Mesoprimary decomposition of binomial submodules

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A R T I C L E   I N F O

A B S T R A C T

Recent results of Kahle and Miller give a method of constructing primary decompositions of binomial ideals by first constructing "mesoprimary decompositions" determined by their underlying monoid congruences. Mesoprimary decompositions are highly combinatorial in nature, and are designed to parallel standard primary decomposition over Noetherean rings. In this paper, we generalize mesoprimary decomposition from binomial ideals to "binomial submodules" of certain graded modules over a monoid algebra, analogous to the way primary decomposition of ideals over a Noetherean ring R generalizes to R-modules. The result is a combinatorial method of constructing primary decompositions that, when restricting to the special case of binomial ideals, coincides with the method introduced by Kahle and Miller.

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1. Introduction

Fix a field $\Bbbk$ and a commutative monoid Q. A *binomial ideal* in the monoid algebra $\Bbbk[Q]$ is an ideal I whose generators have at most two terms. The quotient $\Bbbk[Q]/I$ by a

binomial ideal identifies, up to scalar multiple, any monomials appearing in the same
binomial in I. This induces a *congruence* $\sim_I$ on the monoid Q (an equivalence relation
perserving additivity), and the quotient module $\Bbbk[Q]/I$ is naturally graded with a decom-
position into 1-dimensional $\Bbbk$-vector spaces, at most one per $\sim_I$-class. In [4], Kahle and
Miller introduce *mesoprimary decompositions*, which are combinatorial approximations
of primary decompositions of I constructed from the congruence $\sim_I$.

   Mesoprimary decomposition of binomial ideals is motivated by combinatorially con-
structed primary decompositions of monomial ideals. Any monomial ideal I in the
monoid algebra $\Bbbk[Q]$ is uniquely determined by the monomials it contains. Taking the
quotient $\Bbbk[Q]/I$ amounts to setting these monomials to 0, and the monomials that lie
outside of I naturally grade the quotient $\Bbbk[Q]/I$ with a decomposition into 1-dimensional
$\Bbbk$-vector spaces.

   An irreducible decomposition for a monomial ideal I whose components are themselves
monomial ideals can be constructed by locating *witness monomials* $\mathbf{x}^w$ whose annihilator
modulo I is prime, and then constructing for each witness monomial $\mathbf{x}^w$ the primary
monomial ideal that contains all monomials not lying below $\mathbf{x}^w$. The intersection of these
ideals (one per witness monomial) equals I, and the witnesses are readily identified from
the grading on $\Bbbk[Q]/I$. See [8, Chapter 5] for a full treatment of monomial irreducible
decomposition.

   Combinatorially constructed irreducible decompositions of monomial ideals have also
been shown to live within a larger categorical setting. Much in the way primary de-
composition of ideals over a Noetherean ring R generalizes to R-modules, combina-
torial methods for constructing primary decompositions of monomial ideals can be
generalized to certain modules whose gradings resemble the fine gradings of mono-
mial quotients. See [7] for an overview of these constructions and [3,6] for conse-
quences.

   Kahle and Miller use congruences to extend the above construction from mono-
mial ideals to binomial ideals [4]. Given a binomial ideal I, they pinpoint a collec-
tion of monomials in $\Bbbk[Q]/I$ that behave like witnesses. For each witness $\mathbf{x}^w$, they
construct the *coprincipal component* at $\mathbf{x}^w$, a binomial ideal containing I whose quo-
tient has $\mathbf{x}^w$ as the unique greatest nonzero monomial. The resulting collection of
ideals, one for each witness, decomposes I, and each component admits a canonical
primary decomposition. In this way, mesoprimary decompositions act as a bridge to
primary components of a binomial ideal from the combinatorics of its induced congru-
ence.

   Mesoprimary decompositions are constructed in two settings: first for monoid con-
gruences, and then for binomial ideals; both are designed to parallel standard pri-
mary decomposition in a Noetherian ring R. This motivated Kahle and Miller to
pose Problems 1.1 and 1.2 below, which appeared as [4, Problem 17.11] and [4, Prob-
lem 17.13], respectively. These problems, in turn, serve to motivate the results in this
paper.

Problem 1.1. Generalize mesoprimary decomposition of monoid congruences to congruences on monoid modules.

Problem 1.2. Develop a notion of binomial module over a commutative monoid algebra, and generalize mesoprimary decomposition of binomial ideals to this setting.

One of the largest tasks in generalizing the results of [4] to monoid modules is to separate which constructions should happen in the monoid and which should happen in the module, since these coincide for monoid congruences. See Remarks 3.2 and 3.11 for specific instances of this distinction.

In the first part of this paper (Sections 2–4), we introduce the category Q-Mod of modules over a monoid Q (Definition 2.1) and generalize nearly every result from [4] on monoid congruences to congruences on monoid modules. We define primary and mesoprimary monoid module congruences (Definition 3.3) and give equivalent conditions for these congruences in terms of associated objects and witnesses (Theorems 3.8 and 3.12). We then construct a mesoprimary decomposition, with one component per key witnesses, for any monoid module congruence (Theorem 4.6). The resulting theory completely answers Problem 1.1.

The second part of this paper (Sections 5–7) answers Problem 1.2. We introduce the category $\mathcal{B}_Q$ (Definition 5.6), whose objects are tightly graded modules (Definition 5.1) over the monoid algebra $\Bbbk[Q]$ graded by monoid modules in Q-Mod. It is in this setting that we define binomial submodules (Definition 6.1). We define mesoprimary submodules (Definition 6.3) and associated mesoprime ideals (Definition 6.5), developing a theory of mesoprimary decomposition (Theorem 7.4) that parallels results in [4]. In particular, the binomial submodules of the free module $\Bbbk[Q]$ are precisely the binomial ideals; see Example 6.4.

To conclude the paper, we demonstrate in Section 8 how a binomial primary decomposition may be recovered from a mesoprimary decomposition when the underlying field is algebraically closed.

1.1. Notation

Throughout this paper, assume Q is a finitely generated commutative monoid and $\Bbbk$ is a field. Denote by $\Bbbk[Q]$ the monoid algebra over Q with coefficients in $\Bbbk$. Unless otherwise stated, all $\Bbbk[Q]$-modules are assumed to be finitely generated.

2. The category of monoid modules

In this section, we define the category Q-Mod of modules over a commutative monoid Q and extend some of the fundamental concepts and results from monoid ideals and congruences to the objects of this category. The content of this section (as well as

Section 5) is motivated by Example 2.14. First, we record some preliminary definitions (see [2] for more detail).

Definition 2.1. Fix a commutative monoid Q.

(1) A *Q-module* $(T, \cdot)$ is a set T together with a left action by Q that satisfies $0 \cdot t = t$ and $(q + q') \cdot t = q \cdot (q' \cdot t)$ for all $t \in T$, $q, q' \in Q$. A subset $T' \subset T$ is a *submodule* of T if it is closed under the Q-action, that is, $Q \cdot T' \subset T'$. The submodule of T *generated by elements* $t_1, \ldots, t_r \in T$ is $\langle t_1, \ldots, t_r \rangle = \bigcup_{i=1}^{r} Q \cdot t_i$.
(2) A map $\psi : T \to U$ between Q-modules T and U is a *Q-module homomorphism* if $\psi(q \cdot t) = q \cdot \psi(t)$ for all $t \in T, q \in Q$. The set of Q-module homomorphisms from T to U is denoted by $\mathrm{Hom}_Q(T, U)$, and is naturally a Q-module with action $q \cdot \psi$ given by $(q \cdot \psi)(t) = \psi(q \cdot t)$.
(3) The *category of Q-modules*, denoted Q-Mod, is the category whose objects are Q-modules and whose morphisms are Q-module homomorphisms.

