Some remarks about Mishchenko–Fomenko subalgebras

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A R T I C L E   I N F O

A B S T R A C T

We discuss and compare two different approaches to the notion of Mishchenko–Fomenko subalgebras in Poisson–Lie algebras of finite-dimensional Lie algebras. One of them, commonly accepted by the algebraic community, uses polynomial Ad*-invariants. The other is based on formal Ad*-invariants and allows one to deal with arbitrary Lie algebras, not necessarily algebraic. In this sense, the latter is more universal.

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1. Introduction

This note is primarily motivated by the paper by A. Ooms [10] in which, among other interesting results, the author constructs a counterexample to my completeness criterion for Mishchenko–Fomenko subalgebras [2]. I do not intend to disprove this statement by Ooms. My point is that the example by A. Ooms and the completeness criterion from [2] are both correct. The confusion is caused by the fact that the definitions of Mishchenko–Fomenko subalgebras used in [10] and [2] are different. The purpose of the

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present note is to clarify this issue and perhaps to convince the reader that the definition from [2] is, in some respect, better.

2. Formal Ad*-invariants

Originally, Mishchenko–Fomenko subalgebras appeared in the context of integrable Hamiltonian systems on Lie algebras, real or complex. However this construction is purely algebraic and in what follows we consider finite-dimensional Lie algebras over an algebraically closed field $\mathbb{K}$ of characteristic zero. Here we recall and slightly modify the results of [6]. Basically, we want to develop some algebraic techniques allowing us to deal with arbitrary Lie algebras $\mathfrak{g}$, not necessarily algebraic. So we do not assume the existence of any polynomial and even rational Ad*-invariants. Moreover, we never use the Lie group G associated with $\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{g}^*$ its dual space and $P(\mathfrak{g})$ denote the algebra of polynomials[1] on $\mathfrak{g}^*$. The algebra $P(\mathfrak{g})$ is endowed with the standard Lie–Poisson bracket

$$\{f(x), g(x)\} = \langle x, [df(x), dg(x)] \rangle, \quad x \in \mathfrak{g}^*, \ df(x), dg(x) \in \mathfrak{g}, \tag{1}$$

and we will refer to $P(\mathfrak{g})$ as the Lie–Poisson algebra associated with $\mathfrak{g}$.

Our goal is to construct a "big" commutative subalgebra in $P(\mathfrak{g})$. The argument shift method suggested by A. Mishchenko and A. Fomenko [9] is based on some nice properties of Ad*-invariants. In general, however, polynomial (and even rational) invariants do not necessarily exist. To avoid this problem one can use formal invariants which can be defined in the following way.

Definition 1. Let $\mathsf{F} = \sum_{k=1}^{\infty} f^{(k)}$ be a formal power series where $f^{(k)} \in P(\mathfrak{g})$ is a homogeneous polynomial of degree $k \in \mathbb{N}$. We say that F is a *formal* Ad*-*invariant* at a point $a \in \mathfrak{g}^*$, if the following (formal) identity holds for all $\xi \in \mathfrak{g}$:

$$\langle d\mathsf{F}(x), \mathrm{ad}_\xi^*(a + x) \rangle = 0. \tag{2}$$

From the differential-geometric point of view this condition simply means that the differential of F at the point $a + x$ vanishes on the tangent space of the coadjoint orbit through this point. Thus, the above relation can be understood as the standard definition of an invariant function F where F is replaced by its Taylor expansion F at the point $a \in \mathfrak{g}^*$. The formal identity (2) amounts to the following infinite sequence of polynomial relations:

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[1] $P(\mathfrak{g})$, as a set, is of course the same as the symmetric Lie algebra $S(\mathfrak{g})$, but we use a slightly different point of view thinking of $P(\mathfrak{g})$ as a Poisson algebra and of its elements as functions on the vector space $\mathfrak{g}^*$.

$$\langle df^{(1)}(x), \mathrm{ad}_\xi^* a \rangle = 0,$$

$$\langle df^{(2)}(x), \mathrm{ad}_\xi^* a \rangle = -\langle df^{(1)}(x), \mathrm{ad}_\xi^* x \rangle,$$

$$\langle df^{(3)}(x), \mathrm{ad}_\xi^* a \rangle = -\langle df^{(2)}(x), \mathrm{ad}_\xi^* x \rangle,$$

$$\dots$$

$$\langle df^{(k)}(x), \mathrm{ad}_\xi^* a \rangle = -\langle df^{(k-1)}(x), \mathrm{ad}_\xi^* x \rangle,$$

$$\dots$$

where $\xi \in \mathfrak{g}$ is arbitrary, or equivalently

$$
\begin{aligned}
&\mathrm{ad}_{df^{(1)}(x)}^* a = 0, \\
&\mathrm{ad}_{df^{(2)}(x)}^* a = -\mathrm{ad}_{df^{(1)}(x)}^* x, \\
&\mathrm{ad}_{df^{(3)}(x)}^* a = -\mathrm{ad}_{df^{(2)}(x)}^* x, \\
&\qquad \dots \\
&\mathrm{ad}_{df^{(k)}(x)}^* a = -\mathrm{ad}_{df^{(k-1)}(x)}^* x, \\
&\qquad \dots
\end{aligned}
\tag{3}
$$

The first relation means that the differential $df^{(1)}$ of the first term belongs to the ad^*-stationary subalgebra of $a \in \mathfrak{g}^*$ or equivalently:

$$f^{(1)} \in \mathrm{Ann}\,(a) = \{\eta \in \mathfrak{g} \mid \mathrm{ad}_\eta^* a = 0\},$$

as $f^{(1)}$ is a linear function and hence we may identify $f^{(1)}$ with $df^{(1)}$.

