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Maximal group actions on compact oriented surfaces

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ABSTRACT

Suppose S is a compact oriented surface of genus $\sigma \geq 2$ and C_p is a group of orientation preserving automorphisms of S of prime order $p \geq 5$. We show that there is always a finite supergroup $G > C_p$ of orientation preserving automorphisms of S except when the genus of S/C_p is minimal (or equivalently, when the number of fixed points of C_p is maximal). Moreover, we exhibit an infinite sequence of genera within which any given action of C_p on S implies C_p is contained in some finite supergroup and demonstrate for genera outside of this sequence the existence of at least one C_p-action for which C_p is not contained in any such finite supergroup (for sufficiently large σ).

1. Introduction

A finite group G is said to act in an *orientation preserving manner* on a compact oriented surface S of genus $\sigma \geq 2$ if there is an injection

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$$\epsilon\colon G \hookrightarrow \mathrm{Homeo}^+(S)$$

from G into the group of orientation preserving homeomorphisms. We denote such an action by the ordered pair (G, ϵ), though when unambiguous we write simply G. Two actions (G, ϵ_1), (G, ϵ_2) are said to be *topologically equivalent* if their images $\epsilon_1(G)$ and $\epsilon_2(G)$ are conjugate in $\mathrm{Homeo}^+(S)$.

In the following, we determine when a cyclic group C_p of prime order $p \geq 5$ of orientation preserving homeomorphisms of a surface S is *finitely maximal*, meaning there is no proper finite supergroup $G \leq \mathrm{Homeo}^+(S)$ containing C_p. We show that when such an action exists the genus of S/C_p is minimal (or equivalently, the number of fixed points of the C_p-action is maximal). Following this we show that, for sufficiently large genus, there exists a finitely maximal C_p-action on a surface of genus σ if and only if $\sigma \not\equiv \frac{p-3}{2} \left(\mathrm{mod}\, \frac{p-1}{2}\right)$.

Though an interesting problem in its own right, there are a number of other motivations for this work. For example, in the context of the moduli space $\mathcal{M}_\sigma$ of compact Riemann surfaces of genus σ, there is widespread interest in describing the branch locus, $\mathcal{B}_\sigma$, which is the subset of $\mathcal{M}_\sigma$ of surfaces with non-trivial automorphisms. We define $\mathcal{M}_\sigma^{(G,\epsilon)} \subset \mathcal{M}_\sigma$ to be the set of surfaces whose full group of conformal automorphisms is topologically equivalent to (G, ϵ), and $\overline{\mathcal{M}}_\sigma^{(G,\epsilon)}$ to be the set of surfaces whose full group of conformal automorphisms contains (G, ϵ). In [5], Broughton showed that the sets $\{\mathcal{M}_\sigma^{(G,\epsilon)}\}$ form a stratification of $\mathcal{B}_\sigma$ known as the *equisymmetric stratification*. A first step in describing this stratification is distinguishing between $\mathcal{M}_\sigma^{(G,\epsilon)}$ and $\overline{\mathcal{M}}_\sigma^{(G,\epsilon)}$; the following results represent a significant step in this direction for $G = C_p$ as well as extending current work ([2]) on identifying the isolated strata of $\mathcal{B}_\sigma$. For further reading on the branch locus of moduli space, see also [1,3,10,11,14].

This work also has implications for the connections between topological group actions and subgroups of the mapping class group. Specifically, if $\mathfrak{M}_\sigma$ denotes the mapping class group in genus σ, then there is a natural one-to-one correspondence between conjugacy classes of finite subgroups of $\mathfrak{M}_\sigma$ and equivalence classes of finite topological group actions on a smooth oriented surface of genus σ. Moreover, if $H < G$ both act on a surface of genus σ, then we have the corresponding containment in $\mathfrak{M}_\sigma$. As such, our results allow one to determine when a given conjugacy class in $\mathfrak{M}_\sigma$ of subgroups isomorphic to C_p is finitely maximal in $\mathfrak{M}_\sigma$. See [7,19] for other recent work in this area.

Perhaps the most important consequence of the following work is also the most direct one: it contributes significantly to the eventual goal of a complete classification of finitely maximal C_p-actions. Specifically, it was shown in [4] that for sufficiently large σ, the number of distinct quotient genera S/C_p for C_p-actions on a surface S of genus σ is linear in σ (though this can also be derived from Theorem 4 below). Theorem 5 therefore implies that when classifying maximal actions one need only consider a single quotient genus, thereby greatly reducing the complexity of the problem.

Finally, we believe this work paves the way for some interesting new problems. For example, our results provide at least the initial tools to develop a lower bound for the asymptotic growth rate in terms of the genus of the number of finitely maximal actions (currently it appears that this rate is bounded below by the growth rate of the prime numbers).