Direct sums, direct products, and tensor products exist in the category Q-Mod. We now state their constructions explicitly.

Definition 2.2. Fix two Q-modules T and U.

(1) The *direct sum* $T \oplus U$ is the disjoint union $T \coprod U$ as sets, with the natural Q-action on each component.
(2) The *direct product* $T \times U$ is the cartesian product of T and U as a set, with componentwise Q-action.
(3) The *tensor product* $T \otimes_Q U$ is the collection of formal elements $t \otimes u$ for $t \in T$ and $u \in U$ modulo the equivalence relation generated by

$$t \otimes (q \cdot u) \sim (q \cdot t) \otimes u \text{ for } t \in T \text{ and } u \in U$$

The action of Q is given by $q \cdot (t \otimes u) = (q \cdot t) \otimes u$ for $q \in Q$, $t \in T$ and $u \in U$.

Definition 2.3. Fix a Q-module T. A *congruence on* T is an equivalence relation $\sim$ on T that satisfies $t \sim t' \Rightarrow q \cdot t \sim q \cdot t'$ for all $q \in Q$ and $t, t' \in T$. The *quotient module* $T/\sim$ is the set of equivalence classes of T under $\sim$. The congruence condition on $\sim$ ensures that $T/\sim$ has a well defined action by Q.

Definition 2.4. A subset $T \subset Q$ is an *ideal* if it is a Q-submodule of Q, that is, $Q + T \subset T$. An ideal $P \subset Q$ is *prime* if its complement in Q is a submonoid of Q.

Definition 2.5. Fix a Q-module T, a prime ideal $P \subset Q$, and set $F = Q \setminus P$. The *localization of* T *at* P, denoted T_P, is the set $T \times F$ modulo the equivalence relation $\sim$ that sets $(t, f) \sim (t', f')$ whenever $w \cdot f' \cdot t = w \cdot f \cdot t'$ for some $w \in Q$. The localization

Q_P is naturally a monoid, and T_P is naturally a Q_P-module. Write $t - f$ to denote the class of the element $(t, f) \in T \times F$ in T_P.

Remark 2.6. Any congruence $\sim$ on a Q-module T induces a congruence on T_P.

Definition 2.7. Fix a Q-module T. *Green's preorder* on T sets $t \preceq t'$ whenever $\langle t \rangle \supseteq \langle t' \rangle$. *Green's relation* on T sets $t \sim t'$ whenever $\langle t \rangle = \langle t' \rangle$.

Green's preorder on a monoid orders its elements by divisibility, and this notion extends to Q-modules.

Lemma 2.8. *Green's relation $\sim$ on a Q-module T is a congruence on T, and the quotient $T/\sim$ is partially ordered by divisibility.*

Proof. For $t, t' \in T$ and $q \in Q$, we can see $\langle t \rangle = \langle t' \rangle$ implies $\langle q \cdot t \rangle = \langle q \cdot t' \rangle$. Each element of the quotient $T/\sim$ generates a distinct submodule, so the divisibility preorder is antisymmetric, and thus a partial order. $\square$

We now generalize the notion of a nil element of a monoid.

Definition 2.9. An element $\infty \in T$ in a Q-module T is called a *nil* if it is absorbing, that is, $Q \cdot \infty = \{\infty\}$. The *basin* of a nil $\infty \in T$ is the set

$$B(\infty) = \{t \in T : qt = \infty \text{ for some } q \in Q\}$$

of elements of T that can be sent to ∞ under the action of Q. The *nil set* of T, denoted $N(T)$, is the collection of all nil elements in T.

Definition 2.10. Fix a subset $U \subset T$ of a Q-module T. A *Q-orbit* of U is a connected component of the undirected graph whose vertices are elements of U and whose edges connect two vertices $s, t \in U$ whenever $q \cdot s = t$ for some $q \in Q$. T is *connected* if it has at most one Q-orbit, and T is *properly connected* if $T \setminus N(T)$ has at most one Q-orbit.

Example 2.11. Let T and U be connected Q-modules with nils ∞_T and ∞_U, respectively. If $T \setminus \{\infty_T\}$ and $U \setminus \{\infty_U\}$ are both nonempty, the module $(T \coprod U)/\langle \infty_T \sim \infty_U \rangle$ is connected and has a single nil, but it is not properly connected, since removing the nil produces two distinct Q-orbits.

Remark 2.12. Unlike a monoid, a Q-module may have more than one nil element. However, by Lemma 2.13, each Q-orbit can have at most one nil element.

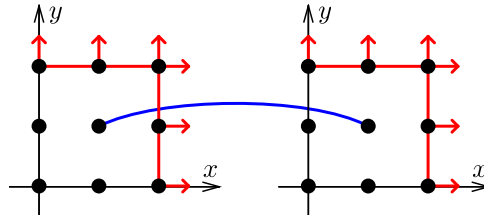**Lemma 2.13.** *The basin of a nil element $\infty \in T$ in a Q-module T is the Q-orbit of T containing ∞.*

Fig. 1. The monoid module that grades M in Example 2.14.

Proof. The basin of ∞ is clearly contained in its Q-orbit, and whenever $qt = s$ for $q \in Q$ and $s, t \in T$, we have $t \in B(\infty)$ if and only if $s \in B(\infty)$. $\quad \square$

Example 2.14. Let $Q = \mathbb{N}^2$, $I = \langle x^2, y^2 \rangle \subset \mathbb{k}[Q]$, $R = \mathbb{k}[Q]/I$, and

$$M = (R \oplus R)/\langle xye_1 - xye_2 \rangle,$$

where e_1 and e_2 generate the free $\mathbb{k}[Q]$-module $R \oplus R$. R is graded by the quotient monoid $Q/\sim_I$, and M is graded by two disjoint copies of $Q/\sim_I$ with both copies of xy and the nil elements identified. Unlike the monoid that grades R, this grading does not have a natural monoid structure. It does, however, have a natural action by Q, corresponding to the action on M by monomials in $\mathbb{k}[Q]$. See Fig. 1 for an illustration.

There is also a notion of decomposition of Q-modules into indecomposables.

Lemma 2.15. *Every Q-module T has a unique decomposition $T = \bigoplus_i T_i$ as a direct sum of connected modules.*

Proof. Any Q-module is the disjoint union of its Q-orbits. $\quad \square$

Remark 2.16. Kernels, in the categorical sense, do not exist in the category Q-Mod. However, there is still a notion of kernel of a Q-module homomorphism as a congruence; see Definition 2.17. This definition is justified by Theorem 2.18, a Q-module analogue of the first isomorphism theorem for groups.

Definition 2.17. Fix a homomorphism $\phi : T \to U$. The *kernel of ϕ*, denoted $\ker(\phi)$, is the congruence $\sim$ on T that sets $t \sim t'$ whenever $\phi(t) = \phi(t')$ for $t, t' \in T$.