Let us denote the space of all formal Ad^*-invariants at $a \in \mathfrak{g}^*$ by $Y_{\mathrm{formal}}(\mathfrak{g}, a)$. It is easy to see that this set is closed under addition and multiplication (clearly, the usual multiplication of formal power series is well defined in our case). Thus, any polynomial $p(\mathsf{F}_1, \dots, \mathsf{F}_s)$ in formal invariants $\mathsf{F}_1, \dots, \mathsf{F}_s \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$ is still a formal invariant. Moreover, if we consider a formal power series

$$\mathsf{P}(\mathsf{F}_1, \dots, \mathsf{F}_s) = \sum_{k=1}^\infty p^{(k)}(\mathsf{F}_1, \dots, \mathsf{F}_s)$$

of formal invariants $\mathsf{F}_1, \dots, \mathsf{F}_s \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$, then $\mathsf{P}(\mathsf{F}_1, \dots, \mathsf{F}_s) \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$.

The next theorem is a *formal* analog of some well-known facts about local invariants of a smooth action of a Lie group at a generic point.

As usual, we say that $a \in \mathfrak{g}^*$ is *regular*, if $\dim \mathrm{Ann}\,(a)$ is minimal and is equal to $s = \mathrm{ind}\,\mathfrak{g}$.

Theorem 1. *Let $a \in \mathfrak{g}^*$ be regular and $\xi_1, \dots, \xi_s$ be a basis of $\mathrm{Ann}\,(a)$. Then there exist formal Ad^*-invariants $\mathsf{F}_1, \dots, \mathsf{F}_s$ such that their linear terms $f_1^{(1)}, \dots, f_s^{(1)}$ coincide*

with $\xi_1, \ldots, \xi_s$ respectively. Moreover, any other formal invariant $\mathsf{F} \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$ can uniquely be represented as a formal power series of $\mathsf{F}_1, \ldots, \mathsf{F}_s$.

Proof. The "existence part" of Theorem 1 was proved in [6]. We only need to comment on the second part. This statement immediately follows from

Lemma 1. *Suppose that $\mathsf{F} \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$ starts with a term of degree m, i.e., $\mathsf{F} = \sum_{k=m}^{\infty} f^{(k)}$. Then there is a homogeneous polynomial $p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_s)$ of degree m such that $\mathsf{F} - p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_s)$ starts with a term of degree $m+1$.*

Proof. Since $\mathsf{F} = \sum_{k=m}^{\infty} f^{(k)}$ is a formal invariant, we have a sequence of relations similar to (3) but these relations start with the identity

$$\langle df^{(m)}(x), \mathrm{ad}_\xi^* a \rangle = 0 \quad \text{for all } \xi \in \mathfrak{g},$$

or equivalently,

$$\mathrm{ad}_{df^{(m)}(x)}^* a = 0,$$

i.e., $df^{(m)}(x) \in \mathrm{Ann}\,(a)$. In other words, $f^{(m)}$ is a homogeneous polynomial of degree m on $\mathfrak{g}^*$ whose differential at every point $x \in \mathfrak{g}^*$ belongs to $\mathrm{Ann}\,(a)$. But this condition obviously means that $f^{(m)}$ can, in fact, be written as a polynomial of the basis elements $\xi_1, \ldots, \xi_s \in \mathrm{Ann}\,(a)$, i.e. $f^{(m)} = p^{(m)}(\xi_1, \ldots, \xi_s)$.

Since ξ_i's are the linear terms of F_i's, we see that the m-terms of F and $p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_m)$ coincide so that the power series $\mathsf{F} - p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_m)$ starts with a term of degree $m+1$, as required. $\quad\square$

Thus, Lemma 1 says the following. Given a formal invariant $\mathsf{F} = \sum f^{(k)} \in Y_{\mathrm{formal}}(\mathfrak{g}, a)$ we can, step by step, kill all of its homogeneous terms by subtracting a suitable polynomial $p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_s)$, $m = 1, 2, \ldots$, in other words, $\mathsf{F} = \sum p^{(m)}(\mathsf{F}_1, \ldots, \mathsf{F}_s)$ as required.

The uniqueness of such an expansion follows from the independence of $\mathsf{F}_1, \ldots, \mathsf{F}_s$ so that no nontrivial polynomial in $\mathsf{F}_1, \ldots, \mathsf{F}_s$ may vanish identically. $\quad\square$

Remark 1. A similar result holds true for an arbitrary linear representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ (see [6]). The only difference is that $\mathrm{Ann}\,(a)$ should be replaced by the orthogonal complement $T_a^\perp \subset V^*$ to the "ρ-orbit" $T_a = \{\rho(\xi)a, \; \xi \in \mathfrak{g}\} \subset V$. If $\rho \simeq \mathrm{ad}^*$, we use the natural identification $\mathrm{Ann}\,(a) \simeq T_a^\perp$ which disappears in the general case but the construction of formal invariants at $a \in V$ still makes sense.

Notice that finding a formal invariant (up to any order) is a problem of Linear Algebra. Indeed, we only need to solve successively the sequence of relations (3). The "existence part" of Theorem 1 tells us that the systems of linear equations we obtain at each step are all consistent and, moreover, the proof given in [6] explains how to make the choice of a solution $f^{(k)}$ unique.

