest possible to p among all such points. Because p lies on some facet of some chamber, Lemma 7 shows that such an α' exists, so the definition of q makes sense. It also shows that $d(p, q)$ is at most $\cosh^{-1} \sqrt{4/3}$.

Together, α and α' span an A_2 root system, of which $\alpha'' := \alpha - \alpha'$ is another root. To prove $0 < \beta \cdot \alpha'$ and $0 < \beta \cdot \alpha''$, suppose otherwise, say $\beta \cdot \alpha' = m \leq 0$. The idea is to work out p and q explicitly and find that $d(p, q)$ is larger than $\cosh^{-1} \sqrt{4/3}$, which is a contradiction. The inner product matrix of α, α', β is

$$\begin{pmatrix} 2 & 1 & k \\ 1 & 2 & m \\ k & m & 2 \end{pmatrix}$$

and (a vector representing) p is the projection of b to α^\perp , i.e., $p = \beta - \alpha(\beta \cdot \alpha)/2$. Now, q is the projection of p to $\alpha^\perp \cap \alpha'^\perp$, or equivalently $\alpha^\perp \cap (\alpha'' - \alpha')^\perp$. The advantage of the latter description is that α is orthogonal to $\alpha'' - \alpha'$. Since p is already orthogonal to α , projecting it to this codimension 2 subspace is the same as projecting it to $(\alpha'' - \alpha')^\perp$. For calculations in our basis we note that $\alpha'' - \alpha'$ has norm 6 and we rewrite it as $\alpha - 2\alpha'$. So $q = p - (\alpha - 2\alpha')(p \cdot (\alpha - 2\alpha'))/6$. Calculation reveals

$$d(p, q) = \cosh^{-1} \sqrt{\frac{4}{3} \cdot \frac{3 + km - k^2 - m^2}{4 - k^2}}$$

By applying $\partial/\partial m$ to the radicand and using $m \leq 0$ and $k \geq 3$, one checks that the right side is decreasing as a function of m . Therefore $d(p, q)$ is at least what one would get by plugging in 0 for m :

$$d(p, q) \geq \cosh^{-1} \sqrt{\frac{4}{3} \cdot \frac{3 - k^2}{4 - k^2}}$$

By differentiating one shows that the right side is decreasing in k . So $d(p, q)$ is larger than the limit as $k \rightarrow \infty$, which is $\cosh^{-1} \sqrt{4/3}$. This is a contradiction, proving the claim. \square

Remark. The origin of the proof was picture-drawing in the hyperbolic plane associated to the span of α, α', β . We recommend sketching the configuration of $\alpha^\perp, \alpha'^\perp, \alpha''^\perp$ and β^\perp when $m = 0$, and contemplating how it would change if m were negative. When $m = 0$, the quadrilateral spanned by p, q and their projections to β^\perp converges to a $(2, 3, \infty)$ triangle as $k \rightarrow \infty$. So the constant $\cosh^{-1} \sqrt{4/3}$ is the length of the short edge of the $(2, 3, \infty)$ triangle in H^2 . It is curious that this bound in Lemma 7, which was optimal, is only barely sufficient for the proof of Theorem 8.

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