Theorem 2.18. *If $\phi : T \to U$ is a Q-module homomorphism, then $T/\ker(\phi) \cong \mathrm{Im}(\phi)$.*

Proof. The homomorphism ϕ is surjective onto its image, and the quotient of T by $\ker(\phi)$ identifies elements with the same image under ϕ. This ensures that the map $T/\ker(\phi) \longrightarrow \mathrm{Im}(\phi)$ is both injective and surjective. $\quad \square$

Corollary 2.19. *Any finitely generated Q-module T is isomorphic to a quotient of a direct sum of finitely many copies of Q.*

Proof. Fix a finitely generated Q-module $T = \langle t_1, \ldots, t_r \rangle$. Let $\phi : \bigoplus_{i=1}^{r} Q \longrightarrow T$, where the map on the i-th summand is given by $Q \to \langle t_i \rangle$. This map is surjective, so Theorem 2.18 implies $T \cong (\bigoplus_{i=1}^{r} Q)/\ker(\phi)$. □

3. Primary and mesoprimary monoid modules

Mesoprimary decomposition of monoid congruences models primary decomposition of ideals in a Noetherian ring R, with mesoprimary congruences playing the role of primary ideals and prime congruences playing the role of prime ideals. In this section, we generalize the notion of mesoprimary monoid congruences to congruences on monoid modules (Definition 3.3), analogous to the way primary decomposition of ideals in R generalizes to finitely generated R-modules. The main result is Theorem 3.12, which generalizes [4, Theorem 6.1] and characterizes mesoprimary monoid module congruences in terms of their associated prime congruences (Definition 3.10).

Definition 3.1. Fix a Q-module T. For each $q \in Q$, let ϕ_q denote the map $T \xrightarrow{\cdot q} T$ given by action by q.

- An element $q \in Q$ acts *cancellatively on T* if ϕ_q is injective.
- An element $q \in Q$ acts *nilpotently on T* if for each $t \in T$, $(nq) \cdot t \in N(T)$ for some nonnegative integer n.
- An element $t \in T$ is *partly cancellative* if whenever $a \cdot t = b \cdot t \notin N(T)$ for $a, b \in Q$ that act cancellatively on T, the morphisms ϕ_a and ϕ_b coincide.

Remark 3.2. Each term in Definition 3.1 is also defined in [4, Definition 2.9] for monoid elements. However, we are forced to make a distinction between monoid elements and monoid module elements (these objects coincide in the setting of [4]). In particular, "cancellative" and "nilpotent" (Definition 3.1) are properties of monoid elements, whereas "partly cancellative" is a property of monoid module elements. Roughly speaking, "cancellative" and "nilpotent" describe how a particular $q \in Q$ acts on different module elements, whereas "partly cancellative" describes how different monoid elements act on a particular $t \in T$.

Definition 3.3. A Q-module T is

- *primary* if each $q \in Q$ is either cancellative or nilpotent on T.
- *mesoprimary* if it is primary and each $t \in T$ is partly cancellative.

A congruence $\sim$ on T is *primary* (respectively, *mesoprimary*) if $T/\sim$ is a primary (respectively, mesoprimary) Q-module.

Lemma 3.4. *Fix a congruence $\sim$ on Q. The Q-module $T = Q/\sim$ is (meso)primary in the sense of Definition 3.3 if and only if $\sim$ is a (meso)primary monoid congruence in the sense of [4, Definition 2.12].*

Proof. For $q \in Q$ let $\overline{q}$ denote the image of q modulo $\sim$. An element $q \in Q$ acts cancellatively on T if and only if its image modulo $\sim$ is cancellative, and q acts nilpotently on T if and only if it has nilpotent image modulo $\sim$. This proves that T is a primary Q-module if and only if $\sim$ is primary as a monoid congruence. Lastly, assuming $\sim$ is P-primary, notice that for $a, b \notin P$, $\phi_a = \phi_b$ if and only if $\overline{a} = \overline{b} \in T$, so each $\overline{q} \in T$ is partly cancellative as a monoid element if and only if it is partly cancellative as an element of a Q-module. This completes the proof. $\square$

Remark 3.5. We now generalize witnesses and key witnesses from [4] to the setting of monoid module congruences. Definition 3.6, while complex, closely resembles [4, Definition 4.7]; see the original text for several motivating examples. We use key witnesses to construct the mesoprimary components (Definition 4.3) appearing in the monoid module congruence decomposition in Theorem 4.6.

Definition 3.6. Let T be a Q-module, $P \subset Q$ a prime ideal, and $\sim$ a congruence on T. For $t \in T$, let $\overline{t}$ denote the image of t in $\overline{T}_P$, and for $p \in P$, let $\phi_p : \overline{T}_P \to \overline{T}_P$ denote the morphism given by the action of p.

(1) An element $w \in T$ is *exclusively maximal* in a set $A \subset \overline{T}_P$ if $\overline{w}$ is the unique maximal element of A under Green's preorder.
(2) An element $w \in T$ with non-nil image in $\overline{T}_P$ is a $\sim$-*witness* for P if for each generator $p \in P$, the class of $\overline{w}$ is non-singleton under $\ker(\phi_p)$ and $\overline{w}$ is not exclusively maximal in that class.
(3) An element $w' \in T$ is an *aide* for a $\sim$-witness w for P and a generator $p \in P$ if w and w' have distinct images in $\overline{T}_P$ but are not distinct under $\ker(\phi_p)$.
(4) An element w with non-nil image in $\overline{T}_P$ is a *key $\sim$-witness for P* if $\overline{w}$ is non-singleton under $\bigcap_{p \in P} \ker(\phi_p)$ and $\overline{w}$ is not exclusively maximal in this non-singleton class.
(5) The prime P is *associated to T* if T has a witness for P, or if $P = \emptyset$ and T has a Q-orbit with no nil.

Remark 3.7. Prior to [4], primary decomposition of monoid congruences was developed by Grillet [1], but these decompositions are too course to effectively recover primary components at the level of binomial ideals [4, Example 2.22]. Nevertheless, in an effort to create a more complete picture, we also generalize primary congruences to our current setting of monoid module congruences.

Theorem 3.8. *A finitely generated Q-module T is primary if and only if it has exactly one associated prime ideal.*

Proof. Suppose T is primary. The set of elements with nilpotent action on T is a prime ideal $P \subset Q$. Since P is finitely generated, some non-nil element $w \in T$ satisfies $P \cdot w \subset N(T)$. This means w is a witness for P, so P is associated to T. Since $Q \setminus P$ acts cancellatively on T, any prime associated to T is contained in P. Moreover, localizing T at any prime P' contained in P identifies any element $w \in T$ with the nil in its orbit, since some $p \in P \setminus P'$ gives $p \cdot w \in N(T)$. Thus, any associated prime must also contain P, which implies P is the only associated prime.

Now suppose T has only one associated prime $P \subset Q$. If $P = \emptyset$, then every element of Q acts cancellatively on T. Now suppose P is nonempty, and fix $t \in T$. The submodule $\langle t \rangle$ is isomorphic to Q modulo some congruence. Since each witness in $\langle t \rangle$ is a witness for P, $\langle t \rangle$ is P-primary by [4, Corollary 4.21]. This means each $p \in P$ acts nilpotently on $\langle t \rangle$ and each $f \in Q \setminus P$ acts cancellatively on $\langle t \rangle$. Since t is arbitrary, each $p \in P$ acts nilpotently on T and each $f \in Q \setminus P$ acts cancellatively on T, meaning T is P-primary. □

Lemma 3.9 generalizes [4, Lemma 2.19] and is central to several proofs, including Theorems 3.12 and 4.6.