3. Mishchenko–Fomenko subalgebras: two versions

We first recall the definition used in [1,2].

Definition 2. Let $a \in \mathfrak{g}^*$ be regular and $\mathsf{F}_1 = \sum f_1^{(k)}, \ldots, \mathsf{F}_s = \sum f_s^{(k)} \in Y_{\text{formal}}(\mathfrak{g}, a)$ be an arbitrary basis of formal Ad^*-invariants at the point $a \in \mathfrak{g}^*$ as in Theorem 1. The *algebra of polynomial shifts* $F_a(\mathfrak{g})$ is defined to be the subalgebra in $P(\mathfrak{g})$ generated by the homogeneous polynomials $f_i^{(k)}$, $i = 1, \ldots, s = \mathrm{ind}\,\mathfrak{g}$, $k \in \mathbb{N}$.

Remark 2. In [1,2], instead of formal Ad^*-invariants we considered the Taylor expansions of smooth or analytic (local) invariants F which always exist in a neighbourhood of a regular point $a \in \mathfrak{g}^*$ (for real or complex Lie algebras):

$$F(a + tx) \simeq \sum_{k=0}^{\infty} t^k f^{(k)}(x).$$

The above definition is just a straightforward extension (or algebraic reformulation) of this construction to the case of an arbitrary field of characteristic zero. The term "algebra of polynomial shifts" was used in our recent paper [4] to emphasise the difference from "standard shifts" $F(x + ta)$ which are not necessarily polynomial in x (e.g., if F itself is not a polynomial). We consider $F_a(\mathfrak{g})$ as the first version of a Mishchenko–Fomenko subalgebra but avoid using this terminology to avoid further confusion with another type of Mishchenko–Fomenko subalgebras discussed below.

The following proposition summarises the main properties of the algebra of polynomial shifts.

Theorem 2. *Let $a \in \mathfrak{g}^*$ be an arbitrary regular element and $F_a(\mathfrak{g})$ the corresponding algebra of polynomial shifts. Then the following properties hold:*

1. *$F_a(\mathfrak{g})$ does not depend on the choice of the basis formal invariants $\mathsf{F}_1, \ldots, \mathsf{F}_s \in Y_{\text{formal}}(\mathfrak{g}, a)$.*
2. *The linear polynomials from $F_a(\mathfrak{g})$ are elements of $\mathrm{Ann}\,(a)$. In other words, $F_a(\mathfrak{g}) \cap \mathfrak{g} = \mathrm{Ann}\,(a)$.*
3. *$F_a(\mathfrak{g})$ is commutative w.r.t. to the standard Lie–Poisson bracket (1).*
4. *$F_a(\mathfrak{g})$ is commutative w.r.t. to the Poisson a-bracket*

$$\{f, g\}_a = \langle a, [df(x), dg(x)] \rangle, \quad x \in \mathfrak{g}^*, \ df(x), dg(x) \in \mathfrak{g}.$$

5. *$F_a(\mathfrak{g})$ is complete, i.e., $\mathrm{tr.deg.}\,F_a(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g})$, if and only if $\mathrm{codim}\,\mathsf{Sing} \geq 2$, where*

$$\mathsf{Sing} = \{y \in \mathfrak{g}^* \mid \dim \mathrm{Ann}\,y > \mathrm{ind}\,\mathfrak{g}\} \subset \mathfrak{g}^*$$

is the set of singular points in $\mathfrak{g}^*$.

6. *In general, the number of algebraically independent polynomials in F_a is*

$$\text{tr.deg.}\, F_a(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{p}_{\mathfrak{g}},$$

where $\mathsf{p}_{\mathfrak{g}}$ is the fundamental semi-invariant of $\mathfrak{g}$.

Proof. Items 2, 3, 4 and 5 have been discussed in many papers (see, for instance, [1,2, 4–6]). The item 5 is the completeness criterion from [2]. The counterexample from [10] uses a different definition of a Mishchenko–Fomenko algebra and does not contradict to item 5 (see Example 1 below). The item 6 was recently proved in [8] for the Mishchenko–Fomenko algebras $Y_a(\mathfrak{g})$ in the sense of Definition 3 below, but it is still true for $F_a(\mathfrak{g})$ if a is regular. Theorem 3 proved below immediately implies both 5 and 6.

We only need to explain item 1 which is fairly easy. Let $\mathsf{F} = \sum f^{(m)} \in Y_{\text{formal}}(\mathfrak{g}, a)$ be an arbitrary formal Ad*-invariant. It is sufficient to show that each term $f^{(m)}$ of this formal series belongs to the Mishchenko–Fomenko subalgebra $F_a(\mathfrak{g})$. We know from Theorem 1 that F can be written as a formal power series in $\mathsf{F}_1, \ldots, \mathsf{F}_s$. But this immediately implies that every term $f^{(m)}$ admits a polynomial representation via $f_i^{(k)}$, $i = 1, \ldots, s = \operatorname{ind} \mathfrak{g}$, $k \le m$, and hence belongs to $F_a(\mathfrak{g})$ as required. $\square$

The next definition of Mishchenko–Fomenko subalgebra is used in [8,10,11] and seems to be more common in the algebraic literature.