2. Preliminaries

We approach the study of topological group actions via surface kernel epimorphisms and generating vectors, as in Broughton [6]. A surface S of genus $\sigma \geq 2$ is topologically equivalent to a quotient of the upper half plane $\mathbb{H}/\Lambda$ where Λ is any torsion free Fuchsian group isomorphic to the fundamental group of S, also called a *surface group* for S. A finite group G acts on S if and only if $G = \Gamma/\Lambda$ for some Fuchsian group Γ containing such a Λ as a normal subgroup of index $|G|$. We call the map $\rho\colon \Gamma \to G$ a *surface kernel epimorphism*.

We define the *signature* of the action of G to be the tuple $(g; m_1, \ldots, m_r)$ where g is the genus of the quotient surface S/G (which we call the orbit genus of the signature) and the quotient map $\pi\colon S \to S/G$ is branched over r points with orders $m_1, \ldots, m_r$ (which we call the periods of the signature). For conciseness, when it appears in a signature we declare $(x)^k$ to mean x listed k separate times, whereas x^k denotes a single entry with x raised to the kth power. The structure of Γ is completely determined by the signature of G. Namely, if G has signature $(g; m_1, \ldots, m_r)$ and $[\ ,\ ]$ denotes commutator, then a presentation for Γ is

$$\Gamma = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \,\middle|\, c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{r} c_i \prod_{j=1}^{g} [a_j, b_j] \right\rangle \tag{1}$$

where

$$\sigma = 1 + |G|(g-1) + \frac{|G|}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right).$$

We call the first $2g$ generators of Γ *hyperbolic generators* and the last r *elliptic generators*. Note that the map ρ is completely determined by the images of the generators of Γ so a convenient way of representing a surface kernel epimorphism is through so-called generating vectors, defined as follows.

Definition. A vector of group elements $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \eta_1, \ldots, \eta_r)$ belonging to a finite group G is called a $(g; m_1, \ldots, m_r)$*-generating vector* for G with genus σ if all of the following hold:

1. $G = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \eta_1, \ldots, \eta_r \right\rangle$,
2. $\prod_{i=1}^{g} [\alpha_i, \beta_i] \cdot \prod_{j=1}^{r} \eta_j = e$, the identity of G,

3. $O(\eta_i) = m_i$, where $O(\ )$ denotes element order,
4. The Riemann–Hurwitz formula holds:

$$\sigma - 1 = |G| \left(g - 1 + \frac{1}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right).$$

We adopt the terminology for hyperbolic and elliptic generators in generating vectors as inherited from the corresponding Fuchsian groups and again we adopt the notation $(\alpha)^k$ to mean k copies of α and α^k to mean a single α raised to the kth power. Also, since our primary goal is to determine when a given C_p-action is finitely maximal we adopt this term for generating vectors themselves. That is, when we say a generating vector is (or is not) finitely maximal, it is understood that the corresponding topological group action is (or is not) finitely maximal.

Since we are describing group actions via generating vectors, we need to determine when two G-actions given by distinct generating vectors define the same action up to topological equivalence. Clearly two topologically equivalent G-actions have the same signature, but the converse is not necessarily true and determining whether two generating vectors define the same action for an arbitrary G is in general a very difficult problem. However, we are only considering group actions of cyclic groups, and the topological classification of such actions are known, see [16] (see also [12, Lemma 2] for cyclic prime group actions and [9, Theorem 7] for all cyclic groups). We summarize and translate these results into the language of generating vectors below.

Theorem 1. *Fix a prime p and let C_p denote a cyclic group of order p. For any $g \geq 2$, there exists precisely one C_p-action up to topological equivalence with signature $(g; -)$. If $r > 1$, then two $(g; m_1, \ldots, m_r)$-generating vectors*

$$(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \eta_1, \ldots, \eta_r) \quad \textit{and} \quad (\alpha'_1, \beta'_1, \ldots, \alpha'_g, \beta'_g, \eta'_1, \ldots, \eta'_r)$$

for G define topologically equivalent group actions if and only if there exists a permutation $\chi \in S_r$ and $\tau \in \mathrm{Aut}(C_p)$ such that

$$(\tau(\eta_{\chi(1)}), \ldots, \tau(\eta_{\chi(r)})) = (\eta'_1, \ldots, \eta'_r)$$

i.e., *the last r generators differ by permutation and/or automorphism of C_p.*

In order to analyze finitely maximal actions, we shall first try to understand when they are not maximal. The first step in this process is determining the signature of a subgroup H from a group G, a problem originally solved in [17]. Since we are only interested in when H is cyclic of prime order, we translate this result into the language of generating vectors and specialize to this case:

Theorem 2. *Let $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \eta_1, \dots \eta_r)$ be a $(g; m_1, \dots m_r)$-generating vector for G. For $C_p \leq G$ let $\Phi\colon G \to S_{G/C_p} \cong S_{[G:C_p]}$ denote the map induced by action of G on the left cosets of C_p. Then the signature of C_p is*

$$(h; (p)^{n_1}, (p)^{n_2}, \dots, (p)^{n_r})$$

where n_i is the number of cycles of length m_i/p in $\Phi(\eta_i)$, and h is found by solving the equation

$$|G| \cdot \left(2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right) = p\left(2h - 2 + \sum_{i=1}^{r} n_i \left(1 - \frac{1}{p}\right)\right).$$

In the special case where C_p is normal, all cycles of $\Phi(\eta_i)$ have the same length, meaning $n_i = 0$ if $\Phi(\eta_i)$ has order m_i, and $n_i = |G|/m_i$ otherwise.