Lemma 3.9. *Fix a connected, P-primary Q-module T, and set $F = Q \setminus P$. Let T/F denote the quotient of T by the congruence*

$$t \sim t' \text{ whenever } f \cdot t = g \cdot t' \text{ for } f, g \in F$$

Then Green's preorder on T/F is a partial order, and T/F is finite.

Proof. Since T is P-primary, the morphisms $T \xrightarrow{\cdot f} T$ are injective for all $f \in F$, so $\sim$ is a well-defined congruence. If $\langle t \rangle = \langle t' \rangle$, then $f \cdot t = t'$ and $g \cdot t' = t$ for some $f, g \in Q$. This means $f \cdot g \cdot t = t$, so f and g are not nilpotent and lie in F, meaning t and t' are identified in T/F. This proves Green's preorder is antisymmetric.

Now, the remaining statement is trivial if $P = \emptyset$, so suppose P is nonempty. T must have a nil ∞ since Q contains elements with nilpotent action on T. The image of ∞ in T/F remains nil as well. Thus, since Q and T are both finitely generated, T/F must be finite. □

Definition 3.10. Fix a Q-module T, a monoid prime $P \subset Q$, and a non-nil $w \in T$.

(1) Let $G_P \subset Q_P$ denote the unit group of Q_P, and let $K_w^P \subset G_P$ denote the stabilizer of $\overline{w} \in T_P$ under the action of G_P.
(2) Let $\approx$ denote the congruence on Q_P that sets $a \approx b$ whenever
    (a) a and b lie in P_P, or
    (b) a and b lie in G_P and $a - b \in K_w^P$.
(3) The *P-prime congruence of T at w* is given by $\ker(Q \to Q_P/ \approx)$.
(4) The P-prime congruence at w is *associated* to T if w is a key witness for T.

Remark 3.11. In Definition 3.10, we are forced to make another distinction between T and Q: should an associated prime congruence of T be a congruence on T or on Q? The condition for a monoid congruence $\sim$ to be P-mesoprimary can be characterized in terms of the congruence on $Q \setminus P$ induced by its action on $Q/\sim$ [4, Corollary 6.7]. The partly cancellative condition is what ensures that each $t \in T$ induces the same congruence, which in our setting is a condition on elements of T.

Next, we characterize mesoprimary Q-modules in terms of their associated prime congruences, generalizing [4, Theorem 6.7] and [4, Corollary 6.7].

Theorem 3.12. *For a Q-module T, the following are equivalent.*

(1) *T is mesoprimary.*
(2) *T has exactly one associated prime congruence.*
(3) *T is P-primary, and for $F = Q \setminus P$,*

$$\ker(F \to \langle t \rangle) = \ker(F \to \langle t' \rangle)$$

for each non-nil $t, t' \in T$.

Proof. From any of these conditions, we conclude that T is primary, say with associated prime P. Notice that $\ker(F \to \langle t \rangle)$ is the prime congruence at t restricted to F. If these congruences coincide for all $t \in T$, then in particular they coincide for all witnesses, so T has exactly one associated prime congruence. This proves $(3) \Rightarrow (2)$.

Now suppose T is mesoprimary, and fix $t, t' \notin N(T)$. Then since t and t' are both partly cancellative, $a \cdot t = b \cdot t$ if and only if $a \cdot t' = b \cdot t'$ for $a, b \notin P$. This means the kernels $\ker(F \to \langle t \rangle)$ and $\ker(F \to \langle t' \rangle)$ coincide. This proves $(1) \Rightarrow (3)$.

Lastly, suppose T has exactly one associated prime congruence, and fix $t \in N(T)$. Fix $a, b \notin P$ and let $\phi_a, \phi_b : T \to T$ denote the actions of a and b on T, respectively. By Theorem 2.18, $\langle t \rangle \cong Q/\sim$ for some congruence $\sim$. Since T has only one associated prime congruence, so does $\sim$, so by [4, Theorem 6.1], $\sim$ is mesoprimary. This means $a \cdot t = b \cdot t$ if and only $a \cdot w = b \cdot w$ for any witness $w \in \langle t \rangle$. Since T has only one associated prime congruence, these actions also coincide for all witnesses in T, meaning $\phi_a = \phi_b$. This proves $(2) \Rightarrow (1)$, thus completing the proof. $\square$

We conclude this section with Theorem 3.13, which ensures that the mesoprimary decomposition constructed in Theorem 4.6 has finitely many components.

Theorem 3.13. *Any finitely generated Q-module T has only finitely many Green's classes of key witnesses.*

Proof. Fix a generating set $g_1, \ldots, g_k$ for T. For each g_i, consider the map $\phi_i : Q \to \langle g_i \rangle$ and let $\sim_i = \ker \phi_i$. The induced isomorphism $Q/\sim_i \to \langle g_i \rangle$ gives a bijection between key
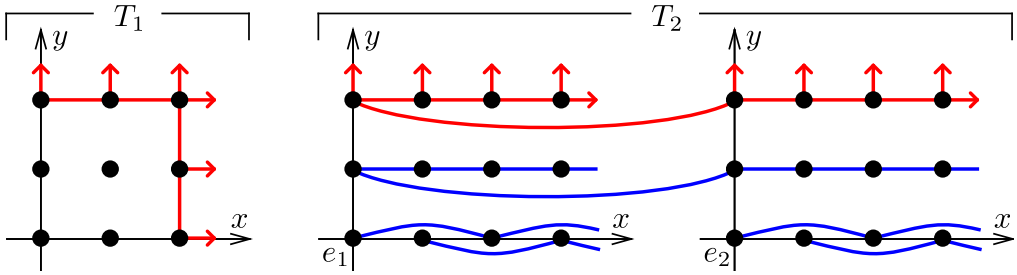
Fig. 2. The monoid module $T = T_1 \oplus T_2$ in Example 4.1.

T-witness and key $\sim_i$-witnesses, and by [4, Theorem 5.6], each congruence $\sim_i$ has only finitely many Green's classes of key witnesses. Since $g_1, \ldots, g_k$ generate T, this bounds the number of Green's classes of key T-witnesses. $\square$

4. Mesoprimary decomposition of monoid modules

In this section, we construct a mesoprimary decomposition for any monoid module congruence $\sim$ (with one caveat; see Remark 4.8). First, we construct a mesoprimary component for each $\sim$-witness.

Example 4.1. Let $Q = \mathbb{N}^2$ and write $R = \mathbb{k}[Q] = \mathbb{k}[x, y]$. Let T_1 denote the Q-module grading $M_1 = R/\langle x^2, y^2 \rangle$, let T_2 denote the Q-module grading the R-module

$$M_2 = (R \oplus R)/\langle (x^2 - 1)e_1, (x^2 - 1)e_2, (x - 1)ye_1, y(e_1 - e_2), y^2 e_1 \rangle,$$

where e_1 and e_2 denote the generators of $M \oplus M$. Write ∞_1 and ∞_2 for the nil elements of T_1 and T_2, respectively, and let $T = T_1 \oplus T_2$. See Fig. 2 for a depiction.