Definition 3. Let $Y(\mathfrak{g}) = P(\mathfrak{g})^{\mathfrak{g}} \subset P(\mathfrak{g})$ be the algebra of Ad*-invariant polynomials or, equivalently, the centre of $P(\mathfrak{g})$. For $f \in Y(\mathfrak{g})$, $a \in \mathfrak{g}^*$ and $t \in \mathbb{K}$ consider the expansion

$$f(x + ta) = \sum f_{a,m}(x)t^m \tag{4}$$

into the powers of t. The polynomials $f_{a,m}(x)$ are called the a-shifts of f. The *Mishchenko–Fomenko algebra $Y_a(\mathfrak{g})$* is defined as the subalgebra in $P(\mathfrak{g})$ generated by the a-shifts $f_{a,k}$ of all $f \in Y(\mathfrak{g})$ (or equivalently of the generators of $Y(\mathfrak{g})$).

We first notice that if $a \in \mathfrak{g}^*$ is regular, then $Y_a(\mathfrak{g})$ is a subalgebra of $F_a(\mathfrak{g})$. Indeed, without loss of generality we may assume that the generators $f \in Y(\mathfrak{g})$ are homogeneous, then $f(x + ta) = t^d f(a + t^{-1}x)$, where $d = \deg f$ and the expansion (4) is, in fact, equivalent to the Taylor expansion of $f(x)$ at the point a:

$$f(a + x) = \sum_{m=0}^{d} f^{(m)}, \quad \text{where } f^{(m)} = f_{a,d-m}.$$

Since $\sum_{m=0}^{d} f^{(m)}$ is a formal invariant at the point $a \in \mathfrak{g}^*$ in the sense of Definition 1, all the a-shifts $f_{a,k}$ belong to $F_a(\mathfrak{g})$ and consequently $Y_a(\mathfrak{g}) \subset F_a(\mathfrak{g})$.

On the other hand, the algebra of polynomial Ad^*-invariants $Y(\mathfrak{g})$ might be trivial even if $\mathrm{ind}\,g = s > 0$. In such a case, the Mishchenko–Fomenko algebra $Y_a(\mathfrak{g})$ is trivial too in contrast to $F_a(\mathfrak{g})$ that remains non-trivial since independent formal Ad^*-invariants $\mathsf{F}_1, \ldots, \mathsf{F}_s$ always exist (Theorem 1).

To illustrate this phenomenon and to show how to describe $F_a(\mathfrak{g})$ in practice, we consider *Counterexample to Bolsinov's assertion* from [10].

Example 1. Consider the solvable Lie algebra $\mathfrak{g}$ of dimension 8 and index 2 defined by the relations:

$$[x_0, x_1] = 5x_1, \ [x_0, x_2] = 10x_2, \ [x_0, x_3] = -13x_3, \ [x_0, x_4] = -8x_4, \ [x_0, x_5] = -3x_5,$$

$$[x_0, x_6] = 2x_6, \ [x_0, x_7] = 7x_7, \ [x_1, x_3] = x_4, \ [x_1, x_4] = x_5, \ [x_1, x_5] = x_6, \ [x_1, x_6] = x_7,$$

$$[x_2, x_3] = x_5, \ [x_2, x_4] = x_6, \ [x_2, x_5] = x_7.$$

The algebra of polynomial Ad^*-invariants is trivial, i.e., $Y(\mathfrak{g}) = \{\mathbb{K}\}$ and therefore the Mishchenko–Fomenko subalgebra $Y_a(\mathfrak{g})$ is trivial too. The singular set Sing has codimension 3 and is defined by three linear equations $\{x_5 = x_6 = x_7 = 0\}$. Thus, according to the completeness criterion from [2] (*Bolsinov's assertion*), the algebra of polynomial shifts $F_a(\mathfrak{g})$, $a \notin \mathsf{Sing}$, is complete, i.e., $\mathrm{tr.deg.}F_a(\mathfrak{g}) = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g}) = 5$.

The Lie algebra $\mathfrak{g}$ possesses two independent rational Ad^*-invariants. If they are given explicitly, then the coefficients of their Taylor expansions at the point $a \in \mathfrak{g}$ can be taken as generators of $F_a(\mathfrak{g})$. However, even if we do not have any information about them, we can still use formal Ad^*-invariants to construct five algebraically independent polynomial shifts. As an example, take $a \in \mathfrak{g}^*$ such that $x_7(a) = 1$ and $x_i(a) = 0$, $i = 0, \ldots, 6$. The stationary subalgebra $\mathrm{Ann}\,(a)$ is generated by x_3 and x_4 and therefore according to Theorem 1 there exist formal invariants of the form:

$$\mathsf{F} = x_3 + f^{(2)} + f^{(3)} + \ldots$$

$$\mathsf{H} = x_4 + h^{(2)} + h^{(3)} + \ldots$$

The "higher" terms can easily be found successively by solving relations (3). Moreover, the solution is unique if in addition we require that $f^{(i)}$ and $h^{(i)}$ vanish identically on the two-dimensional subspace defined by $x_0 = x_1 = x_2 = x_5 = x_6 = x_7 = 0$. Here is the result of a straightforward computation:

$$f^{(2)} = \frac{13}{7}x_3x_7 - x_4x_6 - \frac{1}{2}x_5^2, \quad f^{(3)} = \frac{39}{49}x_3x_7^2 - \frac{3}{7}x_5^2x_7 - \frac{6}{7}x_4x_6x_7 + x_5x_6^2$$

$$h^{(2)} = \frac{8}{7}x_4x_7 - x_5x_6, \qquad h^{(3)} = \frac{4}{49}x_4x_7^2 - \frac{1}{7}x_5x_6x_7 + \frac{1}{3}x_6^3.$$

There is no need to continue this process, as we have already found 5 algebraically independent polynomial shifts: x_3, x_4, $f^{(2)}$, $h^{(2)}$ and one of $f^{(3)}$, $h^{(3)}$. Thus, the algebra

$F_a(\mathfrak{g})$ so obtained is complete despite the fact that $Y(\mathfrak{g})$ and therefore $Y_a(\mathfrak{g})$ are both trivial.