We finish by fixing some notation. Henceforth C_p will denote a cyclic group of prime order $p \geq 5$ acting on a compact Riemann surface S of genus $\sigma \geq 2$ and, for a given group G, e will denote its identity element.

3. Normal extensions of C_p

In this section, we show how to extract a generating vector for $C_p \leq G$ given a generating vector for G. On the level of Fuchsian groups, if Λ is a unformizing Fuchsian group for S, then there are corresponding Fuchsian groups Γ_G, Γ_{C_p} and surface kernel epimorphisms $\rho_G\colon \Gamma_G \to G$ and $\rho_{C_p}\colon \Gamma_{C_p} \to C_p$ where $\rho_G|_{\Gamma_{C_p}} = \rho_{C_p}$. Therefore, given a generating vector for G, we can determine a corresponding generating vector for C_p by considering how the generators of the group Γ_{C_p} relate to the generators of the group Γ_G.

Given an arbitrary group G and subgroup H, this process can be very difficult. However, in Theorem 6 we show that if C_p is not finitely maximal, then it is necessarily a normal subgroup of either the cyclic group C_{pq} of order pq (with q a prime not necessarily distinct from p), a semi-direct product $C_p \rtimes C_q$ (q a prime different from p), or a direct product $C_p \times C_p$. Thus, when considering whether or not C_p is finitely maximal, we only need decide whether or not it is contained normally in one of these groups. We may then invoke [20, Theorem 7.1] (also Proposition 2 of [15] or Theorem 1 of [18]) to find the elliptic elements of the generating vector of C_p from the elliptic elements of the generating vector of the supergroup, simplifying things substantially (see also [8] where a similar process is used). Using these observations with Theorems 1 and 2, we obtain the following.

Lemma 1. *Suppose $C_p \trianglelefteq C_{pq}$. Let $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, d_1, \dots, d_m, c_1, \dots, c_n, f_1, \dots, f_k)$ be a $(g; (q)^m, (p)^n, (pq)^k)$-generating vector for C_{pq}, where $n + k > 0$ (and $m = 0$ if $q = p$). Then C_p has signature $(h; (p)^r)$ where $h = gq + (k + m - 2)\left(\frac{q-1}{2}\right)$ and $r = nq + k$, and its corresponding generating vector is equivalent to $\left((e)^{2h}, (c_1)^q, \dots, (c_n)^q, f_1^q, \dots, f_k^q\right)$.*

Lemma 2. *Suppose $C_p \trianglelefteq (C_p \rtimes C_q)$. Let $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, c_1, \ldots, c_n, d_1, \ldots, d_m)$ be a $(g; (p)^n, (q)^m)$-generating vector for $C_p \rtimes C_q$ where $n + m > 0$ and let $a \neq 1$ be an integer such that $a^q \equiv 0 \pmod{p}$. Then C_p has signature $(h; (p)^r)$ where $h = gq + (m-2)\left(\frac{q-1}{2}\right)$ and $r = nq$, and its corresponding generating vector is equivalent to $((e)^{2h}, c_1, c_1^a, \ldots, c_1^{a^{q-1}}, c_2, c_2^a, \ldots c_n^{a^{q-1}})$.*

Lemma 3. *Suppose $C_p \trianglelefteq (C_p \times C_p)$. Let $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, c_1, \ldots, c_n, d_1, \ldots, d_m)$ be a $(g; (p)^{n+m})$-generating vector for $C_p \times C_p$, where $c_1, \ldots, c_n \in C_p$ and $d_1, \ldots, d_m \notin C_p$ for some fixed subgroup C_p, and suppose $n + m > 0$. Then C_p has signature $(h; (p)^r)$ where $h = gp + (m-2)\left(\frac{p-1}{2}\right)$ and $r = np$, and its corresponding generating vector is equivalent to $((e)^{2h}, (c_1)^p, \ldots (c_n)^p)$.*

We illustrate an application of these results by proving the non-maximality of two different C_p-actions with certain special signatures, a result we will need later.

Theorem 3. *Suppose that $\vec{v}$ is an $(h; (p)^r)$-generating vector for the cyclic group C_p of prime order p for $r \leq 2$. Then $\vec{v}$ is never maximal.*

Proof. First note that there are no generating vectors for C_p when $r = 1$, so we only need consider $r = 0$ and $r = 2$. For both $r = 0$ and $r = 2$, by Theorem 1, there is a unique generating vector $\vec{v}$ for C_p (up to topological equivalence) so in each case if we can exhibit the existence of a generating vector $\vec{u}$ for some group G containing C_p as a normal subgroup acting with signature $(h; -)$ or $(h; p, p)$, then $\vec{v}$ must coincide with this vector.