The mesoprimary decomposition of T constructed in Theorem 4.6 has 4 components: one component is the identity congruence on T_1 and identifies every element of T_2 with ∞_1; the remaining 3 components (one for each key witness $e_1, e_2, ye_1 \in T_2$) decompose T_2 and each identifies every element of T_1 with ∞_2. The only non-identity relation in the common refinement of all 4 congruences relates ∞_1 and ∞_2.

Definition 4.2. Fix a Q-module T. A *cogenerator* of T is a non-nil element $t \in T$ with $q \cdot t \in N(T)$ for every nonunit $q \in Q$. A Q-module T is *coprincipal* if it is P-mesoprimary and all its cogenerators lie in the same Green's class in T_P. A congruence $\sim$ on T is *coprincipal* if $T/\sim$ is a coprincipal Q-module.

Definition 4.3. Fix a Q-module T, a prime $P \subset Q$, and a witness $w \in T$ for P. Let $\overline{q}$ denote the image of $q \in Q$ in Q_P, and $\overline{t}$ denote the image of $t \in T$ in T_P.

- The *order ideal* $T_{\preceq w}^P$ *cogenerated by w at P* consists of those $a \in T$ whose image $\overline{a} \in T_P$ precedes $\overline{w}$ under Green's preorder.

- The *congruence cogenerated by w along P* is the equivalence relation $\sim_w^P$ on T that sets all elements outside of $T_{\preceq w}^P$ equivalent and sets $a \sim_w^P b$ whenever $\overline{a}$ and $\overline{b}$ differ by a unit in T_P and $q \cdot \overline{a} = q \cdot \overline{b} = \overline{w} \in T_P$ for some $q \in Q_P$.

Lemma 4.4 justifies the nomenclature in Definition 4.3.

Lemma 4.4. *The congruence cogenerated by w along P is a coprincipal congruence on T cogenerated by w. Furthermore, $T/\sim_w^P$ is properly connected, and if $T \setminus T_{\preceq w}^P$ is nonempty, then it is the nil class of $T/\sim_w^P$.*

Proof. Let $T' = T/\sim_w^P$. Every non-nil element of T' has the image of w as a multiple, so T' is properly connected, and it is clear that the image of $T \setminus T_{\preceq w}^P$ is nil modulo $\sim_w^P$ as long as it is nonempty. Furthermore, w cogenerates $\sim_w^P$ since the result of acting by any $p \in P$ lies outside $T_{\preceq w}^P$, and any $t \in T$ with non-nil image in T' satisfies $q \cdot t = w$ for some $q \in Q$, so every cogenerator for $\sim_w^P$ lies in the Green's class of w in T_P.

It remains to show that T' is mesoprimary. By Lemma 3.9, $T_{\preceq w}^P$ has finitely many Green's classes in T_P, so each $p \in P$ acts nilpotently on T' and thus T' is P-primary. Furthermore, for each $t \in T$ and for $a, b \in Q \setminus P$, we have $a \cdot t \sim_w^P b \cdot t$ if and only if $a \cdot w \sim_w^P b \cdot w$. In particular, the P-prime congruences at the non-nil elements of T' coincide, so by Theorem 3.12, T' is mesoprimary. $\square$

Definition 4.5. Fix a Q-module T and a congruence $\sim$ on T.

(1) An expression of $\sim$ as the common refinement of finitely many mesoprimary congruences $\sim_1, \sim_2, \ldots$ is a *mesoprimary decomposition* if, for each component $\sim_i$ with associated prime ideal $P_i \subset Q$, the P_i-prime congruences of $\sim$ and $\sim_i$ at each cogenerator for $\sim_i$ coincide.
(2) A mesoprimary decomposition of $\sim$ is *key* if, for each P_i-mesoprimary component $\sim_i$, every cogenerator for $\sim_i$ is a key P_i-witness for $\sim$.

We are now ready to give the main result of this paper. Theorems 3.13 and 4.6 together imply, as a special case, that every monoid module with at most one nil element admits a key mesoprimary decomposition (see Remark 4.8).

Theorem 4.6. *Fix a congruence $\sim$ on a Q-module T. The common refinement of the coprincipal congruences cogenerated by the key witnesses of $\sim$ identifies only the nil elements of $T/\sim$.*

Proof. The nil class of the congruence cogenerated by a witness $w \in T$ for P contains the nil in the connected component of w (if one exists), as well as every element outside of this connected component. This means any P-coprincipal component identifies all of the nil elements of T.

Now, fix distinct $a, b \in T$ and assume a is not nil. If a and b lie in distinct connected components, then any cogenerated congruence whose order ideal contains a does not identify a and b. Assuming a and b lie in the same connected component, it suffices to find a monoid prime $P \subset Q$ and a key witness $w \in T$ for P such that a and b are not equivalent under $\sim_w^P$. Fix a prime P minimal among those containing the ideal $I = \{q \in Q : q \cdot a = q \cdot b\}$. Notice that I (and thus P) must be nonempty since a and b lie in the same connected component.

Since P contains I, the elements a and b have distinct images $\overline{a}$ and $\overline{b}$ in T_P, and each $\overline{q} \in I_P$ also satisfies $\overline{q} \cdot \overline{a} = \overline{q} \cdot \overline{b}$. By minimality of P_P over I_P, there is a maximal Green's class among the elements $\{\overline{q} \in Q_P : \overline{q} \cdot \overline{a} \neq \overline{q} \cdot \overline{b}\}$. Pick an element $q \in Q$ such that $\overline{q}$ lies in this Green's class, and set $w = q \cdot a \in T$. Then w is a key witness for P by construction, and the localization of $\sim_w^P$ does not equate $\overline{a}$ and $\overline{b}$ in T_P, so $\sim_w^P$ does not equate a and b in T. This completes the proof. $\quad\square$

Corollary 4.7. *Fix a Q-module T and a congruence $\sim$ on T. If $T/\sim$ has at most one nil element, then $\sim$ admits a key mesoprimary decomposition.*

Proof. Apply Theorem 4.6 and Lemma 4.4 to $T/\sim$. $\quad\square$

Remark 4.8. Theorem 4.6 states that mesoprimary decomposition of monoid modules fails to distinguish nil elements from one another, and that this is the only obstruction to constructing mesoprimary decompositions in this setting. Fortunately, for the purposes of decomposing graded modules over a monoid algebra, these elements all correspond to zero in the module and thus are indistinguishable.

5. The category of tightly graded $\Bbbk[Q]$-modules

Section 2 defined the category Q-Mod of Q-modules, the setting in which mesoprimary decomposition of monoid congruences is generalized in the prior sections of this paper. In this section, we define the category $\mathcal{B}_Q$ of tightly graded $\Bbbk[Q]$-modules (Definition 5.6), the objects of which are graded by objects of Q-Mod. It is to these graded modules that we generalize mesoprimary decomposition of binomial ideals in the subsequent sections of this paper.

Definition 5.1. Fix a Q-module T and a $\Bbbk[Q]$-module M.

- M is *graded by T* (or just *T-graded*) if there exists a collection of finite dimensional vector spaces $\{M_t\}_{t \in T}$ such that $M \cong \bigoplus_{t \in T} M_t$ as Abelian groups, and for each $q \in Q$, $\mathbf{x}^q \cdot M_t \subset M_{q \cdot t}$.
- The grading of M by T is *fine* (or M is *finely-graded by T*) if $\dim_{\Bbbk} M_t \leq 1$ for each $t \in T$.
- A fine grading of M by T is *tight* (or M is *tightly-graded by T*) if

- $M_t \neq 0$ for each non-nil $t \in T$,
- the orbit of each $\infty \in N(T)$ with $M_\infty = 0$ is properly connected, and
- whenever $m \in M_t$ is nonzero with $\mathbf{x}^q \cdot m = 0$, we have $\dim_{\Bbbk} M_{q \cdot t} = 0$.