This phenomenon was well understood long ago and, in fact, was the main reason for us to slightly modify the original construction by A. Mishchenko and A. Fomenko in order to avoid the problem with non-existence of polynomial invariants and construct a commutative subalgebra of $P(\mathfrak{g})$ as large as possible. To the best of my knowledge this modification is due to Andrey Brailov who explained this construction to me in 1986 when I was a PhD student. I am not sure, however, if he ever published this important remark.

The following proposition gives an obvious necessary and sufficient condition for $F_a(\mathfrak{g})$ and $Y_a(\mathfrak{g})$ to coincide.

Proposition 1. *The following conditions are equivalent:*

1. Ann (a) *is generated by the differentials* $df(a)$, $f \in Y(\mathfrak{g})$,
2. $a \in \mathfrak{g}^*$ *is regular and* $F_a(\mathfrak{g}) = Y_a(\mathfrak{g})$.

Proof. Notice that Condition 1 implies that a is regular. Furthermore, if homogeneous invariants $f_1, \ldots, f_s \in Y(\mathfrak{g})$ are such that $df_1(a), \ldots, df_s(a)$ form a basis of Ann (a), then we may consider the Taylor expansions of $f_1, \ldots, f_s$ at $a \in \mathfrak{g}$

$$f_i(a + x) = \sum_m f_i^{(m)}, \quad i = 1, \ldots, s = \operatorname{ind} \mathfrak{g},$$

as a basis in $Y_{\text{formal}}(\mathfrak{g}, a)$. Since the homogeneous terms $f_i^{(m)}$ in these expansions are the same as the a-shifts of f_i, we immediately conclude that $F_a(\mathfrak{g}) \subset Y_a(\mathfrak{g})$ and hence, $F_a(\mathfrak{g}) = Y_a(\mathfrak{g})$.

On the other hand, assume that $a \in \mathfrak{g}^*$ is regular and $F_a(\mathfrak{g}) = Y_a(\mathfrak{g})$. Let us compare the linear functions contained in $F_a(\mathfrak{g})$ and $Y_a(\mathfrak{g})$. According to item 2 of Theorem 2, the linear functions of $F_a(\mathfrak{g})$ are exactly the elements of Ann (a). On the other hand, the linear functions from $Y_a(\mathfrak{g})$ are the differentials $df(a)$, $f \in Y(\mathfrak{g})$. Since $F_a(\mathfrak{g}) = Y_a(\mathfrak{g})$, we get the desired conclusion. $\square$

There are many examples of $\mathfrak{g}$ and $a \in \mathfrak{g}^*$ for which the above condition is fulfilled. The most important of them are semisimple (reductive) Lie algebras.

If $\operatorname{tr.deg.} Y(\mathfrak{g}) < \operatorname{ind} \mathfrak{g}$, then $Y_a(\mathfrak{g})$ is strictly smaller than $F_a(\mathfrak{g})$. On the contrary, if $\operatorname{tr.deg.} Y(\mathfrak{g}) = \operatorname{ind} \mathfrak{g}$, then $Y_a(\mathfrak{g})$ and $F_a(\mathfrak{g})$ coincide for almost all regular $a \in \mathfrak{g}$. However, if a is regular but the differentials of the polynomial invariants do not generate Ann (a), then we have proper inclusion $Y_a(\mathfrak{g}) \subsetneq F_a(\mathfrak{g})$ (although $\operatorname{tr.deg.} Y_a(\mathfrak{g}) = \operatorname{tr.deg.} F_a(\mathfrak{g})$).

Example 2. Consider, for instance, the seven-dimensional nilpotent Lie algebra $\mathfrak{g}$ with relations (this is the Lie algebra $\mathfrak{g}_{7,1.1(i_\lambda),\lambda=1}$ with number 155 from the list presented in [10][2]):

$$[x_1, x_2] = x_3, \ [x_1, x_3] = x_4, \ [x_1, x_4] = x_5, \ [x_1, x_5] = x_6, \ [x_1, x_6] = x_7,$$
$$[x_2, x_3] = x_5, \ [x_2, x_4] = x_6, \ [x_3, x_4] = x_7.$$