First suppose $r = 0$ and let $D_p = \langle x, y \mid x^2 = y^p = e, xyx = y^{-1} \rangle$ denote the dihedral group of order $2p$. Since we must have $h \geq 1$, the vector $\vec{u} = (xy, xy, (x)^{2h})$ is clearly a generating vector for D_p with signature $(0; (2)^{2h+2})$. Application of Lemma 2 implies that the C_p subgroup has signature $(h; -)$, and by uniqueness, its corresponding generating vector must be equivalent to $\vec{v}$.

For $r = 2$ let $C_{2p} = \langle y \mid y^{2p} = e \rangle$ denote the cyclic group of order $2p$. The vector $\vec{u} = ((y^p)^{2h}, y, y^{-1})$ is clearly a generating vector for C_{2p} with signature $(0; (2)^{2h}, (2p)^2)$. Application of Lemma 2 implies that the C_p subgroup has signature $(h; p, p)$, and by uniqueness, its corresponding generating vector must be equivalent to $\vec{v}$. $\square$

4. Signatures of C_p-actions

Before we develop our main results, we first describe the possible signatures of C_p that can occur for a fixed genus σ. Since it is important in the description of C_p-actions, for a given σ and p we henceforth let κ denote the integer with $0 \leq \kappa < (p-1)/2$ and $\kappa = \sigma \left(\bmod \frac{p-1}{2}\right)$.

Theorem 4. *For $\sigma \geq 2$, with the single exception where $\frac{2(\sigma - 1 - p(\kappa - 1))}{p-1} - \delta p = 1$, the valid signatures for a C_p-action on a surface of genus σ are*

$$\left(\kappa + \delta\left(\frac{p-1}{2}\right); (p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1} - \delta p}\right)$$

where δ *runs over the integers satisfying* $0 \leq \delta \leq \frac{2(\sigma-1-p(\kappa-1))}{p(p-1)}$.

Proof. First, it is easy to verify that each of these signatures satisfies the Riemann–Hurwitz formula for genus σ. Now suppose that $(h; (p)^r)$ is any other signature that satisfies the Riemann–Hurwitz formula for genus σ, so

$$\sigma - 1 = p(h-1) + r\left(\frac{p-1}{2}\right).$$

Since $\left(\kappa; (p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1}}\right)$ also satisfies the Riemann–Hurwitz formula for genus σ, we also have

$$\sigma - 1 = p(\kappa-1) + \frac{2(\sigma-1-p(\kappa-1))}{p-1}\left(\frac{p-1}{2}\right).$$

Subtracting the first equation from the second, we get

$$0 = p(\kappa-h) + \left(\frac{2(\sigma-1-p(\kappa-1))}{p-1} - r\right)\left(\frac{p-1}{2}\right).$$

Since $(p-1)/2$ is relatively prime to p, this equation can only hold if $\frac{2(\sigma-1-p(\kappa-1))}{p-1} - r = \delta p$ for some δ, or equivalently, $r = \frac{2(\sigma-1-p(\kappa-1))}{p-1} - \delta p$. Substituting back into the equation, we get $h - \kappa = \delta\left(\frac{p-1}{2}\right)$ or $h = \kappa + \delta\left(\frac{p-1}{2}\right)$. It follows that $(h; (p)^r)$ must be one of the signatures specified in the statement of the Theorem.

The case $\frac{2(\sigma-1-p(\kappa-1))}{p-1} - \delta p = 1$ is excluded since this would result in a generating vector of the form $(h; p)$, which is never valid for a C_p-action. For the remaining signatures, Harvey's Theorem ([13]) shows each is a valid signature for a C_p-action on a surface of genus σ. $\square$

We note that for a fixed σ, Theorem 4 provides an algorithm to construct all possible signatures for a C_p-action on a surface of genus σ. Specifically, start with the signature $\left(\kappa; (p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1}}\right)$ and then add and subtract multiples of $(p-1)/2$ and p, respectively, to κ and to $(p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1}}$ until the number of p's is less than p (with the single exception being when $r = p+1$, and for this case, we terminate at this point). We also observe that the signature $\left(\kappa; (p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1}}\right)$ is the signature exhibiting the smallest orbit genus (which is necessarily between 0 and $(p-3)/2$), and any other orbit genus must be at least $(p-1)/2$. This signature also has the largest number of periods.

5. The signature of a maximal action

We now have the necessary tools to prove our first main result: that most of the signatures with which C_p can act result in actions that are not finitely maximal.

Theorem 5. *Suppose that $\vec{v}$ is an $(h;(p)^r)$-generating vector for the cyclic group C_p of prime order p where $r \geq 3$. If $h \geq (p-3)/2$, then $\vec{v}$ is not maximal.*

Proof. Letting $C_p = \langle x \rangle$, we shall show that each such vector is the restriction of a generating vector of $C_{2p} = \langle y \rangle$ where $x = y^2$.