Remark 5.2. A tight grading of a $\Bbbk[Q]$-module M by a Q-module T ensures that we can determine enough of the structure of M from the grading. The first condition ensures that T does not have any unnecessary elements, and the second ensures each connected component has its own nil (see Proposition 5.7). Example 5.3 demonstrates what can cause the third condition to fail.

Example 5.3. The $\Bbbk[x]$-module $M = \langle x^2 \rangle \oplus (\Bbbk[x]/\langle x^2 \rangle)$ is finely graded by $\mathbb{N}$, but since $x \cdot (0, x)$ is zero, this grading is not tight. However, M is tightly graded by the disjoint union of $\langle 2 \rangle \subset \mathbb{N}$ (which tightly grades $\langle x^2 \rangle \subset \Bbbk[x]$) and $\mathbb{N}/\langle 2 \rangle$ (which tightly grades $\Bbbk[x]/\langle x^2 \rangle$). This grading more accurately reflects the algebraic structure of M.

In order to study finely graded $\Bbbk[Q]$-modules, it suffices to consider tight gradings. In particular, every tight grading is fine, and Theorem 5.4 shows that a tight grading can be recovered from any fine grading by choosing an appropriate Q-module.

Theorem 5.4. *Fix a $\Bbbk[Q]$-module M finely graded by a Q-module T. Then there exists a Q-module that tightly grades M.*

Proof. We construct the desired Q-module in two steps. First, define a Q-module T' that, as a set, consists of those $t \in T$ for which $\dim_{\Bbbk} M_t = 1$, along with a distinguished element ∞. Given $t \in T'$ and $q \in Q$, define $q \cdot t \in T'$ by

$$q \cdot t = \begin{cases} q \cdot t \in T & \mathbf{x}^q M_t \neq 0 \\ \infty & \text{otherwise.} \end{cases}$$

That is, the result of acting on t by q in T if $\mathbf{x}^q M_t \neq 0$, and $\infty \in T'$ otherwise. The Q-module T' also finely grades M since each nonzero M_t for $t \in T$ has a corresponding degree in T'. Moreover, the only degree $t \in T'$ with $\dim_{\Bbbk} M_t = 0$ is $t = \infty$, and whenever $\mathbf{x}^q M_t = 0$, we have $q \cdot t = \infty$. In particular, T' satisfies the first and third conditions for a tight grading in Definition 5.1.

Next, let $T_1', \ldots, T_r'$ denote the distinct Q-orbits of $T' \setminus \{\infty\}$, and let T'' denote the disjoint union of $T_1', \ldots, T_r'$ along with distinguished elements $\infty_1, \ldots, \infty_r$. Define a Q-module structure on T'' so that ∞_i is nil for each $i \leq r$, and for $t \in T_i'$ and $q \in Q$,

$$q \cdot t = \begin{cases} \infty_i & q \cdot t = \infty \\ q \cdot t \in T_i' & \text{otherwise.} \end{cases}$$

Since the orbit of each nil ∞_i of T'' with trivial support in M is properly connected, T'' tightly grades M. This completes the proof. $\square$

Definition 5.5. Suppose M and N are $\Bbbk[Q]$-modules, graded by Q-modules T and U, respectively.

- A homomorphism $\phi : M \to N$ is said to be *graded with degree* $\psi \in \operatorname{Hom}_Q(T, U)$ if for each $t \in T$, we have $\phi(M_t) \subset N_{\psi(t)}$.
- Let $\underline{\operatorname{Hom}}_R(M, N)_\psi$ denote the set of morphisms $\phi : M \to N$ of degree ψ, and

$$\underline{\operatorname{Hom}}_R(M, N) = \bigoplus_{\psi \in \operatorname{Hom}_Q(T, U)} \underline{\operatorname{Hom}}_R(M, N)_\psi$$

  for the set of *graded homomorphisms* from M to N. $\underline{\operatorname{Hom}}_R(M, N)$ is naturally a $\operatorname{Hom}_Q(T, U)$-graded $\Bbbk[Q]$-module with ring action $(\mathbf{x}^q \cdot \phi)(m) = \phi(\mathbf{x}^q \cdot m)$ for each $m \in M$ and $q \in Q$.
- A homomorphism $\phi \in \underline{\operatorname{Hom}}_R(M, N)$ is *homogeneous* if it is a sum of homomorphisms with homogeneous degree in $\operatorname{Hom}_Q(T, U)$.

Definition 5.6. The *category of tightly graded $\Bbbk[Q]$-modules* is the category $\mathcal{B}_Q$ whose objects are pairs (M, T) consisting of a Q-module T together with a $\Bbbk[Q]$-module M tightly graded by T, and whose morphisms are graded $\Bbbk[Q]$-module homomorphisms. When there is no confusion, we often write $M \in \mathcal{B}_Q$ to denote the $\Bbbk[Q]$ module and use T_M to denote the Q-module which tightly grades M.

Proposition 5.7. *The category $\mathcal{B}_Q$ is closed under taking direct sums and tensor products. More precisely, given two $\Bbbk[Q]$-modules M and N tightly graded by Q-modules T and U, respectively, the direct sum $M \oplus N$ is naturally graded by $T \oplus U$, and the tensor product $M \otimes_{\Bbbk[Q]} N$ is naturally graded by $T \otimes_Q U$.*

Proof. The homogeneous elements of $M \oplus N$ have the form $(m, 0), (0, n)$ for homogeneous $m \in M_t$, $n \in N_u$, and the degree map is given by $\deg(m, 0) = t \in T \oplus U$, $\deg(0, n) = u \in T \oplus U$. The homogeneous elements of $M \otimes_{\Bbbk[Q]} N$ have the form $m \otimes n$ for homogeneous $m \in M_t$, $n \in N_u$, and the degree map is given by $\deg(m \otimes n) = t \otimes u$. Notice that

$$\deg(m \otimes (\mathbf{x}^q \cdot n)) = t \otimes (q \cdot u) = (q \cdot t) \otimes u = \deg((\mathbf{x}^q \cdot m) \otimes n),$$

so this degree map is well defined. $\quad\square$

6. Mesoprimary $\Bbbk[Q]$-modules

In this section, we define binomial submodules of tightly graded $\Bbbk[Q]$-modules (Definition 6.1), generalizing the concept of "binomial ideal". We define mesoprimary binomial submodules (Definition 6.3), which, like mesoprimary binomial ideals, are characterized by their unique associated mesoprime ideal (Theorem 6.7).

Definition 6.1. Fix a tightly T-graded $\Bbbk[Q]$-module M and a nonzero element $m \in M$.

- The element m is a *monomial* if it is homogeneous under the T-grading.
- The element m is a *binomial* if it is a sum of at most two monomial elements.
- A submodule $N \subset M$ is *monomial* (resp. *binomial*) if it is generated by monomial (resp. binomial) elements.