It is straightforward to verify that $\operatorname{ind} \mathfrak{g} = 3$ and the singular set $\mathsf{Sing} \subset \mathfrak{g}^*$ is defined by three equations $\{x_5 = x_6 = x_7 = 0\}$ so that $\operatorname{codim} \mathsf{Sing} = 3$. The algebra $Y(\mathfrak{g})$ of polynomial invariants is generated by four polynomials (see [10]):

$$x_7, \quad f = x_6^2 - 2x_5 x_7, \quad g = 2x_6^5 - 10x_5 x_6^3 x_7 + 15 x_5^2 x_6 x_7^2 - 15 x_4 x_5 x_7^3 + 15 x_3 x_6 x_7^3 - 15 x_2 x_7^4,$$

and

$$h = (4f^5 - g^2)/x_7^3 = -225 x_2^2 x_7^5 + 450 x_2 x_3 x_6 x_7^4 - 450 x_2 x_4 x_5 x_7^4 +$$
$$450 x_2 x_5^2 x_6 x_7^3 - 300 x_2 x_5 x_6^3 x_7^2 + 60 x_2 x_6^5 x_7 - 225 x_3^2 x_6^2 x_7^3 + 450 x_3 x_4 x_5 x_6 x_7^3 -$$
$$450 x_3 x_5^2 x_6^2 x_7^2 + 300 x_3 x_5 x_6^4 x_7 - 60 x_3 x_6^6 - 225 x_4^2 x_5^2 x_7^3 + 450 x_4 x_5^3 x_6 x_7^2 -$$
$$300 x_4 x_5^2 x_6^3 x_7 + 60 x_4 x_5 x_6^5 - 128 x_5^5 x_7^2 + 95 x_5^4 x_6^2 x_7 - 20 x_5^3 x_6^4,$$

which satisfy one relation $4f^5 - g^2 - hx_7^3 = 0$.

Since $\operatorname{tr.deg.} Y(\mathfrak{g}) = \operatorname{ind} \mathfrak{g}$, the differentials $df(a)$, $f \in Y(\mathfrak{g})$ generate $\operatorname{Ann}(a)$ for almost all regular points $a \in \mathfrak{g}^*$ but not for all in this case. From the point of view of the theory of integrable Hamiltonian systems, it is natural to think of the generators x_7, f, g and h as first integrals of a Hamiltonian system on $\mathfrak{g}^*$, and consider the momentum mapping $\Phi = (x_7, f, g, h) : \mathfrak{g}^* \to \mathbb{K}^4$. At a generic point the differential of this map has rank 3 and it makes sense to introduce the set of critical points of Φ

$$\mathsf{Crit} = \{ y \in \mathfrak{g}^* \mid \dim \operatorname{span}\{df(y), \ f \in Y(\mathfrak{g})\} < \operatorname{ind} \mathfrak{g} \}.$$

In the notation from [8], the complement to this set Crit can be written as $\mathfrak{g}^*_{\mathrm{Reg}}$ in contrast to $\mathfrak{g}^*_{\mathrm{reg}} = \mathfrak{g}^* \backslash \mathsf{Sing}$. In our example, Crit is defined by two equations $\{x_6 = x_7 = 0\}$ so that Crit is larger than Sing (or equivalently, $\mathfrak{g}^*_{\mathrm{Reg}}$ is smaller than $\mathfrak{g}^*_{\mathrm{reg}}$).

This means that there are regular elements $a \in \mathfrak{g}^*$ for which $Y_a(\mathfrak{g}) \subsetneq F_a(\mathfrak{g})$, namely it is so for every $a \in \mathfrak{g}^*$ with $x_6(a) = x_7(a) = 0$, $x_5(a) \neq 0$. For instance, if we take $a \in \mathfrak{g}^*_{\mathrm{reg}} \backslash \mathfrak{g}^*_{\mathrm{Reg}} = \mathsf{Crit} \backslash \mathsf{Sing}$ such that $x_5(a) = 1$, $x_i(a) = 0$, $i \neq 5$, then it is easy to verify that $Y_a(\mathfrak{g})$ contains only one linear function, namely, x_7, whereas $F_a(\mathfrak{g})$ contains three: x_5, x_6 and x_7 (a basis of $\operatorname{Ann}(a)$).

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[2] There is nothing special in this example. My choice was more or less random within a sub-list of Lie algebras with some suitable properties.

4. Infinitesimal properties of $F_a(\mathfrak{g})$ and $Y_a(\mathfrak{g})$

For any Poisson subalgebra $A \subset P(\mathfrak{g})$, we define $dA(x)$ as the subspace of $\mathfrak{g}$ generated by the differentials of $f \in A$ at point $x \in \mathfrak{g}^*$:

$$dA(x) = \text{span}\{df(x), \ f \in \mathcal{A}\} \subset \mathfrak{g}.$$

A description of such subspaces turns out to be important for various problems in algebra, differential geometry and Hamiltonian mechanics and in this section we discuss and compare them for the subalgebras $F_a(\mathfrak{g})$ and $Y_a(\mathfrak{g})$ following our recent paper [4].

The description of $dF_a(x)$ is very simple and can be given in terms of the pencil of skew-symmetric forms generated by the forms $\mathcal{A}_x, \mathcal{A}_a : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined as

$$\mathcal{A}_x(\xi, \eta) = \langle x, [\xi, \eta] \rangle \quad \text{and} \quad \mathcal{A}_a(\xi, \eta) = \langle a, [\xi, \eta] \rangle.$$

The following statement is well known [2,5,6].

Proposition 2. $dF_a(x) = \sum \text{Ker}\, \mathcal{A}_{x+\lambda a} = \sum \text{Ann}\,(x + \lambda a)$, *where the sum is taken over all* [3] $\lambda \in \mathbb{K}$ *such that* $x + \lambda a \notin \textsf{Sing}$.

Recall that a pair of skew-symmetric forms can simultaneously be reduced to an elegant Jordan–Kronecker canonical form [12] playing an important role in the theory of compatible Poisson brackets [3,5,7,13]. Here we formulate one straightforward and simple corollary of the Jordan–Kronecker decomposition theorem referring to [4] for details.