By pairing like elements, we first observe that for any generating vector $\vec{v}$ of C_p there exist integers n and k with $r = 2n+k$ and $k \leq p-1$ such that after applying appropriate transformations from Theorem 1 we have

$$\vec{v} = \left(e, ..., e, x^{c_1}, x^{c_1}, x^{c_2}, x^{c_2}, ..., x^{c_n}, x^{c_n}, x^{d_1}, .., x^{d_k}\right) \tag{2}$$

(note that $2c_1 + \ldots 2c_n + d_1 + \ldots + d_k \equiv 0 (\operatorname{mod} p)$). Letting $m = 2h + 2 - k$, since we are assuming $h \geq \frac{p-3}{2}$ and we know $k \leq p-1$, we have $m \geq 0$ since

$$m = 2h + 2 - k \geq 2\left(\frac{p-3}{2}\right) + 2 - (p-1) = 0.$$

Therefore, $(0; (2)^m, (p)^n, (2p)^k)$ is a valid signature for a C_{2p}-action whose C_p-subgroup exhibits signature $(h;(p)^r)$ by Lemma 1. As it will be important later, we note that m and k have the same parity.

To prove non-maximality, we show the generating vector

$$\vec{u} = \left((y^p)^m, y^{2c_1}, ...y^{2c_n}, y^{d_1+\chi(d_1)p}, \ldots, y^{d_k+\chi(d_k)p}\right)$$

where $\chi\colon C \to \{0,1\}$ is the characteristic function on the even integers is a $(0; 2^m, p^n, (2p)^k)$-generating vector for C_{2p} whose restriction to C_p is $\vec{v}$.

If $k > 0$, clearly the elements $(y^p)^m, y^{2c_1}, ...y^{2c_n}, y^{d_1+\chi(d_1)p}, \ldots, y^{d_k+\chi(d_k)p}$ generate C_{2p} (since at least one of them has order $2p$). If $k = 0$, then $m = 2h + 2 \geq 2$, and by assumption we know $r \geq 2$ so $(y^p)^m, y^{2c_1}, ...y^{2c_n}$ must also generate C_{2p} (since it contains an element of order 2 and an element of p). So in all cases, the elements of $\vec{u}$ generate C_{2p}. Next we check they satisfy the necessary relation. We have

$$\prod_{i=1}^{m} y^p \prod_{i=1}^{n} y^{2c_i} \prod_{i=1}^{k} y^{d_i+\chi(d_i)p} = y^{mp+2c_1+\ldots+2c_n+d_1+\chi(d_1)p\ldots+d_k+\chi(d_k)p} = y^N.$$

Since $2c_1 + \ldots 2c_n + d_1 + \ldots + d_k \equiv 0 \,(\operatorname{mod} p)$, we must also have $2c_1 + \ldots 2c_t + d_1 + \chi(d_1)p \ldots + d_k + \chi(d_k)p \equiv 0 \,(\operatorname{mod} p)$ and thus y^N is either the identity or has order 2. However, for each i, $d_i + \chi(d_i)p$ is odd and the parity of m and k are the same, so it follows that $mp + d_1 + \chi(d_1)p \ldots + d_k + \chi(d_k)p$ is even. In particular, y^N is the identity and thus $\vec{u}$ is a generating vector for C_{2p}.

To see that $\vec{v}$ is the restriction of $\vec{u}$, we simply apply Lemma 1. $\square$

The following result is immediate.

Corollary 1. *Fix a genus $\sigma \geq 2$. If $\kappa = \frac{p-3}{2}$ then there are no finitely maximal actions of C_p on a surface of genus σ. Otherwise, the signature of any finitely maximal C_p-action is*

$$\left(\kappa; (p)^{\frac{2(\sigma-1-p(\kappa-1))}{p-1}}\right).$$

In particular, S/C_p has the smallest possible quotient genus, and the action of C_p has the maximal number of fixed points over all possible C_p-actions on a surface of genus σ.

Proof. By Theorem 5, the orbit genus h of any signature $(h; (p)^r)$ which might result in a finitely maximal generating vector for C_p must satisfy $h < (p-3)/2$. Theorem 4 then proves the result. $\square$

6. The existence of maximal actions

Next we show that, for sufficiently large genus, there exists a finitely maximal action if and only if $\kappa \neq \frac{p-3}{2}$. In order to do this, we first note that Corollary 1 ensures that the orbit genus of the signature $(h; (p)^r)$ of a finitely maximal action must satisfy $h < \frac{p-3}{2}$, so we henceforth assume this to be the case. To exhibit the existence of a maximal action, we shall build generating vectors which cannot possibly extend to a larger group. Though this would typically be very difficult, we shall first show that such a C_p is necessarily contained in one of C_{pq}, $C_p \rtimes C_q$ or $C_p \times C_p$, as defined in Section 3. This significantly simplifies calculations as we only need consider whether a generating vector $\vec{v}$ for C_p is the restriction of a generating vector of one of the groups that contains C_p, and for this we can invoke Lemmas 1, 2 and 3.