Lemma 6.2. *Fix a tightly T-graded $\Bbbk[Q]$-module M and a binomial submodule $N \subset M$. Let $\sim_N$ denote the equivalence relation on T which sets $a \sim_N b$ whenever $m_a + m_b \in N$ for some nonzero $m_a \in M_a, m_b \in M_b$. Then $\sim_N$ is a congruence on T, and $\overline{M} = M/N$ is tightly graded by $\overline{T} = T/\sim_N$.*

Proof. It is clear that $\sim_N$ is a congruence on T, and that $\overline{T}$ finely grades $\overline{M}$. If $\bar{t} \in \overline{T}$ is non-nil, then each representative $t \in T$ for $\bar{t}$ is non-nil, meaning $\dim_\Bbbk \overline{M}_{\bar{t}} = 1$. Additionally, if $\mathbf{x}^q \cdot \overline{m} = 0$ for some nonzero $\overline{m} \in \overline{M}_{\bar{t}}$, then any nonzero $m \in M$ whose image in $\overline{M}$ equals $\overline{m}$ satisfies $\mathbf{x}^q \cdot m = 0$. Since T tightly grades M, this means $\dim_\Bbbk M_{q \cdot t} = 0$, so $\dim_\Bbbk \overline{M}_{q \cdot \bar{t}} = 0$. Lastly, if $\bar{t} \in \overline{T}$ is nil and $\dim_\Bbbk \overline{M}_{\bar{t}} = 0$, then each representative $t \in T$ of $\bar{t}$ is nil and satisfies $\dim_\Bbbk M_t = 0$. As such, N cannot contain any nonzero binomials whose monomials have image of degree $\bar{t}$ in $\overline{M}$, so $\overline{T}$ tightly grades $\overline{M}$, as desired. $\square$

Definition 6.3. A tightly T-graded $\Bbbk[Q]$-module M is *mesoprimary* if T is a mesoprimary Q-module and $M_\infty = 0$ for each nil $\infty \in T$. A binomial submodule $N \subset M$ is *mesoprimary* if M/N is a mesoprimary $\Bbbk[Q]$-module.

Example 6.4. If $I \subset \Bbbk[Q]$ is a binomial ideal, then $\Bbbk[Q]/I$ is tightly T-graded for $T = Q/\sim_I$. Moreover, $\Bbbk[Q]/I$ is mesoprimary when T is mesoprimary and I is maximal among binomial ideals inducing the congruence $\sim_I$. By Lemma 3.4, this is precisely when I is mesoprimary; see [4, Definition 10.4].

Definition 6.5 generalizes [4, Definition 12.1].

Definition 6.5. Fix a tightly T-graded $\Bbbk[Q]$-module M, and a binomial submodule $N \subset M$. Fix a monoid prime $P \subset Q$, and let G denote the unit group of Q_P.

- The *monomial localization of M at P*, denoted M_P, is the $\Bbbk[Q]_P$-module obtained by adjoining to M the inverses of all monomials outside of the monomial ideal $\mathfrak{m}_P = \langle \mathbf{x}^p : p \in P \rangle$.
- An element $w \in T$ is an *N-witness for P* if w is a $\sim_N$-witness for P on T, and w is *essential* if a nonzero element of M_w is minimal (under Green's preorder) among the monomials of some element $m \in M$ annihilated by $\mathfrak{m}_P$ in M_P/N_P. A nonzero monomial $m_w \in M_w$ is called a *monomial P-witness for N*.
- Fix a monomial N-witness $m \in M_w$ for P. The *stabilizer* of w along a prime $P \subset Q$ is the subgroup $K_w^P \subset G_P$ fixing the class of w in T_P. The *character at m* is the

homomorphism $\rho : K_w^P \to \Bbbk^*$ such that $(I_{\rho,P})_P = \mathrm{ann}(\overline{m}) + \mathfrak{m}_P$, where $\overline{m}$ denotes the image of m in M_P. The *P-mesoprime of M at m* is the mesoprime ideal $I_{\rho,P}$.

- If $m \in M_w$ is an essential N-witness for P, we say the mesoprime $I_{\rho,P}$ is *associated to N*, and m is an *N-witness for $I_{\rho,P}$*.

Proposition 6.6 generalizes [4, Lemma 12.4], and can be proven using a similar argument to Theorem 3.13.

Proposition 6.6. *Any binomial submodule of a tightly graded $\Bbbk[Q]$-module has finitely many essential witnesses.*

Theorem 6.7 generalizes [4, Proposition 12.10], and may fail to hold if the grading Q-module is not properly connected; see Example 6.8.

Theorem 6.7. *Fix a tightly T-graded $\Bbbk[Q]$-module M with T properly connected, and fix a binomial submodule $N \subset M$. Then N is mesoprimary if and only if N has exactly one associated mesoprime.*

Proof. Let $\overline{M} = M/N$ and $\overline{T} = T/\sim_N$. First, suppose N has exactly one associated mesoprime $I_{\rho,P}$. This means the P-prime congruences agree at all N-witnesses, so $\overline{T}$ has exactly one associated prime congruence, and thus is mesoprimary. Furthermore, if $\overline{M}_\infty \neq 0$ for some nil $\infty \in \overline{T}$, then the associated mesoprime at any nonzero element of $\overline{M}_\infty$ differs from the associated mesoprime in any non-nil degree $t \in T$ with $q \cdot \overline{t} = \infty$ for some $q \in Q$.

Next, suppose N is mesoprimary. Fix N-witnesses $w, w' \in T$ with associated mesoprimes $I_{\rho,P}$ and $I_{\rho',P}$, respectively, and suppose $w = q \cdot w'$. Since T has exactly one associated prime congruence, multiplication by $\mathbf{x}^q$ induces an isomorphism $I_{\rho,P} \to I_{\rho',P}$, that is, the associated mesoprimes at w and w' coincide. Since T is properly connected, this shows that the associated mesoprimes at every M-witness coincide. $\quad\square$

Example 6.8. Resuming notation from Theorem 6.7, the result may fail to hold in general if T is not properly connected. Let $I = \langle x-1, y \rangle, J = \langle x-2, y \rangle \subset \Bbbk[x,y]$, $M = \Bbbk[x,y]/I \oplus \Bbbk[x,y]/J$, and $T = Q/\sim_I \oplus Q/\sim_J$ with nils identified. Even though M is mesoprimary, it has two distinct associated mesoprimes, one for each connected component of $Q/\sim_I \oplus Q/\sim_J$.

7. Mesoprimary decomposition of binomial submodules

In this section, we use mesoprimary submodules (Definition 6.3) to construct a mesoprimary decomposition of any binomial submodule of a tightly graded $\Bbbk[Q]$-module (Theorem 7.4), thus completing our answer to Problem 1.2.

Definition 7.1. Denote by M a tightly T-graded $\Bbbk[Q]$-module, $N \subset M$ a binomial submodule, $P \subset Q$ a prime, and $m_w \in M_w$ a monomial N-witness for P.