Let A and B be two skew-symmetric forms on a finite-dimensional vector space V, we will think of them as just two skew-symmetric matrices. Let $r = \max_{\lambda \in \mathbb{K}} \text{rank}\,(A + \lambda B)$ be the rank of the pencil of skew-symmetric forms $\mathcal{P} = \{A + \lambda B\}$. Without loss of generality we assume that B is regular in this pencil, i.e. $\text{rank}\, B = r$.

Consider the Pfaffians of all $r \times r$ diagonal minors of $A + \lambda B$ as polynomials in λ and denote by $\textsf{p}$ their greatest common divisor. Notice that $\textsf{p} = 1$ if and only if the rank of $A + \lambda B$ never drops, i.e., equals r for each $\lambda \in \mathbb{K}$. The following formula is a corollary of the Jordan–Kronecker decomposition theorem.

Proposition 3. *Let* $L = \sum \text{Ker}\,(A + \lambda B)$ *where the sum is taken over all* $\lambda \in \mathbb{K}$ *such that* $\text{rank}\,(A + \lambda B) = r$. *Then* $\dim L = \frac{1}{2}(\dim V + \text{corank}\,\mathcal{P}) - \deg \textsf{p}$.

Let us transfer and apply this formula to our pencil of skew-symmetric forms $\mathcal{P} = \{\mathcal{A}_{x+\lambda a}\}$ on $\mathfrak{g}$. Obviously, $\dim V = \dim \mathfrak{g}$ and $\text{corank}\,\mathcal{P} = \text{ind}\,\mathfrak{g}$ (here we use the fact that $a \in \mathfrak{g}^*$ is regular). So we only need to clarify the meaning of $\textsf{p}$. This (kind of a) polynomial is known as the fundamental semi-invariant $\textsf{p}_\mathfrak{g}$ of $\mathfrak{g}$. To define $\textsf{p}_\mathfrak{g}$ consider the Pfaffians

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[3] It is sufficient to consider finitely many values of λ in this sum. For example, one can arbitrarily choose distinct rational numbers $\lambda_1, \ldots, \lambda_k$ with $k = \dim \mathfrak{g}$.

$p_1, \ldots, p_N$ of all $r \times r$ diagonal minors of the matrix $\mathcal{A}_y = \left(c_{ij}^k y_k \right)$, $r = \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$. Then $\mathsf{p}_{\mathfrak{g}}$ is the greatest common divisor of $p_1, \ldots, p_N$ (all these polynomials are now considered as elements of $P(\mathfrak{g})$, i.e., as polynomials in $y_1 \ldots, y_n$). Thus we have,

$$p_1(y) = \mathsf{p}_{\mathfrak{g}}(y) \cdot h_1(y)$$

$$\cdots$$

$$p_N(y) = \mathsf{p}_{\mathfrak{g}}(y) \cdot h_N(y)$$

where $h_1(y), \ldots, h_N(y)$ do not have any non-constant common factors. This implies, by the way, that the singular set Sing is the union of two subsets

$$\mathsf{Sing}_0 = \{\mathsf{p}_{\mathfrak{g}} = 0\} \quad \text{and} \quad \mathsf{Sing}_1 = \{\mathsf{h}_1(y) = 0, \ldots, h_N(y) = 0\}.$$

Thus, there are three possibilities:

- $\mathsf{p}_{\mathfrak{g}} = 1$ and then $\mathsf{Sing}_0 = \emptyset$, $\mathsf{Sing} = \mathsf{Sing}_1$ and $\operatorname{codim} \mathsf{Sing} \geq 2$,
- $h_i = \operatorname{const} \in \mathbb{K}$ and then $\mathsf{Sing} = \mathsf{Sing}_0$, $\operatorname{codim} \mathsf{Sing} = 1$ and $\mathsf{Sing}_1 = \emptyset$,
- both $\mathsf{p}_{\mathfrak{g}}$ and h_i are non-constant, then both Sing_0 and Sing_1 are non-empty and $\operatorname{codim} \mathsf{Sing}_0 = 1$ and $\operatorname{codim} \mathsf{Sing}_1 \geq 2$.

Replacing y by $x + \lambda a$ we obtain two possibilities: either $\mathsf{p}_{\mathfrak{g}}(x + \lambda a)$ is still a greatest common divisor of $p_1(x + \lambda a), \ldots, p_N(x + \lambda a)$ (now we consider them as polynomials in one single variable λ), or the greatest common divisor $\mathsf{p}_{x,a}(\lambda)$ is "bigger". The latter condition simply means that $h_1(x + \lambda a), \ldots, h_N(x + \lambda a)$ have a non-trivial common factor, or in geometric terms, that the straight line $x + \lambda a$, $\lambda \in \mathbb{K}$, intersects the set Sing_1.

Thus, we come to the following conclusion which is similar to the Joseph–Shafrir formula (Section 7.2 in [8]). Notice that this is a straightforward corollary of the Jordan–Kronecker decomposition theorem.