We start by making some preliminary observations about a group C_p that is not contained in any of C_{pq}, $C_p \rtimes C_q$ or $C_p \times C_p$. Now clearly if $G \geq C_p$ and C_p is normal in some subgroup H of G, then it is necessarily contained in a subgroup isomorphic to one of these three groups. Therefore, we may restrict to groups in which C_p is not normal in any subgroup.

Since C_p is not normal in any subgroup of G, it must be a Sylow subgroup, so we can use the Sylow theorems to determine certain information about the structure of G and the signature of its action. First, we know that $N = [G : C_p] = sp + 1$ for some s, and there are precisely $sp + 1$ subgroups of order p, all of which are conjugate in G. It also follows that if $x \in G$ is an element whose order is divisible by p, then it must have order p. Thus G has signature of the form $(g; (p)^k, m_1, \ldots, m_t)$ where $p \nmid m_i$ for all i and for some k. The following Lemma states that we can specify this signature much further.

Lemma 4. *Suppose that C_p acts with signature $(h;(p)^r)$ and $G > C_p$ where C_p is not a proper normal subgroup in any subgroup H of G. Then the signature of G is $(g;(p)^r, m_1, \ldots, m_t)$ where $p \nmid m_i$ for all i.*

Proof. By our observations above, we already know that G has signature $(g;(p)^k, m_1, \ldots, m_t)$ for some k, so we just need to show that $k = r$. In order to do this, we use Theorem 2 and the notation introduced in that result. If ζ is an element of a generating vector of order p for G, then $\Phi(\zeta)$ can only have cycles of length p and length 1. Since $N = sp+1$, there must be at least one cycle of length 1 and hence for each element in a generating vector $\vec{v}$ of order p, it must induce at least one element in a generating vector for C_p; *i.e.*, $r \leq k$.

Now suppose that ζ is an element of a generating vector for G of order p. Given a coset gC_p, $\zeta g C_p = gC_p$ if and only if $g^{-1}\zeta g C_p = C_p$. This means $g^{-1}\zeta g \in C_p$, or $\zeta \in gC_pg^{-1}$. Suppose that ζ stabilizes two distinct cosets g_1C_p and g_2C_p. Then it follows that $\zeta \in g_1C_pg_1^{-1}$ and $\zeta \in g_2C_pg_2^{-1}$. Since both are cyclic of prime order, it follows that $g_1C_pg_1^{-1} = g_2C_pg_2^{-1}$ or $g_2^{-1}g_1C_pg_1^{-1}g_2 = C_p$, so $g_2^{-1}g_1$ normalizes C_p. However, C_p is its own normalizer, so $g_2^{-1}g_1 \in C_p$ or $g_1 \in g_2C_p$, so $g_1C_p = g_2C_p$, a contradiction. It follows that ζ stabilizes at most one coset of C_p and thus $k \leq r$. It follows that $r = k$. $\square$

Theorem 6. *Suppose that C_p acts with signature $(h;(p)^r)$ and extends to a group G but is not normal in any subgroup of G. Then $h \geq \frac{p-3}{2}$.*

Proof. Under these assumptions, Lemma 4 gives the signature of G. Using the same notation, applying Theorem 2 and simplifying, we have

$$sp+1 = \frac{2(h-1) + r\left(\frac{p-1}{p}\right)}{2(g-1) + r\left(\frac{p-1}{p}\right) + \sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right)}$$

for $s \geq 1$. This means that

$$p+1 \leq \frac{2(h-1) + r\left(\frac{p-1}{p}\right)}{2(g-1) + r\left(\frac{p-1}{p}\right) + \sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right)}.$$

Rewriting, we get

$$(p+1)\left(2(g-1) + r\left(\frac{p-1}{p}\right) + \sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right)\right) \leq 2(h-1) + r\left(\frac{p-1}{p}\right).$$

Simplifying, we get

$$p(g-1) + g + \frac{r(p-1)}{2} + \frac{p+1}{2}\sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right) \leq h.$$

By Theorem 3, we may assume $r \geq 3$, and we know that $t \geq 0$ and $g \geq 0$. Thus

$$\frac{p-3}{2} = -p + \frac{3(p-1)}{2} \leq p(g-1) + g + \frac{r(p-1)}{2} + \frac{p+1}{2}\sum_{i=1}^{t}\left(1 - \frac{1}{m_i}\right) \leq h. \qquad \square$$

Theorem 6 proves that if C_p acts with signature $(h;(p)^r)$ where $h < (p-3)/2$ and C_p is contained in some larger group G, then it must be normal in some subgroup of G. In particular, it will be a subgroup of either C_{pq}, $C_p \rtimes C_q$, or $C_p \times C_p$. We now show finitely maximal actions exist by constructing generating vectors for C_p which are not restrictions of generating vectors for any of these groups.

Theorem 7. *If $h \leq (p-5)/2$ and $r > p+7$, there exists a maximal generating vector $\vec{v}$ of C_p with signature $(h;(p)^r)$.*

Proof. By our previous remarks, we only need to determine whether or not a generating vector for C_p arises from the action of a supergroup – C_{pq}, $C_p \rtimes C_q$, or $C_p \times C_p$ – in which C_p is normal. We first make a couple of observations about such generating vectors.