- The *monomial submodule cogenerated by w along P* is the submodule $M_w^P(N)$ of M generated in those degrees $u \in T$ that lie outside of the order ideal $T_{\leq w}^P$ cogenerated by w along P under the congruence $\sim_N$.
- The *P-mesoprime component of N cogenerated by w* is the preimage $W_w^P(I)$ in M of the submodule $N_P + C_w^P(N) + M_w^P(N) \subset M_P$, where

$$C_w^P(N) = \langle m_a - m_b : \mathbf{x}^q(m_a - m_b) \in M_w \rangle.$$

Example 7.2. Resuming notation from Definition 7.1, if $M = \Bbbk[Q]$ and $N = I$ is a binomial ideal, then $C_w^P(I) = I_{\rho,P} \subset M$, so in this case Definition 7.1 is equivalent to [4, Definition 12.13]. In general, we have $I_{\rho,P}M_P \subset C_w^P(I)$, but equality need not hold. Let $Q = \mathbb{N}^2$, $M = (\Bbbk[x,y]/\langle y - xy, y^2 \rangle)^{\oplus 2}$, and T the Q-module that tightly grades M. Write e_1 and e_2 for the generators of the summands of M, and let $N = \langle ye_1 - ye_2 \rangle$, $w = ye_1$. Then the associated mesoprime at w is $I_{\rho,P} = \langle 1 - x, y \rangle$ and $W_w^P(N) = \langle e_1 - e_2, e_1(1-x), e_2(1-x) \rangle$. Here, $W_w^P(N)$ must contain the binomial $e_1 - e_2$ in order to induce the desired coprincipal congruence on T, and this is not captured in the combinatorial data of $I_{\rho,P}$ alone.

Proposition 7.3. *Denote by M a tightly T-graded $\Bbbk[Q]$-module, $N \subset M$ a binomial submodule, $P \subset Q$ a prime, and $w \in T$ an N-witness for P. The submodule $W_w^P(N)$ is mesoprimary with associated mesoprime $I_{\rho,P}$. In particular, if N induces the congruence $\sim$ on T, then $W_w^P(N)$ induces the coprincipal congruence $\sim_w^P$.*

Proof. Let $\approx$ denote the congruence induced by $W_w^P(N)$. We can see from the definitions that $\sim_w^P$ refines $\approx$, so it remains to show that no further relations are added. Since the congruences induced by N_P and $C_w^P(N)$ both refine $\sim_w^P$, it suffices to show that the nil class of $\approx$ is identical to that of $\sim_w^P$. That is, we must check that whenever $a \sim b$ and $a \approx b$ for non-nil $a, b \in T$, these relations are induced by the same binomial elements in N_P and $C_w^P(N)$. Suppose $m_a - m_b \in N$ for nonzero $m_a \in M_a$, $m_b \in M_b$ such that $a \approx b$ but $m_a, m_b \notin M_w^P(N)$. Since $a, b \in T_{\leq w}^P(\sim)$, we can find $q \in Q$ so that $q \cdot a$ and $q \cdot b$ are Green's equivalent to w in T_P. This means $\mathbf{x}^q m_a - \mathbf{x}^q m_b \in M_w \cap N$, and since w is not in the nil class of $\sim$, we must have $\mathbf{x}^q m_a - \mathbf{x}^q m_b = 0$. In particular, this means $m_a - m_b \in C_w^P(N)$, as desired. $\square$

Theorem 7.4. *Any binomial submodule N of a tightly T-graded $\Bbbk[Q]$-module M equals the intersection of the coprincipal components cogenerated by its essential witnesses.*

Proof. Pick an element $m \in M$ outside of N. The goal is to find an essential witness w and a monoid prime P such that m lies outside of the coprincipal component $W_w^P(N)$.

First, suppose the image of m lies outside of the localization N_P along a maximal prime P of Q. Replacing m with a monomial multiple of m, it suffices to assume that $\mathfrak{m}_P m \subset N$, that is to say, m is annihilated (modulo N) by the maximal ideal $\mathfrak{m}_P$. This means some monomial of m has as its graded degree an essential witness w for P. By the minimality of w, the image of m modulo $W_w^P(N)$ lies in the image of M_w modulo $W_w^P(N)$, which is nonzero by Proposition 7.3.

Next, suppose the image of m under localization along some non-maximal monoid prime P lies outside of N_P. The above argument implies that the localized image of m lies outside of some P-coprincipal component of N_P, which by Definition 7.1 equals the localization W_P of a P-coprincipal component W of N. Since localizing W along P is injective, this completes the proof. $\square$

8. Primary decomposition of binomial submodules

In this section, we extend the results of [4, Section 15] to construct a primary decomposition for any binomial submodule of a tightly graded $\Bbbk[Q]$-module. More specifically, the results presented in this section directly parallel those found in [4, Proposition 15.1], [4, Corollary 15.2], and [4, Theorem 15.4], used to construct primary decompositions of mesoprimary binomial ideals over an algebraically closed field. Corollary 8.3, together with Theorem 7.4, yields a combinatorial method of primarily decomposing a binomial submodule whenever $\Bbbk = \overline{\Bbbk}$ is algebraically closed.

Proposition 8.1. *Fix a tightly T-graded $\Bbbk[Q]$-module M and a mesoprimary binomial submodule $N \subset M$. The associated primes of N are precisely the associated primes of its unique associated mesoprime $I_{\rho,P}$. In particular, N is primary if and only if its associated mesoprime is prime.*

Proof. Suppose $\overline{T}_P = T_P/\sim_N$ has unit group G. Notice that localizing along P induces an injection $M/N \hookrightarrow (M/N)_P$ since the monomials outside of $\mathfrak{m}_P$ are nonzerodivisors on the quotient modulo any P-mesoprimary ideal. Moreover, by Theorems 6.7 and 3.12, $(M/N)_P$ has finitely many nonzero $(\overline{T}_P/G)$-graded pieces, all isomorphic to $(\Bbbk[Q]/I_{\rho,P})_P$. The partial order on $\overline{T}_P/G$ afforded by Lemma 3.9 induces a filtration of $(M/N)_P$ by M_P-submodules, each free of finite rank as a module over $(\Bbbk[Q]/I_{\rho,P})_P$. $\square$

Theorem 8.2. *Suppose $\Bbbk = \overline{\Bbbk}$ is algebraically closed. Fix a tightly T-graded $\Bbbk[Q]$-module M and a mesoprimary binomial submodule $N \subset M$. If $I_{\rho,P}$ is the unique associated mesoprime of N and $I_{\rho,P} = \bigcap_\sigma I_{\sigma,P}$ is the unique primary decomposition of $I_{\rho,P}$ from [4, Proposition 11.9], then*

$$N = \bigcap_\sigma (N + I_{\sigma,P} M)$$

is the unique minimal primary decomposition of N.

Proof. Each submodule $N + I_{\sigma,P}M \subset M$ is binomial and mesoprimary, and thus primary by Proposition 8.1. The intersection $\bigcap_\sigma (N + I_{\sigma,P}M)$ certainly contains N, and the converse follows from the equality $N = N + I_{\rho,P}M$. □

Corollary 8.3. *Suppose $\Bbbk = \overline{\Bbbk}$ is algebraically closed. Every binomial submodule $N \subset M$ of a tightly T-graded $\Bbbk[Q]$-module M admits a primary decomposition in which each component is again binomial.*

Proof. Apply Theorem 7.4 to construct a mesoprimary decomposition for N, then apply Theorem 8.2 to each mesoprimary component. □

Remark 8.4. In [5], mesoprimary decomposition is used to combinatorially construct irreducible decompositions of binomial ideals, using "soccular decomposition" as an intermediate step. It remains an interesting question to extend soccular decomposition to tightly graded modules; we record this here.

Problem 8.5. Extend soccular decomposition [5] to tightly graded modules.

Acknowledgments

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