Theorem 3. *Let $a \in \mathfrak{g}^*$ be regular and $dF_a(x) = \operatorname{span}\{df(x), \ f \in F_a(\mathfrak{g})\} \subset \mathfrak{g}$, $x \in \mathfrak{g}^*$. Then*

$$\dim dF_a(x) = \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{p}_{x,a},$$

where $\mathsf{p}_{x,a}(\lambda)$ is the greatest common divisor of the Pfaffians $p_1(x + \lambda a), \ldots, p_N(x + \lambda a)$ of all $r \times r$ diagonal minors of the matrix $\mathcal{A}_{x + \lambda a} = \left(c_{ij}^k (x_k + \lambda a_k) \right)$, $r = \dim \mathfrak{g} - \operatorname{ind} \mathfrak{g}$. In particular,

$$\dim dF_a(x) \leq \frac{1}{2}(\dim \mathfrak{g} + \operatorname{ind} \mathfrak{g}) - \deg \mathsf{p}_{\mathfrak{g}},$$

with equality if and only if the straight line $x + \lambda a$, $\lambda \in \mathbb{K}$, does not intersect the subset $\mathsf{Sing}_1 \subset \mathsf{Sing}$.

A similar formula holds true for Mishchenko–Fomenko subalgebras $Y_a(\mathfrak{g})$ after some additional amendments. For each $x \in \mathfrak{g}^*$ consider the subspace $dY_a(x) = \mathrm{span}\{df(x), \ f \in Y_a(\mathfrak{g})\} \subset \mathfrak{g}$. As noticed above, for regular $a \in \mathfrak{g}^*$ we have the inclusion $Y_a(\mathfrak{g}) \subset F_a(\mathfrak{g})$ and therefore $dY_a(x) \subset dF_a(x)$ for any $x \in \mathfrak{g}^*$. A sufficient condition for these two subspaces to coincide is very simple (cf. Proposition 1).

Proposition 4. *Let $a \in \mathfrak{g}^*$ be regular. If the straight line $x + \lambda a$ does not belong to* Crit, *then $dY_a(x) = dF_a(x)$.*

Proof. Indeed, if $y = x + \lambda a \notin \mathsf{Crit} \cup \mathsf{Sing}$, then the differentials of the shifted invariants $f_\lambda(x) = f(x + \lambda a) \in Y_a(\mathfrak{g})$, $f \in Y(\mathfrak{g})$, generate $\mathrm{Ker}\,\mathcal{A}_{x + \lambda a}$. Thus, $\mathrm{Ker}\,\mathcal{A}_{x + \lambda a} \subset dY_a(x)$ for infinitely many λ's and in view of Proposition 2, we have the converse inclusion $dF_a(x) \subset dY_a(x)$. $\square$

On the other hand, $Y_a(\mathfrak{g})$ is well defined for any $a \in \mathfrak{g}^*$ both regular and singular, whereas $F_a(\mathfrak{g})$ in general makes no sense for singular $a \in \mathfrak{g}^*$. Nevertheless, the description of the subspace $dY_a(x)$ is easy to obtain if we notice that $dY_a(x) = dY_x(a)$ and more generally this subspace $dY_x(a)$ depends only of the two-dimensional subspace of $\mathfrak{g}^*$ generated by a and x so that $dY_a(x) = dY_{a'}(x')$ if $\mathrm{span}(a', x') = \mathrm{span}(a, x)$. In particular, if the straight line $x + \lambda a$ does not belong to the singular set, we may assume without loss of generality that x is regular. Then we have

Proposition 5. *Let $x \in \mathfrak{g}^*$ be regular and assume the straight line $a + \lambda x$ does not belong to* Crit. *Then $dY_a(x) = dF_x(a)$.*

Hence we immediately obtain the following version of Theorem 3 for the Mishchenko–Fomenko subalgebras $Y_a(\mathfrak{g})$ (simply by interchanging x and a).

Theorem 4. *Let $x \in \mathfrak{g}^*$ be regular and $dY_a(x) = \mathrm{span}\{df(x), \ f \in Y_a(\mathfrak{g})\} \subset \mathfrak{g}$. Assume that $\mathrm{tr.deg.}Y(\mathfrak{g}) = \mathrm{ind}\,\mathfrak{g}$ and the straight line $a + \lambda x$ does not belong to* Crit. *Then*

$$\dim dY_a(x) = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g}) - \deg \mathsf{p}_{a,x},$$

where $\mathsf{p}_{a,x}(\lambda)$ is the greatest common divisor of the Pfaffians $p_1(a + \lambda x), \ldots, p_N(a + \lambda x)$ of all $r \times r$ diagonal minors of the matrix $\mathcal{A}_{a + \lambda x} = \left(c_{ij}^k(a_k + \lambda x_k)\right)$, $r = \dim \mathfrak{g} - \mathrm{ind}\,\mathfrak{g}$.
Furthermore,

$$\dim dY_a(x) \leq \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g}) - \deg \mathsf{p}_\mathfrak{g},$$

with equality if and only if the straight line $a + \lambda x$, $\lambda \in \mathbb{K}$, does not intersect the subset $\mathsf{Sing}_1 \subset \mathsf{Sing}$.

For a fixed $a \in \mathfrak{g}^*$ such a line exists if and only if $a \notin \mathsf{Sing}_1$. In particular,

$$\mathrm{tr.deg.} Y_a(\mathfrak{g}) \leq \frac{1}{2}(\dim \mathfrak{g} + \mathrm{ind}\,\mathfrak{g}) - \deg \mathsf{p}_\mathfrak{g},$$

with equality if and only if $a \notin \mathsf{Sing}_1$.

Remark 3. The latter statement of this theorem is the Joseph–Shafrir formula (Section 7.2 in [8]). In particular, Sing_1 must coincide with the set $\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{wreg}}$ introduced in [8] in a different way.

Acknowledgments

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