By Lemma 2, if $g \in C_p$ occurs exactly t times as an elliptic element in a generating vector $\vec{u}$ for $C_p \rtimes C_q$, $t \geq 1$, then there exists at least one other group element different from g that occurs exactly t times in $\vec{v}$ too. Second, by Lemma 3 if $g \in C_p$ appears as an elliptic element in a generating vector $\vec{v}$ for $C_p \times C_p$, then it must occur tp times for some $t \geq 1$.

Now if $r \not\equiv 1 \bmod p$, for s satisfying $2s \equiv -(r-(p+1)) \bmod p$, let

$$\vec{v}_1 = \left((e)^{2h}, x, x^2, \ldots x^{p-2}, x^{p-1}, (x)^{r-(p+1)}, (x^s)^2\right)$$

and if $r \equiv 1 \bmod p$, for s satisfying $4s \equiv -(r-(p+3)) \bmod p$, let

$$\vec{v}_2 = \left((e)^{2h}, x, x^2, \ldots x^{p-2}, x^{p-1}, (x)^{r-(p+3)}, (x^s)^4\right).$$

Clearly both are generating vectors for C_p with signature $(h;(p)^r)$ (note that such s's always exist since $p \geq 5$).

Since there are at least $p-3$ elliptic elements occurring exactly once in both $\vec{v}_1$ and $\vec{v}_2$, neither of these could be the restriction of a generating vector $\vec{u}$ of $C_p \times C_p$. Next, assuming $x^s \neq x$, x^s appears precisely three times in $\vec{v}_1$ and five times in $\vec{v}_2$. Since $r > p+7$, x appears at least 8 times in $\vec{v}_1$ and at least and 6 times in $\vec{v}_2$. In both cases all other powers of x appear exactly once. In particular, in either generating vector, no other element appears the same number of times as x^s, so neither are the restriction of a generating vector $\vec{u}$ of $C_p \rtimes C_q$. If $x^s = x$, then x is the only element to appear multiple times in either $\vec{v}_1$ or $\vec{v}_2$, and so again, neither are the restriction of a generating vector $\vec{u}$ of $C_p \rtimes C_q$.

To finish, we need to show that neither $\vec{v}_1$ nor $\vec{v}_2$ are the restriction of a generating vector $\vec{u}$ of C_{pq}. If one were, then it would be of the form given in Lemma 1 and there would exist k, m, g such that $h = gq + \frac{(k+m-2)(q-1)}{2}$. Now, if $q \neq 2$, since $p-3$ elements are never repeated in either $\vec{v}_1$ or $\vec{v}_2$, we must have $k \geq p-3$ so

$$h = gq + \frac{(k+m-2)(q-1)}{2} \geq \frac{(p-5)(q-1)}{2} > \frac{p-5}{2},$$

contradicting our initial assumption on h. If $q = 2$ and $x = x^s$, then there are $p-2$ elements that are never repeated in either $\vec{v}_1$ or $\vec{v}_2$, so we must have $k \geq p-2$. If $x \neq x^s$, then x^s appears an odd number of times in both $\vec{v}_1$ and $\vec{v}_2$, and so we must also have $k \geq p-2$. Therefore, in both cases, we have

$$h = 2g + \frac{(k+m-2)}{2} \geq \frac{(p-4)}{2} > \frac{p-5}{2},$$

again contradicting our initial assumption on h. It follows that neither $\vec{v}_1$ nor $\vec{v}_2$ are the restriction of a generating vector $\vec{u}$ of C_{pq}. $\square$

The following result is immediate.

Corollary 2. *For sufficiently large σ, there exists a finitely maximal C_p-action on a surface of genus σ if and only if $\kappa \neq \frac{p-3}{2}$.*

Proof. Suppose that C_p acts on a surface of genus σ with $(h;(p)^r)$. Theorem 3 implies if $r \leq 2$, then C_p is not finitely maximal and Theorem 5 implies if $r \geq 3$ and $h \geq (p-3)/2$, then C_p is not finitely maximal. Theorem 7 implies that if $h \leq (p-5)/2$ and $r > p+7$, then there always exists a finitely maximal C_p-action. The only signatures excluded from these results are of the form $(h;(p)^r)$ with $0 \leq h \leq (p-5)/2$ and $3 \leq r \leq p+7$. However, there are only finitely many such excluded signatures, so there exists a genus σ_0 (which can be found using the Riemann–Hurwitz formula) such that for $\sigma > \sigma_0$, C_p does not act with any of these excluded signatures. Therefore, for $\sigma > \sigma_0$ there exists a finitely maximal C_p-action on a surface of genus σ if and only if $\kappa \neq \frac{p-3}{2}$. $\square$

We finish by noting that based on computations for small primes, in the case of the excluded signatures, there are instances where there are finitely maximal actions and there are instances where there are no finitely maximal actions, see Examples 1 and 2. However, the arguments to prove this become increasingly ad-hoc as there does not appear to be any predictable pattern on the signatures that do exhibit finitely maximal actions; we feel this would be an interesting topic for future study.

Example 1. The vectors (x, x, x^5) and (x, x^2, x^4) are the only distinct generating vectors (up to topological equivalence) for $C_7 = \langle x \rangle$ with signature $(0;(7)^3)$. The first of these vectors extends to an action of C_{14} using Lemma 1, and the second extends to an action of $C_7 \rtimes C_3$ using Lemma 2. In particular, there are no finitely maximal C_7 actions with signature $(0; (7)^3)$.

Example 2. The vector $\vec{v} = (x, x^2, x^3, x^4, x^4)$ is a generating vector for $C_7 = \langle x \rangle$ with signature $(0; (7)^5)$. Since x^4 is the only element appearing twice in $\vec{v}$, by Lemmas 2 and 3, it can only extend to a cyclic group. However, if it extends to C_{7q} with signature $(g; (q)^m, (7)^n, (7q)^k)$, q a prime, then $k \geq 3$ since three elements of $\vec{v}$ are never repeated. By Lemma 1, this would mean

$$0 = gq + \frac{(k+m-2)(q-1)}{2} \geq \frac{1}{2}$$

a contradiction. Thus, $\vec{v}$ corresponds to a maximal action on some surface.

References

[1] G. Bartolini, A.F. Costa, M. Izquierdo, On the connectivity of branch loci of moduli spaces, Ann. Acad. Sci. Fenn. Math. 38 (1) (2013) 245–258.

[2] G. Bartolini, A.F. Costa, M. Izquierdo, On isolated strata of p-gonal Riemann surfaces in the branch locus of moduli spaces, Albanian J. Math. 6 (1) (2012) 11–19.

[3] G. Bartolini, M. Izquierdo, On the connectedness of the branch locus of the moduli space of Riemann surfaces of low genus, Proc. Amer. Math. Soc. 140 (1) (2012) 35–45.

[4] S. Bozlee, A. Wootton, Asymptotic equivalence of group actions on surfaces and Riemann–Hurwitz solutions, Arch. Math. (Basel) 102 (6) (2014) 565–573.

[5] S.A. Broughton, The equisymmetric stratification of the moduli space and the Krull dimension of the mapping class group, Topology Appl. 37 (1990) 101–113.

[6] S.A. Broughton, Classifying finite group actions on surfaces of low genus, J. Pure Appl. Algebra 69 (1990) 233–270.

[7] S.A. Broughton, A. Wootton, Finite Abelian subgroups of the mapping class group, Algebr. Geom. Topol. 7 (2007) 1651–1697.

[8] E. Bujalance, F.J. Cirre, M.D.E. Conder, On extendability of group actions on compact Riemann surfaces, Trans. Amer. Math. Soc. 355 (2003) 1537–1557.

[9] M. Carvacho, Nonequivalent families of group actions on Riemann surfaces, J. Pure Appl. Algebra 217 (12) (2013) 2345–2355.

[10] A.F. Costa, M. Izquierdo, On the existence of connected components of dimension one in the branch locus of moduli spaces of Riemann surfaces, Math. Scand. 111 (1) (2012) 53–64.

[11] A.F. Costa, M. Izquierdo, Equisymmetric strata of the singular locus of the moduli space of Riemann surfaces of genus 4, in: Geometry of Riemann Surfaces, in: London Math. Soc. Lecture Note Ser., vol. 368, Cambridge Univ. Press, Cambridge, 2010, pp. 120–138.

[12] J. Gilman, On conjugacy classes in the Teichmüller modular group, Michigan Math. J. 23 (1976) 53–63.

[13] W.J. Harvey, Cyclic groups of automorphisms of a compact Riemann surface, Quart. J. Math., Oxford Ser. (2) 17 (1966) 86–97.

[14] M. Izquierdo, D. Ying, Equisymmetric strata of the moduli space of cyclic trigonal Riemann surfaces of genus 4, Glasg. Math. J. 51 (1) (2009) 19–29.

[15] A. Kontogeorgis, The group of automorphisms of cyclic extensions of rational function fields, J. Algebra 216 (2) (1999) 665–706.

[16] J. Nielsen, Die Struktur periodischer Transformationen von Flächen, Danske Vid Selsk. Mat-Fys. Medd. 1 (1937) 1–77.

[17] D. Singerman, Subgroups of Fuchsian groups and finite permutation groups, Bull. Lond. Math. Soc. 2 (1970) 319–323.

[18] M. Streit, J. Wolfart, Characters and Galois invariants of regular dessins, Rev. Mat. Complut. 13 (1) (2000) 49–81.

[19] M. Stukow, Conjugacy classes of finite subgroups of certain mapping class groups, Turkish J. Math. 28 (2004) 101–110.
[20] A. Wootton, Defining equations for cyclic prime covers of the Riemann sphere, Israel J. Math. 157 (2007) 103–122.